Point Sets and Dynamical Systems in the Autocorrelation Topology

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Abstract

This paper is about the topologies arising from statistical coincidence on locally finite point sets in locally compact Abelian groups G. The first part defines a uniform topology (autocorrelation topology) and proves that, in effect, the set of all locally finite subsets of G is complete in this topology. Notions of statistical relative denseness, statistical uniform discreteness, and statistical Delone sets are introduced.

The second part looks at the consequences of mixing the original and autocorrelation topologies, which together produce a new Abelian group, the autocorrelation group. In particular the relation between its compactness (which leads then to a G-dynamical system) and pure point diffractivity is considered. Finally for generic regular model sets it is shown that the autocorrelation group can be identified with the associated compact group of the cut and project scheme that defines it. For such a set the autocorrelation group, as a G- dynamical system, is a factor of the dynamical local hull.

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1 Introduction

The use of dynamical systems in the study of the internal order of discrete point sets in real spaces $\mathbb{R}^d$ has been remarkably effective. The basic idea, which probably has its roots in statistical mechanics, was explicitly formulated by Radin and Wolff in [7]. Let $\Lambda \subset \mathbb{R}^d$ be a point set. We will always assume that our point sets are **locally finite**, meaning that their intersections with compact subsets of $\mathbb{R}^d$ are finite (equivalently they are discrete and closed). The dynamical hull $\mathbb{X} = \mathbb{X}(\Lambda)$ of Λ is the closure of the $\mathbb{R}^d$-translation orbit of Λ in some suitable topology.

The commonly used topology, which is the one advocated in [7], declares that two point sets, Λ_1, Λ_2 are close if their restrictions to some large ball around 0 are close in the Hausdorff metric. The resulting space $\mathbb{X}$ is compact and $(\mathbb{R}^d, \mathbb{X})$ is a topological dynamical system. A variation of this topology is to require instead that the restrictions of the two sets to some large open ball around 0 are coincident after some small overall translation. If the sets have **finite local complexity** [1] then the two topologies are the same. In any case we will refer to either of these two as **local topologies**, since they depend on the local structure of the point set.

The importance of the concept is that several fundamental geometrical properties of point sets have equally fundamental interpretations in terms of their dynamical hulls , notably *repetitivity* $\leftrightarrow$ *minimality* and *uniform cluster frequencies* $\leftrightarrow$ *unique ergodicity*. Some of the deepest results in the study of point sets and also in tiling theory have come by utilizing the machinery of dynamical systems through this connection.

One of the most interesting and diagnostic manifestations of the long-range internal order of a point set Λ is the existence of a diffraction pattern with a prominent component of Bragg peaks. In fact many of the most famous examples (e.g. the vertices of a Penrose tiling) are pure point diffractive, that is, there is nothing but Bragg peaks. The exact definitions are not necessary for what follows, but pure point diffraction is a result of the existence of many **ϵ-almost-periods** for every positive ϵ, that is, translations t that almost perfectly

[1] A set Ω has finite local complexity if, for each compact set K in $\mathbb{R}^d$, there are, up to translation, only finitely many classes of points that can appear in the form $\Omega \cap (a + K)$ as a runs over $\mathbb{R}^d$

match up Λ with itself in an average or statistical sense:

$$\lim_{R\to\infty} \frac{\sharp((t+\Lambda)\bigtriangleup\Lambda)\cap B_R(0)}{\text{vol }(B_R(0))} < \epsilon\,, \tag{1}$$

where $\bigtriangleup$ is the symmetric difference operator.

Now this suggests quite a different notion of closeness which reflects a low average discrepancy between the two sets or, to put is another way, high statistical coincidence. This can be supplemented to include small translations: two point sets are close if after a small translation they are statistically almost the same. This is the **autocorrelation topology**. We can again form the dynamical hull of a point set Λ, say $\mathbb{A} = \mathbb{A}(\Lambda)$.

There is no reason to expect $\mathbb{X}$ and $\mathbb{A}$ to be in any way related, and indeed this is in general what happens. But it is a striking fact that it is the local topology that captures the fundamental geometric properties of the set and the autocorrelation that holds the keys to the diffractive properties. Since most of the famous examples of aperiodic point sets have very beautiful local structure and are also pure point diffractive, it comes as no surprise that for these examples $\mathbb{X}$ and $\mathbb{A}$ are related, namely $\mathbb{A}$ is a factor of $\mathbb{X}$. In fact this result holds for all Λ which are regular generic model sets. In the final section of the paper we prove that for a regular model set, $\mathbb{A}(\Lambda)$ is isomorphic to the "torus" $\mathbb{T}$ of its cut and project scheme, thus laying down the connection to the paper of Schlottmann [10] which shows the existence of a mapping $\mathbb{X} \longrightarrow \mathbb{T}$.

This paper is about the topologies arising by statistical coincidence. The first part is about statistical coincidence alone (no translations included) and centres on a completeness result for locally finite sets in this topology. The second part adds in translations and leads to some results on $\mathbb{A}$ (which is actually an Abelian group), when it is compact and when it is pure point diffractive.

The results of the paper do not depend very much on the special properties of $\mathbb{R}^d$ other than it is a σ-compact locally compact Abelian group. Thus the paper is set in the more general context of a σ-compact locally compact Abelian group G (written additively) and its Haar measure ω, unique up to a positive factor. Autocorrelation depends on averaging over something and for that purpose we fix once and for all an averaging sequence $\mathcal{A} = \{A_n\}_{n\in\mathbb{N}}$ satisfying

i) each A_n is a compact subset of G;

ii) for all n, $A_n \subset A^\circ_{n+1}$;

iii) $\bigcup_{n\in\mathbb{N}} A_n = G$;

iv) the van Hove condition.

Intuitively the van Hove condition says that the surface to bulk ratio of the A_n tends to 0 as n tends to infinity. Precisely this is written as: for all compact sets $K \subset G$,

$$\limsup_{n\to\infty} \frac{\omega\left(\partial^K(A_n)\right)}{\omega(A_n)} \;=\; 0\,, \tag{2}$$

where the K-boundary $\partial^K(A)$ of any compact set A is defined by

$$\partial^K(A) \;= ((K+A)\backslash A^\circ) \cup ((-K + \overline{G\backslash A}) \cap A) \;=\; 0\,, \tag{3}$$

and $^\circ$ and $\overline{\{\}}$ are interiors and closures respectively. [2]

Since $G = \bigcup_{n\in\mathbb{N}} A^\circ_{n+1}$, we see that for any compact subset $K \subset G$, there is a finite cover of it sets from $\mathcal{A}$, and then $K \subset A_n$ for some n. In particular, for any $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ so that $A_m + K \subset A_n$.

In this paper we shall use the words *van Hove sequence* to mean a sequence satisfying the conditions itemized above.

