

Characterization of model sets by dynamical systems

MICHAEL BAAKE[†], DANIEL LENZ[‡] and ROBERT V. MOODY[§]

[†] *Fakultät für Mathematik, Universität Bielefeld, Postfach 100131,
33501 Bielefeld, Germany
(e-mail: mbaake@math.uni-bielefeld.de)*

[‡] *Fakultät für Mathematik, TU Chemnitz, 09107 Chemnitz, Germany
(e-mail: dlenz@mathematik.tu-chemnitz.de)*

[§] *Department of Mathematics and Statistics, University of Victoria, Victoria,
British Columbia V8W 3P4, Canada
(e-mail: rmoody@uvic.ca)*

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Abstract. It is shown how regular model sets can be characterized in terms of the regularity properties of their associated dynamical systems. The proof proceeds in two steps. First, we characterize regular model sets in terms of a certain map β and then relate the properties of β to those of the underlying dynamical system. As a by-product, we can show that regular model sets are, in a suitable sense, as close to periodic sets as possible among repetitive aperiodic sets.

1. Introduction

Delone sets provide an important model class for the description of aperiodic order. In particular, they can be viewed as a mathematical abstraction of the set of atomic positions of a physical quasicrystal (at zero temperature, or at a given instant of time). Many of the rather intriguing spectral properties of quasicrystals can be formulated, in a simplified manner, on the basis of Delone sets. The latter contain the important class of model sets (see below for definitions), which is our main topic in this paper.

Since the discovery of quasicrystals [34], model sets have been a particular focus of attention because they are, except under extreme conditions, pure point diffractive [14, 24, 36]. This property remains true also under certain equivariant perturbations, which turn them into deformed model sets [14, 8, 3], and extend the applicability of these sets considerably [37].

Model sets are discrete point sets that arise by the (partial) projection of a lattice from some ‘higher-dimensional’ or ‘super’ space. To avoid misunderstandings, and to accommodate situations where the concept of dimension is not available,

01 we call this super space the *embedding space* below. Model sets have been found useful in
 02 numerous studies both by experimentalists modelling quasicrystals and by mathematicians
 03 studying aperiodic order and diffraction. One principal difficulty has been to find good
 04 characterizations of them. In particular, what are the intrinsic properties of a point set that
 05 permits its description as a projection from (parts of) some higher-dimensional lattice?

06 Another major ingredient in the study of aperiodic point sets (and tilings) has been the
 07 use of dynamical systems. Given a (suitably discrete) point set $\Lambda \subset \mathbb{R}^d$, for example,
 08 one associates with it a space which is the closure of its \mathbb{R}^d -translation orbit, the closure
 09 taken in a topology that compares point sets for a more or less exact match in local regions
 10 around the origin. This is called the dynamical hull, or *local hull* in this paper (since we
 11 shall meet other hulls that are dynamical systems as well). The major objective of this
 12 paper is to characterize model sets in terms of the properties of their local hulls.

13 As the theory of model sets and related mathematics has developed, it has become clear
 14 that the properties of the ambient space that are required are sufficiently weak that the
 15 group \mathbb{R}^d may be replaced by any σ -compact locally compact Abelian (LCA) group G ,
 16 without increasing the complexity of the proofs. In fact, this additional generality is to
 17 some extent necessary to understand model sets, as we shall see. In this paper, we take this
 18 more general setting.

19 The main theorem of the paper is the following.

20 THEOREM 1. *Let G be a σ -compact LCA group and (\mathbb{X}, G) a point set dynamical system
 21 on G . Then, for (\mathbb{X}, G) to be the dynamical system associated to a repetitive regular model
 22 set it is necessary and sufficient for the following four conditions to be satisfied.*

- 23 (1) *All elements of \mathbb{X} are Meyer sets.*
- 24 (2) *(\mathbb{X}, G) is strictly ergodic, i.e. uniquely ergodic and minimal.*
- 25 (3) *(\mathbb{X}, G) has pure point dynamical spectrum with continuous eigenfunctions.*
- 26 (4) *The eigenfunctions of (\mathbb{X}, G) separate almost all points of \mathbb{X} (i.e. the set
 27 $\{\Gamma \in \mathbb{X} : \text{there exists } \Gamma' \neq \Gamma \text{ with } f(\Gamma) = f(\Gamma') \text{ for all eigenfunctions } f\}$
 28 has measure 0).*

29
 30 The necessity of the conditions is already known [35, 36], so our main task is to deal
 31 with the converse, the four listed properties characterize repetitive regular model sets,
 32 although we end up proving the necessity again in the process.

33 The proof is broken into three main parts. The first part is to use the properties (3) and
 34 (4) to identify elements of \mathbb{X} that cannot be separated by the continuous eigenfunctions.
 35 This results in a new dynamical system (\mathbb{E}, G) , where \mathbb{E} is actually a compact Abelian
 36 group, and a surjective G -mapping of \mathbb{X} onto \mathbb{E} . Although this new group need not
 37 be a torus, it is nonetheless useful to simply call such a map a torus map or *torus*
 38 *parametrization*, in analogy to [1].

39 The second part is to show that \mathbb{E} can be identified with another dynamical hull \mathbb{A}
 40 of Λ , this time determined not by the local topology, but rather by a topology called the
 41 *autocorrelation topology*. This topology compares point sets globally for statistical match.

42 The third step, which actually appears first in the paper, is to show that a torus mapping
 43 of \mathbb{X} onto \mathbb{A} assures that we are in the situation of model sets: we can explicitly construct
 44 the embedding space, the lattice and the mechanism which controls the projection down

01 into the ambient space. This is really the heart of the matter. Given a Meyer set Λ , we have
02 its two hulls $\mathbb{X}(\Lambda)$ and $\mathbb{A}(\Lambda)$. These are quite natural objects. The mapping β between
03 them, when it exists, is the most natural mapping possible. It is really nothing but looking
04 at the same elements of $\mathbb{X}(\Lambda)$, but in another topology. The assumption of the existence
05 of the map is the same as saying that this change of topology is continuous, which in turn
06 is the same as requiring that the local and global topologies are consistent with each other.
07 It is this consistency that effectively characterizes the cut and project formalism.

08 The existence of windows for realizing the elements of \mathbb{X} as model sets (or inter
09 model sets) emerges as we require more out of the mapping β : first that it is one-to-one
10 somewhere, and finally that it is one-to-one almost everywhere. If we go so far as to assume
11 that it is one-to-one everywhere, we collapse into the crystallographic case (Theorem 10).
12 Thus, condition (4) of Theorem 1 seems to contain the essence of aperiodicity (at least in
13 the context of Meyer sets). This gives another instance for the intuition that regular model
14 sets are a very natural generalization of crystallographic (i.e. fully periodic) point sets, and
15 that aperiodic model sets are, in this sense, as close to periodic sets as possible among
16 (repetitive) aperiodic Meyer sets.

17 Section 2 introduces the basic definitions and concepts used throughout the paper.
18 In particular, in §§2.1, 2.2 and 2.3 we establish the basic notions about the point sets and
19 dynamical hulls that appear in the paper. Section 2.4 deals with cut and project schemes
20 (CPSs) and model sets. Section 2.5 introduces the notion of a torus parametrization.
21 While the material of that section is essentially known, the point of view taken there is
22 of fundamental importance for our considerations.

23 Beyond the main theorem, there are a number of intermediate results that are interesting
24 in their own right and are part of the overall proof. Section 3 serves the purpose of detailing
25 these results and indicating the logical flow of the paper. The paper proper then begins with
26 the consequences of a torus parametrization $\beta: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$, gradually refining what
27 can be learned from it as further conditions are added.

28 Model sets, as one sees them in the literature, come with varying definitions and side
29 conditions, depending on the requirements of the moment. However, our results require
30 quite precise notions of what constitutes a CPS, which windows are permitted and how
31 they relate to the CPS. Much, but not all, of this appears in the work of Schlottmann cited
32 above. To make things clear, particularly the important ideas of irredundancy, which is not
33 standard, and inter model sets, which are new [19], we have reworked this material and
34 included it in the paper. Our attitude is that the main purpose of the paper is to prove the
35 sufficiency direction of the Theorem 1, whence we have written the paper so that it moves
36 in that direction from the very beginning. By the time that we have proved sufficiency, we
37 actually know enough to prove necessity rather easily.

38 The paper has been delayed in reaching its final form by various circumstances
39 around the lives of its authors. Nonetheless, its results have been announced in several
40 places [26, 27]. Meanwhile, based on this paper, an extension of part of this theory to multi-
41 colour sets has been worked out [19], and this has been effectively used in establishing the
42 equivalence of pure pointedness and model sets for substitution tilings and point sets [18],
43 a result that, for the case of unimodular Pisot substitutions in one dimension, has recently
44 also been discussed in a slightly different context [16].

01 2. Basic definitions and hulls

02 This paper is a study of the relationship between various concepts in the regime of aperiodic
03 order, formulated in terms of point sets in LCA groups. Let us first introduce the concepts.

05 2.1. *Aperiodic order and diffraction theory: the general setting.* Let G be an
06 LCA group, with Haar measure θ_G (normalized as $\theta_G(G) = 1$ if G is compact). We assume
07 that G is σ -compact (also called countable at infinity). This is equivalent to the existence
08 of an *averaging sequence* $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ of open, relatively compact sets $A_n \subset G$
09 with $\overline{A_n} \subset A_{n+1}$ and $G = \bigcup_{n \geq 1} A_n$. In fact, the averaging sequence can be chosen to be
10 a *van Hove sequence*, see [36] for details. This means that, for every compact set $K \subset G$,

$$12 \lim_{n \rightarrow \infty} \frac{\theta_G(((K + A_n) \setminus A_n^\circ) \cup ((-K + \overline{G \setminus A_n}) \cap A_n))}{\theta_G(A_n)} = 0,$$

14 where the bar (circle) denotes the closure (interior) of a set. In effect, this rather technical
15 looking condition states that for each compact subset K of G , the K -boundary of the
16 averaging sequence becomes negligible (in the sense of measure) to the sequence itself as
17 $n \rightarrow \infty$. Note that general van Hove sequences need not be nested.

19 A subset Λ of G is called *U -uniformly discrete* if, for the open neighbourhood U of
20 0 in G and for all $x \in \Lambda$, $(x + U) \cap \Lambda = \{x\}$. We say that Λ is *uniformly discrete* if a
21 neighbourhood U exists for which Λ is U -uniformly discrete. By the σ -compactness of G ,
22 every uniformly discrete set in G is countable. The set of all uniformly discrete subsets
23 of G is denoted by $\mathcal{D} = \mathcal{D}(G)$ and the set of U -uniformly discrete subsets by \mathcal{D}_U .

24 Uniformly discrete sets can have various further regularity properties. A uniformly
25 discrete subset Λ of G is called *Delone* if it is also relatively dense, i.e. if there exists a
26 compact set K in G with $G = \Lambda + K$.

27 Now, let Λ be an arbitrary uniformly discrete set. Then, Λ is of *finite local complexity*
28 (FLC) if the set of K -clusters,

$$29 \{(-x + \Lambda) \cap K : x \in \Lambda\},$$

31 is finite for every compact $K \subset G$. This is equivalent to $\Lambda - \Lambda$ being discrete and
32 closed [36]. If $\Lambda \subset G$ is a Delone set and there exists a finite set $F \subset G$ with
33 $\Lambda - \Lambda \subset \Lambda + F$, then Λ is called a *Meyer set*. Evidently, $\Lambda - \Lambda$ is uniformly discrete
34 whenever Λ is Meyer.

35 A point set $\Lambda \subset G$ of finite local complexity is called *repetitive* if for every compact K
36 in G the set of repetitions of $\Lambda \cap K$

$$37 \{t \in G : (-t + \Lambda) \cap K = \Lambda \cap K\}$$

39 is relatively dense in G . Λ is said to have *uniform patch frequencies* (some people say
40 *uniform cluster frequencies*) if, for each finite subset P of G and for all $a \in G$,

$$42 \frac{\text{card}\{t \in G : t + P \subset \Lambda \cap (a + B_n)\}}{\theta_G(B_n)}$$

44 converges uniformly in a along every van Hove sequence $\{B_n : n \in \mathbb{N}\}$, see [36] for details.

01 The diffraction pattern of a solid modelled by Λ can be described as follows [10, 14].
 02 For $x \in G$, let δ_x denote the normalized point (or Dirac) measure at $x \in G$. If the limit
 03 (taken in the vague topology)

$$04 \quad \gamma := \lim_{n \rightarrow \infty} \frac{1}{\theta_G(A_n)} \sum_{x, y \in \Lambda \cap A_n} \delta_{x-y}$$

05
 06 exists, it is called the *autocorrelation measure* of Λ relative to the averaging sequence \mathcal{A} .
 07 If Λ has uniform patch frequencies, the limit exists and does not depend on the choice
 08 of \mathcal{A} (as long as it is van Hove). The autocorrelation measure is positive definite and hence
 09 transformable, i.e. we can take its Fourier transform $\widehat{\gamma}$. This is a positive measure on the
 10 dual group \widehat{G} , called the *diffraction measure*. For $G = \mathbb{R}^n$, it describes the outcome of a
 11 diffraction experiment, compare [10] for details.
 12

13
 14 **2.2. The local hull.** In this and the next section, we introduce two topologies on the
 15 set \mathcal{D} of all uniformly discrete subsets of G . The interplay of these two topologies is a
 16 main feature of the paper.

17 The so-called *local topology* (LT) on \mathcal{D} is defined via the uniform structure given by the
 18 entourages

$$19 \quad U_{\text{LT}}(K, V) := \{(\Gamma, \Gamma') \in \mathcal{D} \times \mathcal{D} : (v + \Gamma) \cap K = \Gamma' \cap K \text{ for some } v \in V\}$$

20
 21 for $K \subset G$ compact and V a neighbourhood of 0 in G . Thus, two uniformly discrete
 22 sets are close if they agree on a ‘large’ compact set up to a ‘small’ (global) translation.
 23 For definitions, terminology and basic theorems on uniformities, see [9, 31].

24 As is immediate from the definition of the local topology, the canonical action of G on \mathcal{D}
 25 given by

$$26 \quad G \times \mathcal{D} \longrightarrow \mathcal{D}, \quad (t, \Lambda) \mapsto -t + \Lambda,$$

27 is continuous. In particular, if $\mathbb{X} \subset \mathcal{D}$ is compact in the local topology and invariant under
 28 this action, then (\mathbb{X}, G) is a topological dynamical system. Such a dynamical system will
 29 be called a *point set dynamical system*.

30 The hull of an element $\Lambda \in \mathcal{D}$ in the local topology (i.e. the closure of the orbit
 31 $G + \Lambda = \{x + \Lambda : x \in G\}$) is denoted by $\mathbb{X}(\Lambda)$.

32
 33 **FACT 1. [36]** *If Λ is a Delone set, the hull $\mathbb{X}(\Lambda)$ is LT-compact if and only if Λ is of finite
 34 local complexity, i.e. if and only if $\Lambda - \Lambda$ is discrete and closed.*

35 In this case, $\mathbb{X}(\Lambda)$ gives rise to a point set dynamical system $(\mathbb{X}(\Lambda), G)$. This dynamical
 36 system is a basic object in the study of the long-range order of discrete point sets because of
 37 its ability to reflect important geometric properties in the language of dynamical systems.

38
 39 **FACT 2. [36]** *Let Λ be a Delone set of finite local complexity. Then, the dynamical system
 40 $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic (i.e. there exists precisely one G -invariant probability
 41 measure on $\mathbb{X}(\Lambda)$) if and only if Λ has uniform patch frequencies.*

42 Two Delone sets Λ, Λ' are *locally indistinguishable* (LI) if each cluster of Λ (i.e. each
 43 set of the form $\Lambda \cap K$ with $K \subset G$ compact) is a translate $-x + (\Lambda' \cap (x + K))$ of a cluster
 44 of Λ' and *vice versa*. This equivalence relation defines the so-called LI classes.

01 FACT 3. Let G be an LCA group and $\Lambda \subset G$ a Delone set of finite local complexity.
 02 Then, the following properties are equivalent.

- 03 (1) The set Λ is repetitive.
 04 (2) The hull $\mathbb{X}(\Lambda)$ is the LI class of Λ .
 05 (3) The dynamical system $(\mathbb{X}(\Lambda), G)$ is minimal.

06 *Proof.* This is a variant of Gottschalk's theorem, see [36] for details. \square

07
 08 The definition of closeness in the LT has a special consequence for translates of
 09 Meyer sets.

10
 11 FACT 4. Let Λ be a Meyer set. Then, for all suitably small neighbourhoods V of 0 in G
 12 and all $C \subset G$ compact with $C \cap G \neq \emptyset$, the equality $\Lambda \cap C = (-x + \Lambda) \cap C$ holds
 13 whenever $(-x + \Lambda, \Lambda) \in U_{\text{LT}}(C, V)$ for $x \in \Lambda - \Lambda$.

14 *Proof.* As Λ is Meyer, one has $\Lambda - \Lambda \subset \Lambda + F$ with F a finite set. Clearly, also
 15 $(\Lambda - \Lambda) + (\Lambda - \Lambda) \subset \Lambda + F'$, with F' still finite, so that uniform discreteness persists
 16 to $\Lambda - \Lambda + (\Lambda - \Lambda)$. Thus, there exists an open neighbourhood V of 0 in G so small that
 17 $V \cap ((\Lambda - \Lambda) + (\Lambda - \Lambda)) = \{0\}$. Now, $(-x + \Lambda, \Lambda) \in U_{\text{LT}}(C, V)$ for $x \in G$ implies

$$18 \quad (v - x + \Lambda) \cap C = \Lambda \cap C$$

19
 20 for some $v \in V$. Now, if $x \in \Lambda - \Lambda$, $\Lambda \cap C \neq \emptyset$ yields $v \in V \cap ((\Lambda - \Lambda) + (\Lambda - \Lambda)) = \{0\}$
 21 and the fact is proved. \square

22
 23 Let (\mathbb{X}, G) be a point set dynamical system which is uniquely ergodic. In this case, there
 24 is a canonical Hilbert space associated to (\mathbb{X}, G) , the space $L^2(\mathbb{X}, \mu)$ of square integrable
 25 functions on \mathbb{X} (with respect to the unique G -invariant probability measure μ). The action
 26 of G on \mathbb{X} gives rise to a unitary representation T of G on this space via

$$27 \quad T_t : L^2(\mathbb{X}, \mu) \longrightarrow L^2(\mathbb{X}, \mu), \quad (T_t f)(\Lambda) := f(-t + \Lambda),$$

28
 29 for $f \in L^2(\mathbb{X}, \mu)$ and $t \in G$. An $f \in L^2(\mathbb{X}, \mu)$ is called an *eigenfunction* of T with
 30 *eigenvalue* $\hat{s} \in \widehat{G}$ (the dual group) if $T_t f = (\hat{s}, t)f$ for every $t \in G$, where (\hat{s}, \cdot) denotes
 31 the character defined by \hat{s} . An eigenfunction (to \hat{s} , say) is called *continuous* if it has a
 32 continuous representative f with $f(-t + \Lambda) = (\hat{s}, t)f(\Lambda)$, for all $\Lambda \in \mathbb{X}$ and $t \in G$.
 33 The representation T is said to have *pure point spectrum* if the set of eigenfunctions is
 34 total in $L^2(\mathbb{X}, \mu)$. One then also says that the dynamical system (\mathbb{X}, G) has *pure point*
 35 *dynamical spectrum*.

36
 37
 38 2.3. *The autocorrelation hull.* The *upper density* of a point set $\Lambda \subset G$ is defined by

$$39 \quad \overline{\text{dens}}(\Lambda) := \limsup_{n \rightarrow \infty} \frac{\text{card}(\Lambda \cap A_n)}{\theta_G(A_n)}$$

40
 41
 42 with respect to the averaging van Hove sequence \mathcal{A} chosen before. The lower density,
 43 $\underline{\text{dens}}(\Lambda)$, is defined analogously. If $\overline{\text{dens}}(\Lambda) = \underline{\text{dens}}(\Lambda)$, this is called the *density* of Λ ,
 44 denoted by $\text{dens}(\Lambda)$. We usually suppress the explicit reference to \mathcal{A} .

