MATHEMATICAL QUASICRYSTALS: A TALE OF TWO TOPOLOGIES

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In quasicrystals, both mathematical and physical, there is an interplay of two notions of order: a local order and a long-range order manifested through diffraction. These two types of order lead to two very different notions of closeness, the local and autocorrelation topologies, which are in principle unrelated. Given a discrete point set in $\mathbb{R}^d$, each one leads to a dynamical hull. It is exactly the context of the cut and project formalism that marries these concepts. In particular we give a new characterization of regular model sets in terms of a mapping relating the two dynamical hulls.

1. Introduction

The distinguishing feature of physical crystals and quasicrystals is their point-like diffraction. This phenomenon is a measure of their internal long-range order. It is particularly remarkable for quasicrystals in as much as they do not seem to be based on repetition of a fundamental cell.

By contrast, the physical properties and processes of formation of quasicrystals seem to be largely dependent on their local structure and one can suppose that similar local environments must result in similar physics.

These two forms of order – long-range and local – can be put into a setting in which they are seen as a manifestation of two very different topologies. In quasicrystals, physical and mathematical, they are married in a remarkable way. This paper is about these two topologies and what is means for them to live in harmonious co-existence.

From the very beginning of quasicrystals, it was observed that indexing the Bragg peaks required more than the expected number of parameters: for example, in the icosahedral cases, six rather than the familiar three for crystals,. This suggested that the atomic positions of these substances might be modeled by projecting from a lattice lying in a higher dimensional space. Thus arose the cut and project formalism, which has been a mainstay in attempts to understand and model quasicrystals. The very same mechanism has been seen to be an alternative way to describe many of the well-known tiling models and has played an important role in the development of the mathematics of diffraction in aperiodic sets.

In all these cut and project systems two things happen: the two topologies are in co-existence and the resulting structures satisfy the Meyer condition.[a] In this paper we will

[a]In the case of modulated model sets, the Meyer property may be destroyed.

see that in fact these two properties characterize the cut and project formalism.

In more detail, beginning with a discrete subset $\varLambda$ in real Euclidean space $\mathbb{R}^d$ we construct two dynamical hulls: $\mathbb{X}(\varLambda)$ and $\mathbb{A}(\varLambda)$, derived from the local and long-range topologies respectively, both of which carry a $\mathbb{R}^d$ action. The connection between them, when it exists, is the existence of a continuous mapping $\beta : \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$. In the case that $\varLambda$ is Meyer, this mapping exists and is 1-1 almost everywhere if and only if $\varLambda$ is a regular model set (see Propositions 4.2,6.1 for the precise statements). In this case $\varLambda$ is also pure point diffractive.

Perhaps the most remarkable point about this is the intricacy of the almost $1-1$-ness. Except for crystals – in which case β is an isomorphism - the set of points at which the map is not $1-1$ is dense in $\mathbb{A}(\varLambda)$.

2. Uniformly discrete sets and two topologies

We start by assembling the basic definitions for discrete point sets. We work in $\mathbb{R}^d$ with the standard metric.[b] For $c \in \mathbb{R}^d$, we let $B_r(c)$ be the ball of radius r about c, with $B_r := B_r(0)$.

Definition 2.1. A subset $\varLambda \subset \mathbb{R}^d$ is **locally finite** if its intersection with every ball $B_r(c)$ is finite.

Let $r > 0$. A subset $\varLambda \subset \mathbb{R}^d$ is called r-**uniformly discrete** if all the sets $x + B_r$, $x \in \varLambda$, are mutually disjoint.

We will denote by $\mathbf{D_r}$ the set of all r-uniformly discrete subsets of $\mathbb{R}^d$. This space admits an action by $\mathbb{R}^d$ – namely the translation action. For $\varLambda \in \mathbf{D_r}$ and $x \in \mathbb{R}^d$ we denote the x-translate of $\varLambda$ by $x + \varLambda$.

A locally finite subset $\varLambda \subset \mathbb{R}^d$ has **finite local complexity** if for all $R > 0$, as c runs over $\mathbb{R}^d$, there are, up to translation, only finitely many sets of the form $\varLambda \cap B_R(c)$.