Definition: A set is **locally finite** if its intersection with every compact set is finite.

2 $(\mathcal{D}, d)$ as a complete metric space

Definition: Let $\Lambda, \Lambda' \subset G$ be two locally finite sets. Define

$$d(\Lambda,\Lambda') \;:=\; \limsup_{n\to\infty} \frac{\sharp((\Lambda \,\triangle\, \Lambda') \cap A_n)}{\omega(A_n)}\,, \tag{4}$$

where $\sharp$ means the cardinality of the set. This is a pseudometric. We obtain a metric by defining the equivalence relation

$$\Lambda \equiv \Lambda' \Leftrightarrow d(\Lambda, \Lambda') = 0$$

and factoring d through it:

$$\mathcal{D} \;:=\; \{\Lambda \subset G \mid \Lambda \quad \text{locally finite}\}/\equiv \quad \text{and} \quad d : \mathcal{D}\times\mathcal{D} \longrightarrow \mathbb{R}_{\geq 0}\,. \tag{5}$$

[2] We do not consider here the question of the existence of such a sequence. For compactly generated locally compact Abelian groups one can use the structure theorem to explicitly construct such sequences.

Proposition 2.1 *$(\mathcal{D}, d)$ is a complete metric space.*

Proof: Let $\{\Lambda_m\}$ be a sequence of locally finite subsets of G which form a Cauchy sequence when regarded in $\mathcal{D}$. We will construct a locally finite subset Λ of G to which this sequence converges when considered in $\mathcal{D}$.
(**First Case:** $\lim_{n\to\infty} \omega(A_n) = \infty$)

We can pick a subsequence $\{\Lambda_{k_m}\}$ such that $d(\Lambda_{k_m}, \Lambda_{k_{m+1}}) \leq 4^{-m}$ for each $m \geq 0$.

Since

$$d(\Lambda_{k_m}, \Lambda_{k_{m+1}}) = \limsup_{n\to\infty} \frac{\sharp((\Lambda_{k_m} \triangle \Lambda_{k_{m+1}}) \cap A_n)}{\omega(A_n)} \leq \frac{1}{4^m}$$

there exists $n_m > 0$ such that for all $n \geq n_m$ we have

$$\frac{\sharp((\Lambda_{k_m} \triangle \Lambda_{k_{m+1}}) \cap A_n)}{\omega(A_n)} \leq \frac{1}{2^m} .$$

We may assume that the sequence n_m is increasing (since we can replace each n_m with any larger natural number).

We define now

$$\Lambda'_1 = \Lambda_{n_1}$$

and inductively

$$\Lambda'_{m+1} = \Lambda_{n_{m+1}} \triangle \left((\Lambda'_m \triangle \Lambda_{n_{m+1}}) \cap A_{n_m}\right) .$$

In fact

$$\begin{aligned} \Lambda'_{m+1} \cap A_{n_m} &= \Lambda'_m \cap A_{n_m} \quad \text{and} \\ \Lambda'_{m+1} \cap (G \backslash A_{n_m}) &= \Lambda_{n_{m+1}} \cap (G \backslash A_{n_m}) . \end{aligned}$$

Since $\lim_{n\to\infty} \omega(A_n) = \infty$ we have : $\Lambda'_{m+1} \equiv \Lambda_{n_{m+1}}$. By construction we have $\sharp((\Lambda'_m \triangle \Lambda'_{m+1}) \cap A_n)/\omega(A_n) \leq 2^{-m}$ for each m and n.

Let $1 \leq k < l$ be integers and n be arbitrary. Then

$$\begin{aligned} \frac{\sharp((\Lambda'_k \triangle \Lambda'_l) \cap A_n)}{\omega(A_n)} &= \frac{\sharp(\triangle_{i=k}^{l-1}(\Lambda'_i \triangle \Lambda'_{i+1}) \cap A_n)}{\omega(A_n)} \leq \frac{\sharp(\cup_{i=k}^{l-1}(\Lambda'_i \triangle \Lambda'_{i+1}) \cap A_n)}{\omega(A_n)} \\ &\leq \frac{\sum_{i=k}^{l-1} \sharp((\Lambda'_i \triangle \Lambda'_{i+1}) \cap A_n)}{\omega(A_n)} \leq \sum_{i=k}^{l-1} \frac{1}{2^i} \leq \frac{1}{2^{k-1}} . \end{aligned} \tag{6}$$

Let n be arbitrary and let $l(n) := 2 + \lfloor \log_2 \omega(A_n) \rfloor$ where $\lfloor \quad \rfloor$ means the integer part. Let $m, k \geq l(n)$. Then by (6):

$$\frac{\sharp((\Lambda'_m \,\Delta\, \Lambda'_k) \cap A_n)}{\omega(A_n)} \leq \frac{1}{2^{\min\{m,k\}-1}} \leq \frac{1}{2^{l(n)-1}} < \frac{1}{\omega(A_n)}$$

since by the definition of $l(n)$ we have $2^{l(n)-1} > \omega(A_n)$. Hence for each $m, k \geq l(n)$ we have:

$$\begin{aligned} \sharp((\Lambda'_m \,\triangle\, \Lambda'_k) \cap A_n) < 1 \Rightarrow \sharp((\Lambda'_m \,\Delta\, \Lambda'_k) \cap A_n) = 0 \\ \Rightarrow (\Lambda'_m \,\Delta\, \Lambda'_k) \cap A_n = \emptyset \Rightarrow \Lambda'_m \cap A_n = \Lambda'_k \cap A_n \,. \end{aligned} \tag{7}$$

We are now able to define a new set Λ by

$$\Lambda \cap A_n = \Lambda'_{l(n)} \cap A_n \tag{8}$$

for all n. This is well defined since for $n < n'$, $l(n) \leq l(n')$ and hence by (7) we have

$$\Lambda'_{l(n)} \cap A_n = \Lambda'_{l(n')} \cap A_n = (\Lambda'_{l(n')} \cap A_{n'}) \cap A_n \,.$$

Now, Λ is our required limit. First of all we note that for any compact $K \subset G$, $K \subset A_n$ for some n. Now $\Lambda \cap A_n = \Lambda'_{l(n)} \cap A_n$ and $\Lambda'_{l(n)}$ is made up from subsets of $\Lambda_1, \ldots, \Lambda_{l(n)}$. In turn, each of these contains only finitely many points from K since each $\Lambda_k \in \mathcal{D}$. Thus $\Lambda \cap K$ is finite, showing that $\Lambda \in \mathcal{D}$.