01 The *mixed autocorrelation topology* (mACT) on \mathcal{D} is defined via the uniform structure
02 given by the entourages

$$03 \quad U_{\text{mACT}}(V, \varepsilon) := \{(\Gamma, \Gamma') \in \mathcal{D} \times \mathcal{D} : d(v + \Gamma, \Gamma') \leq \varepsilon \text{ for some } v \in V\},$$

05 for every neighbourhood V of 0 in G and every $\varepsilon > 0$, where the pseudo-metric d on \mathcal{D} is
06 defined by the upper density of the symmetric difference of sets:

$$07 \quad d(\Gamma, \Gamma') := \overline{\text{dens}}(\Gamma \Delta \Gamma'). \quad (1)$$

09 Note that the triangle inequality follows from the fact that $\Gamma \Delta \Gamma' \subset (\Gamma \Delta \Gamma'') \cup$
10 $(\Gamma' \Delta \Gamma'')$, for arbitrary point sets $\Gamma'' \subset \mathcal{D}$. With this definition, d is G -invariant
11 (i.e. $d(t + \Gamma, t + \Gamma') = d(\Gamma, \Gamma')$ for all $t \in G$ and all $\Gamma, \Gamma' \in \mathcal{D}$), because \mathcal{A} has the
12 van Hove property. We call mACT the *mixed autocorrelation topology* because it mixes the
13 ordinary topology of G with the topology introduced by the pseudo-metric d . The topology
14 induced by d itself, in turn, ultimately arises from the autocorrelation (see below) and we
15 thus call it the *autocorrelation topology*.

16 Note that d contains information on the statistical coincidence of the global structure.
17 Thus, two sets are close in the mACT if their global structures are statistically close up to
18 a small translation.

19 It is obvious that, for a general LCA group G , d does not define a metric on \mathcal{D} . However,
20 it does permit the construction of a completion where d becomes a metric. To see this,
21 fix an open neighbourhood U of 0 in G and consider the restriction of the pseudo-metric d
22 (still denoted by d) to \mathcal{D}_U . Introduce an equivalence relation \equiv on \mathcal{D}_U by setting $\Gamma \equiv \Gamma'$
23 if and only if $d(\Gamma, \Gamma') = 0$, with d as defined in (1). The quotient of \mathcal{D}_U by this
24 equivalence relation is denoted by \mathcal{D}_U^{\equiv} . By construction, the pseudo-metric d on \mathcal{D} induces
25 a metric on \mathcal{D}_U^{\equiv} , which we again call d . Then, d is a G -invariant metric on \mathcal{D}_U^{\equiv} , and \mathcal{D}_U^{\equiv}
26 is complete as a metric space, although neither of these two facts is obvious, compare
27 [28, Corollary 3.10].

28 We give \mathcal{D}_U^{\equiv} the uniform topology induced by U_{mACT} and again call it the mACT. Again,
29 this is *not* the same as the metric topology induced by d itself, since it takes small shifts in
30 the sets into account in order to make the action of G continuous.

31 PROPOSITION 1. \mathcal{D}_U^{\equiv} is complete in the mACT.

32 *Proof.* For $\Lambda, \Lambda' \in \mathcal{D}_U$, if $\varepsilon > 0$ and $d(\Lambda, \Lambda') < \varepsilon$, one finds

$$33 \quad \begin{aligned} 34 \quad \varepsilon &> d(\Lambda, \Lambda') = \overline{\text{dens}}(\Lambda \Delta \Lambda') \\ 35 \quad &= \overline{\text{dens}}((\Lambda \setminus (\Lambda \cap \Lambda')) \cup (\Lambda' \setminus (\Lambda \cap \Lambda'))) \\ 36 \quad &\geq \overline{\text{dens}}(\Lambda \setminus (\Lambda \cap \Lambda')) \geq \overline{\text{dens}}(\Lambda) - \underline{\text{dens}}(\Lambda \cap \Lambda'). \end{aligned}$$

37 By symmetry, one also has $\varepsilon > \overline{\text{dens}}(\Lambda') - \underline{\text{dens}}(\Lambda \cap \Lambda')$ and, hence, one obtains
38 $|\overline{\text{dens}}(\Lambda) - \overline{\text{dens}}(\Lambda')| < 2\varepsilon$.

39 Now, let $\{\Lambda_i\} \subset \mathcal{D}_U$ be a Cauchy net with respect to the mACT. We want to prove that
40 the net converges when seen in \mathcal{D}_U^{\equiv} .

41 For any $\varepsilon > 0$ and any open neighbourhood V of 0, there is some n so that for all
42 $i, j \succcurlyeq n$ (with \succcurlyeq referring to the partial order on the index set) and for suitable $v_{ij} \in V$,

01 one has $d(v_{ij} + \Lambda_i, \Lambda_j) < \varepsilon$. Then, by the above calculation, $|\overline{\text{dens}}(v_{ij} + \Lambda_i) - \overline{\text{dens}}(\Lambda_j)|$
 02 $< 2\varepsilon$. Since $\overline{\text{dens}}(v_{ij} + \Lambda_i) = \overline{\text{dens}}(\Lambda_i)$, we see that $\{\overline{\text{dens}}(\Lambda_i)\}$ is a Cauchy net in \mathbb{R} and
 03 so converges to some limit $c \geq 0$. If $c = 0$, then $\{\Lambda_i\} \rightarrow \emptyset \in \mathcal{D}_U^{\equiv}$, and we are done. So we
 04 only need to consider the case that $c > 0$.

05 Returning to the Cauchy net $\{\Lambda_i\} \subset \mathcal{D}_U$, choose any open neighbourhood V of 0 so
 06 that $-V + V + V \subset U$, and any ε that satisfies $0 < \varepsilon < c/3$. Fix n so that $i, j \succ n$ implies
 07 that $(\Lambda_i, \Lambda_j) \in U_{\text{mACT}}(V, \varepsilon)$ and $\overline{\text{dens}}(\Lambda_i) > c/2$.

08 We know that, for all $j, k \succ n$,

$$09 \quad d(v_{jk} + \Lambda_j, \Lambda_k) < \varepsilon, \quad \text{for some } v_{jk} \in V.$$

11 Then, for all $j, k \succ n$, $d(v_{jk} + v_{nj} + \Lambda_n, v_{nk} + \Lambda_n) < 3\varepsilon$ or, using translation invariance,
 12 $d(-v_{nk} + v_{jk} + v_{nj} + \Lambda_n, \Lambda_n) < 3\varepsilon$. However, for $x \in \Lambda_n$,

$$14 \quad \{-v_{nk} + v_{jk} + v_{nj} + x\} \cap \Lambda_n = \begin{cases} \{x\}, & \text{if } -v_{nk} + v_{jk} + v_{nj} = 0, \\ \emptyset, & \text{otherwise,} \end{cases}$$

17 because $-V + V + V \subset U$ and all of the sets Λ_ℓ lie in \mathcal{D}_U . So, if $-v_{nk} + v_{jk} + v_{nj} \neq 0$,
 18 then $d(-v_{nk} + v_{jk} + v_{nj} + \Lambda_n, \Lambda_n) = 2\overline{\text{dens}}(\Lambda_n) \geq 2(c/2) > 3\varepsilon$, a contradiction. Thus,
 19 $v_{jk} + v_{nj} = v_{nk}$ and $v_{jk} = -v_{kj}$ for all j, k .

20 In principle, v_{jk} depends on V and ε . However, the same little argument shows that it is
 21 actually unique in the sense that it will be the same element for any $V' \subset V$, $\varepsilon \leq \varepsilon'$.

22 Let $\Lambda'_j := -v_{nj} + \Lambda_j \in \mathcal{D}_U$, for all $j \succ n$. Then, for all $j, k \succ n$,

$$24 \quad d(\Lambda'_j, \Lambda'_k) = d(-v_{nj} + \Lambda_j, -v_{nk} + \Lambda_k) = d(v_{jk} + \Lambda_j, \Lambda_k) < \varepsilon.$$

25 This shows that $\{\Lambda'_n\}$ is a Cauchy net in \mathcal{D}_U , with respect to the metric topology defined
 26 by d . By [28, Corollary 3.9], it converges to some $\Lambda \in \mathcal{D}_U^{\equiv}$. It is easy to see that also $\{\Lambda_n\}$
 27 converges to Λ , which completes the argument. \square

29 Denote the equivalence class of $\Lambda \in \mathcal{D}_U$ by $[\Lambda]$, and let β be the canonical mapping
 30 from \mathcal{D}_U to \mathcal{D}_U^{\equiv} , i.e.

$$31 \quad \beta : \mathcal{D}_U \longrightarrow \mathcal{D}_U^{\equiv}, \quad \Lambda \mapsto [\Lambda].$$

32 Each Λ in \mathcal{D}_U gives rise to the *autocorrelation hull* $\mathbb{A}(\Lambda)$ defined as the closure of the
 33 orbit $G + \beta(\Lambda)$ in the mACT. By construction, one may as well consider $\mathbb{A}(\Lambda)$ to be
 34 the Hausdorff completion of G with respect to the uniform topology on G that is given
 35 by pulling back the autocorrelation topology from \mathcal{D}_U . In detail, define a pseudo-metric
 36 (relative to Λ) on G by

$$38 \quad d_G(s, t) := d(t + \Lambda, s + \Lambda) = d(t - s + \Lambda, \Lambda). \quad (2)$$

39 Then, the uniformity on G is described by the sets

$$41 \quad \{(t, s) \in G \times G : d(v + t + \Lambda, s + \Lambda) < \varepsilon\} \quad (3)$$

43 where $v \in V$, and V and ε run over all neighbourhoods of 0 and all non-negative real
 44 numbers, respectively.

This can be written in a more suggestive way via the set of ε -almost periods of Λ ,

$$P_\varepsilon := \{t \in \Lambda - \Lambda : d_G(t, 0) < \varepsilon\}. \quad (4)$$

Then, the entourages (3) are just the sets

$$\{(t, s) \in G \times G : t - s \in V + P_\varepsilon\}. \quad (5)$$

These entourages are evidently G -invariant. This has an important consequence: $\mathbb{A}(\Lambda)$, now being the completion of the Abelian group G with respect to the invariant uniformity as defined by (5), carries a natural Abelian group structure. Moreover, G acts minimally on $\mathbb{A}(\Lambda)$ through the translation action. This is the second topological dynamical system for our group G . Of course, this construction depends entirely (and crucially) on the starting set Λ . In the following, we often shift back and forth between the two views of $\mathbb{A}(\Lambda)$: as a subset of $\mathcal{D}_U^{\overline{\overline{}}}$ and as a completion of G .

If we start with a set $\Lambda \in \mathcal{D}_U$, we can form the two hulls $\beta(\mathbb{X}(\Lambda))$ and $\mathbb{A}(\Lambda)$. In general, these are *not* related in any obvious way. In particular, neither is contained in the other. If, however, β is continuous, then obviously $\beta(\mathbb{X}(\Lambda)) \subset \mathbb{A}(\Lambda)$. We refer to this mapping $\beta : \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$ as the *canonical torus map*. Moreover, if β is continuous and $\mathbb{X}(\Lambda)$ is compact, then $\beta(\mathbb{X}(\Lambda)) = \mathbb{A}(\Lambda)$, as $\beta(\mathbb{X}(\Lambda))$ is then a compact, and hence closed, set containing $G + \beta(\Lambda)$. In this case, $\mathbb{A}(\Lambda)$ becomes a compact topological group. We shall have more to say about this situation.

2.4. Cut and project schemes and model sets. Here, we introduce model sets and discuss some of their basic features. For further details and proofs, we refer to [24, 35, 36].

Model sets arise as (partial) projections from a high-dimensional periodic structure to a lower-dimensional subspace. This is formalized in the following notion.

A *cut and project scheme* (CPS) is a triple (G, H, \mathcal{L}) consisting of LCA groups G and H , with G also being σ -compact, and a lattice \mathcal{L} in $G \times H$ such that the two natural projections $\pi_1 : G \times H \rightarrow G, (t, h) \mapsto t$ and $\pi_2 : G \times H \rightarrow H, (t, h) \mapsto h$ of the scheme

$$\begin{array}{ccc} G & \xleftarrow{\pi_1} & G \times H & \xrightarrow{\pi_2} & H \\ & & \cup & & \\ & & \mathcal{L} & & \end{array} \quad (6)$$

satisfy the following properties.

- The restriction $\pi_1|_{\mathcal{L}}$ of π_1 to \mathcal{L} is injective.
- The image $\pi_2(\mathcal{L})$ is dense in H .

Let $L := \pi_1(\mathcal{L})$ and $(\cdot)^* : L \rightarrow H$ be the mapping $\pi_2 \circ (\pi_1|_{\mathcal{L}})^{-1}$. Note that \cdot^* is indeed well defined on L and that it can often be extended to a larger subgroup of G (such as the rational span $\mathbb{Q}L$ in the Euclidean case), but not to all of G .

Moreover, as \mathcal{L} is a discrete and co-compact subgroup of $G \times H$, the quotient

$$\mathbb{T} := (G \times H)/\mathcal{L}$$

is a compact Abelian group. In the standard cut and project setting with Euclidean spaces only, this group is a torus, compare [1]. There is an obvious action of G on \mathbb{T} given by

$$x + ((t, h) + \mathcal{L}) := (x + t, h) + \mathcal{L}, \quad x \in G.$$

01 Then, (\mathbb{T}, G) is minimal and hence uniquely ergodic as well (as \mathbb{T} is a compact Abelian
02 group).

03 Given a CPS (6) and a subset $S \subset H$, we define $\lambda(S)$ by

$$04 \quad \lambda(S) := \{x \in L : x^* \in S\}.$$

05
06 Then, $\lambda(S)$ is relatively dense if the interior of S is non-empty and it is uniformly discrete
07 if the closure of S is compact, see [24] for details.

08 A *model set*, associated with the CPS (6), is a non-empty subset Λ of G of the form

$$09 \quad \Lambda = x + \lambda(y + W),$$

10
11 where $x \in G$, $y \in H$ and $W \subset H$ is a non-empty compact set with $W = \overline{W^\circ}$. A model set
12 $\Lambda = x + \lambda(y + W)$ is called *regular* if $\theta_H(\partial W) = 0$. A (regular) model set of the above
13 form is called *generic* if $(y + \partial W) \cap L^* = \emptyset$. Any model set is a Delone set. Namely,
14 it is uniformly discrete as W is compact and relatively dense as W has non-empty interior.
15 In fact, they are even Meyer sets, because $\Lambda - \Lambda \subset \lambda(W - W)$ and $W - W$ is also
16 compact, and they are thus also FLC sets. Moreover, a regular model set has uniform patch
17 frequencies (i.e. the associated dynamical system is uniquely ergodic) and a generic model
18 set is repetitive.

19 Our prime concern are model sets and their dynamical systems. It turns out that the
20 dynamical system associated with the model set $\lambda(W)$ may contain sets Λ' which are
21 not model sets themselves with respect to the given CPS. It is hard to determine their
22 precise structure in terms of the window. However, under a condition called *irredundancy*
23 (see below for more), all of these sets Λ' satisfy

$$24 \quad t + \lambda(c + W^\circ) \subset \Lambda' \subset t + \lambda(c + W) \tag{7}$$

25
26 with suitable $t \in G$ and $c \in H$. This suggests to work right from the start with sets of
27 the form $t + \lambda(W^\circ) \subset \Lambda \subset t + \lambda(W)$. This approach is also taken in [19] in order to
28 characterize multi-component model sets. We call such sets *inter model sets* (IMSs).

29 The condition we need reads as follows (see [19] and §§5 and 9).

30 *Definition 1.* Let (G, H, \mathcal{L}) be a CPS. A subset S of H is called *irredundant* (with respect
31 to the given CPS), if its stabilizer in H is trivial, i.e. if the equation $c + S = S$ holds only
32 for $c = 0 \in H$.

33 To state our results, we also need the following definition.

34
35 *Definition 2.* A dynamical system (\mathbb{X}, G) is said to be associated with a (regular) model
36 set if there exists a (regular) model set Λ such that $\mathbb{X} = \mathbb{X}(\Lambda)$.

37
38 **2.5. The torus parametrization: abstract results.** In this section, we look briefly at
39 factors of dynamical systems (\mathbb{X}, G) in which the factors are of the form of compact
40 Abelian groups with minimal G -actions. These results are essentially known. Throughout,
41 G will be an LCA group (although most of this works for other groups as well).
42 The situations that we have in mind are special actions of G by translations on point set
43 dynamical systems. These actions generalize concepts from [1] and [36] known as torus
44 parametrizations, and we retain this terminology here.

01 *Definition 3.* Let \mathbb{X} be a compact space and (\mathbb{X}, G) a topological dynamical system under
 02 the action of G . A continuous G -map $\rho: \mathbb{X} \rightarrow \mathbb{K}$ into a compact Abelian group \mathbb{K} on
 03 which G acts minimally is called *torus parametrization*.

04 *Definition 4.* Let $\rho: \mathbb{X} \rightarrow \mathbb{K}$ be a torus parametrization. For $\xi \in \mathbb{K}$, we call the inverse
 05 image $\rho^{-1}(\{\xi\})$ the *fibre* over ξ . Then, $\Gamma \in \mathbb{X}$ is called *singular* if the fibre over $\rho(\Gamma)$
 06 consists of more than one element. Otherwise, Γ is called *non-singular*. In this case,
 07 $\{\Gamma\} = \rho^{-1}(\rho(\Gamma))$ is called a *singleton fibre*.
 08

09 LEMMA 1. *If $\rho: \mathbb{X} \rightarrow \mathbb{K}$ is a torus parametrization, ρ is onto.*

10
 11 *Proof.* As \mathbb{X} is compact and ρ continuous, the image $\rho(\mathbb{X})$ is compact. Let Γ be an
 12 arbitrary element of \mathbb{X} . As ρ is a G -map, $\rho(\mathbb{X})$ contains the orbit of $\rho(\Gamma)$. As G acts
 13 minimally on \mathbb{K} , this orbit is dense in \mathbb{K} . Thus, $\rho(\mathbb{X})$ is a dense compact subset of \mathbb{K} ,
 14 hence agrees with \mathbb{K} . \square

15 Let us continue with an interesting property of the torus parametrization. Namely, each
 16 torus parametrization induces a minimal subsystem of the original dynamical system.
 17

18 PROPOSITION 2. *Let $\rho: \mathbb{X} \rightarrow \mathbb{K}$ be a torus parametrization. If the set*

$$R(\mathbb{X}) := \{\Gamma \in \mathbb{X} : \Gamma \text{ is non-singular}\}$$

19
 20
 21 *is non-empty, it is G -invariant and G acts minimally on its closure $\mathbb{X}_R := \overline{R(\mathbb{X})}$.*
 22

23 *Proof.* The G -invariance of \mathbb{X}_R is clear, as ρ is a G -map; it remains to show minimality.
 24 To do so, let an arbitrary $\Gamma \in R(\mathbb{X})$ be given, and consider some $\Lambda' \in \mathbb{X}_R$. Let $\mathbb{X}(\Lambda')$
 25 be the closure of its G -orbit in \mathbb{X} . Of course, the restriction $\rho_{\mathbb{X}(\Lambda')}: \mathbb{X}(\Lambda') \rightarrow \mathbb{K}$ of ρ
 26 to $\mathbb{X}(\Lambda')$ is a torus parametrization as well. In particular, it is onto. Thus, we can find
 27 $\Gamma' \in \mathbb{X}(\Lambda')$ with $\rho(\Gamma') = \rho(\Gamma)$. By $\Gamma \in R(\mathbb{X})$, we infer $\Gamma = \Gamma' \in \mathbb{X}(\Lambda')$. As $\Gamma \in R(\mathbb{X})$
 28 was arbitrary, this implies $R(\mathbb{X}) \subset \mathbb{X}(\Lambda')$. As $\mathbb{X}(\Lambda') \subset \mathbb{X}_R$ is clear anyway, we obtain,
 29 after taking closures,

$$\mathbb{X}_R \subset \mathbb{X}(\Lambda') \subset \mathbb{X}_R.$$

30
 31
 32 As $\Lambda' \in \mathbb{X}_R$ was arbitrary, the statement follows. \square

33 We now discuss continuity properties of the inverse of a torus parametrization.
 34 While these results are not particularly hard to prove, they are a crucial ingredient behind
 35 the reconstruction of the window given in Lemma 3 in §5.
 36

37 PROPOSITION 3. *Let $\rho: \mathbb{X} \rightarrow \mathbb{K}$ be a torus parametrization. Let $\alpha: \mathbb{K} \rightarrow \mathbb{X}$ be any
 38 section of ρ (i.e. $\rho \circ \alpha$ is the identity on \mathbb{K}). Then, α is continuous at all points which are
 39 images of non-singular points, i.e. at all points of $\rho(R(\mathbb{X}))$.*

40
 41 The proof of this proposition is an immediate consequence of the following lemma.