A subset of $\varLambda \in \mathbf{D_r}$ is called **Delone** if it is relatively dense: that is, there is an $R > 0$ so that for all $c \in \mathbb{R}^d$, $\varLambda \cap B_R(c) \neq \emptyset$.

The uniform discreteness is the hard-shell condition for atoms, which is very reasonable. It implies local finiteness. Finite local complexity is also natural enough, though one should note that it imposes restrictions on orientational order. Finite local complexity can also be characterized by the property that

$$\varLambda - \varLambda \quad \text{is locally finite.} \tag{1}$$

A strong form of this, which we will need and discuss later, is the Meyer condition:

Definition 2.2. $\varLambda \subset \mathbb{R}^d$ satisfies the **Meyer condition** if

$$\varLambda - \varLambda \quad \text{is uniformly discrete (for some } r > 0\text{).} \tag{2}$$

It is a **Meyer set** if in addition it is relatively dense.

We now come to the two notions of closeness of point sets that are the fundamental concepts of the paper. We will define two topologies, each defined by means of a uniformity – a description of the sets of point set pairs that are to be considered close to one another.

[b]Almost everything in this paper carries over to the setting of separable locally compact Abelian groups with minor changes.

Definition 2.3. For each pair $(s, B_R(c))$, consisting of a positive real number and a ball, define

$$U(s, B_R(c)) := \{(\Lambda, \Lambda') \in \mathbf{D_r} \times \mathbf{D_r} : (v + \Lambda) \cap B_R(c) = \Lambda' \cap B_R(c), \text{for some } \ v \in B_s\} \quad (3)$$

These sets form a fundamental system for a uniform structure on $\mathbf{D_r}$ whose topology has the sets

$$U(s, B_R(c))[\Lambda] := \{\Lambda' \in \mathbf{D_r} : (\Lambda, \Lambda') \in U(s, B_R(c))\} \quad (4)$$

as a neighbourhood basis of Λ.

This uniformity can also be described by a metric, though it is not particularly natural. The intuition is that two sets are close if after a small shift they agree on a large ball. One should note that the uniformity is certainly invariant under translations, but the individual entourages $U(s, B_R(c))$ are not. The resulting topology has as a basis of open sets the sets of the form:

$$U(s, B_R(c))[\Lambda] := \{\Lambda' : (\Lambda, \Lambda') \in U(s, B_R(c))\}\,.$$

We call it the **local topology** on $\mathbf{D_r}$.

The second notion of closeness is based on average matching of the two sets.

Definition 2.4. Let $\Lambda, \Lambda' \in \mathbf{D_r}$.

$$d(\Lambda, \Lambda') \ := \limsup_{r\to\infty} \frac{\sharp((\Lambda \ \triangle \ \Lambda') \cap B_r)}{\mathrm{vol}(B_r)}\,, \quad (5)$$

where $\triangle$ is the symmetric difference operator. It is rather easy to see that d is a pseudo-metric on $\mathbf{D_r}$.

We obtain a metric by defining the equivalence relation

$$\Lambda \equiv \Lambda' \Leftrightarrow d(\Lambda, \Lambda') = 0$$

and factoring d through it:

$$\mathbf{D_r}^{\equiv} \ := \ \mathbf{D_r}/\equiv \quad \text{and} \quad d : \mathbf{D_r}^{\equiv} \times \mathbf{D_r}^{\equiv} \longrightarrow \mathbb{R}_{\geq 0} \quad (6)$$

and note that the translation action of $\mathbb{R}^d$ is retained on $\mathbf{D_r}^{\equiv}$.

We call the resulting topology on $\mathbf{D_r}$ **coarse autocorrelation topology** and denoted by β the natural mapping (which is an $\mathbb{R}^d$-mapping):

$$\beta : \mathbf{D_r} \longrightarrow \mathbf{D_r}^{\equiv}\,. \quad (7)$$

Prop 2.1.

i) $\mathbf{D_r}$ is a complete metric space in the local topology [14, 17].
ii) $\mathbf{D_r}^{\equiv}$ is a complete space and the Hausdorff completion of $\mathbf{D_r}$ in the coarse autocorrelation topology [12].