Second, we prove that $d(\Lambda, \Lambda'_m) \leq 2^{-(m-1)}$ for each m. Let $n \in \mathbb{N}$ be arbitrary, and let $k \geq \max\{m, l(n)\}$. Then by (7)

$$\Lambda \cap A_n = \Lambda'_{l(n)} \cap A_n = \Lambda'_k \cap A_n \,.$$

Hence

$$\frac{\sharp((\Lambda'_m \,\Delta\, \Lambda) \cap A_n)}{\omega(A_n)} = \frac{\sharp((\Lambda'_m \,\Delta\, \Lambda'_k) \cap A_n)}{\omega(A_n)} \leq \frac{1}{2^{m-1}}$$

because of (6) and

$$d(\Lambda, \Lambda'_m) = \limsup_{n\to\infty} \frac{\sharp((\Lambda \,\Delta\, \Lambda'_m) \cap A_n)}{\omega(A_n)} \leq \limsup_{n\to\infty} \frac{1}{2^{m-1}} = \frac{1}{2^{m-1}}\,,$$

showing that

$$\lim_{m\to\infty} \Lambda'_m \,=\, \Lambda\,.$$

However, $\Lambda'_m \equiv \Lambda_{k_m}$ by construction. Hence:

$$\lim_{m\to\infty} \Lambda_{k_m} = \Lambda .$$

So we started with an arbitrary Cauchy sequence and we proved that this has a converging subsequence. This prove that our space is complete.
(**Second Case:** $\lim_{n\to\infty} \omega(A_n) = c < \infty$) Let $\{\Lambda_m\}$ be a Cauchy sequence in $\mathcal{D}$. $\{\Lambda_m\}$ is a Cauchy sequence, hence there exists a m_0 so that $\forall m, l > m_0$ we have $d(\Lambda_m, \Lambda_n) < (2c)^{-1}$.

Let now $m, l > m_0$ be arbitrary. Since

$$\limsup_{n\to\infty} \frac{\sharp((\Lambda_m \,\Delta\, \Lambda_l) \cap A_n)}{\omega(A_n)} < \frac{1}{2c}$$

there exists an n_0 such that for all $n > n_0$ we have :

$$\frac{\sharp((\Lambda_m \,\Delta\, \Lambda_l) \cap A_n)}{\omega(A_n)} < \frac{1}{c} .$$

But the sequence $\{\omega(A_n)\}$ is increasing and convergent to c , hence $\omega(A_n) \leq c$ for all n. This implies that:

$$\frac{\sharp((\Lambda_m \,\Delta\, \Lambda_l) \cap A_n)}{c} \leq \frac{\sharp((\Lambda_m \,\Delta\, \Lambda_l) \cap A_n)}{\omega(A_n)} < \frac{1}{c} .$$

It follows that $\sharp((\Lambda_m \,\Delta\, \Lambda_l) \cap A_n) < 1$, so

$$\Lambda_m \cap A_n = \Lambda_l \cap A_n \,,\, \forall n > n_0 .$$

Finally $\Lambda_m = \Lambda_l$ so $\Lambda_m \equiv \Lambda_l$, the sequence is constant from m_0 on, and hence it is convergent. □

Remark 2.2 Note that if we have n'_m an increasing sequence of natural numbers with the property that $n'_m \geq n_m \, \forall m$, then in the previous proof we can replace $\{n_m\}_m$ by $\{n'_m\}_m$. We will use this fact in the following results.

Remark 2.3 In the second case of the proof of Prop. 2.1 (when the measure of G is finite), we have proved that in fact d induces the discrete topology on $\mathcal{D}$. In this case all the results of the next section become trivial.

Remark 2.4 Since n_m is increasing we have the following description of Λ'_m:

$$\begin{aligned}\Lambda'_m \cap A_{n_1} &= \Lambda_{k_1} \cap A_{n_1} \\ \Lambda'_m \cap (A_{n_i} \backslash A_{n_{i-1}}) &= \Lambda_{k_i} \cap (A_{n_i} \backslash A_{n_{i-1}}),\ 2 \le i \le m \\ \Lambda'_m \cap (G \backslash A_{n_m}) &= \Lambda_{k_m} \cap (G \backslash A_{n_m})\end{aligned}$$

and hence the following description of Λ:

$$\begin{aligned}\Lambda \cap A_{n_1} &= \Lambda_{k_1} \cap A_{n_1} \\ \Lambda \cap (A_{n_i} \backslash A_{n_{i-1}}) &= \Lambda_{k_i} \cap (A_{n_i} \backslash A_{n_{i-1}}),\ i \ge 2\,.\end{aligned}$$

Remark 2.5 Neither the pseudometric d nor the metric d inherited from it is necessarily G-invariant. Invariance has to be derived from the van Hove property of our sequence. However, the van Hove property is a statement about boundary to bulk ratios in terms of measure, whereas the metric is involved with actual counting of points. Only when the points actually "eat up volume" is it possible to link the two ideas. Later, when we introduce uniform discreteness we will be able to do this and then obtain G-invariance on the smaller spaces $\mathcal{D}_V$ (see Corollary 3.10).

With the notation from the proof of Prop. 2.1 we have $\lim_{n\to\infty} \Lambda'_n = \Lambda$ in the local topology. However, in general there is no connection between these two topologies, as the following example shows.

Example 2.6 Let $\Lambda_m = \mathbb{Z}\backslash\{-m, -m+1, \ldots, m\}$ and let $A_n := [-n, n]$. Then in the local topology

$$\lim_{n\to\infty} \Lambda_n = \emptyset\,,$$

whereas in the autocorrelation topology we have

$$\lim_{n\to\infty} \Lambda_n = \mathbb{Z}\,.$$

More generally let $\{\Lambda_m\}$ be any sequence of locally finite subsets of G and Λ any other locally finite set subset of G. Let A_n be any van Hove sequence with the property that

$$\begin{aligned}\lim_{n\to\infty} \omega(A_n) &= \infty\,, \\ \cup_{n\in\mathbb{N}} A_n &= G\,.\end{aligned}$$

Define $\{\Lambda'_m\}$ by

$$\begin{aligned}\Lambda'_m \cap A_m &:= \Lambda \cap A_m \\ \Lambda'_m \cap (G\backslash A_m) &:= \Lambda_m \cap (G\backslash A_m)\,.\end{aligned}$$

(so we replace the points Λ_m which inside A_m by those of Λ). Then we have $\lim\{\Lambda'_m\} = \lim\{\Lambda_m\}$ in the autocorrelation topology (assuming that the limit exists), but in the local topology $\lim\{\Lambda'_m\} = \Lambda$.

3 Stable geometric properties under convergence

As above, G is a σ-compact locally compact abelian group, $\mathcal{A} = \{A_n\}$ is a fixed van Hove sequence, and d is the metric defined by this van Hove sequence on $\mathcal{D}$.