42 LEMMA 2. *Let K_1 and K_2 be compact spaces and $\sigma: K_1 \rightarrow K_2$ continuous. Let $\xi_1 \in K_1$
 43 and $\xi_2 \in K_2$ be given such that $\{\xi_1\} = \sigma^{-1}(\{\xi_2\})$. Then, a net (ξ_i) in K_1 converges to ξ_1
 44 whenever $(\sigma(\xi_i))$ converges to ξ_2 .*

01 *Proof.* By the compactness of K_1 , the net (ξ_t) has converging subnets. Thus, it suffices to
 02 show that every converging subnet converges to ξ_1 . So, consider a converging subnet.
 03 Without loss of generality, we may assume that this converging subnet is (ξ_t) itself.
 04 Let ξ'_1 be its limit. Then, by the continuity of σ , we have $\sigma(\xi'_1) = \lim_t \sigma(\xi_t) = \xi_2$.
 05 As, by assumption, $\{\xi_1\} = \sigma^{-1}(\{\xi_2\})$, we infer $\xi'_1 = \xi_1$, and the proof (both of Lemma 2
 06 and Proposition 3) is complete. \square

08 3. Outline of the paper and summary of the main theorems

09 The overall objective of the paper is to prove Theorem 1, particularly in the direction
 10 of sufficiency. The basic setting is that of a Meyer set Λ for which the local hull
 11 $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic. We are interested in continuous G -mappings from the
 12 local hull to the autocorrelation hull, and particularly in those that are non-singular almost
 13 everywhere. Simply the existence of such a mapping $\beta_{\mathbb{A}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$ produces
 14 the first prerequisite for the appearance of model sets, a CPS. This is described in §4.2.
 15 Any CPS has a compact Abelian group \mathbb{T} associated with it: the quotient of the product of
 16 the ambient group and the internal group by the associated lattice. A key feature of the CPS
 17 that arises in our situation is that the mapping $\beta_{\mathbb{A}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$ can be viewed as a
 18 mapping $\mathbb{X}(\Lambda) \rightarrow \mathbb{T}$.

19 **THEOREM 2.** *Let Λ be a Meyer set for which $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic. Suppose
 20 that there exists a continuous G -map $\beta_{\mathbb{A}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$. Then, there is a CPS
 21 (G, H, \mathcal{L}) with associated compact Abelian group \mathbb{T} for which $\mathbb{A} \simeq \mathbb{T}$ via a topological
 22 isomorphism which is a G -map that sends $\Lambda \in \mathbb{A}$ to $0 \in \mathbb{T}$. In particular, there is a G -map
 23 $\beta_{\mathbb{T}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{T}$.*

25 Having constructed a cut and project set, we next need a window to be in the regime
 26 of model sets. As studied in §5, the crucial condition to provide a window is the non-
 27 singularity of the G -map $\beta_{\mathbb{A}}$. To avoid technical difficulties, we state the result here in a
 28 slightly simplified form.

29 **THEOREM 3A.** *Let Λ be a Meyer subset of G such that $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic.
 30 Assume that there exists a continuous G -map $\beta_{\mathbb{A}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$ which is one-to-one at
 31 least at one point. Then, there is a minimal dynamical subsystem $(\mathbb{X}(\Lambda)_R, G)$ of $(\mathbb{X}(\Lambda), G)$
 32 that is associated with a repetitive model set. In particular, if Λ is repetitive, $(\mathbb{X}(\Lambda), G)$
 33 itself is associated with a model set.*

35 The previous theorem does not assert that the constructed model set is regular, i.e. that
 36 the measure of the boundary of the window is 0. Concerning this topic, our result is
 37 Theorem 5. It shows that the boundary has Haar measure 0 if and only if the map $\beta_{\mathbb{A}}$
 38 is one-to-one almost everywhere. In fact, if the canonical map $\beta: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$ is
 39 one-to-one almost everywhere, we can go further.

40 **THEOREM 6.** *Let G be a σ -compact LCA group and Λ a Meyer subset of G such that the
 41 canonical map $\beta: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$ is continuous and one-to-one almost everywhere, with
 42 respect to the Haar measure on $\mathbb{A}(\Lambda) = \beta(\mathbb{X}(\Lambda))$. Then, $\mathbb{X}(\Lambda)$ is uniquely ergodic and
 43 Λ agrees with a regular model set up to a set of density 0. Furthermore, if Λ is repetitive,
 44 $\mathbb{X}(\Lambda)$ is actually associated to a regular model set.*

01 So far, we have assumed existence of a continuous G -map $\beta_{\mathbb{A}} : \mathbb{X}(\Lambda) \longrightarrow \mathbb{A}(\Lambda)$.
 02 However, what conditions are required to obtain such a map? This is studied in §6.
 03 Our main answer is the following.

04 **THEOREM 7.** *Let Λ be a Meyer subset of G such that $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic.
 05 Then, the following assertions are equivalent.*

- 06 (a) *There exists a continuous G -map $\beta_{\mathbb{A}} : \mathbb{X}(\Lambda) \longrightarrow \mathbb{A}(\Lambda)$.*
 07 (b) *$(\mathbb{X}(\Lambda), G)$ has pure point dynamical spectrum with continuous eigenfunctions.*
 08 *In this case, $\Gamma, \Gamma' \in \mathbb{X}(\Lambda)$ satisfy $\beta_{\mathbb{A}}(\Gamma) = \beta_{\mathbb{A}}(\Gamma')$ if and only if $f(\Gamma) = f(\Gamma')$ for
 09 every eigenfunction f .*

11 The proof of the implication (b) \implies (a) of this theorem requires an intermediate
 12 step. From the assumptions on $(\mathbb{X}(\Lambda), G)$, we create a new dynamical system (\mathbb{E}, G)
 13 by identifying elements of \mathbb{X} which are indistinguishable by means of the continuous
 14 eigenfunctions. This new space \mathbb{E} can be given the structure of a compact Abelian group.
 15 This new group is then shown to be just $\mathbb{A}(\Lambda)$. This is discussed in §7 and, in particular,
 16 in Theorem 8.

17 Theorems 3, 5 and 7 establish the sufficiency part of our main Theorem 1 (and most of
 18 the necessity too). This is discussed in §8. The link back is provided in §9 via the following
 19 result.

20 **THEOREM 9.** *Let a CPS (G, H, \mathcal{L}) and a non-empty window $W \subset H$ with $W = \overline{W^\circ}$
 21 and $\theta_H(\partial W) = 0$ be given. If $\Lambda \subset G$ satisfies $t + \wedge(W^\circ) \subset \Lambda \subset t + \wedge(W)$ for some
 22 $t \in G$, then the canonical map $\beta : \mathbb{X}(\Lambda) \longrightarrow \mathbb{A}(\Lambda)$ is continuous and one-to-one almost
 23 everywhere.*

24 Let us make a short comment here. During the process of proving the above results,
 25 we encounter groups \mathbb{A} and \mathbb{T} and maps $\beta_{\mathbb{A}}$ and $\beta_{\mathbb{T}}$ from $\mathbb{X}(\Lambda)$ into these groups. We show
 26 that these groups are isomorphic and that, in this sense, $\beta_{\mathbb{A}}$ and $\beta_{\mathbb{T}}$ agree. In fact,
 27 in retrospect, we can then even show that these maps agree with the canonical map β
 28 introduced above. However, this is not at all clear at the respective times of appearance
 29 and, for this reason, we carefully distinguish these maps and groups.

31 Finally, our results also imply an interesting characterization of the fully periodic case
 32 as discussed in §10:

33 **Definition 5.** A set $\Lambda \subset G$ is called *crystallographic* (or fully periodic) if its set of periods

$$\text{per}(\Lambda) := \{t \in G : t + \Lambda = \Lambda\}$$

36 forms a *lattice*, i.e. a co-compact discrete subgroup of G .

38 **THEOREM 10.** *Let G be an LCA group and Λ a uniformly discrete subset of G . Then, the
 39 following assertions are equivalent.*

- 40 (i) *Λ is crystallographic.*
 41 (ii) *Λ is Meyer and the map $\beta_{\mathbb{A}} : \mathbb{X}(\Lambda) \longrightarrow \mathbb{A}(\Lambda)$ is continuous and injective.*
 42 (iii) *All of the following conditions hold:*
 43 (1) *all elements of $\mathbb{X}(\Lambda)$ are Meyer sets;*
 44 (2) *$(\mathbb{X}(\Lambda), G)$ is uniquely ergodic;*

- 01 (3) $(\mathbb{X}(\Lambda), G)$ has pure point dynamical spectrum with continuous eigenfunctions;
 02 (4) the eigenfunctions separate all points of $\mathbb{X}(\Lambda)$.

03 In this case, $(\mathbb{X}(\Lambda), G)$ is also minimal, hence strictly ergodic.

04 The paper revolves around the important concept of Meyer sets. We have defined a
 05 set $\Lambda \subset G$ to be Meyer if it is a Delone set and $\Lambda - \Lambda$ is contained in a finite number
 06 of translates of Λ . We already noted that this implies that $\Lambda - \Lambda$ is also a Delone set
 07 (the important point being that it is uniformly discrete). For $G = \mathbb{R}^d$, this is an equivalence,
 08 and in fact the most common definition of a Meyer set is a Delone set whose set of
 09 differences is uniformly discrete. This result is due to Lagarias [17]. In Appendix A,
 10 we show that the two concepts are equivalent if G is compactly generated. We also show
 11 that, in this case, the requirement that $\Lambda - \Lambda$ be uniformly discrete is equivalent to the
 12 apparently weaker statement that for each compact subset K of G , the number of points of
 13 $(t + K) \cap (\Lambda - \Lambda)$ is finite and uniformly bounded as t runs over G (Theorem 11).
 14

15 4. Consequences of a continuous G -mapping $\beta_{\mathbb{A}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$: construction of a 16 cut and project scheme

17 Let Λ be a Meyer subset of G such that the associated dynamical system $(\mathbb{X}(\Lambda), G)$ is
 18 uniquely ergodic. As Λ is Meyer, there is an open neighbourhood U of 0 in G so that Λ is
 19 U -uniformly discrete, i.e., $\Lambda \in \mathcal{D}_U$. Moreover, it also follows that each element of $\mathbb{X}(\Lambda)$
 20 is U -uniformly discrete. As discussed in §2.3, Λ gives rise to the autocorrelation hull \mathbb{A} ,
 21 which is an Abelian group.

22 In this section, we assume that $\mathbb{A}(\Lambda)$ is compact and that there exists a torus
 23 parametrization $\beta_{\mathbb{A}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$. We do *not* assume that the map $\beta_{\mathbb{A}}$ is given by
 24 the canonical projection β .

25 Our objective in this section is to create a CPS out of this torus mapping and to show that
 26 $\mathbb{A}(\Lambda)$ is G -isomorphic with the torus \mathbb{T} of the associated CPS. Section 5 then shows how
 27 non-singularity of the torus parametrization can be used to provide and study a window.
 28

29 In the following, we freely use notation from §2 and, in particular, §2.4.

30
 31 4.1. *Establishing Axioms A1–A4 of [5].* Let Λ be a Meyer subset of G such that the
 32 associated dynamical system $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic. We assume the existence of
 33 a torus parametrization $\beta_{\mathbb{A}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$.

34 In order to create a CPS from these data, we rely on the construction described in [5],
 35 based on the Dirac comb δ_{Λ} of our point set Λ . It is defined by $\delta_{\Lambda} := \sum_{x \in \Lambda} \delta_x$.
 36 The construction now requires that the four assumptions (A1), (A2), (A3⁺) and (A4) of [5]
 37 hold for the measure δ_{Λ} . Let us fix an averaging sequence \mathcal{A} of van Hove type; the result
 38 will not depend on this choice, due to the unique ergodicity of $(\mathbb{X}(\Lambda), G)$.

39 As Λ is Meyer, the measure $\delta_{\Lambda} = \sum_{x \in \Lambda} \delta_x$ is translation bounded, i.e. for all compact
 40 $K \subset G$, there exists a constant C_K with $\sup_{t \in G} \delta_{\Lambda}(t + K) \leq C_K$. This is just the validity
 41 of (A1) for the measure δ_{Λ} .

42 As $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic, the autocorrelation

$$43 \gamma := \lim_{n \rightarrow \infty} \frac{1}{\theta_G(A_n)} \sum_{x, y \in \Gamma \cap A_n} \delta_{x-y} \quad (8)$$

01 exists for every $\Gamma \in \mathbb{X}(\Lambda)$, does not depend on Γ , and is equal to $\sum_{x \in \Delta} \eta(x) \delta_x$, with
 02 $\Delta = \Lambda - \Lambda$ and a suitable positive definite function $\eta: G \rightarrow \mathbb{C}$. This is assumption (A2)
 03 for δ_Λ .

04 Note that $\eta(0) = \text{dens}(\Lambda)$, and $\eta(x) = 0$ whenever $x \notin \Lambda - \Lambda$. In fact, the function η
 05 is closely connected to the metric d described above in (1) and (2). More precisely, a direct
 06 calculation gives

$$07 \quad d(s + \Lambda, t + \Lambda) := \lim_{n \rightarrow \infty} \frac{\text{card}(((s + \Lambda) \Delta (t + \Lambda)) \cap A_n)}{\theta_G(A_n)} = 2(\eta(0) - \eta(t - s)). \quad (9)$$

09 The set $\{x \in G : \eta(x) \neq 0\}$ is clearly a subset of Δ and hence uniformly discrete, as Λ is
 10 Meyer, and this is assumption (A3⁺).

11 Finally, as β_Δ is continuous, its image $\mathbb{A}(\Lambda)$ is compact. By [28], this implies (see also
 12 Lemma 5) that $\widehat{\gamma}$ is a pure point measure on \widehat{G} . This in turn means that, for each $\varepsilon > 0$,
 13 the set of ε -almost periods defined in (4) is relatively dense in G , compare [5]. This is
 14 assumption (A4).
 15

16 We close this section by noting that the ε -almost periods do not depend on Λ , but only
 17 on $\mathbb{X}(\Lambda)$. More precisely, by the uniform existence of the autocorrelation (8) and (9), for
 18 every $\Gamma \in \mathbb{X}(\Lambda)$, the identities

$$19 \quad P_\varepsilon = \{t \in \Lambda - \Lambda : d_G(t, 0) < \varepsilon\} = \{t \in G : d_G(t, 0) < \varepsilon\} = \{t \in \Gamma - \Gamma : d(t, 0) < \varepsilon\} \quad (10)$$

20 hold whenever $\varepsilon < 2\eta(0)$.
 21

22
 23 4.2. *Creating a cut and project scheme.* Here, we use the method of [5] to construct a
 24 CPS out of γ and Λ . This is possible since we have just established the validity of the
 25 necessary conditions (A1), (A2), (A3⁺) and (A4).
 26

27 Let L be the group generated by the set $\Delta = \Lambda - \Lambda$. Clearly, the pseudo-metric d
 28 discussed in (2) restricts to L and gives a pseudo-metric d_L by

$$29 \quad d_L(s, t) := d_G(s, t) = d(s + \Lambda, t + \Lambda) = 2(\eta(0) - \eta(t - s)), \quad (11)$$

30 where the last equality follows from (9). The topology on L defined by this is again called
 31 the *autocorrelation topology*. It makes L into a topological group.
 32

33 A fundamental system of neighbourhoods of 0 in L is given by the P_ε , $\varepsilon > 0$, defined
 34 above in (4). Let H be the Hausdorff completion of L under the autocorrelation topology
 35 and let $\phi: L \rightarrow H$ be the corresponding completion map. It should be noted that ϕ is not
 36 injective in general. In fact, if Λ is a lattice, one finds $H = \{0\}$.

37 In any case, let \mathcal{L} be the subgroup $\{(t, \phi(t)) \mid t \in L\}$. Then, this subgroup is a lattice
 38 in $G \times H$ and we arrive at a CPS (G, H, \mathcal{L}) as shown in (6). The pseudo-metric d_L on L
 39 induces a corresponding metric d_H on H . Let B_ε^H denote the corresponding open ball of
 40 radius ε in H . Then,

$$41 \quad P_\varepsilon = \phi^{-1}(\phi(L) \cap B_\varepsilon^H). \quad (12)$$

42 **PROPOSITION 4.** *Let $\Lambda \subset G$ be a Meyer set such that $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic.
 43 Then, $\Delta = \Lambda - \Lambda$ is totally bounded (or precompact) in the autocorrelation topology.
 44 In particular, $\overline{\phi(\Delta)}$ and $\overline{\phi(\Lambda)}$ are compact subsets of H .*

01 *Proof.* The subsets P_ε , $0 < \varepsilon < 2\eta(0)$, form a fundamental system of neighbourhoods
 02 for 0 in L . Fix one of them. It is relatively dense in G and hence there is a compact K
 03 with $G = P_\varepsilon + K$. Let $s \in \Delta$ and write $s = t + k$, with $t \in P_\varepsilon$ and $k \in K$. Then,
 04 $s - t \in (\Delta - \Delta) \cap K$ which is a finite set F since Δ is a Meyer set (so, $\Delta - \Delta$ is uniformly
 05 discrete, see Appendix A). Finally, $s = t + s - t \in P_\varepsilon + F$, so $\Delta \subset P_\varepsilon + F$, showing that
 06 Δ is totally bounded. \square

07 Let $\mathbb{T} = \mathbb{T}(\Lambda) := (G \times H)/\mathcal{L}$ be the corresponding compact Abelian quotient group.
 08 There is a natural action of G on \mathbb{T} , defined by letting $x \in G$ act as $(u, v) + \mathcal{L} \mapsto$
 09 $(x + u, v) + \mathcal{L} \in \mathbb{T}$ for all $(u, v) \in G \times H$. In this way, \mathbb{T} becomes a dynamical system for
 10 G , both measure theoretically (using the Haar measure $\theta_{\mathbb{T}}$) and topologically. The G -orbit
 11 of $0 \in \mathbb{T}$ is dense in \mathbb{T} , as is every other orbit. The homomorphism $\iota: G \rightarrow \mathbb{T}$ provided by
 12 this orbit is not injective in general: its kernel is $\ker(\phi) \subset L$, the set of *statistical periods*
 13 of Λ . Clearly, ϕ plays the role of the \star -map, wherefore we once again write t^\star rather than
 14 $\phi(t)$ from now on.

15 Now, the important fact is that the compact group \mathbb{T} we have just constructed agrees
 16 with $\mathbb{A}(\Lambda)$ defined in §2.3. More precisely, we have the following result from [28], which
 17 follows from the definition of $\mathbb{A}(\Lambda)$ and the characterization of \mathbb{T} as the completion of G
 18 in the so-called mixed topology given in [5]. For the convenience of the reader, we sketch
 19 a proof.
 20

21 **PROPOSITION 5.** *Let $\Lambda \subset G$ be a Meyer set such that $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic, and*
 22 *let $\beta_{\mathbb{A}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$ be the corresponding torus parametrization. Then, $\mathbb{T} \simeq \mathbb{A}(\Lambda)$,*
 23 *and this isomorphism is a G -map when both spaces are given their natural G -actions.*
 24

25 *Proof.* Let $\alpha: L \rightarrow G \times L$ be the diagonal map. Then, $\alpha(L)$ is discrete in $G \times L$
 26 and $(G \times L)/\alpha(L)$ becomes a topological group in the usual way. Furthermore, we
 27 have $G \simeq (G \times L)/\alpha(L)$ via the canonical embedding $x \mapsto (x, 0) + \alpha(L)$, and
 28 we provide G with a new topology this way, called the *mixed topology*. There is a
 29 homomorphism of $(G \times L)/\alpha(L)$ into the compact group $\mathbb{T} = (G \times H)/\mathcal{L}$ defined
 30 by $(x, t) + \alpha(L) \mapsto (x, t^\star) + \mathcal{L}$. In [5], it is shown that, via this map, \mathbb{T} is the
 31 Hausdorff completion of $(G \times L)/\alpha(L)$. Therefore, \mathbb{T} may be identified with the Hausdorff
 32 completion of G in the mixed topology and $\iota(G) \subset \mathbb{T}$ is the Hausdorff space *associated*
 33 *with G* . Given this construction of \mathbb{T} , we are left with the task to relate the mixed topology
 34 to the autocorrelation topology.