If $\omega(G) < \infty$ all the results in this section are trivial since, as we have pointed out above, the metric then induces the discrete topology. For this reason in all the proofs we study only the case $\omega(G) = \infty$. In particular

$$\lim_{n\to\infty} \omega(A_n) = \infty\,.$$

Definition 3.1 *Let $\Lambda \subset G$ be a locally finite set.*

- *For $K \subset G$ a compact set, Λ is* K**-relatively dense** *if for all $x \in G$, $(x+K) \cap \Lambda \neq \emptyset$.*
- *For a neighborhood V of $\{0\}$, Λ is* V**-uniformly discrete** *if for all $x \in G$ we have $(x+V) \cap (\Lambda\backslash\{x\}) = \emptyset$.*
- *Λ is* **weakly-uniformly discrete** *if for every compact K in G there exists a constant c_K such that for any $t \in G$*

$$\sharp(\Lambda \cap (t+K)) \leq c_K\,.$$

- *For K a compact set and V a neighborhood of 0, Λ is a (K,V)-***Delone set** *if Λ is K-relatively dense and V-uniformly discrete.*

Remark 3.2 When we don't need the parameters we say only uniformly discrete, relatively dense or Delone set.

Definition 3.3 *Let $\{\Lambda_\alpha\}_\alpha \subset G$ be a family of locally finite sets. We say that this family is:*

i) **equi-uniformly discrete** *if there exists a neighborhood V of $\{0\}$ such that Λ_α is V- uniformly discrete for all α.*

ii) **equi-relatively dense** *if there exists a compact set K such that Λ_α is K- relatively dense for all α.*

iii) **equi-weakly-uniformly discrete** *if for any compact K in G there exists a constant c_K such that for all α and for all $t \in G$, $\sharp(\Lambda_\alpha \cap (t+K)) \leq c_K$*

iv) **equi-Delone** *if the family is equi-relatively dense and equi-uniformly discrete.*

Remark 3.4 If a family is V -uniformly discrete then it is W-uniformly discrete for some neighborhood W of $\{0\}$ with compact closure. If we have a family of V-equi-uniformly discrete sets then we can chose the same W for the entire family.

Definition 3.5 *We say that a set Λ has a certain $\mathcal{A}$- statistical property if we can find a set Λ' which has that property and $d(\Lambda, \Lambda') = 0$.*

Lemma 3.6 *Let $A, K \subset G$ with $0 \in K$ and K compact. Let V be a compact neighborhood of 0 in G with $V = -V$. Then*

$$V + \partial^K(A) \subset \partial^{V+K}(A) \,.$$

Proof: Let $x \in \partial^K(A)$, $v \in V$. We need to show that $v + x \in \partial^{V+K}(A)$.

Suppose that $x \in (K + A)\backslash A^\circ$. Then $v + x \in V + K + A$, and if $v + x \notin A^\circ$, we have what we wish. If $v + x \in A^\circ \subset A$, then from $x \in G\backslash A^\circ = \overline{G\backslash A}$,

$$v + x \in (V + \overline{G\backslash A}) \cap A \subset \partial^{V+K}(A) \,,$$

as required.

On the other hand, if $x \in (-K + \overline{G\backslash A}) \cap A$ then $v + x \in V - K + \overline{G\backslash A}$, and if $v + x \in A$ we have what we need. If $v + x \notin A$ then from $x \in A$ we have $v + x \in V + A \subset V + K + A$, so

$$v + x \in (V + K + A)\backslash A^\circ \subset \partial^{V+K}(A) \,.$$

□

Proposition 3.7 *Let $\Lambda \subset G$ be statistically relatively dense and statistically uniformly discrete. Then Λ is a statistically Delone set.*

Proof: Λ is statistically relatively dense means that there exists $B \subset G$ and a compact K such that B is K-relatively dense and $d(\Lambda, B) = 0$. Λ is statistically uniformly discrete means that there exists $C \subset G$ and a neighborhood of zero V such that C is V-uniformly discrete and $d(\Lambda, C) = 0$. Without loss of generality we may assume that V has compact closure.

Let $\mathcal{P} = \{E \mid C \subset E \subset B \cup C \text{ and } E \text{ is } V-\text{uniformly discrete}\}$ and order it by inclusion. Since $C \in \mathcal{P},\ P \neq \emptyset$.

Let $\mathcal{T} \subset \mathcal{P}$ be non-empty and totally ordered. Let $M = \cup\{E \mid E \in \mathcal{T}\}$. Obviously $C \subset M \subset B \cup C$. Suppose by contradiction that M is not V-uniformly discrete. Then there exists $x \in M$ so that

$$(x + V) \cap (M \backslash \{x\}) \neq \emptyset \,.$$

Let $y \in ((x + V) \cap (M \backslash \{x\})$. Since $x, y \in M$, there exists $E_1, E_2 \in \mathcal{T}$ such that $x \in E_1$ and $y \in E_2$. But $\mathcal{T}$ is totally ordered, so $E_1 \subset E_2$ or $E_2 \subset E_1$. So we can find $E \in \mathcal{T}$ such that

$$x, y \in E \Rightarrow y \in ((x + V) \cap (E \backslash \{x\})) \,.$$

Hence E is not V-uniformly discrete, contradicting the fact that $E \in \mathcal{T}$.

By Zorn's Lemma we know that there exists a maximal element $Z \in \mathcal{P}$. In particular Z is V-uniformly discrete. We prove that Z is K'-relatively dense , where $K' = K + \overline{V}$ is compact.

Suppose by contradiction that Z is not K'-relatively dense. Then there exists $x \in G$ such that

$$(x + K') \cap Z = \emptyset \,.$$

Since B is K-relatively dense, there exists $y \in (x + K) \cap B$.

Let $N = Z \cup \{y\}$. Then $y \notin Z$ and Z is maximal in $\mathcal{P}$ implies that $N \notin \mathcal{P}$. But $C \subset Z \subset N \subset B \cup C$ and $N \notin \mathcal{P}$ implies that N is not V-uniformly discrete.

Hence there exists $z \in (y + V) \cap (N \backslash \{y\})$, from which $z \in Z$; and also $z \in (y+V)$ and $y \in (x+K)$ from which $z \in (x+K+V) \cap Z \subset (x+K') \cap Z = \emptyset$. This contradiction proves that Z is K' relatively discrete.

Now

$$C \subset Z \subset B \cup C \Rightarrow 0 = d(C, \Lambda) \leq d(Z, \Lambda) \leq d(B \cup C, \Lambda) \leq d(B, \Lambda) + d(C, \Lambda) = 0$$

Hence $d(Z,\Lambda) = 0$. □

Lemma 3.8 *Given an arbitrary compact set K, we can construct $\{n_m\}$ in Prop. 2.1 such that:*

$$\lim_{k\to\infty} \frac{\sum_{m=1}^{k} \omega(\partial^K(A_{n_m}))}{\omega(A_{n_k})} = 0 \,.$$

Proof:

For this to be true is enough to have:

$$\omega(\partial^K(A_{n_m})) < \omega(A_{n_m}) \,\text{for all}\, m \,,$$

and

$$k^2\omega(A_{n_m}) < \omega(A_{n_k}) \;\; \text{for all}\; m < k \,.$$

We have to prove two things:

i) the two conditions imply the result of the lemma

ii) we can chose m_n in the proof of 2.1 to satisfy these conditions.

i):

$$\begin{aligned} \frac{\sum_{m=1}^{k} \omega(\partial^K(A_{n_m}))}{\omega(A_{n_k})} &= \frac{\omega(\partial^K(A_{n_k})) + \sum_{m=1}^{k-1} \omega(\partial^K(A_{n_m}))}{\omega(A_{n_k})} \\ &\leq \frac{\omega(\partial^K(A_{n_k})) + \sum_{m=1}^{k-1} \omega((A_{n_m}))}{\omega(A_{n_k})} \\ &\leq \frac{\omega(\partial^K(A_{n_k}))}{\omega(A_{n_k})} + \frac{\sum_{m=1}^{k-1} (1/k^2)\omega((A_{n_k}))}{\omega(A_{n_k})} \\ &\leq \frac{\omega(\partial^K(A_{n_k}))}{\omega(A_{n_k})} + \frac{k-1}{k^2} \,. \end{aligned} \tag{9}$$

Since both terms on the right side of the inequality go to zero we get the result.
ii): The key for this is the fact that in the proof of Prop. 2.1 , as long as $n_m > n_{m-1}$, we can replace each n_m by any larger number. Since

$$\lim_{n\to\infty} \frac{\omega(\partial^K(A_n))}{\omega(A_n)} = 0 \,,$$

there exists a j such that $\omega(\partial^K(A_n)) < \omega(A_n)$ for all $n > j$. By taking $n_1 > j$ the first condition is satisfied.

Proceeding inductively, let $C(k) := \max_{1\leq m\leq k-1} m^2\omega(A_{n_m})$. At the beginning of the section we showed that we can assume $\lim_{n\to\infty}\omega(A_n) = \infty$. Find $n(k)$ so that $n \geq n(k)$ implies $\omega(A_n) > C(k)$. Choose any $n_k \geq n(k)$. Then for all $m < k$, $\omega(A_{n_k}) > C(k) \geq k^2\omega(A_{n_m})$. The second condition is satisfied. □

Proposition 3.9 *Let $\{\Lambda_m\}$ be a convergent sequence of locally finite sets.*

a) *If Λ_n are equi-uniformly discrete then the limit is statistically uniformly discrete.*

b) *If all Λ_n are equi-Delone sets then the limit is statistically Delone set.*

c) *If all Λ_n are equi-relatively dense then the limit is statistically relatively dense.*

Proof:

a) Choose V in the definition of the uniform discreteness so that its closure is compact and $V = -V$. Let $K = \overline{V+V}$ and let $\{n_m\}$ be as in the previous lemma. We may also assume that $A_{n_m} + K + K \subset A_{n_{m+1}}$. Let $K' = \overline{V}$. Let Λ be the set constructed in Prop. 2.1 with this $\{n_m\}$ and let

$$B := \Lambda \cap \bigcup_{m\in\mathbb{N}} \partial^{K'}(A_{n_m}) .$$

We prove that $\Lambda\backslash B$ is V-uniformly discrete and B has density zero.

If $x \in \Lambda\backslash B$ then there exists some m such that $x \in A_{n_m}\backslash A_{n_{m-1}}$. Then from the construction of B, $(x+V)\cap\Lambda \subset (A_{n_m}\backslash A_{n_{m-1}})\cap\Lambda \subset \Lambda_{n_m}$, which itself is V-uniformly discrete. This shows that $\Lambda\backslash B$ is V-uniformly discrete.

On the other hand, for $x \in \Lambda\cap\partial^{K'}(A_{n_m})$ we have, by Lemma 3.6, $x+V \subset \partial^K(A_{n_m})$. We show now that each set $x+V$ contains at most two points from Λ.

Let r be minimal such that $(x+V)\cap(A_{n_r}\backslash A_{n_{r-1}}) \neq \emptyset$. Let $y \in (x+V)\cap(A_{n_r}\backslash A_{n_{r-1}})$. Then $y \in x+V$. Since $V = -V$ we get $x \in y+V$, so

$$x+V \subset y+V+V \subset y+K \subset A_{n_r}+K \subset A_{n_{r+1}} .$$

Thus $x+V \subset A_{n_{r+1}}$.

We show now that $(x+V)\cap A_{n_{r-1}} = \emptyset$:

Suppose by contradiction that $(x+V)\cap A_{n_{r-1}} \neq \emptyset$. From the minimality of r we get that

$$(x+V)\cap(A_{n_{r-1}}\backslash A_{n_{r-2}}) = \emptyset .$$

Thus $(x+V)\cap A_{n_{r-1}} \subset (x+V)\cap A_{n_{r-2}}$, so $(x+V)\cap A_{n_{r-2}} \neq \emptyset$.

Let $y \in (x+V)\cap A_{n_{r-2}}$. As above,

$$x+V \subset y+K \subset A_{n_{r-2}}+K \subset A_{n_{r-1}} \,,$$

contrary to $x+V\cap (A_{n_r}\backslash A_{n_{r-1}}) \neq \emptyset$.

Now, since $x+V \subset A_{n_{m+1}}$ and $(x+V)\cap A_{n_{m-1}} = \emptyset$ we get that $x+V \subset (A_{n_{m+1}}\backslash A_{n_{m-1}})$, thus $x+V$ can meet only $(A_{n_{m+1}}\backslash A_{n_m})$ and $A_{n_m}\backslash A_{n_{m-1}}$.

Since each set $x+V$ contains at most two points from Λ we get

$$\omega(V)\sharp(\Lambda\cap\partial^{K'}(A_{n_m})) \leq 2\,\omega(\partial^{K}(A_{n_m}))\,.$$

Now the previous lemma gives $d(B,\emptyset)=0$.

b) We know from a) that Λ is statistically uniformly discrete. We prove now that it is statistically relatively dense. Let K be given by the equi-relative density. We can assume that $0\in K$ and $K=-K$.

Let $\{n_m\}$ be as in the previous lemma. We can also ask that $A_{n_m}+K+K \subset A_{n_{m+1}}$. Let Λ be the set constructed in Prop. 2.1 with this $\{n_m\}$ and set

$$B := \bigcup_{m=1}^{\infty} (\Lambda_{n_m}\cup\Lambda_{n_{m+1}})\cap\partial^{K''}(A_{n_m})\,.$$

In the same way as above we can prove that $\Lambda\cup B$ is K-relatively dense and B has density zero.

c) Let K be defined by the relative density . Let V be a compact neighborhood of $\{0\}$. Let $K' := K+\overline{V}$. We make the same construction as in b). The only problem is that B may not have density zero.