35 By definition, a basis for the open neighbourhoods of 0 in G , in the mixed topology,
 36 consists of the sets of the form $V + P_\varepsilon$, V an open neighbourhood of 0 in the original
 37 topology of G , $\varepsilon > 0$ (as these are precisely the sets in G which correspond to the
 38 sets $V \times P_\varepsilon + \alpha(L) \subset (G \times L)/\alpha(L)$ under our isomorphism). On the other hand,
 39 as discussed in §2, the autocorrelation completion \mathbb{A} of G comes about by supplying G
 40 with the uniformity induced from \mathcal{D} which has the entourage sets $U_{\text{mACT}}(V, \varepsilon) =$
 41 $\{(\Lambda', \Lambda'') : \exists v \in V \text{ with } d(v + \Lambda', \Lambda'') < \varepsilon\}$. The corresponding neighbourhoods of
 42 0 in G are then
 43

$$44 \quad U_{\text{mACT}}(V, \varepsilon)(0) = \{x \in G : \exists v \in V \text{ with } d(v + \Lambda, x + \Lambda) < \varepsilon\}.$$

01 Now, the definition of P_ε implies

$$02 \quad U_{\text{mACT}}(V, \varepsilon)(0) = V + P_\varepsilon,$$

03
04 and the proof is complete. \square

05
06 The key consequence of Proposition 5 is that our map $\beta_{\mathbb{A}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$ can be
07 interpreted as a continuous G map $\beta_{\mathbb{T}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{T}$. This gives the following theorem.

08
09 **THEOREM 2.** *Let Λ be a Meyer set for which $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic. Suppose that*
10 *there exists a continuous G map $\beta_{\mathbb{A}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$. Then, there is a CPS (G, H, \mathcal{L})*
11 *with associated compact Abelian group \mathbb{T} for which $\mathbb{A} \simeq \mathbb{T}$ via a topological isomorphism*
12 *which is a G -map that sends $\Lambda \in \mathbb{A}$ to $0 \in \mathbb{T}$. In particular, there is a torus parametrization*
13 *$\beta_{\mathbb{T}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{T}$.*

14 15 5. Consequences of the existence of non-singular elements: the window

16 We continue to assume that Λ is Meyer such that the associated dynamical system
17 $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic and that there exists a torus parametrization, i.e. a
18 continuous G -map $\beta_{\mathbb{A}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$. In this section, we investigate some
19 consequences, first that $\beta_{\mathbb{A}}$ is non-singular at least at one element, and second that $\beta_{\mathbb{A}}$
20 is non-singular almost everywhere.

21
22
23 **5.1. Existence of a non-singular element.** Assume that $\mathbb{X}(\Lambda)$ has at least one non-
24 singular element, see Definition 4. Thus, we have a dynamical subsystem $\mathbb{X}(\Lambda)_R$ that is
25 the closure of the set of non-singular elements $R(\mathbb{X})$ of $\mathbb{X}(\Lambda)$, as defined in Proposition 2.

26 In the previous section, we have constructed a CPS from Λ as well as a continuous map
27 $\beta_{\mathbb{T}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{T}$. In this section, we aim at proving the following.

28
29 **THEOREM 3.** *Let $\Lambda \subset G$ be a Meyer set such that $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic. Assume*
30 *that there exists a continuous G -map $\beta_{\mathbb{A}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$, which is one-to-one at least*
31 *at one point. Then, there is an irredundant CPS (G, H, \mathcal{L}) associated with $\mathbb{X}(\Lambda)$ and a*
32 *subset $W \subset H$, $W = \overline{W^\circ}$ compact, so that every non-singular element of $\mathbb{X}(\Lambda)$ is of the*
33 *form*

$$34 \quad \Gamma = x + \lambda(-h + W^\circ) = x + \lambda(-h + W)$$

35 for some $(x, h) \in G \times H$.

36 Each element of $\mathbb{X}(\Lambda)_R$ is repetitive and an IMS for the window W . If Λ itself is
37 repetitive, one has $\mathbb{X}(\Lambda) = \mathbb{X}(\Lambda)_R$.

38
39 The proof requires some preparation. The following lemma is one of the cornerstones
40 of the present work. It says that the mACT, which is defined by statistical information
41 at infinity, is actually compatible with the local topology, which is defined by local
42 information, whenever a certain condition is met. This condition is that Γ is non-singular
43 relative to $\beta_{\mathbb{A}}$. As mentioned above, we always assume in this section that $\Lambda \subset G$ is a
44 Meyer set such that $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic.

01 LEMMA 3. *Let \mathcal{A} be an averaging sequence for G as introduced above, and let $\Gamma \in \mathbb{X}(\Lambda)$*
 02 *be non-singular. Given any positive integer M , there is an $\varepsilon = \varepsilon(M) > 0$ so that*

$$03 \quad t \in P_\varepsilon \quad \implies \quad (t + \Gamma) \cap A_M = \Gamma \cap A_M.$$

04 *Proof.* By Proposition 5, $\mathbb{T} \simeq \mathbb{A}(\Lambda)$. Now, the statement can be concluded from Lemma 2
 05 after noticing that $d(\beta_{\mathbb{A}}(t + \Gamma), \beta_{\mathbb{A}}(\Gamma)) < \varepsilon$ whenever $t \in P_\varepsilon$. Namely, Lemma 2 then
 06 implies that $t + \Gamma$ and Γ are arbitrarily close in the local topology if ε is sufficiently small.
 07 As Γ is Meyer and $P_\varepsilon \subset \Gamma - \Gamma$ by (10), Fact 4 implies that Γ and $t + \Gamma$ actually agree
 08 on arbitrarily large compact sets, such as A_M , if $\varepsilon > 0$ is chosen accordingly. \square

09
 10 As a consequence of Lemma 3, and extending an argument used before in [6], we can
 11 show that every non-singular element of $\mathbb{X}(\Lambda)$ is a model set.

12 PROPOSITION 6. *If Γ is a non-singular element of $\mathbb{X}(\Lambda)$ with $0 \in \Gamma$, one has $\Gamma =$*
 13 *$\lambda(W^\circ) = \lambda(W)$, where $W := \overline{\Gamma^*}$ and $W = \overline{W^\circ}$.*

14
 15 *Proof.* By $0 \in \Gamma$, we have $\Gamma \subset \Gamma - \Gamma \subset \Lambda - \Lambda \subset L$. Now, let $x_0 \in \Gamma$. Choose a
 16 positive integer M so that $x_0 \in A_M$. Choose $\varepsilon(M)$ according to Lemma 3.

17 Let $y \in L$ and suppose that $y^* \in x_0^* + B_{\varepsilon(M)}^H$. Then, $y^* - x_0^* \in L^* \cap B_{\varepsilon(M)}^H = P_{\varepsilon(M)}^*$,
 18 which implies $y - x_0 \in P_{\varepsilon(M)}$. Then, $x_0 - y \in P_{\varepsilon(M)}$ and, by Lemma 3, $(x_0 - y + \Gamma) \cap A_M =$
 19 $\Gamma \cap A_M$. This implies $x_0 - y + u = x_0$ for some $u \in \Gamma$. Then, $y = u \in \Gamma$, so
 20 $\Gamma \supset \lambda(x_0^* + B_{\varepsilon(M)}^H)$ and $x_0^* + B_{\varepsilon(M)}^H \subset W$. This shows that

$$21 \quad \lambda(x_0^* + B_{\varepsilon(x_0)}^H) \subset \Gamma \quad \text{for all } x_0 \in \Gamma, \quad (13)$$

22
 23 where $\varepsilon(x_0)$ is the $\varepsilon(M)$ of the previous lemma. Now,

$$24 \quad W := \overline{\Gamma^*} = \overline{\bigcup_{x_0 \in \Gamma} (x_0^* + B_{\varepsilon(x_0)}^H)} \supset \bigcup_{x_0 \in \Gamma} (x_0^* + B_{\varepsilon(x_0)}^H) =: V.$$

25
 26 Obviously, V is open and contains Γ^* . Thus, $\overline{W^\circ} \supset \overline{V} \supset \overline{\Gamma^*} = W$ and $W = \overline{V} = \overline{W^\circ}$.

27 By (13), $\Gamma = \lambda(V)$. As Γ belongs to $\mathbb{X}(\Lambda)$, a restriction gives a continuous torus
 28 parametrization

$$29 \quad \beta_{\mathbb{T}}|_{\Gamma} : \mathbb{X}(\Gamma) \subset \mathbb{X}(\Lambda) \longrightarrow \mathbb{T}.$$

30
 31 As Γ is non-singular, the torus parametrization $\beta_{\mathbb{T}}$, and then even more the torus
 32 parametrization $\beta_{\mathbb{T}}|_{\Gamma}$, is one-to-one at Γ .

33 We next show $\partial V \cap L^* = \emptyset$. If $p \in G$ satisfies $p^* \in \partial V \cap L^*$, then, by the denseness
 34 of L^* , we can find a net $(t_i) \in L$ with (t_i^*) in $V \cap L^*$ and $t_i^* \rightarrow p^*$. Without loss of
 35 generality, we may assume that $p - t_i + \Gamma = \lambda(p^* - t_i^* + V)$ converges to some element
 36 $\Gamma' \in \mathbb{X}(\Gamma)$. Then, $\Gamma \neq \Gamma'$ as one contains p and the other does not. On the other hand,
 37 for some $(a, b) \in G \times H$,

$$38 \quad \beta_{\mathbb{T}}|_{\Gamma}(\Gamma) = (a, b) + L^* = \lim_i (a, b + p^* - t_i^*) + L^* = \lim_i \beta_{\mathbb{T}}|_{\Gamma}(p - t_i + \Gamma) = \beta_{\mathbb{T}}|_{\Gamma}(\Gamma'),$$

39
 40 contradicting the non-singularity of Γ .

41 By $\partial V \cap L^* = \emptyset$, we have

$$42 \quad \Gamma = \lambda(V) = \lambda(V \cup \partial V) = \lambda(\overline{V}) = \lambda(W).$$

43
 44 As $\Gamma = \lambda(V)$ and $V \subset W^\circ$, we infer $\Gamma = \lambda(W^\circ)$ as well, and the proof is complete. \square

01 We can use Proposition 6 to show that the CPS we have just created is irredundant,
02 and also to determine that each element in the orbit closure of the non-singular elements,
03 i.e. in $X(\Lambda)_R$, is an IMS for some translate of the same window W .

04 **PROPOSITION 7.** *Let a CPS (G, H, \mathcal{L}) be given, together with a window $W \subset H$ that is
05 non-empty, compact and satisfies $W = \overline{W^\circ}$. Consider an IMS Λ with $\lambda(W^\circ) \subset \Lambda \subset$
06 $\lambda(W)$. With $\mathbb{T} = (G \times H)/\mathcal{L}$ as above, the following assertions are equivalent.*

- 07 (i) *There exists a continuous G -map $\beta_{\mathbb{T}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{T}$ with $\beta_{\mathbb{T}}(\Lambda) = (0, 0) + \mathcal{L}$*
08 (ii) *The window W is irredundant, i.e. $W = c + W$ implies $c = 0$.*

09 *In this case, $\Lambda' \in \mathbb{X}(\Lambda)$ satisfies $\beta_{\mathbb{T}}(\Lambda') = (x, h) + \mathcal{L}$ if and only if $x + \lambda(-h + W^\circ) \subset$
10 $\Lambda' \subset x + \lambda(-h + W)$ holds.*

12 *Proof.* The implication (ii) \implies (i) follows by the argument given in [36] to prove the case
13 $\Lambda = \lambda(W)$ (see [19] as well).

14 The implication (i) \implies (ii) and the last statement will be proved together. This will
15 be done in three steps. To this end, let $\beta_{\mathbb{T}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{T}$ be continuous with $\beta_{\mathbb{T}}(\Lambda) =$
16 $(0, 0) + \mathcal{L}$, and consider an arbitrary $\Lambda' \in \mathbb{X}(\Lambda)$.

18 *Step 1.* $\beta_{\mathbb{T}}(\Lambda') = (x, h) + \mathcal{L}$ implies $x + \lambda(-h + W^\circ) \subset \Lambda' \subset x + \lambda(-h + W)$.

19 Let (x, h) be given with $\beta_{\mathbb{T}}(\Lambda') = (x, h) + \mathcal{L}$, and let $y \in G$ be chosen so that
20 $0 \in \Lambda'' := -y + \Lambda'$. Let $\{t_n + \Lambda\}_n$, $t_n \in G$, be a net converging to Λ'' in $\mathbb{X}(\Lambda)$.
21 Without loss of generality, we may assume that $0 \in t_n + \Lambda$ for all n . Then, in particular,
22 $t_n \in -\Lambda$ and therefore $t_m^* - t_n^* \in W - W$ for all n, m . As $W - W$ is compact, we may
23 assume that $\{t_n^*\}_n \rightarrow -k \in H$, possibly after restricting to a subnet.

24 Now, $\beta_{\mathbb{T}}(t_n + \Lambda) = \iota(t_n) + \beta_{\mathbb{T}}(\Lambda)$, where, since $\beta_{\mathbb{T}}$ is a G -map, $\iota(t_n) = (t_n, 0) + \mathcal{L} =$
25 $(0, -t_n^*) + \mathcal{L}$, which converges to $(0, k) + \mathcal{L}$ in \mathbb{T} . Thus, by the continuity of $\beta_{\mathbb{T}}$,
26 $\beta_{\mathbb{T}}(\Lambda'') = (0, k) + \mathcal{L}$ and $\beta_{\mathbb{T}}(\Lambda') = \beta_{\mathbb{T}}(y + \Lambda'') = (y, k) + \mathcal{L}$. As, by assumption,
27 $\beta_{\mathbb{T}}(\Lambda') = (x, h) + \mathcal{L}$, we infer $(y, k) + \mathcal{L} = (x, h) + \mathcal{L}$. This gives

$$29 \quad x + \lambda(-h + W^\circ) = y + \lambda(-k + W^\circ) \quad \text{and} \quad x + \lambda(-h + W) = y + \lambda(-k + W). \quad (14)$$

31 Consider an arbitrary $z \in \lambda(-k + W^\circ)$, so that $z^* + k \in W^\circ$. Then, for all large n ,
32 $z^* - t_n^* \in W^\circ$ and

$$33 \quad z \in \lambda(t_n^* + W^\circ) = t_n + \lambda(W^\circ) \subset t_n + \Lambda.$$

34 Thus, $z \in \Lambda''$ and $\lambda(-k + W^\circ) \subset \Lambda''$ follows. Adding y , and invoking (14), we end up
35 with

$$37 \quad x + \lambda(-h + W^\circ) \subset \Lambda'.$$

38 Conversely, if $z \in \Lambda''$, then $z \in t_n + \Lambda$ for sufficiently large n , so that $z^* - t_n^* \in W$ and,
39 in the limit, $z^* \in -k + W$, i.e. $z \in \lambda(-k + W)$, which implies $\Lambda' \subset y + \lambda(-k + W)$.
40 Again, using (14), we obtain

$$42 \quad \Lambda' \subset x + \lambda(-h + W).$$

44 *Step 2.* $c + W = W$ implies $c = 0$, i.e. condition (ii) holds.

01 Note that $c + W = W$ implies $c + W^\circ = W^\circ$. As $W = \overline{W^\circ}$, the boundary of W is
 02 nowhere dense. By the Baire category theorem, there exists then a $d \in H$ with

$$03 \quad \lambda(d + W^\circ) = \lambda(d + W).$$

05 Moreover, $\beta_{\mathbb{T}}$ is onto by Lemma 1. Thus, there exist $\Lambda', \Lambda'' \in \mathbb{X}(\Lambda)$ with

$$07 \quad \beta_{\mathbb{T}}(\Lambda') = (0, -d) + \mathcal{L}, \quad \beta_{\mathbb{T}}(\Lambda'') = (0, -d - c) + \mathcal{L}. \quad (15)$$

08 By the result of Step 1, this implies

$$10 \quad \lambda(d + W^\circ) \subset \Lambda' \subset \lambda(d + W) \quad \text{as well as} \quad \lambda(d + c + W^\circ) \subset \Lambda'' \subset \lambda(d + c + W).$$

12 By our choice of d , and because we both have $c + W = W$ and $c + W^\circ = W^\circ$, we can
 13 infer $\Lambda' = \Lambda''$. This, in turn, implies $\beta_{\mathbb{T}}(\Lambda') = \beta_{\mathbb{T}}(\Lambda'')$, and $c = 0$ follows from (15).

14 *Step 3.* $x + \lambda(-h + W^\circ) \subset \Lambda' \subset x + \lambda(-h + W)$ implies $\beta_{\mathbb{T}}(\Lambda') = (x, h) + \mathcal{L}$.

16 Let (y, f) with $\beta_{\mathbb{T}}(\Lambda') = (y, f) + \mathcal{L}$ be given. By Step 1, we then have

$$18 \quad y + \lambda(-f + W^\circ) \subset \Lambda' \subset y + \lambda(-f + W).$$

19 Adding $-x$ yields

$$21 \quad y - x + \lambda(-f + W^\circ) \subset \Lambda' - x \subset y - x + \lambda(-f + W). \quad (16)$$

23 On the other hand, the assumption on (x, h) gives

$$25 \quad \lambda(-h + W^\circ) \subset \Lambda' - x \subset \lambda(-h + W). \quad (17)$$

26 These inclusions show that $(y - x)$ belongs to L and we can rewrite (16) as

$$28 \quad \lambda((y - x)^* - f + W^\circ) \subset \Lambda' - x \subset \lambda((y - x)^* - f + W). \quad (18)$$

30 Now, a combination of (17) and (18) gives

$$31 \quad \lambda((y - x)^* - f + W^\circ) \subset \lambda(-h + W) \quad \text{and} \quad \lambda(-h + W^\circ) \subset \lambda((y - x)^* - f + W),$$

33 which, in turn, implies

$$35 \quad ((y - x)^* - f + W^\circ) \cap L^* \subset -h + W \quad \text{and} \quad (-h + W^\circ) \cap L^* \subset (y - x)^* - f + W.$$

36 Taking closures and using $\overline{W^\circ} = W$ as well as the denseness of L^* in H , we obtain

$$38 \quad (y - x)^* - f + W \subset -h + W \quad \text{and} \quad -h + W \subset (y - x)^* - f + W.$$

40 These inclusions yield $f - h - (y - x)^* + W = W$ and, by Step 2,

$$42 \quad f - h - (y - x)^* = 0.$$

43 This, however, means $(y, f) + \mathcal{L} = (x, h) + \mathcal{L} = \beta_{\mathbb{T}}(\Lambda')$, and the proof of Step 3, and
 44 also of the entire claim, is complete. \square

01 5.2. *The proof of Theorem 3.* To prove Theorem 3, consider a non-singular element Γ
 02 of $\mathbb{X}(\Lambda)$. By translating Γ , we may assume $0 \in \Gamma$ without loss of generality. Proposition 6
 03 then implies $\Gamma = \lambda(W^\circ) = \lambda(W)$, where $W := \overline{\Gamma^*}$ and $W = \overline{W^\circ}$ is compact.
 04 By Proposition 5, $\mathbb{A}(\Lambda) \simeq \mathbb{T}$.

05 Assume $\Gamma = \Lambda$ for the moment. Then, by Proposition 7, every $\Lambda' \in \mathbb{X}(\Lambda)$ is an IMS
 06 of the form that we require. If, on the other hand, Λ is singular, these results apply to
 07 all of the elements of $\mathbb{X}(\Lambda)_R$, since it contains all of the non-singular elements and is the
 08 closed hull of any of its elements. As pointed out in Fact 2, the elements of $\mathbb{X}(\Lambda)_R$ are all
 09 repetitive.

10 This finishes the proof of Theorem 3.

11 *Remark 1.* There is very little that one can say about the generator Λ of the hull $\mathbb{X}(\Lambda)$
 12 being a model set, or even an IMS, if repetitivity or some other consistency property is
 13 not assumed. One can, for instance, take a model set, add some finite set of spurious
 14 points, and take the hull of the resulting set. That destroys the set as a model set, but
 15 does not destroy the properties of the minimal part $\mathbb{X}(\Lambda)_R$ of the hull, which will not have
 16 been altered. However, with the assumption of non-singularity almost everywhere, we can
 17 obtain information up to sets of density 0.
 18

19 5.3. *Consequences of non-singularity almost everywhere.* In this section, we shall
 20 prove Theorem 6. To do so, we need some preparation around the regularity of the window
 21 in the CPS. To do so, we assume the following setting.

22 **(S)** (G, H, \mathcal{L}) is a CPS, $W \subset H$ is a non-empty, compact set with $W = \overline{W^\circ}$ and Λ is an
 23 arbitrary IMS for it, i.e. $\lambda(W^\circ) \subset \Lambda \subset \lambda(W)$. There exists a continuous G -map
 24 $\beta_{\mathbb{T}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{T}$ with $\beta_{\mathbb{T}}(\Lambda) = (0, 0) + \mathcal{L}$.