As in Prop.3.7 we construct B' a maximal V-uniformly discrete subset of B. Then B' has density zero and, exactly as in Prop.3.7, $\Lambda\cup B'$ is K'-relatively dense. □

$\mathcal{D}_V$ be the set of equivalence classes of V-uniformly discrete subsets of G. We let d_V denote the restriction of the d both to the set of V-uniformly discrete subsets of G and to their equivalence classes $\mathcal{D}_V$. Restriction to $\mathcal{D}_V$ brings with it the property of G-invariance which we will need in the next section.

Corollary 3.10 *Let $V=-V$ be a compact symmetric neighborhood of $\{0\}$ in G. Then*

a) *d_V is a G-invariant on the set of V-uniformly discrete subsets of G;*

b) $\mathcal{D}_V$ *is complete and G-invariant.*

Proof: a): Let Λ, Λ' be V-uniformly discrete sets and let $t \in G$. Let $W = -W$ be a compact symmetric neighborhood of $\{0\}$ satisfying $W + W \subset V$. Then for all $x, y \in \Lambda$ with $x \neq y$, $(x+W) \cap (y+W) = \emptyset$. Now

$$\begin{aligned} d(t+\Lambda, t+\Lambda') &= \limsup_{n\to\infty} \frac{\sharp(((t+\Lambda) \bigtriangleup (t+\Lambda')) \cap A_n)}{\omega(A_n)} \\ &= \limsup_{n\to\infty} \frac{\sharp((\Lambda \bigtriangleup \Lambda') \cap (-t+A_n))}{\omega(A_n)} . \end{aligned}$$

Comparing this with $d(\Lambda, \Lambda')$ we see that the difference is due to $(-t+A_n)\backslash A_n$ and $A_n \backslash (-t+A_n)$ both of which are in $\partial^K(A_n)$ for $K := \{0, t, -t\}$; and in magnitude the difference is bounded by the sum of

$$\limsup_{n\to\infty} \frac{\sharp(\Lambda \cap \partial^K(A_n))}{\omega(A_n)}$$

and the corresponding value for Λ'. However, for each $x \in \Lambda \cap \partial^K(A_n)$, $x + W \subset \partial^{W+K}(A_n)$, by Lemma 3.6, and so, taking into account the V-uniformness of Λ,

$$\sharp(\Lambda \cap \partial^K(A_n)) \leq \frac{\omega(\partial^{W+K}(A_n))}{\omega(W)} .$$

There is a similar expression for Λ'. Now the van Hove property shows that the limits are 0, and so $d_V(\Lambda, \Lambda') = d_V(t+\Lambda, t+\Lambda')$ as required.
b): The set of V-uniformly discrete subsets of G is G-invariant, and by a) so is the pseudo-metric d_V on it. Thus d_V induces a G-invariant metric on $\mathcal{D}_V$. Prop. 3.9 (and its proof) show that $\mathcal{D}_V$ is complete. □

Remark 3.11 i.) Let $\Lambda = \mathbb{Z} \backslash \bigcup_{n=1}^{\infty} \{2^n, 2^n+1, ..., 2^n+n\}$. Then Λ is not relatively dense, but $d(\Lambda, \mathbb{Z}) = 0$.

ii.) Let $\Lambda' = \mathbb{Z} \cup \bigcup_{n=1}^{\infty} \{2^n + \frac{1}{n}\}$. Then Λ' is not uniformly discrete , but $d(\Lambda', \mathbb{Z}) = 0$.

iii.) Let now $\Lambda'' = \Lambda' \backslash \bigcup_{n=1}^{\infty} \{2^n+1, ..., 2^n+n\}$. Then Λ'' is neither relatively dense or uniformly discrete , but $d(\Lambda'', \mathbb{Z}) = 0$.

4 The autocorrelation group $\mathbb{A}(\Lambda)$

Let $\Lambda \subset G$ be any Delone set.

Definition 4.1 We define a pseudo-metric on G: $d_\Lambda(t, t') = d(t + \Lambda, t' + \Lambda)$.

d_Λ is a G-invariant pseudo-metric (see Corollary 3.10). The interest in this pseudo-metric stems from its connection with the autocorrelation of Λ. For $t \in G$,

$$\eta(t) := \lim_{n\to\infty} \frac{\sharp(\Lambda \cap (t + \Lambda) \cap A_n)}{\omega(A_n)}$$

is the t-**autocorrelation coefficient** of Λ, and

$$\eta := \sum \eta(t)\delta_t$$

is the **autocorrelation (measure)**. If the autocorrelation exists, then in fact for all $t \in G$,

$$d_\Lambda(t, 0) = 2(\eta(0) - \eta(t)) .$$

For more on this, see [2].

Note that d_Λ is not in general a metric on G: for $t, t' \in G$,

$$d_\Lambda(t, t') = 0 \Leftrightarrow d_\Lambda(t - t', 0) = 0 \Leftrightarrow d(t - t' + \Lambda, +\Lambda) = 0$$

that is, $t - t'$ is a **statistical period** of Λ.

Definition 4.2 For each open neighborhood V of 0 and each $\epsilon > 0$ define

$$U(V, \epsilon) := \{(x, y) \in G \times G \mid \exists v \in V \text{ such that } d_\Lambda(-v + x, y) < \epsilon\} .$$

The set of all of these $U(V, \epsilon)$ form a fundamental set of entourages for a uniformity $\mathcal{U}$ on G. Moreover, since each $U(V, \epsilon)$ is G-invariant, we obtain in this way a new topological group structure on G, called the **mixed topology** of G.

Let $\mathbb{A} = \mathbb{A}(\Lambda)$ denote the completion of G in this new topology, which is a new topological group called the **autocorrelation completion** of G.

For each $y \in G$ and each $U \in \mathcal{U}$ define $U[y] := \{x \in G \mid (x, y) \in U\}$.

Definition 4.3 For each $\epsilon > 0$, define the ϵ**-almost periods** of Λ: $P_\epsilon := \{t \in G \mid d_\Lambda(t, 0) < \epsilon\}$.

For each $\epsilon > 0$ and V a neighborhood of $\{0\}$ we have:

$$U(V, \epsilon)[0] = P_\epsilon + V \, .$$

Remark 4.4 Let $\epsilon_0 := 2d(\Lambda, \emptyset)$. Then for all $\epsilon > \epsilon_0$, $P_\epsilon = G$, and if. V is a neighborhood of $\{0\}$ then $U(V, \epsilon)[0] = G$.

Recall that a uniform space X is said to be **precompact** if and only if its Hausdorff completion $\widehat{X}$ is compact or, equivalently, for each entourage U of X there exists finite cover of X with U-small sets ([3],Thm. 4.2.3).