25 Proposition 7 has the following consequence.

26
 27 **PROPOSITION 8.** *Let (S) be valid, with an IMS Λ . For any $c \in H$, the following properties
 28 are equivalent.*

- 29 (i) $\lambda(-c + W^\circ) = \lambda(-c + W)$.
 30 (ii) $\partial(-c + W) \cap L^* = \emptyset$.
 31 (iii) *The fibre over $(0, c) + \mathcal{L}$ is non-singular.*

32 *In this case, $\lambda(-c + W^\circ) = \lambda(-c + W)$ constitutes the fibre over $(0, c) + \mathcal{L}$, and one
 33 has the inclusion $\mathbb{X}(\lambda(-c + W)) \subset \mathbb{X}(\Lambda)$.*

34 *Proof.* The equivalence of properties (i) and (ii) is obvious. Also, **(S)** allows us to use
 35 Proposition 7, whence we see that (i) implies (iii).

36 It remains to show that (iii) implies (ii), or its contraposition. To this end, let us assume
 37 that there is some $p \in L$ with $p^* \in \partial(-c + W)$. The $\beta_{\mathbb{T}}$ -fibre over $(0, c) + \mathcal{L}$ is non-empty
 38 and consists of the elements $\Lambda' \in \mathbb{X}(\Lambda)$ such that

$$39 \lambda(-c + W^\circ) \subset \Lambda' \subset \lambda(-c + W)$$

40
 41 by Proposition 7. We claim that there are at least two elements on this fibre, one of which
 42 contains p while the other does not.

43 Take any Λ' on the fibre. Suppose first that $p \notin \Lambda'$. Since p^* is on the boundary of
 44 $-c + W$, there is a net $\{\ell_n\}$ in L with $\{\ell_n^*\} \rightarrow c$, such that $p \in \lambda(-\ell_n^* + W^\circ)$ for all n .

01 Then, on the fibre over $(0, \ell_n^*) + \mathcal{L}$, there is a set $\Lambda_n \in \mathbb{X}(\Lambda)$ with $\wedge(-\ell_n^* + W^\circ) \subset \Lambda_n \subset$
 02 $\wedge(-\ell_n^* + W)$. By the compactness of $\mathbb{X}(\Lambda)$, there is a convergent subnet of $\{\Lambda_n\}$ which
 03 we may assume to be $\{\Lambda_n\}$ itself. Let $\{\Lambda_n\} \rightarrow \Lambda'' \in \mathbb{X}(\Lambda)$. Then, $p \in \Lambda_n$ for all n
 04 implies $p \in \Lambda''$. Also, $\beta_{\mathbb{T}}(\Lambda'') = \lim_n \beta_{\mathbb{T}}(\Lambda_n) = \lim_n (0, \ell_n^*) + \mathcal{L} = (0, c) + \mathcal{L}$, so Λ'' is
 05 on the same fibre as Λ' , but it contains p whereas Λ' does not.

06 The argument for the case when $p \in \Lambda'$ is similar. This time, choose a net $\{\ell_n\}$ in L
 07 with $\{\ell_n^*\} \rightarrow c$, $p \notin \wedge(-\ell_n^* + W)$, for all n . We then find Λ_n on the fibre over $(0, \ell_n^*) + \mathcal{L}$,
 08 with $p \notin \Lambda_n$, and get $\Lambda'' \in \mathbb{X}(\Lambda)$ on the fibre over $(0, c) + \tilde{L}$, also with $p \notin \Lambda''$.

09 The last statement of the proposition is obvious. \square

10 Next, let us relate the properties of W versus ∂W to the injectivity of $\beta_{\mathbb{T}}$.

11
 12 **THEOREM 4.** [25] *Let (G, H, \mathcal{L}) be a CPS. Let M be a measurable, relatively compact*
 13 *set in H . Then,*

$$14 \quad \text{dens}(x + \wedge(M - h)) := \lim_{n \rightarrow \infty} \frac{\text{card}((x + \wedge(M - h)) \cap A_n)}{\theta_G(A_n)} = \text{dens}(\mathcal{L}) \theta_H(M),$$

16 *which is valid for all $(x, h) \in G \times H$ if $\theta_H(\partial M) = 0$, and otherwise for $\theta_G \times \theta_H$ -almost*
 17 *every $(x, h) \in G \times H$.*

19 **LEMMA 4.** *Let $M \subset \mathbb{T}$ be any measurable subset whose preimage in $G \times H$ is contained*
 20 *in a subset of the form $G \times B$ with $\theta_H(B) = 0$. Then, $\theta_{\mathbb{T}}(M) = 0$.*

21 *Proof.* Observe first that $\theta_{G \times H}(A \times B) = \theta_G(A) \theta_H(B) = 0$ for any relatively compact
 22 measurable set $A \subset G$. Since G is σ -compact, we may now employ the averaging
 23 sequence $\mathcal{A} = \{A_n\}$ of §2.1, with $A_n \subset A_{n+1}$ and $G = \bigcup_n A_n$, to conclude that also
 24 $\theta_{G \times H}(G \times B) = 0$.

25 Let now $\pi : G \times H \rightarrow \mathbb{T}$ be the canonical projection. Define, for $\xi \in \mathbb{T}$, the measure
 26 ν_ξ on $G \times H$ by

$$27 \quad \nu_\xi := \sum_{z \in \pi^{-1}(\xi)} \delta_z.$$

29 Standard desintegration (e.g. using a fundamental domain) shows that $\theta_{G \times H} = \theta_{\mathbb{T}} \circ \nu$,
 30 i.e. $\int f(z) d\theta_{G \times H}(z) = \int \nu_\xi(f) d\theta_{\mathbb{T}}(\xi)$ for any measurable non-negative f on $G \times H$.
 31 This gives

$$32 \quad 0 \leq \theta_{\mathbb{T}}(M) = \int 1_M(\xi) d\theta_{\mathbb{T}}(\xi) \leq \int \nu_\xi(1_M \circ \pi) d\theta_{\mathbb{T}}(\xi)$$

$$33 \quad \leq \int \nu_\xi(1_{G \times B}) d\theta_{\mathbb{T}}(\xi) = \theta_{G \times H}(G \times B)$$

34
 35
 36
 37 and the proof is finished because the last term vanishes as shown above. \square

38
 39 **THEOREM 5.** *Let (S) be in place. Then, the boundary of W has measure 0 if and only if*
 40 *$\beta_{\mathbb{T}}$ is one-to-one almost everywhere.*

41 *Proof.* By Proposition 8, $\Lambda' \in \mathbb{X}(\Lambda)$ is non-singular if and only if one has $\Lambda' =$
 42 $x + \wedge(-h + W^\circ) = x + \wedge(-h + W)$ and $L^* \cap (-h + \partial W) = \emptyset$ for some $(x, h) \in G \times H$.
 43 In this case, one has $x + \wedge(-h + \partial W) = \emptyset$. We thus have $\text{dens}(x + \wedge(-h + \partial W)) = 0$
 44 at this point.

01 If $\beta_{\mathbb{T}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{T}$ is one-to-one \mathbb{T} -almost everywhere, we also have this relation
 02 $G \times H$ -almost everywhere, due to $\theta_{G \times H} = \theta_{\mathbb{T}} \circ \nu$ (see the proof of the previous
 03 lemma). Consequently, by Theorem 4 and because $\text{dens}(\mathcal{L}) \neq 0$, we may conclude that
 04 $\theta_H(\partial W) = 0$.

05 Conversely, suppose that $\theta_H(\partial W) = 0$. By Proposition 8,

$$\begin{aligned} 06 \quad F &:= \{\xi \in \mathbb{T} : \text{the fibre over } \xi \text{ contains more than one element}\} \\ 07 &= \{(x, h) + \mathcal{L} \in \mathbb{T} : \lambda(-h + W^\circ) \neq \lambda(-h + W)\}. \end{aligned}$$

08 This gives

$$\begin{aligned} 09 \quad \theta_{\mathbb{T}}(F) &= \theta_{\mathbb{T}}(\{(x, h) + \mathcal{L} \in \mathbb{T} : \lambda(-h + W^\circ) \neq \lambda(-h + W)\}) \\ 10 &= \theta_{\mathbb{T}}(\{(x, h) + \mathcal{L} \in \mathbb{T} : h \in L^* + \partial W\}) \\ 11 &= \theta_{\mathbb{T}}(G \times (L^* + \partial W) \bmod \mathcal{L}) = 0, \end{aligned}$$

12 where we used Lemma 4 in the last step together with the fact that L^* is countable. \square

13 We can now proceed to the final result of this section.

14 **THEOREM 6.** *Let G be a σ -compact LCA group and Λ a Meyer subset of G such that the
 15 canonical map $\beta: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$ is continuous and one-to-one almost everywhere, with
 16 respect to the Haar measure on $\mathbb{A}(\Lambda) = \beta(\mathbb{X}(\Lambda))$. Then, $\mathbb{X}(\Lambda)$ is uniquely ergodic and
 17 Λ agrees with a regular model set up to a set of density 0. Furthermore, if Λ is repetitive,
 18 $\mathbb{X}(\Lambda)$ is actually associated to a regular model set.*

19 *Proof.* We are given a Meyer set Λ and assume that $\beta: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$ is continuous
 20 and one-to-one almost everywhere. We first want to show that Λ differs from a model
 21 set up to a set of points of density 0. As β is continuous, $\mathbb{A}(\Lambda)$ is compact. Moreover,
 22 G acts minimally on $\mathbb{A}(\Lambda)$ by definition. Thus, $(\mathbb{A}(\Lambda), G)$ is uniquely ergodic with the
 23 Haar measure $\theta_{\mathbb{A}}$ on $\mathbb{A}(\Lambda)$. We show that $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic as well.

24 As β is one-to-one almost everywhere, there exists a subset $\mathbb{A}' \subset \mathbb{A}(\Lambda)$ of full measure
 25 such that β is one-to-one on $\mathbb{X}' := \beta^{-1}(\mathbb{A}')$ and the complement of \mathbb{X}' is mapped into the
 26 complement of \mathbb{A}' by β . Now, note that, by Proposition 3, any inverse of β is continuous
 27 when restricted to \mathbb{A}' . Thus, extending this continuous function, say by setting it constant
 28 on $\mathbb{A}(\Lambda) \setminus \mathbb{A}'$, we find a measurable $\alpha: \mathbb{A} \rightarrow \mathbb{X}(\Lambda)$, which is an inverse to β on \mathbb{A}' .

29 Let μ be any G -invariant probability measure on $\mathbb{X}(\Lambda)$. As $(\mathbb{A}(\Lambda), G)$ is uniquely
 30 ergodic, $\beta^*(\mu)$ is the Haar measure $\theta_{\mathbb{A}}$ on $\mathbb{A}(\Lambda)$. In particular, $\mu(\beta^{-1}(M)) = 0$ whenever
 31 M is a subset of $\mathbb{A}(\Lambda)$ of measure 0. In particular, $\mu(\mathbb{X}(\Lambda) \setminus \mathbb{X}') = 0$. Let f be any
 32 measurable bounded function on $\mathbb{X}(\Lambda)$. Then, f and $f \circ \alpha \circ \beta$ only differ on $\beta^{-1}(\mathbb{A}(\Lambda) \setminus \mathbb{A}')$,
 33 which has μ -measure 0. This implies

$$34 \quad \mu(f) = \mu(f \circ \alpha \circ \beta) = \beta^*(\mu)(f \circ \alpha) = \theta_{\mathbb{A}}(f \circ \alpha) = \alpha^*(\theta_{\mathbb{A}})(f).$$

35 Thus, μ is uniquely determined and the unique ergodicity of $(\mathbb{X}(\Lambda), G)$ follows.

36 Now, the assumptions of Theorems 2 and 3 are satisfied, and we find both a CPS such
 37 that $\mathbb{A}(\Lambda) \simeq \mathbb{T}$ and a dynamical system $(\mathbb{X}(\Lambda)_R, G)$ associated to a model set $\Gamma = \lambda(W)$
 38 inside of $(\mathbb{X}(\Lambda), G)$ with irredundant W and (metrizable) internal group H .

01 From the previous results, and Theorem 5 in particular, we know that the almost one-
 02 to-oneness of β forces the boundary of W to have measure 0. Consider the fibre lying over
 03 $(x, h) + \mathcal{L}$. If the fibre is non-singular, the single element of $\mathbb{X}(\Lambda)$ is $x + \lambda(h + W)$, which
 04 is a *regular* model set. Even if the fibre is singular, set(s) lying there differ by density 0
 05 from the regular model set $x + \lambda(h + W)$, since $\text{dens}(x + \lambda(h + \partial W)) = 0$ by Theorem 4.

06 Of course, if Λ is repetitive, $\mathbb{X}(\Lambda)$ is generated by any of its elements, and so $\mathbb{X}(\Lambda)$ is
 07 actually associated to a regular model set. \square

09 6. *Existence of a continuous $\beta_{\mathbb{A}}$ and pure point spectrum with continuous eigenfunctions*
 10 Let Λ be a Meyer set such that $(\mathbb{X}(\Lambda), G)$ is a uniquely ergodic dynamical system.
 11 The existence of a torus parametrization $\beta_{\mathbb{A}} : \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$ has proved to be the key to
 12 linking Λ to the realm of model sets. In this section, we connect the existence of a torus
 13 parametrization with properties of the dynamical system $\mathbb{X}(\Lambda)$ itself. These properties are
 14 pure pointedness of the spectrum and continuity of the eigenfunctions.
 15

16 THEOREM 7. *Let Λ be a Meyer subset of G such that $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic.
 17 Then, the following assertions are equivalent.*

18 (a) *There exists a continuous G -map $\beta_{\mathbb{A}} : \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$.*

19 (b) *$(\mathbb{X}(\Lambda), G)$ has pure point dynamical spectrum with continuous eigenfunctions.*

20 *In this case, $\Gamma, \Gamma' \in \mathbb{X}(\Lambda)$ satisfy $\beta_{\mathbb{A}}(\Gamma) = \beta_{\mathbb{A}}(\Gamma')$ if and only if $f(\Gamma) = f(\Gamma')$ for
 21 every eigenfunction f .*
 22

23 This and the next section of the paper are devoted to the proof of this result. In this
 24 section, we prove Theorem 7 in the direction (a) \implies (b). In the following section,
 25 we prove the converse.
 26

27 6.1. *The proof of (a) \implies (b) of Theorem 7.* Let Λ be a Meyer set with associated
 28 uniquely ergodic dynamical system $(\mathbb{X}(\Lambda), G, \mu)$ and let T be the corresponding unitary
 29 representation of G on $L^2(\mathbb{X}(\Lambda), \mu)$.

30 Recall that the eigenvalues of this dynamical system form a subgroup of \widehat{G} , which we
 31 denote by $P(T)$. In addition, we need to consider the diffraction measure $\widehat{\nu}$, which is
 32 constant on $\mathbb{X}(\Lambda)$ due to the unique ergodicity. For any measure ν on \widehat{G} , we introduce
 33 the set
 34

$$35 P(\nu) := \{k \in \widehat{G} : \nu(\{k\}) \neq 0\}, \quad (19)$$

36 which is a countable set, and the subgroup of \widehat{G} that it generates, denoted by $\langle P(\nu) \rangle$.

37 We recall the following result that has already been established in the literature.
 38

39 LEMMA 5. *Let $\Lambda \subset G$ be a Meyer set. If $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic, the following
 40 assertions are equivalent.*

41 (i) *$\mathbb{A}(\Lambda)$ is compact.*

42 (ii) *$\widehat{\nu}$ is a pure point measure.*

43 (iii) *$(\mathbb{X}(\Lambda), G)$ has pure point dynamical spectrum.*

44 *In this case, the dynamical spectrum $P(T)$ of $(\mathbb{X}(\Lambda), G)$ satisfies $P(T) = \langle P(\widehat{\nu}) \rangle$.*

01 *Proof.* The equivalence of assertions (i) and (ii) is shown in [28]. The equivalence of
 02 assertions (ii) and (iii) is proved in [20, Theorem 3.2]. The last statement is proved in
 03 [2, Theorem 9]. \square

04 If there exists a continuous G -map $\beta_{\mathbb{A}} : \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$, Theorem 2 tells us
 05 that we have a CPS (G, H, \mathcal{L}) and a compact group $\mathbb{T} = (G \times H)/\mathcal{L}$. Moreover,
 06 $\mathbb{A}(\Lambda) = \mathbb{T}$. Thus, $\beta_{\mathbb{A}}$ induces a continuous map $\beta_{\mathbb{T}}$ between $\mathbb{X}(\Lambda)$ and \mathbb{T} . There is then a
 07 canonical homomorphism $\iota : G \rightarrow \mathbb{T}$ of topological groups with dense range defined by
 08 $x \mapsto (x, 0) + \mathcal{L}$. Dualizing, we obtain an injective homomorphism $\hat{\iota} : \hat{\mathbb{T}} \rightarrow \hat{G}$ of the dual
 09 topological groups. Lemma 5 tells us that $\hat{\gamma}$ is a pure point measure.
 10

11 **LEMMA 6.** *Let Λ be a Meyer set such that $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic and $\hat{\gamma}$ is a pure*
 12 *point measure. Then, $\langle P(\hat{\gamma}) \rangle \subset \hat{\iota}(\hat{\mathbb{T}})$.*

13 *Proof.* Due to unique ergodicity, each element of $\mathbb{X}(\Lambda)$ has the same autocorrelation
 14 measure γ . Let $C_c(G)$ denote the space of continuous complex-valued functions of
 15 compact support on G . For every $c \in C_c(G)$, we define $g_c : G \rightarrow \mathbb{C}$ by $g_c = c * \tilde{c} * \gamma$.
 16 Then, there is a continuous positive definite function $g_c^{\mathbb{T}}$ on \mathbb{T} so that $g_c^{\mathbb{T}} \circ \iota = g_c$
 17 (see [5, §4] as well). In particular, we can expand $g_c^{\mathbb{T}}$ in a uniformly converging Fourier
 18 series

$$19 \quad g_c^{\mathbb{T}}(x) = \sum_{k \in \hat{\mathbb{T}}} a_c(k)(k, x)$$

20 with non-negative numbers $a_c(k)$ that satisfy

$$21 \quad \sum_{k \in \hat{\mathbb{T}}} a_c(k) = g_c(0).$$

22 Composing $g_c^{\mathbb{T}}$ with the homomorphism $\iota : G \rightarrow \mathbb{T}$ and using the definition of $\hat{\iota}$,
 23 we obtain

$$24 \quad g_c(x) = \sum_{k \in \hat{\mathbb{T}}} a_c(k) \hat{\iota}(k, x).$$

25 As the $a_c(k)$ are summable, we can calculate the Fourier transform of g_c to arrive at

$$26 \quad |\hat{c}|^2 \hat{\gamma} = \hat{g}_c = \sum_{k \in \hat{\mathbb{T}}} a_c(k) \delta_{\hat{\iota}(k)},$$

27 which is a finite positive measure on \hat{G} .

28 This shows that

$$29 \quad B := \{k \in \hat{\mathbb{T}} : a_c(k) > 0 \text{ for some continuous } c \text{ with compact support}\}$$

30 is mapped under $\hat{\iota}$ into $P(\hat{\gamma})$ defined in (19). Taking for c an approximate unit, one infers
 31 that B is actually mapped onto $P(\hat{\gamma})$. As $\hat{\iota}(\hat{\mathbb{T}})$ is a subgroup of \hat{G} , which then contains
 32 $\iota(B)$, the desired conclusion follows immediately. \square

33 **PROPOSITION 9.** *Let Λ be a Meyer set in G such that $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic.*
 34 *If there exists a continuous G -mapping $\beta_{\mathbb{A}} : \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$, then $(\mathbb{X}(\Lambda), G)$ has pure*
 35 *point dynamical spectrum with continuous eigenfunctions.*
 36

01 *Proof.* By Lemma 5, the dynamical system $(\mathbb{X}(\Lambda), G)$ has pure point dynamical spectrum.
 02 Moreover, as discussed after the lemma, we can then identify $\mathbb{A}(\Lambda)$ and \mathbb{T} . Thus, $\beta_{\mathbb{A}}$ yields
 03 a continuous G -map from $\mathbb{X}(\Lambda)$ to \mathbb{T} .