Lemma 4.5 *Let V be an open neighborhood of $\{0\}$ with compact closure in the standard topology of G and let $\epsilon > 0$. Then the following are equivalent:*

a) $U(V, \epsilon)[0]$ *is precompact in the mixed topology,*

b) *for all $0 < \epsilon' < \epsilon$ there exists K a compact set in G with the standard topology so that*

$$P_\epsilon \subset P_{\epsilon'} + K \, .$$

Proof: Suppose that $U(V, \epsilon)[0]$ is precompact and let $0 < \epsilon' < \epsilon$. Cover $U(V, \epsilon)[0]$ by finitely many translations of $U(V, \epsilon')[0]$. Then using the previous remark there exist $t_1, ..., t_n$ such that:

$$P_\epsilon + V \subset \bigcup_{i=1}^{n} (t_i + P_{\epsilon'} + V) \, .$$

Since V has compact closure, $K := \overline{\bigcup_{i=1}^{n} t_i + V}$ is compact. Hence:

$$P_\epsilon \subset P_\epsilon + V \subset \bigcup_{i=1}^{n} (t_i + P_{\epsilon'} + V) \subset P_{\epsilon'} + K \, .$$

Conversely, let U' be an open neighborhood of $\{0\}$ for G in the mixed topology. We need to cover $U(V, \epsilon)[0]$ with finitely many translates of U'. For this purpose we can assume that $U' = U(V', \epsilon')[0]$ for some open neighborhood V' of $\{0\}$ and some $\epsilon' < \epsilon$. By assumption there exists compact K in the standard topology so that $P_\epsilon \subset P_{\epsilon'} + K$. Then

$$U(V, \epsilon)[0] = P_\epsilon + V \subset P_{\epsilon'} + K + V \subset P_{\epsilon'} + K + \overline{V} \, .$$

Since $K + \overline{V}$ is compact there exist $t_1, ..., t_n$ such that $K + \overline{V} \subset \bigcup_{i=1}^{n} (t_i + V')$, so we obtain

$$U(V,\epsilon)[0] \subset P_{\epsilon'}+K+\overline{V} \subset \bigcup_{i=1}^{n}(t_i+P_{\epsilon'}+V') = \bigcup_{i=1}^{n}(t_i+U(V',\epsilon')[0]) \subset \bigcup_{i=1}^{n}(t_i+U')\,.$$

This proves that $U(V,\epsilon)[0]$ is precompact. □

Proposition 4.6 $\mathbb{A}$ *is compact if and only if for all* $\epsilon > 0$, P_ϵ *is relatively dense in* G *(in the standard topology).*

Proof: Suppose that $\mathbb{A}$ is compact. Let $\epsilon > 0$. Choose $\epsilon' > \max\{\epsilon, \epsilon_0\}$ Since $\mathbb{A}$ is compact, G is precompact. Let V be an arbitrary open neighborhood of $\{0\}$ with compact closure. Then $U(V,\epsilon')[0] = G$ is precompact hence there exists K, compact in G such that

$$G = P_{\epsilon'} \subset P_\epsilon + K\,.$$

Hence P_ϵ is relatively dense.

Conversely, fix any $\epsilon > \epsilon_0$. Let $0 < \epsilon' < \epsilon$. Since $P_{\epsilon'}$ is relatively dense in G then exists K compact such that $P_\epsilon \subset P_{\epsilon'} + K$. Hence for any V open neighborhood of $\{0\}$ with compact closure we have by Lemma 4.5 that $G = U(V,\epsilon)[0]$ is precompact. □

Corollary 4.7 *Let* G *be a* σ*-compact locally compact Abelian group. Let* $\Lambda \subset G$ *be a locally finite with a well-defined* $\mathcal{A}$*-autocorrelation. Assume that* $\Lambda - \Lambda$ *is uniformly discrete. Then the following are equivalent:*

a) P_ϵ *is relatively dense for all* $\epsilon > 0$*;*

b) Λ *is pure point diffractive;*

c) $\mathbb{A}(\Lambda)$ *is compact.*

Proof: See ([2], Thm. 5). □

Proposition 4.8 *Let* Λ *be a Delone subset of the locally compact Abelian group* G*. The following are equivalent:*

a) $\mathbb{A}$ *is locally compact;*

b) *There exists an* $\epsilon > 0$ *such that for all* $0 < \epsilon' < \epsilon$ *there exists compact* K *with* $P_\epsilon \subset P_{\epsilon'} + K$*.*

Proof: Suppose that $\mathbb{A}$ is locally compact. Let $\varphi : G \to \mathbb{A}$ be the uniformly continuous map which defines the completion.

Let U' be a compact neighborhood of $\{0\}$ in $\mathbb{A}$. Then we can find $\epsilon > 0$ and V an open neighborhood of $\{0\}$ in G such that $\varphi(U(V,\epsilon))[0] \subset U'$. Then $U(V,\epsilon)[0]$ is precompact, so we can apply Lemma 4.5.

Conversely, let V be an open neighborhood of $\{0\} \in G$ with compact closure. Again by Lemma 4.5, $U(V,\epsilon)$ is precompact. □

Remark 4.9 The completion mapping $\varphi : G \to \mathbb{A}$ provides a natural G-action on $\mathbb{A}$, If $\mathbb{A}$ is compact we have a dynamical system, both topologically and measure theoretically (using Haar measures). Compact or not, the action of G on $\mathbb{A}$ is **minimal** in the sense that every G-orbit is dense in $\mathbb{A}$.

As pointed out in the introduction, Λ has an associated local dynamical hull obtained from the closure of its G-orbit in the local topology. In general, one should not expect any nice relationship between $\mathbb{X}$ and $\mathbb{A}$. However, for model sets, there is a strong connection between the two, as we shall see in Sec. 5.

In the case that G is a real space $\mathbb{R}^d$, the use of the Hausdorff metric d_H on subsets of $\mathbb{R}^d$ allows simple reformulations of some of the results above. Note that for $A \subset B \subset \mathbb{R}^d$,

$$d_H(A,B) < \infty \Longleftrightarrow B \subset A + K \text{ for some compact set } K \subset \mathbb{R}^d\,.$$

Now the following are obvious:

Corollary 4.10 *The following are equivalent in* $\mathbb{R}^d$*:*

a) $\mathbb{A}$ *is locally compact;*

b) *There exists an* $\epsilon > 0$ *so that for all* $0 < \epsilon' < \epsilon, d_H(P_\epsilon, P_{\epsilon'}) < \infty$ *.*

c) *There exists an* $\epsilon > 0$ *such that for all* $0 < \epsilon', \epsilon'' < \epsilon, d_H(P_{\epsilon'}, P_{\epsilon''}) < \infty$*.*