04 Every element $\lambda \in \widehat{\mathbb{T}}$ gives rise to a continuous eigenfunction

$$05 \quad f_{\lambda} := \lambda \circ \beta_{\mathbb{T}} : \mathbb{X}(\Lambda) \longrightarrow \mathbb{C}$$

06
 07 to the eigenvalue $\hat{i}(\lambda)$, and we infer

$$08 \quad \hat{i}(\widehat{\mathbb{T}}) \subset P(T),$$

09
 10 where the point spectrum $P(T)$ is the set of eigenvalues. Combining this with the results
 11 of Lemmas 5 and 6, we obtain the following chain of inclusions:

$$12 \quad \hat{i}(\widehat{\mathbb{T}}) \subset P(T) = \langle P(\widehat{\gamma}) \rangle \subset \hat{i}(\widehat{\mathbb{T}}).$$

13
 14 Therefore, $\hat{i}(\widehat{\mathbb{T}}) = P(T)$. Thus, the f_{λ} , $\lambda \in \widehat{\mathbb{T}}$, provide eigenfunctions for each eigenvalue.
 15 As each eigenvalue has multiplicity one by ergodicity, we have found a complete system
 16 of eigenfunctions, all of which are continuous. \square

17
 18 This finishes the proof of Theorem 7 in the direction (a) \implies (b). \square

19
 20 *Remark 2.* Under the hypothesis of Proposition 9, $\langle P(\widehat{\gamma}) \rangle = \hat{i}(\widehat{\mathbb{T}})$. This also holds under
 21 the assumptions of Lemma 6 (and, in fact, still in more general situations). This will be
 22 discussed further in [4].

23 24 7. Consequences of unique ergodicity, pure point dynamical spectrum and continuous 25 eigenfunctions

26 The aim of this section is to prove the following theorem. As discussed at the end of this
 27 section, this theorem will provide the proof of the missing direction of Theorem 7.

28
 29 **THEOREM 8.** *Let $\Lambda \subset G$ be a Meyer set and $(\mathbb{X}(\Lambda), G)$ be uniquely ergodic. Suppose that
 30 $(\mathbb{X}(\Lambda), G)$ has pure point dynamical spectrum and continuous eigenfunctions. Let $\mathbb{A}(\Lambda)$
 31 be the autocorrelation hull. Then, there exists a torus parametrization from \mathbb{X} to \mathbb{A} for
 32 which $\Lambda \mapsto 0$.*

33 To prove this result, we proceed as follows. In §7.1, we assume that (\mathbb{X}, G) is an
 34 arbitrary uniquely ergodic dynamical system with pure point spectrum and continuous
 35 eigenfunctions. We then show how to construct a compact topological group \mathbb{E} and a
 36 continuous surjective G -map $\beta_{\mathbb{E}} : \mathbb{X} \longrightarrow \mathbb{E}$. In the subsequent subsections, we return to
 37 the case of (\mathbb{X}, G) being a Meyer dynamical system, assuming now that we have pure point
 38 spectrum and continuous eigenfunctions, and show that the continuous map $\beta_{\mathbb{E}} : \mathbb{X} \longrightarrow \mathbb{E}$
 39 constructed in the first subsection is effectively none other than a torus parametrization
 40 $\beta_{\mathbb{A}} : \mathbb{X} \longrightarrow \mathbb{A}$.

41
 42 *Remark 3.* As investigated by Robinson [33] in the case of $G = \mathbb{R}^d$ and $G = \mathbb{Z}^d$,
 43 continuity of the eigenfunctions is related to uniform existence of certain limits (see [22]
 44 for recent results in the case of general LCA groups G as well).

01 7.1. *A general construction.* Let (\mathbb{X}, G) be a uniquely ergodic dynamical system with
 02 unique G -invariant probability measure μ . This gives rise to a unitary representation
 03 T of G on $L^2(\mathbb{X}, \mu)$.

04 Assume that T has pure point dynamical spectrum with all eigenfunctions continuous.
 05 This means that $L^2(\mathbb{X}, \mu)$ has an orthonormal basis $\{f_\lambda : \lambda \in P(T)\}$ where the point
 06 spectrum $P(T)$ (i.e. the set of eigenvalues of T) is some *subgroup* of \widehat{G} , the character
 07 group of G , and each f_λ is a continuous eigenfunction for the character λ . Note that,
 08 due to ergodicity, all eigenvalues are simple and the corresponding eigenspaces are thus
 09 one-dimensional [40]. We may assume that each f_λ is normalized to 1 (in the L^2 -norm).

10 Define $A' \sim A''$ when $f_\lambda(A') = f_\lambda(A'')$ for all $\lambda \in P(T)$. Let $\mathbb{E} := \mathbb{X}/\sim$ and let $\beta_{\mathbb{E}}$
 11 denote the canonical mapping from \mathbb{X} to \mathbb{E} . Note that the f_λ can be factored through the
 12 equivalence relation. Give the quotient space the uniform structure for which the cylinder
 13 sets given by

$$14 \quad U(F, \varepsilon) := \{(\beta_{\mathbb{E}}(A'), \beta_{\mathbb{E}}(A'')) \in \mathbb{E} \times \mathbb{E} : |f_\lambda(A') - f_\lambda(A'')| < \varepsilon, \lambda \in F\},$$

16 where F runs through all finite subsets of $P(T)$ and ε through the positive reals, are a
 17 fundamental system of entourages. The mapping $\beta_{\mathbb{E}} : \mathbb{X} \rightarrow \mathbb{E}$ is uniformly continuous
 18 because the eigenfunctions are continuous (hence, uniformly continuous, since \mathbb{X} is
 19 compact). Thus, \mathbb{E} is compact and hence complete.

20 Each of the basic entourages of \mathbb{E} is actually G -invariant (since the f_λ are
 21 *eigenfunctions*) and we obtain from this an obvious G -action on \mathbb{E} for which the natural
 22 mapping $\beta_{\mathbb{E}}$ from \mathbb{X} to \mathbb{E} is a G -map. This implies the orbit $\beta_{\mathbb{E}}(G + \Lambda)$ of $\beta_{\mathbb{E}}(\Lambda)$ to be
 23 dense in \mathbb{E} .

24 Pull back the uniformity of \mathbb{E} to G by using the entourages

$$25 \quad \{(s, t) \in G \times G : (\beta_{\mathbb{E}}(s + \Lambda), \beta_{\mathbb{E}}(t + \Lambda)) \in U(F, \varepsilon)\}.$$

28 This new uniformity on G is compatible with the group structure (this comes down again
 29 to the G -invariance of each of the fundamental entourages) and we have a uniformly
 30 continuous mapping of G , equipped with this new topology, into \mathbb{E} . In fact, \mathbb{E} is the
 31 completion of G under this new uniform topology, and since this latter is also an Abelian
 32 group, \mathbb{E} becomes a compact Abelian group. In more detail, since G is getting its structure
 33 by pulling back the induced structure on $\beta_{\mathbb{E}}(G + \Lambda)$ in \mathbb{E} under $t \mapsto \beta_{\mathbb{E}}(t + \Lambda)$, we may
 34 apply [9, Ch. II.3.8, Proposition 17] to see that the Hausdorff space associated with G
 35 under this new uniformity is homeomorphic to $\beta_{\mathbb{E}}(G + \Lambda)$ and, hence, their completions
 36 are also homeomorphic.

37 So, out of the continuity of the eigenfunctions, we obtain a new compact Abelian
 38 group \mathbb{E} and a torus parametrization

$$39 \quad \beta_{\mathbb{E}} : \mathbb{X} \rightarrow \mathbb{E}. \tag{20}$$

41 By construction, we have the following proposition.

42
 43 **PROPOSITION 10.** *For each $\lambda \in P(T)$, there exists a unique continuous function g_λ on \mathbb{E}*
 44 *with $f_\lambda = g_\lambda \circ \beta_{\mathbb{E}}$.*

01 Since $(\mathbb{E}, G, \theta_{\mathbb{E}})$ is pure point with eigenvalues $P(T)$, it must be conjugate to the
 02 G -action on $\mathbb{S} := \widehat{P(T)}$. This, and in fact a more general statement, is known as the
 03 Halmos–von Neumann representation theorem, compare [40, Theorem 5.18]. In the case
 04 at hand, we give a short proof. Explicitly, equip the subgroup $P(T)$ with the discrete
 05 topology, so that its dual group \mathbb{S} is compact. Since G is mapped homomorphically into \mathbb{S}
 06 (namely each $g \in G$ goes to the evaluation map at g of $P(T)$), it follows that \mathbb{S} admits
 07 a canonical (and minimal) action of G . Fix $x_0 \in \mathbb{E}$ and normalize the g_λ by requiring
 08 $g_\lambda(x_0) = 1, \lambda \in P(T)$. Then, $|g_\lambda(x)| = 1$ for all $x \in \mathbb{E}$.

09 **PROPOSITION 11.** *The groups \mathbb{E} and \mathbb{S} are isomorphic as topological groups by the*
 10 *mapping $j : \mathbb{E} \rightarrow \mathbb{S}$ defined by $j(x) : P(T) \rightarrow U(1), \lambda \mapsto g_\lambda(x)$, and thereby the*
 11 *dynamical systems (\mathbb{E}, G) and (\mathbb{S}, G) are topologically conjugate.*

13 *Proof.* By the normalization condition on $g_\lambda(x_0)$, we infer

$$14 \quad g_\lambda(x)g_\mu(x) = g_{\lambda\mu}(x) \quad \text{and} \quad g_{\lambda^{-1}}(x) = \overline{g_\lambda(x)}.$$

16 This implies that $j(x)$ is indeed an element of $\mathbb{S} = \widehat{P(T)}$. Now, the continuity of j follows
 17 directly from the continuity of the g_λ . One checks that j is a G -map. Injectivity of j
 18 follows as the $g_\lambda, \lambda \in P(T)$, separate the points of \mathbb{E} . To show that j is onto, it suffices to
 19 show that the dual j^* of j

$$20 \quad j^* : \widehat{\mathbb{S}} = P(T) \rightarrow \widehat{\mathbb{E}}, \quad j^*(\lambda) := \lambda \circ j$$

22 is injective (since the image of j is closed and the action of G on \mathbb{S} is minimal). This can
 23 be seen as follows. Let $\lambda_1, \lambda_2 \in P(T)$ be given, with

$$24 \quad j^*(\lambda_1) = j^*(\lambda_2).$$

26 Then, $\lambda_1 \circ j = \lambda_2 \circ j$, i.e. $j(x)(\lambda_1) = j(x)(\lambda_2)$ for every $x \in \mathbb{E}$. As $j(x)(\lambda_1) = g_{\lambda_1}(x)$
 27 and, similarly, for λ_2 , we see that this means $g_{\lambda_1} = g_{\lambda_2}$ which, in turn, implies $f_{\lambda_1} = f_{\lambda_2}$,
 28 and finally $\lambda_1 = \lambda_2$.

30 Thus, we see that j is indeed a continuous bijection between compact spaces. Therefore,
 31 the inverse of j is continuous as well.

32 One has to show two more things: that j is compatible with the group action, and that j
 33 is a group homomorphism. Both of them are more or less straightforward calculations. \square

35 **7.2. Pure point dynamical spectrum together with continuous eigenfunctions imply a**
 36 **torus parametrization.** In this section, we specialize the setting of the previous section by
 37 assuming that Λ is Meyer and $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic with pure point dynamical
 38 spectrum and continuous eigenfunctions. The main objective is to prove Theorem 8.
 39 Given (20), it remains to show that \mathbb{E} as constructed above is isomorphic to the hull
 40 $\mathbb{A} = \mathbb{A}(\Lambda)$ of Λ in the mACT. This is done in the next two sections. Here, we provide
 41 some preparation.

42 As in the proof of Lemma 6, let $C_c(G)$ denote the space of continuous complex-valued
 43 functions of compact support on G and define, for $c \in C_c(G)$, the function $\varphi_c : \mathbb{X} \rightarrow \mathbb{C}$
 44 by $\varphi_c(\Lambda) = (c * \delta_\Lambda)(0)$. Let $g_c := c * \tilde{c} * \gamma_\Lambda$ be the corresponding smoothed out

01 autocorrelation of Λ , which is a continuous function on G . From unique ergodicity and
02 Dworkin's argument [11, 20],

$$03 \quad g_c(t) = \langle T_t \varphi_c, \varphi_c \rangle \quad (21)$$

04 where $t \in G$ and $\langle \cdot, \cdot \rangle$ is the inner product on $L^2(\mathbb{X}, \mu)$ whereby it is a Hilbert space. It is
05 not hard to see that the function $t \mapsto \langle T_t f, f \rangle$ is continuous, bounded and positive definite
06 for any $f \in L^2(\mathbb{X}, \mu)$. Thus, by Bochner's theorem (see [23]), there exists a unique finite
07 positive measure σ_f on \widehat{G} with

$$08 \quad \langle T_t f, f \rangle = \int_{\widehat{G}} (\widehat{s}, t) d\sigma_f(\widehat{s})$$

09 (see [2, 3] as well for a further discussion of this). It turns out that, in our context,
10 the spectral measure σ_{φ_c} can be explicitly calculated for $c \in C_c(G)$ in terms of $\widehat{\gamma}$.
11 More precisely,

$$12 \quad \sigma_{\varphi_c} = |\widehat{c}|^2 \widehat{\gamma} \quad (22)$$

13 for any $c \in C_c(G)$, compare [36, 20, 2]. This equation links the dynamical spectrum and
14 the diffraction spectrum.

15
16
17 **PROPOSITION 12.** *Let λ be an eigenvalue of the uniquely ergodic dynamical system*
18 *($\mathbb{X}(\Lambda), G$), with associated normalized eigenfunction f_λ . Let $h \in L^2(\mathbb{X}(\Lambda), \mu)$ be*
19 *arbitrary. Then, $\langle h, f_\lambda \rangle = 0$ if and only if $\sigma_h(\{\lambda\}) = 0$.*

20 *Proof.* By Stone's theorem, compare [23, §36D], there exists a projection valued measure

$$21 \quad E: \text{Borel sets of } \widehat{G} \longrightarrow \text{projections on } L^2(\mathbb{X}(\Lambda), \mu)$$

22 with $E(B \cap C) = E(B)E(C)$ for $B, C \subset \widehat{G}$ measurable and

$$23 \quad \langle T_t f, g \rangle = \int_{\widehat{G}} (\widehat{s}, t) d\sigma_{f,g}(\widehat{s}) \quad (23)$$

24 where the measure $\sigma_{f,g}$ on \widehat{G} is defined by $\sigma_{f,g}(B) := \langle E(B)f, g \rangle$. In particular, we have
25 $\sigma_{f,f} = \sigma_f$.

26 From $E(B)E(\{\lambda\}) = E(B \cap \{\lambda\})$, we infer that $\sigma_{E(\{\lambda\})f,g}$ is concentrated on $\{\lambda\}$ for
27 arbitrary $f, g \in L^2(\mathbb{X}(\Lambda), \mu)$. This easily yields that $T_t E(\{\lambda\})f = (\lambda, t)E(\{\lambda\})f$ for any
28 $f \in L^2(\mathbb{X}(\Lambda), \mu)$.

29 Conversely, if f is an eigenfunction to λ , we infer from the validity of the equation
30 $(\lambda, t)\langle f, g \rangle = \langle T_t f, g \rangle = \int_{\widehat{G}} (\widehat{s}, t) d\sigma_{f,g}(\widehat{s})$, for all $t \in G$, that $\sigma_{f,g}$ is concentrated on λ .
31 This easily gives $E(\{\lambda\})f = f$ for any eigenfunction to the eigenvalue λ .

32 Put together, this means that $E(\{\lambda\})$ is the orthogonal projection onto the eigenspace
33 for the eigenvalue λ . This eigenspace is one-dimensional by ergodicity. Thus, $E(\{\lambda\})h =$
34 $\langle h, f_\lambda \rangle f_\lambda$ and we infer

$$35 \quad |\langle h, f_\lambda \rangle|^2 = \|E(\{\lambda\})h\|^2 = \langle E(\{\lambda\})h, E(\{\lambda\})h \rangle = \sigma_h(\{\lambda\}).$$

36 Now, the statement of the proposition is immediate. \square

37
38
39 We are now ready to prove the isomorphism between \mathbb{E} and $\mathbb{A} = \mathbb{A}(\Lambda)$. Both spaces
40 in question, \mathbb{E} and \mathbb{A} , are obtained by completion of G in uniform topologies for which
41 a fundamental system of G -invariant entourages exist. For this reason, it is actually
42 sufficient to show that the identity mapping from G to itself is bi-continuous at 0 when
43 these two topologies are put on two sides.
44

01 7.3. *Continuity of $\mathbb{A} \rightarrow \mathbb{E}$.* If Λ is a Meyer set, we know that $\Delta = \Lambda - \Lambda$ is uniformly
02 discrete. Consequently, there is a compact neighbourhood K of 0 in G so that, for all $t \in \Delta$,
03 $(t + K) \cap \Delta = \{t\}$.

04 Let $\{x_i\}$ be a net in G which converges to 0 in the autocorrelation topology. Then, there
05 are elements $v_i \in G$ converging to 0 in the original topology of G so that $d(v_i + x_i + \Lambda, \Lambda)$
06 converges to 0. Here, d is the pseudo-metric defined in (1), which, by (9), satisfies

$$07 \quad d(s + \Lambda, t + \Lambda) = \lim_{n \rightarrow \infty} \frac{\text{card}(((s + \Lambda) \Delta (t + \Lambda)) \cap A_n)}{\theta_G(A_n)} = 2(\eta(0) - \eta(s - t)),$$

09 so, for $y_i := v_i + x_i$,

$$10 \quad d(y_i + \Lambda, \Lambda) = 2(\eta(0) - \eta(y_i)) \rightarrow 0. \quad (24)$$

11 This convergence of the $\{y_i + \Lambda\}$ to Λ shows that $y_i \in \Delta$ for all sufficiently large i .
12 If we show that $\{y_i\}$ converges to 0 in the topology of \mathbb{E} , this will also give convergence of
13 the original net $\{x_i\}$, since the topology of $\mathbb{X}(\Lambda)$, and hence \mathbb{E} , is defined so that shifts by
14 small elements of G are small.

15 Now, $\gamma_\Lambda = \sum_{t \in \Delta} \eta(t) \delta_t$. Let $c \in C_c(G)$ with $\text{supp}(c * \tilde{c}) \subset K$. By our choice of K ,
16 0 is then the only element of Δ in $\text{supp}(c * \tilde{c})$. Thus,

$$18 \quad g_c(y_i) = (c * \tilde{c} * \gamma_\Lambda)(y_i) = \sum_{t \in \Delta} \eta(t) (c * \tilde{c})(y_i - t) = \eta(y_i) (c * \tilde{c})(0).$$

20 By (24), this implies $\lim_i g_c(y_i) = g_c(0)$. By (21), this means that

$$21 \quad \langle T_{y_i} \varphi_c, \varphi_c \rangle \rightarrow \langle \varphi_c, \varphi_c \rangle. \quad (25)$$

22 As we have pure point spectrum with the set of eigenvalues $P(T)$ and corresponding
23 normalized eigenfunctions f_λ , $\lambda \in P(T)$, we can write φ_c as a Fourier series, $\varphi_c =$
24 $\sum_{\lambda \in P(T)} a_\lambda f_\lambda$, where the a_λ (which depend on c) are complex numbers.

25 Then, using (25), we find $\sum \lambda(y_i) |a_\lambda|^2 \|f_\lambda\|^2 \rightarrow \sum |a_\lambda|^2 \|f_\lambda\|^2$ which results in

$$26 \quad \sum |a_\lambda|^2 (\lambda(y_i) - 1) \rightarrow 0,$$

27 by normalization of the eigenfunctions. Taking complex conjugates then yields

$$28 \quad \sum |a_\lambda|^2 (\overline{\lambda(y_i)} - 1) \rightarrow 0.$$

29 As λ takes values in $U(1)$, we have

$$30 \quad |\lambda(y_i) - 1|^2 = (1 - \lambda(y_i)) + (1 - \overline{\lambda(y_i)})$$

31 and we obtain

$$32 \quad \sum |a_\lambda|^2 |\lambda(y_i) - 1|^2 = \sum |a_\lambda|^2 (1 - \lambda(y_i) + 1 - \overline{\lambda(y_i)}) \rightarrow 0.$$

33 Thus, $\{\lambda(y_i)\} \rightarrow 1$, whenever $a_\lambda \neq 0$. Now, $a_\lambda \neq 0$ means that φ_c is not orthogonal to f_λ .

34 By Proposition 12, this is equivalent to $\lambda \in P(\sigma_{\varphi_c})$ (see (19) for the definition of $P(\cdot)$).

35 Thus, we have $\{\lambda(y_i)\} \rightarrow 1$ for all $\lambda \in P(\sigma_{\varphi_c})$ and for all $c \in C_c(G)$. As $c \in C_c(G)$ is
36 arbitrary, (22) then shows that $\{\lambda(y_i)\} \rightarrow 1$ for all $\lambda \in P(\widehat{\gamma})$, and then for all $\lambda \in \langle P(\widehat{\gamma}) \rangle$.

37 As $P(\widehat{\gamma}) = P(T)$ by Lemma 5, this means, for all $\lambda \in P(T)$,

$$38 \quad f_\lambda(y_i + \Lambda) = \lambda(y_i) f_\lambda(\Lambda) \rightarrow f_\lambda(\Lambda)$$

39 and this is precisely the meaning of convergence of $\{y_i\}$ to 0 in the $\mathbb{E}(\Lambda)$ -topology.

40 This concludes the continuity argument in the first direction.

01 7.4. *Continuity of $\mathbb{E} \rightarrow \mathbb{A}$.* Let $\lambda \in P(T)$, so $f_\lambda(x + \Lambda) = \lambda(x)f_\lambda(\Lambda)$ for all
 02 $x \in G$. The continuity of f_λ then shows that $|f_\lambda|$ is a non-zero constant function on $\mathbb{X}(\Lambda)$.
 03 Let $\{x_i\} \rightarrow 0$ in the \mathbb{E} -topology on G . Then, $\{f_\lambda(x_i + \Lambda)\} \rightarrow f_\lambda(\Lambda)$ shows that
 04 $\{\lambda(x_i)\} \rightarrow 1$ for all $\lambda \in P(T)$.

05 Let $c \in C_c(G)$ be chosen so that $0 \leq c(x) \leq 1$ for all $x \in G$, with $c(x) = 1 \Leftrightarrow x = 0$.
 06 Moreover, let c be so that $v := c * \tilde{c}$ satisfies $(\text{supp}(v) - \text{supp}(v)) \cap (\Delta - \Delta) = \{0\}$, which
 07 rests upon the Meyer property. Then, $\|c\|_2^2 = v(0) > v(x) \geq 0$ for all $x \in G \setminus \{0\}$ and
 08 $\text{supp}(v) \cap \Delta = \{0\}$.

09 Let $\varphi_c = \sum_{\lambda \in P(T)} a_\lambda f_\lambda$ be the Fourier expansion of φ_c . Choose $\varepsilon > 0$ and find a finite
 10 set $F \subset P(T)$ so that

$$\|\varphi_c - \sum_{\lambda \in F} a_\lambda f_\lambda\|_2 < \varepsilon.$$

13 Choose N in the index set of $\{x_i\}$ so that, for all $i \succcurlyeq N$ and all $\lambda \in F$,

$$|\lambda(x_i) - 1| < \frac{\varepsilon}{1 + \sum_{\lambda \in F} |a_\lambda|}.$$

17 Then,

$$\begin{aligned} 18 \quad \|T_{x_i} \varphi_c - \varphi_c\|_2 &< \left\| T_{x_i} \varphi_c - T_{x_i} \sum_{\lambda \in F} a_\lambda f_\lambda \right\|_2 + \left\| \sum_{\lambda \in F} \lambda(x_i) a_\lambda f_\lambda - \sum_{\lambda \in F} a_\lambda f_\lambda \right\|_2 \\ 19 \quad &+ \left\| \sum_{\lambda \in F} a_\lambda f_\lambda - \varphi_c \right\|_2 \\ 20 \quad &< \varepsilon + \sum_{\lambda \in F} |\lambda(x_i) - 1| |a_\lambda| \|f_\lambda\|_2 + \varepsilon < 3\varepsilon, \end{aligned}$$

25 since the T_x are unitary and $\|f_\lambda\|_2 = 1$.

26 Thus, $\{T_{x_i} \varphi_c\} \rightarrow \varphi_c$ and hence $\{\langle T_{x_i} \varphi_c, \varphi_c \rangle\} \rightarrow \langle \varphi_c, \varphi_c \rangle$ and, using (21) again,

$$\left\{ g_c(x_i) = \sum_{t \in \Delta} v(x_i - t) \eta(t) \right\} \rightarrow g_c(0). \quad (26)$$

31 For each x_i , there is at most one $t_i \in \Delta$ with $x_i - t_i \in \text{supp}(v)$. Thus,

$$g_c(x_i) = \begin{cases} v(x_i - t_i) \eta(t_i), & \text{if } t_i \text{ exists,} \\ 0, & \text{otherwise.} \end{cases}$$

35 Moreover, $g_c(0) = v(0) \eta(0)$.

36 Now, (26) implies that $v(x_i - t_i) \eta(t_i) \rightarrow v(0) \eta(0) \neq 0$ (so, in particular, the t_i must
 37 exist eventually). Since $0 \leq v(x_i - t_i) \leq v(0)$ and $0 \leq \eta(t_i) \leq \eta(0)$, we get $\{\eta(t_i)\} \rightarrow \eta(0)$
 38 and $\{v(x_i - t_i)\} \rightarrow v(0)$. By the choice of c , $\{v_i := t_i - x_i\} \rightarrow 0$.

39 Now, $\{x_i\}$ converges to 0 in the \mathbb{A} -topology since $\{v_i + x_i\} = \{t_i\}$, the $\{v_i\} \rightarrow 0$ in the
 40 original topology of G , and $\{d(t_i + \Lambda, \Lambda) = 2(\eta(0) - \eta(t_i))\} \rightarrow 0$.

41 This implies continuity in the other direction and completes the proof of Theorem 8. \square

44 7.5. *Proof of Theorem 7, (b) \implies (a).* This is immediate from Theorem 8. \square

01 8. *The proof of the sufficiency direction of Theorem 1*

02 *Proof.* The hypotheses of Theorem 1, in the direction of sufficiency, include those of
03 Theorem 7. Thus, we have a torus parametrization $\mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$ coming from the
04 mapping $\mathbb{X}(\Lambda) \rightarrow \mathbb{E}$. This provides us with a CPS according to Theorem 2.

05 By assumption, our torus parametrization is one-to-one almost everywhere. Thus,
06 Theorems 3 and 5 apply. Therefore, we obtain a window by Theorem 3, whose boundary
07 has Haar measure 0 in H by Theorem 5.

08 Repetitivity of Λ is equivalent to the minimality of $\mathbb{X}(\Lambda)$, which is assumed. Thus,
09 $\mathbb{X}(\Lambda)$ is associated with a regular model set according to Theorem 3 as required. \square
10

11 9. *The proof of necessity in Theorem 1*

12 Up to now, the direction of investigation has been from dynamical systems and torus
13 parametrizations to CPSs and model sets. In this section, we derive results going in the
14 other direction and prove that the conditions of Theorem 1 are necessary.
15

16
17 9.1. *Irredundancy.* Assume that we are given an IMS Λ with respect to the CPS
18 (G, H, \mathcal{L}) . We wish to construct a torus parametrization for the local hull of Λ . As we
19 have seen in Proposition 7, the curious property of irredundancy is crucial to the existence
20 of such a map. What happens if we have a CPS together with an IMS for which the
21 window fails irredundancy? Can we modify the CPS (and thereby the torus) to get the
22 irredundancy? The answer is yes. However, it is here that the notion of an *inter* model set
23 becomes important.

24
25 **LEMMA 7.** *Let (G, H, \mathcal{L}) be a CPS. Let $W \subset H$ be a non-empty compact set with*
26 *$\overline{W^\circ} = W$ and $\theta_H(\partial W) = 0$. Let $\Lambda \subset G$ with $\lambda(W^\circ) \subset \Lambda \subset \lambda(W)$ be arbitrary.*
27 *Then, there exists a CPS (G, H', \mathcal{L}') and $W' \subset H'$ compact, non-empty and irredundant,*
28 *with $W' = \overline{W'^\circ}$ and $\theta_{H'}(\partial W') = 0$, such that $\lambda(W'^\circ) \subset \Lambda \subset \lambda(W')$, i.e. that Λ is a*
29 *regular IMS for the new CPS (G, H', \mathcal{L}') with window W' .*

30 *Remark 4.* The proof of the lemma relies on factoring out the stabilizer of W . It is crucial
31 to note that sets of the form $\lambda(W)$ may *not* be representable as sets of the form $\lambda(W')$
32 in the emerging ‘quotient’ scheme. Rather, sets lying between $\lambda(W^\circ)$ and $\lambda(W)$ can be
33 exhibited as sets lying between $\lambda(W'^\circ)$ and $\lambda(W')$. We refer the reader to [19] for further
34 discussion.

35
36 *Proof.* Let (G, H, \mathcal{L}) be the given CPS, with the usual conditions on the projections π_1
37 and π_2 , and with the compact regular window $W = \overline{W^\circ} \neq \emptyset$. Let

$$I := \text{stab}_H(W) = \{t \in H : t + W = W\}$$

38
39
40 be the stabilizer of W , which is a subgroup of H . Clearly, I is closed, and $I \subset W - W$
41 implies that I is compact. Observe that we also have $W^\circ + I = W^\circ$.

42 Define the factor group $H' = H/I$ and let $\rho: H \rightarrow H'$ be the natural map. Moreover,
43 define

$$\mathcal{L}' := \{(x, \rho(x^\star)) : x \in L\} \subset G \times H'$$

01 together with a mapping $\mathcal{L} \rightarrow \mathcal{L}'$ defined by $(x, x^*) \mapsto (x, \rho(x^*))$. This is a group
 02 homomorphism, and surjective. Since $(x, x^*) \mapsto (0, 0) \in \mathcal{L}'$ is only possible for $x = 0$,
 03 we see that also $\rho(x^*) = 0$ in this case. Consequently, the kernel of this homomorphism is
 04 $\{(0, 0)\}$, and $\mathcal{L} \simeq \mathcal{L}'$.

05 Consider the diagram

$$\begin{array}{ccc} G \times H & \xrightarrow{\text{id} \times \rho} & G \times H' \\ \text{nat} \downarrow & & \downarrow \text{nat} \\ (G \times H)/\mathcal{L} & \longrightarrow & (G \times H')/\mathcal{L}' \end{array}$$

11 where the horizontal arrow in the lower line exists in an obvious way, because $(x, x^*) \in \mathcal{L}$
 12 is mapped to $(x, \rho(x^*)) \in \mathcal{L}'$. Since \mathcal{L} is a closed subgroup of $G \times H$, quotient is
 13 Hausdorff. Moreover, the natural mapping $G \times H \rightarrow (G \times H)/\mathcal{L}$ is an open map,
 14 and we get that $(G \times H)/\mathcal{L} \rightarrow (G \times H')/\mathcal{L}'$ is continuous.

15 Consequently, $(G \times H')/\mathcal{L}'$ is compact, whence \mathcal{L}' is co-compact. Consider now a
 16 compact neighbourhood $U \times \rho(V)$ of 0 in $G \times H'$, where V is a compact neighbourhood
 17 of 0 in H and U is a compact neighbourhood of 0 in G . Then,

$$19 \quad \mathcal{L}' \cap (U \times \rho(V)) = \{(x, \rho(x^*)) : x \in U, \rho(x^*) \in V\} = \{(x, x^*) : x \in U, x^* \in V + I\}.$$

20 Since $U \times (V + I)$ is compact and \mathcal{L} is a lattice, the set $\{(x, x^*) : x \in U, x^* \in V + I\}$
 21 is *finite*, and contains $(0, 0^*) = (0, 0)$. Consequently, $\mathcal{L}' \cap (U \times \rho(V))$ is also finite, with
 22 $(0, 0) \in \mathcal{L}' \cap (U \times \rho(V))$. Consequently, $(0, 0)$ is isolated and \mathcal{L}' is (uniformly) discrete.

23 This shows that (G, H', \mathcal{L}') is another CPS, with all of the properties required, for which
 24 we now need a window. To this end, define $W' = \rho(W)$. We note that W' is compact since
 25 W is. Moreover, also W is the complete preimage of $\rho(W)$ since $W = I + W$. As ρ is an
 26 open map, $\rho(W^\circ)$ is open and we have $W'^\circ = \rho(W^\circ)$ and $W' = \overline{W'^\circ} \neq \emptyset$.

27 Note that $\rho: H \rightarrow H'$ induces a mapping from the Haar measure θ_H to a Haar
 28 measure $\theta_{H'}$, with

$$30 \quad \theta_{H'}(\partial W') = \theta_{H'}(W' \setminus W'^\circ) = \theta_H((W + I) \setminus (W^\circ + I)) = \theta_H(W \setminus W^\circ) = 0.$$

31 Finally, if $\lambda(W^\circ) \subset \Lambda \subset \lambda(W)$ for some $\Lambda \subset G$ in the original CPS, then

$$33 \quad \lambda(W'^\circ) \subset \Lambda \subset \lambda(W')$$

34 in the new CPS. This is easy to check because $x \in \lambda(W'^\circ) \iff \rho(x^*) \in W'^\circ \iff$
 35 $x^* \in W^\circ$ and similarly for the remaining details. This completes the argument. \square

37
 38 **9.2. Torus parametrizations for model sets.** Assume that we are given an IMS Λ with
 39 respect to the CPS (G, H, \mathcal{L}) with a window W . According to the previous section,
 40 we may assume that the CPS is irredundant, and then, by Proposition 7, that we have a
 41 torus parametrization of the local hull into the compact group \mathbb{T} of this scheme. We now
 42 convert this to a torus map into $\mathbb{A}(\Lambda)$.

43 **LEMMA 8.** *Let (G, H, \mathcal{L}) be a CPS and let $W \subset H$ be a compact, non-empty and*
 44 *irredundant window, with $W = \overline{W^\circ}$ and $\theta_H(\partial W) = 0$. Then, the following holds.*

- 01 (a) $\theta_H((W - h_n) \triangle W) \longrightarrow 0$, whenever $\{h_n\}$ is a net in H with $h_n \rightarrow 0 \in H$.
 02 (b) If $h \in H$ satisfies $\theta_H((W - h) \triangle W) = 0$, then $h = 0$.

03 *Proof.* (a) Denote the canonical representation of H on $L^1(H, \theta_H)$ by τ^H , i.e., $\tau_h^H f(x) =$
 04 $f(-h+x)$. It is well known that this representation is strongly continuous [32]. This means
 05 that $\tau_h^H f \rightarrow f$ for $h \rightarrow 0$ and all $f \in L^1(H, \theta_H)$. Now, let 1_W be the characteristic
 06 function of the compact set $W \subset H$. Then, 1_W belongs to $L^1(H, \theta_H)$ and therefore
 07

$$08 \quad \theta_H((W - h) \triangle W) = \|\tau_h^H 1_W - 1_W\|_{L^1} \longrightarrow 0, \quad \text{as } h \rightarrow 0.$$

09 This proves (a).

10 (b) Let $h \in H$ be given with $\theta_H((W - h) \triangle W) = 0$. As the window is translationally
 11 fixed only by $0 \in H$, it suffices to show $W - h = W$, i.e. $(W - h) \triangle W = \emptyset$. Assume the
 12 contrary. Then, $(W - h) \triangle W$ actually contains an open set because $W = \overline{W^\circ}$. This implies
 13 $\theta_H((W - h) \triangle W) > 0$ and we arrive at a contradiction. \square
 14

15 **PROPOSITION 13.** Let (G, H, \mathcal{L}) be a CPS with associated torus \mathbb{T} . Let $W \subset H$ be
 16 a compact, non-empty and irredundant window, with $W = \overline{W^\circ}$ and $\theta_H(\partial W) = 0$.
 17 Let $\Lambda \subset G$ be an arbitrary IMS, i.e. $\lambda(W^\circ) \subset \Lambda \subset \lambda(W)$. Then, selecting a
 18 neighbourhood U so that $U \in \mathcal{D}_U$, the mapping $j: \mathbb{T} \rightarrow \mathcal{D}_U^\equiv, (x, h) + \mathcal{L} \mapsto$
 19 $\beta(x + \lambda(W - h))$, is continuous and injective. In particular, j gives an isomorphism
 20 between \mathbb{T} and $\mathbb{A}(\Lambda)$.

21 *Proof.* We first show the continuity of j . Let $\{\xi_i\}$ be a net in \mathbb{T} with $\xi_i \rightarrow \xi \in \mathbb{T}$. Then,
 22 without loss of generality, we may assume that $\xi = (x, h) + \mathcal{L}$, $\xi_i = (x_i, h_i) + \mathcal{L}$ and
 23 $x_i \rightarrow x$, $h_i \rightarrow h$. As the topology of \mathcal{D}_U^\equiv allows for small translations, it suffices to
 24 show that $\beta(\lambda(W - h_i)) \rightarrow \beta(\lambda(W - h))$. By part (a) of Lemma 8, we infer
 25

$$26 \quad \theta_H((W - h_i) \triangle (W - h)) \longrightarrow 0. \quad (27)$$

27 Therefore,

$$\begin{aligned} 28 \quad d(\beta(\lambda(W - h_i)), \beta(\lambda(W - h))) &= \text{dens}(\lambda(W - h_i) \triangle \lambda(W - h)) \\ 29 &= \text{dens}(\lambda((W - h_i) \triangle (W - h))) \\ 30 &\stackrel{\text{by Theorem 4}}{=} \text{dens}(\mathcal{L})\theta_H((W - h_i) \triangle (W - h)) \\ 31 &\stackrel{\text{by (27)}}{\longrightarrow} 0. \end{aligned}$$

32 This proves the continuity statement.

33 We now show injectivity. To do so, let (x, h) and (x', h') in $G \times H$ be given with
 34 $\beta(x + \lambda(W - h)) = \beta(x' + \lambda(W - h'))$. This implies
 35

$$36 \quad \beta(\lambda(z^* + k + W')) = \beta(\lambda(W')),$$

37 where $z = x - x'$, $k = h' - h$ and $W' = -h' + W$. By Proposition 7 and Theorem 4,
 38 we then obtain
 39

$$\begin{aligned} 40 \quad \text{dens}(\mathcal{L})\theta_H((z^* + k + W') \triangle \lambda(W')) &= \text{dens}(\lambda(z^* + k + W') \triangle \lambda(W')) \\ 41 &= d(\beta(\lambda(z^* + k + W')), \beta(\lambda(W'))) = 0. \end{aligned}$$

01 By part (b) of Lemma 8, this gives $0 = k + z^*$ or, put differently, $(x, h) + \mathcal{L} = (x', h') + \mathcal{L}$.
 02 This proves injectivity.

03 The inverse of a continuous injective map on a compact space is continuous as well.

04 So far, we know that j is a continuous bijective map from \mathbb{T} onto $j(\mathbb{T})$. By the
 05 continuity of j and the minimality of the action of G on \mathbb{T} , $j(\mathbb{T})$ is just $\mathbb{A}(\lambda(W))$.
 06 By uniform distribution, $\beta(\lambda(W)) = \beta(\Lambda)$ and $\mathbb{A}(\lambda(W)) = \mathbb{A}(\Lambda)$ follows. This proves
 07 the last statement. \square

08 *Remark 5.* It is known from [28] that \mathbb{T} and $\mathbb{A}(\Lambda)$ are isomorphic for regular model sets,
 09 and our proof is in some sense a variant of the proof in [28]. However, our result here is
 10 more explicit in that β is shown to establish this isomorphism, and this will allow us to
 11 clarify the relationship between β and $\beta_{\mathbb{A}}$.
 12

13 **THEOREM 9.** *Let (G, H, \mathcal{L}) be a CPS, and let a non-empty window $W \subset H$ with
 14 $W = \overline{W^\circ}$ and $\theta_H(\partial W) = 0$ be given. If $\Lambda \subset G$ satisfies $t + \lambda(W^\circ) \subset \Lambda \subset t + \lambda(W)$
 15 for some $t \in G$, then the canonical mapping $\beta: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$ is continuous and one-
 16 to-one almost everywhere.*

17 *Proof.* By Lemma 7, we may assume, without loss of generality, that the CPS is
 18 irredundant. Assume furthermore that $t = 0$. Proposition 7 then gives a torus
 19 parametrization $\beta_{\mathbb{T}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{T}$ with
 20

$$21 \quad \beta_{\mathbb{T}}(\Lambda') = (x, h) + \mathcal{L} \iff x + \lambda(-h + W^\circ) \subset \Lambda' \subset x + \lambda(-h + W). \quad 22$$

23 By $\theta_H(\partial W) = 0$ and Theorem 4, $\beta(\Lambda') = \beta(x + \lambda(-h + W))$ whenever
 24 $x + \lambda(-h + W^\circ) \subset \Lambda' \subset x + \lambda(-h + W)$. Thus, Proposition 13 shows that $\beta = j \circ \beta_{\mathbb{T}}$
 25 with a continuous injective j . Thus, β is a continuous torus parametrization. It remains to
 26 show that it is one-to-one almost everywhere. However, this follows from Theorem 5. \square
 27

28 *Remark 6.* Proposition 13 and the considerations in the proof of the previous theorem
 29 effectively show that the maps $\beta_{\mathbb{A}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}$ and $\beta_{\mathbb{T}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{T}$, yielding a
 30 description of Λ as a regular model set in the previous sections, can be identified with
 31 the canonical map β .
 32

33 9.3. The end of Theorem 1.