Corollary 4.11 *The following are equivalent in* $\mathbb{R}^d$*:*

a) $\mathbb{A}$ *is compact;*

b) *for all* $\epsilon > 0, d_H(P_\epsilon, \mathbb{R}^d) < \infty$*;*

c) *for all* $0 < \epsilon, \epsilon', d_H(P_\epsilon, P_{\epsilon'}) < \infty$*.*

5 Regular model sets

A **cut and project scheme** is a triple $(G, H, \widetilde{L})$ of locally compact Abelian groups in which $\widetilde{L}$ is a lattice in $G \times H$ and for which the natural projections π_1, π_2 satisfy $\pi_1|_{\widetilde{L}}$ is injective, and $\pi_2(\widetilde{L})$ is dense in H:

$$\begin{array}{ccccc} G & \stackrel{\pi_1}{\longleftarrow} & G \times H & \stackrel{\pi_2}{\longrightarrow} & H \quad . \\ & & \bigcup & & \\ & & \widetilde{L} & & \end{array} \tag{10}$$

We let $L := \pi_1(\widetilde{L})$ and $^* : L \longrightarrow H$ be the mapping $\pi_2 \circ (\pi_1)^{-1}|_L$. By hypothesis, the group $\mathbb{T} := (G \times H)/\widetilde{L} = \{(t, t^*)|t \in L\}$ is compact. The obvious G-action on $\mathbb{T}$ makes it into a (minimal, see below) dynamical system.

A **regular model set** (defined by the cut and project scheme (10)) is a non-empty set of the form $\Lambda = x + \{t \in L|t^* \in W\}$ where $W \subset H$ is compact and satisfies the conditions

$$W = \overline{W^\circ} \quad \text{and} \quad \theta_H(\partial W) = 0$$

where θ_H is Haar measure on H. It is possible to replace the cut and project scheme by one with a smaller H if necessary, so that for $u \in H$, $u + W = W$ if and only if $u = 0$ [9]. We will assume that this condition holds in what follows. The regular model set Λ is **generic** if $\partial W \cap L^* = \emptyset$. [3]

Regular model sets are always Delone sets [5, 6] and have well-defined autocorrelations. In particular we can consider the autocorrelation group $\mathbb{A}(\Lambda)$. A key point is that $\mathbb{A}$ and $\mathbb{T}$ are isomorphic, so in fact $\mathbb{T}(\Lambda)$ for a regular model set has a very natural interpretation – namely the completion of the orbit of Λ under the autocorrelation topology.

Proposition 5.1 *Let G be a compactly generated locally compact Abelian group and let Λ be a regular model set of the cut and project scheme* (10). *Then* $\mathbb{A}(\Lambda) \simeq \mathbb{T}(\Lambda)$ *and the isomorphism is also a G-mapping.*

Proof: There is no loss in assuming that $\Lambda = \{t \in L|t^* \in W\}$. The action of G on $\mathbb{T} = (G \times H)/\widetilde{L}$ is defined by $x + (t + \widetilde{L}) = x + t + \widetilde{L}$, and it is easy to see that the image of G in $\mathbb{T}$ under this map is dense. So $\mathbb{T}$ and $\mathbb{A}$ are the *completions* of G under the respective topologies on G induced by the G-orbits of $\{0\}$ in these

[3]Model sets were introduced by Y. Meyer [4] in his study of harmonious sets.

two groups. It suffices to show that these topologies on G coincide. For the $\mathbb{T}$-topology, $x \in G$ is close to zero if and only if there is a small open neighborhood V of G and a pair (t, t^*), where $t \in L$ so that $x - t \in V$ and t^* close to $\{0\} \in H$.

On the other hand, $x \in G$ is close to zero in the $\mathbb{A}$-topology if and only if there is a small open neighborhood V of G, and a small $\epsilon > 0$, and a $t \in G$ so that $x - t \in V$ and $t \in P_\epsilon$. Such a t necessarily lies in $\Lambda - \Lambda \subset L$. So we need to show that for $t \in L$, t^* is close to zero in H if and only if $t \in P_\epsilon$ for some small ϵ.

By uniform distribution (see [6, 9])

$$\begin{aligned} d_\Lambda(t,0) &= \lim_{n\to\infty} \frac{1}{\omega(A_n)} \sum_{x \in (\Lambda \backslash (t+\Lambda)) \cap A_n} 1 + \lim_{n\to\infty} \frac{1}{\omega(A_n)} \sum_{x \in ((t+\Lambda)\backslash \Lambda) \cap A_n} 1 \\ &= \theta_H(W \backslash (t^* + W)) + \theta_H((t^* + W) \backslash W) \\ &= \theta_H(W \backslash (t^* + W)) + \theta_H(W \backslash (-t^* + W)) \,, \end{aligned}$$

since the second term converges to the autocorrelation $d(t+\Lambda, \Lambda) = d_\Lambda(t, 0)$. It remains to prove that $\theta_H(W\backslash(t^* + W))$ converges to 0 if and only if t^* converges to 0.

Now, for all $u \in H$,

$$\theta_H(W \backslash (u + W)) = \theta_H(W) - (\mathbf{1_W} * \widetilde{\mathbf{1_W}})(\mathbf{u}) \,,$$

where $\mathbf{1_W}$ is the indicator function for the set W and $\widetilde{\{\}}$ changes the sign of the argument. This is uniformly continuous in u (for this result on convolutions see [8], Ch. 1), so disposes of the 'if' part.

Conversely, let $\{u_i\}$ be a net in H for which $\{\theta_H(W\backslash(u_i + W))\}$ converges to 0. The u_i eventually lie in $W - W$ which is compact, so we may assume that in fact the u_i converge, say to u_0. Then $\theta_H(\overset{\circ}{W} \backslash (u_0 + W)) = 0$ and $\overset{\circ}{W} \backslash (u_0 + W)$ is open, so $\overset{\circ}{W} \backslash (u_0 + W) = \emptyset$. Thus $\overset{\circ}{W} \subset u_0 + W$, so $W \subset u_0 + W$. A similar argument leads to the reverse inclusion. Then, by our assumptions above, $u_0 = 0$. □

Corollary 5.2 *For any regular generic model set there is a G-invariant surjective continuous mapping $\mathbb{X}(\Lambda) \longrightarrow \mathbb{A}(\Lambda)$. Furthermore, this mapping is $1-1$ almost everywhere with respect to the Haar measure on $\mathbb{A}(\Lambda)$.*

Proof: By [10] there is a unique G-invariant continuous surjective mapping $\mathbb{X}(\Lambda) \longrightarrow \mathbb{T}(\Lambda)$ which maps Λ to $\{0\}$ in $\mathbb{T}$, and it is $1-1$ $\mathbb{T}$-almost everywhere. □

Remark 5.3 Corollary 5.2 in effect characterizes the regular model sets amongst the relatively dense sets Λ satisfying the Meyer property $\Lambda - \Lambda$ is uniformly discrete [1].

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