34 *Proof.* Let Λ be a regular model set. By Lemma 7, we may assume that its CPS is
 35 irredundant and that we are in the situation of Theorem 9. This provides us with an almost
 36 everywhere one-to-one continuous mapping of $\mathbb{X}(\Lambda)$ onto $\mathbb{A}(\Lambda)$. Theorem 6 shows that
 37 $\mathbb{X}(\Lambda)$ is uniquely ergodic and, since Λ is repetitive, $\mathbb{X}(\Lambda)$ is also minimal. By Theorem 7,
 38 we obtain a pure point dynamical spectrum with continuous eigenfunctions that separate
 39 almost all points of $\mathbb{X}(\Lambda)$. \square
 40

42 10. The crystallographic case

43 The aim of this section is to give a proof of the following result on the characterization of
 44 fully periodic Delone sets.

01 THEOREM 10. *Let G be an LCA group and Λ a uniformly discrete subset of G . Then, the*
 02 *following assertions are equivalent.*

- 03 (i) Λ is crystallographic.
 04 (ii) Λ is Meyer and the map $\beta : \mathbb{X}(\Lambda) \longrightarrow \mathbb{A}(\Lambda)$ is continuous and injective.
 05 (iii) All of the following conditions hold:
 06 (1) all elements of $\mathbb{X}(\Lambda)$ are Meyer sets;
 07 (2) $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic;
 08 (3) $(\mathbb{X}(\Lambda), G)$ has pure point dynamical spectrum with continuous eigenfunctions;
 09 (4) the eigenfunctions separate all points of $\mathbb{X}(\Lambda)$.

10 *In this case, $(\mathbb{X}(\Lambda), G)$ is also minimal, hence strictly ergodic.*

11 Recall that Λ is called *crystallographic* (or fully periodic) if its periods

$$12 \text{per}(\Lambda) := \{t \in G : t + \Lambda = \Lambda\}$$

13 form a lattice, i.e. a co-compact discrete subgroup of G .

14 We start by proving the equivalence of claims (i) and (ii) of Theorem 10.

15 LEMMA 9. *The Delone set $\Lambda \subset G$ is crystallographic if and only if Λ is Meyer and the*
 16 *mapping $\beta : \mathbb{X}(\Lambda) \longrightarrow \mathbb{A}(\Lambda)$ is continuous and injective.*

17 *Proof.* The ‘only if’ part is easy: if the Delone set Λ is crystallographic, with lattice of
 18 periods $P = \text{per}(\Lambda)$, it is of the form $\Lambda = F + P$, where F must be a finite set due to the
 19 uniform discreteness of Λ . From $P - P = P$, we have $\Lambda - \Lambda = (F - F) + P$. This is
 20 still uniformly discrete, because $F - F$ is still finite, and Λ is Meyer.

21 Let $\mathbb{T} = G/\text{per}(\Lambda)$ be the compact quotient of G by the set of periods. Then, there is a
 22 natural isomorphism $\mathbb{T} \longrightarrow \mathbb{X}(\Lambda)$. This easily yields the statement about β .

23 We now prove the ‘if’ statement: as β is continuous, $\mathbb{A}(\Lambda)$ is compact and we have pure
 24 point diffraction. In particular, the ε -almost periods P_ε are relatively dense.

25 Obviously, $\text{per}(\Lambda)$ is a subgroup of G . As Λ is a Meyer set, $\text{per}(\Lambda)$, being a subset
 26 of $\Lambda - \Lambda$, is uniformly discrete. Therefore, it suffices to show that the set of periods is
 27 relatively dense (which implies that the quotient $G/\text{per}(\Lambda)$ is compact). We shall show
 28 that the set of periods contains the set of P_ε of ε -almost periods for a suitable $\varepsilon > 0$. As P_ε
 29 is relatively dense, the desired statement follows. We now give the details.

30 As $\mathbb{A}(\Lambda)$ is compact and β is continuous and injective by assumption, it has a
 31 continuous inverse

$$32 \alpha : \mathbb{A}(\Lambda) \longrightarrow \mathbb{X}(\Lambda).$$

33 As $\mathbb{A}(\Lambda)$ is compact, α is even uniformly continuous. Thus, for every compact $C \subset G$ and
 34 open $V \subset G$, there exists an $\varepsilon > 0$ such that

$$35 (\alpha(\xi) + t, \alpha(\xi)) = (\alpha(\xi + t), \alpha(\xi)) \in U_{\text{LT}}(C, V) \quad (28)$$

36 for all $\xi \in \mathbb{A}$ and all $t \in P_\varepsilon$, where we use the addition of t for the translation action on both
 37 spaces for simplicity. Here, of course,

$$38 U_{\text{LT}}(C, V) = \{(\Gamma, \Gamma') : (\Gamma + v) \cap C = \Gamma' \cap C \text{ for a suitable } v \in V\}.$$

01 Now, choose an open neighbourhood V of 0 in G according to Fact 4, meaning that
 02 $V \cap ((\Lambda - \Lambda) + (\Lambda - \Lambda)) = \{0\}$, and a compact set $C \subset G$ such that, for all $t \in G$,
 03 $(t + C) \cap \Gamma \neq \emptyset$.

04 Choose $\varepsilon > 0$ so that (28) holds. As α is onto, we infer that, for every $\Gamma \in \mathbb{X}(\Lambda)$ and
 05 every $t \in P_\varepsilon$,

$$06 \quad (\Gamma + t, \Gamma) \in U_{\text{LT}}(C, V).$$

07 Fact 4 then implies

$$08 \quad (\Gamma + t) \cap C = \Gamma \cap C$$

09 for every $\Gamma \in \mathbb{X}(\Lambda)$ and $t \in P_\varepsilon$. As Γ is arbitrary, we can replace Γ by $s + \Gamma$ with $s \in G$
 10 arbitrary. Thus, we end up with

$$11 \quad (\Gamma + s + t) \cap C = (\Gamma + s) \cap C$$

12 for every $s \in G$. As $C \neq \emptyset$, this shows that every $t \in P_\varepsilon$ is a period of Γ . \square

13 We now show the equivalence of claims (ii) and (iii).

14 LEMMA 10. A set $\Lambda \subset G$ is Meyer and $\beta: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$ is continuous and injective
 15 if and only if the following four conditions hold:

- 16 (1) all elements of $\mathbb{X}(\Lambda)$ are Meyer sets;
- 17 (2) $(\mathbb{X}(\Lambda), G)$ is uniquely ergodic;
- 18 (3) $(\mathbb{X}(\Lambda), G)$ has pure point dynamical spectrum with continuous eigenfunctions;
- 19 (4) the eigenfunctions separate all points of $\mathbb{X}(\Lambda)$.

20 *Proof.* We first show the ‘only if’ part: the validity of (1) is clear as Λ is Meyer.
 21 By assumption on β , $\mathbb{X}(\Lambda)$ is isomorphic to the group $\mathbb{A}(\Lambda)$. Now, $(\mathbb{A}(\Lambda), G)$ is uniquely
 22 ergodic (as the action of G is minimal and $\mathbb{A}(\Lambda)$ is a group) and it has pure point dynamical
 23 spectrum with continuous eigenfunctions given by the characters. As, by assumption
 24 on β , the dynamical system $(\mathbb{X}(\Lambda), G)$ is topologically conjugate to $(\mathbb{A}(\Lambda), G)$, the
 25 assertions (2), (3) and (4) follow.

26 We now prove the ‘if’ part. We can apply Theorem 7 to obtain a continuous mapping
 27 $\beta_{\mathbb{A}}: \mathbb{X}(\Lambda) \rightarrow \mathbb{A}(\Lambda)$ as we have pure point dynamical spectrum with continuous
 28 eigenfunctions. Moreover, again by Theorem 7, the map is actually injective because the
 29 eigenfunctions separate *all* points.

30 It remains to show that $\beta_{\mathbb{A}}$ agrees with β . We first note that $(\mathbb{X}(\Lambda), G)$ is minimal,
 31 because $(\mathbb{A}(\Lambda), G)$ is minimal and $\beta_{\mathbb{A}}$ is injective. By injectivity of $\beta_{\mathbb{A}}$ and Theorem 3,
 32 $(\mathbb{X}(\Lambda), G)$ is associated with a repetitive model set. In fact, the model set is regular by
 33 Proposition 5. Thus, by Theorem 9, the map β is continuous. Consequently, β and $\beta_{\mathbb{A}}$
 34 are continuous G -maps from $\mathbb{X}(\Lambda)$ into \mathcal{D}_U^{\equiv} , for suitable U , which agree on Λ . Therefore,
 35 they agree everywhere. \square

36 A. Appendix. Meyer sets in locally compact Abelian groups

37 Let G be an LCA group and let Λ be a Delone subset of G . We wish to compare the
 38 following two properties that Λ may have.

- 39 **(M1)** $\Lambda - \Lambda \subset \Lambda + F$ for some finite set F .
- 40 **(M2)** $\Lambda - \Lambda$ is uniformly discrete.

01 Often these properties are taken as (equivalent) characterizations of Meyer sets, which
 02 they are for groups of the form \mathbb{R}^d . In fact, it is easy to see that **(M1)** always implies
 03 **(M2)**. The reverse implication for \mathbb{R}^d was proved by Lagarias [17]. Here, we prove it for
 04 all compactly generated LCA groups. The proof goes in two steps. First, we prove that
 05 the group generated by Λ is finitely generated. In fact, this is equivalent to saying that G
 06 is compactly generated. After that, we can basically follow Lagarias' proof in the more
 07 general setting.

08 An apparently weaker concept than uniform discreteness is weak uniform discreteness:

09 *Definition 6.* $S \subset G$ is *weakly uniformly discrete* if for each compact subset K of G
 10 and for all $a \in G$, $\text{card}(S \cap (a + K))$ is bounded by a constant that depends only on K
 11 (not on a).

12 Remarkably, as we shall see, for a Delone subset Λ of a compactly generated group G ,
 13 the difference set $\Delta := \Lambda - \Lambda$ is uniformly discrete if and only if it is weakly uniformly
 14 discrete.
 15

16 **PROPOSITION 14.** *If a Delone set Λ satisfies **(M1)**, it also satisfies **(M2)**.*

17 *Proof.* Since Λ is a Delone set, it is locally finite, i.e. $\Lambda \cap K$ is a finite set (or empty), for
 18 any compact set $K \subset G$. To establish the uniform discreteness of Δ , it is sufficient to show
 19 that 0 is an isolated point of $\Delta - \Delta$. Using **(M1)** twice, one has
 20

$$21 \quad 0 \in \Delta - \Delta \subset (\Lambda + F) - (\Lambda + F) = \Delta + (F - F) \subset \Lambda + F'$$

22 where $F' = F + F - F$ is still a finite set. Consequently, $\Lambda + F'$ is locally finite, and 0
 23 must be an isolated point of it, hence also of $\Delta - \Delta$. This gives **(M2)**. \square
 24

25 **PROPOSITION 15.** *Let G be an LCA group and let Λ be relatively dense in G . Suppose
 26 that $\langle \Lambda \rangle$ (the subgroup of G generated by Λ) is finitely generated. Then, G is compactly
 27 generated.*

28 *Proof.* Let F be a finite set that generates $\langle \Lambda \rangle$. As Λ is relatively dense, there is a compact
 29 set $K \subset G$ with $G = \Lambda + K$. Then, $F \cup K$ is compact and generates G . \square
 30

31 **LEMMA 11.** *Let G be an LCA group of the form $G' \times T$ where T is a compact group.
 32 Then, the projection mapping $G \rightarrow G'$ defined by this splitting maps locally finite sets to
 33 locally finite sets.*

34 *Proof.* Suppose that $S \subset G$ is locally finite, but its projection P' is not. Then, there exists
 35 a compact set $K \subset G'$ with $P' \cap K$ infinite and we have $S \cap (K \times T)$ infinite too. This is
 36 a contradiction because $K \times T$ is compact. \square
 37

38 **PROPOSITION 16.** *Let G be compactly generated. Let $\Lambda \subset G$ be relatively dense and
 39 suppose that $\Delta = \Lambda - \Lambda$ is locally finite. Then, $\langle \Lambda \rangle$ is finitely generated.*

40 *Proof.* By the structure theorem for compactly generated LCA groups [13, Theorem 9.8],
 41 G is isomorphic to $\mathbb{R}^m \times \mathbb{Z}^n \times T$, where T is compact. We identify G with this group and
 42 so can also view it as a subgroup of $\mathbb{R}^m \times \mathbb{R}^n \times T =: G' \times T$. We shall use $'$ to indicate
 43 the projection map of $G' \times T$ onto G' . By Lemma 11, $\Delta' \subset G'$ is locally finite. Also, Λ is
 44 locally finite, due to the corresponding property of Δ .

01 Select a compact set $C \subset G$ so that $\Lambda + C = G$. Since the projection of C into G' is
 02 compact, we can find $R > 0$ so that $\Lambda + (B_R \times T) \supset \mathbb{R}^m \times \mathbb{Z}^n \times T$, where B_R is the open
 03 ball of radius R around 0 in $\mathbb{R}^m \times \mathbb{R}^n$. Increasing R if necessary, we may assume

$$04 \quad \Lambda + (B_R \times T) = \mathbb{R}^m \times \mathbb{R}^n \times T.$$

05
 06 Consider $F := (\Lambda \cup (\Lambda - \Lambda)) \cap (B_{2R} \times T)$, which is finite. It is plain that $F \subset \langle \Lambda \rangle$.
 07 We show that $\langle \Lambda \rangle = \langle F \rangle$. In fact, Λ is contained in the semigroup generated by F .

08 Let $\lambda \in \Lambda$. If $\lambda \in B_{2R} \times T$, then $\lambda \in F$, so suppose $\lambda \notin B_{2R} \times T$. We need to get
 09 closer to 0, using a point of F . To this end, let $B_R(u')$ be the open ball of radius R around
 10 u' in G' , where u' is taken to be the unique point which is at distance R from λ' on the
 11 line segment $[0, \lambda']$ in $\mathbb{R}^m \times \mathbb{R}^n$ joining 0 to λ' . Thus, $\lambda' \in \overline{B_R(u')} \setminus B_R(u') = \partial B_R(u')$.
 12 Let $u := (u', 0) \in G' \times T$. By our choice of R , we can write $u = \mu_1 + b$, where
 13 $\mu_1 \in \Lambda, b \in B_R \times T$, so $\mu_1' = u' - b' \in B_R(u')$.

14 We have (i) $\mu_1 \in \Lambda$, (ii) $|\mu_1'| < |\lambda'|$, (iii) $|\lambda' - \mu_1'| < 2R$, where $|\cdot|$ is the standard
 15 Euclidean norm in \mathbb{R}^{m+n} . Also, $\lambda - \mu_1 \in (B_{2R} \times T) \cap \Delta \subset F$ and so we have

$$16 \quad \lambda = f_1 + \mu_1, \quad \text{with } f_1 \in F, \mu_1 \in \Lambda, |\mu_1'| < |\lambda'|.$$

17
 18 We now continue inductively, getting

$$19 \quad \lambda = f_1 + \cdots + f_k + \mu_k$$

20
 21 where $f_1, \dots, f_k \in F, \mu_k \in \Lambda$ and $|\mu_k'| < |\mu_{k-1}'| < \cdots < |\lambda'|$, until $|\mu_k'| < 2R$. This
 22 must happen for some k since Λ' is locally finite and $\Lambda' \cap \overline{B_{\lambda'}(0)}$ is finite. Then, $\mu_k \in F$
 23 and we have shown that $\lambda \in \langle F \rangle$. \square

24 **THEOREM 11.** *Let G be a compactly generated LCA group. Suppose that Λ is a relatively*
 25 *dense subset of G and that $\Lambda - \Lambda$ is weakly uniformly discrete. Then, Λ satisfies (MI).*

26
 27 The proof of this result is really not different from that given in [17]. The only things to
 28 notice are that the full strength of uniform discreteness of $\Lambda - \Lambda$ is not required and that
 29 we are no longer confined to real spaces.

30 *Proof.* We may assume that $0 \in \Lambda$, translating Λ if necessary. Let $L := \langle \Lambda \rangle$ which
 31 is finitely generated by Proposition 16: say $\langle \Lambda \rangle = \langle e_1, \dots, e_s \rangle$. For all $x \in L$,
 32 $x = \sum_{i=1}^s a_i e_i, a_i \in \mathbb{Z}$, although not necessarily uniquely.

33 Define $\|x\| = \min\{\sum |a_i| : x = \sum a_i e_i\}$. This defines a norm on L (where the
 34 proof of the triangle inequality requires a short calculation). For each $N \in \mathbb{N}$, define
 35 $F(N) := \{x \in L : \|x\| \leq N\}$, which is clearly a finite set.

36 Let $K \subset G$ be a symmetric compact neighbourhood of 0 so that, for every $u \in G$,
 37 we have $(u + K) \cap \Lambda \neq \emptyset$, and also that G is generated as a group by K . The goal is
 38 now to show the existence of finitely many ‘stepping stones’, forming a set F , such that
 39 any difference $x - y$ of points in Λ lies in $\Lambda + F$.

40 To this end, let

$$41 \quad M := \max\{\text{card}((u + 2K) \cap (\Lambda - \Lambda)) : u \in G\},$$

$$42 \quad m := \max\{\|u\| : u \in (\Lambda - \Lambda) \cap (K + K - K)\}.$$

43
 44 The former exists on the basis of the weak uniform discreteness of $\Lambda - \Lambda$.

01 Let $x, y \in \Lambda$ and let $v := y - x$. Now, $x \in \ell K := K + \dots + K$ (ℓ summands) for
02 some ℓ , and we may write

$$03 \quad x = k_1 + \dots + k_\ell, \quad \text{for some } k_i \in K.$$

05 Let $x_0 := x, x_1 := x - k_1, x_2 := x - k_1 - k_2, \dots, x_\ell = 0$ and define the parallel
06 sequence $y_i := x_i + v, 0 \leq i \leq \ell$. Then, $y_i - y_{i+1} = x_i - x_{i+1} = k_{i+1} \in K$, for all
07 $0 \leq i \leq \ell - 1$.

08 Choose $p_i, q_i \in \Lambda$ with $p_i - x_i, q_i - y_i \in K$ with the special choices $p_0 = x, q_0 = y$,
09 $p_\ell = 0 = x_\ell$. Note that $q_0 = y = x + v = x_0 + v = y_0$, so in particular $q_0 - y_0 = 0 \in K$.
10 Then, for each $i \in \{0, 1, \dots, \ell\}$, one has $q_i - p_i - v = q_i - y_i + x_i - p_i \in 2K$. Thus,

$$11 \quad V := \{q_i - p_i : i = 0, \dots, \ell\} \subset \{(v + 2K) \cap (\Lambda - \Lambda)\},$$

13 so $\text{card}(V) \leq M$.

14 Similarly,

$$16 \quad p_i - p_{i+1} = (p_i - x_i) + (x_i - x_{i+1}) + (x_{i+1} - p_{i+1}) \in (K + K - K) \cap (\Lambda - \Lambda)$$

17 so $\|p_i - p_{i+1}\| \leq m$.

18 In the same way, $\|q_i - q_{i+1}\| \leq m$, so

$$20 \quad \|q_i - p_i - (q_{i+1} - p_{i+1})\| \leq 2m.$$

21 Along with the bound on the cardinality of V , this gives rise to

$$23 \quad \|u - u'\| \leq 2mM$$

24 for all $u, u' \in V$.

25 Now, $v = q_0 - p_0 = y - x \in V$ and $q_\ell = q_\ell - p_\ell \in V$, so $\|v - q_\ell\| \leq 2mM$.
26 Since $v, q_\ell \in L, v - q_\ell \in F(2mM)$ and

$$28 \quad y - x = v \in q_\ell + F(2mM) \subset \Lambda + F(2mM).$$

29 Since $x, y \in \Lambda$ were arbitrary, $\Lambda - \Lambda \subset \Lambda + F(2mM)$. □

31 **COROLLARY 1.** *Let G be a compactly generated LCA group. Let $\Lambda \subset G$ be a Delone set.*
32 *Then, $\Lambda - \Lambda$ is uniformly discrete if and only if it is weakly uniformly discrete.*

33 *Proof.* Use Theorems 11 and 14. □

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