The first of these statements is quite intuitive. The second is trickier since it is hard to see how to build up a point set from a Cauchy sequence of sets in which the matching is improving only in density and not in particular patches of space.

A simple example shows why we should not expect β to be particularly nice with respect to the local and autocorrelation topologies.

Example 2.1. Let $\Lambda_n := \mathbb{Z}\backslash\{-n, -n+1, \dots, n\}$. Then $\{\Lambda_n\}$ converges to $\emptyset$ in the local topology and to $\mathbb{Z}$ in the autocorrelation topology. If instead $\Lambda_n := [-n, n] \cup \{\dots, n-4, n-2\} \cup \{n+2, n+4, \dots\}$ then convergence is to $\mathbb{Z}$ in the local topology and the sequence fails to converge in the autocorrelation topology.

The autocorrelation topology needs to be modified to put it on an equal footing with the local topology. The local topology is obtained as large coincidence after a small shift. As it stands Λ, Λ' can be close in the coares autocorrelation topology only if they have a large statistical coincidence – no small shift involved. We introduce the **autocorrelation topology** by defining a uniform structure on $\mathbf{D_r}$ with the fundamental system of entourages

$$U(s, \epsilon) := \{(\Lambda, \Lambda') \in \mathbf{D_r} \times \mathbf{D_r} : d(v + \Lambda, \Lambda') < \epsilon \ \text{ for some } v \in B_s\}$$

.

Prop 2.2. $\mathbf{D_r}$ is complete in the autocorrelation topology.

3. Dynamical hulls

In the dynamics of a physical system one considers a (compact) configuration space or phase space and the time evolution of points of this space. Of special importance is the notion of recurrence, in the sense of the time-orbit returning very closely to the initial configuration.

In the geometry of point sets and tilings a parallel construction on points sets is possible, using the translation action of the underlying space as the group action, instead of time. This provides a powerful way of understanding the internal geometry of a given point set. We begin with $\mathbf{D_r}$ and its translation action by $\mathbb{R}^d$, take a single point set of interest, $\Lambda \in \mathbf{D_r}$, and then construct the closure of its orbit under translation: $\overline{\mathbb{R}^d + \Lambda}$.

Our two topologies provide two separate interpretations of the orbit closure of $\Lambda \in \mathbf{D_r}$:

$$\mathbb{X}(\Lambda) = \overline{\mathbb{R}^d + \Lambda} \quad \text{in the local topology}$$
$$\mathbb{A}(\Lambda) = \overline{\mathbb{R}^d + \Lambda} \quad \text{in the autocorrelation topology}$$

which are the **local and autocorrelation hulls** respectively. Local hulls have been used extensively in tilings and the study of aperiodic sets. The autocorrelation hull is the subject of [12]. It is the interplay between these two that we wish to understand.

It is useful to note that both $\mathbb{X}(\Lambda)$ and $\mathbb{A}(\Lambda)$ can be interpreted as completions of our starting space $\mathbb{R}^d$. In each case one simply provides $\mathbb{R}^d$ with a new topology by pulling back of the uniform topology on $\mathbf{D_r}$ to $\mathbb{R}^d$ using the mapping $x \mapsto x + \Lambda$. In both cases, points that are usually close in $\mathbb{R}^d$ remain so. But now also $x, x' \in \mathbb{R}^d$ are close if $(x + \Lambda, x' + \Lambda)$ are close in the local (resp. autocorrelation) topologies.

It is interesting that $\mathbb{A}(\Lambda)$ has the structure of an Abelian group, a fact that arises from the group structure of $\mathbb{R}^d$ and the translation invariance of the autocorrelation entourages. In particular it has a unique probability Haar measure $\theta_{\mathbb{A}}$. In general $\mathbb{X}(\Lambda)$ does not get a group structure in this way.

We need to know what the basic open sets for this new autocorrelation topology of $\mathbb{R}^d$ look like. It suffices to consider open sets around 0, and for these a basis consists of the sets $U'(s,\epsilon)[0] = B_s \times P_\epsilon$ where

$$P_\epsilon := \{t \in \mathbb{R}^d : d(t+\Lambda, \Lambda) < \epsilon\}\,. \tag{8}$$

The elements of P_ϵ are called the **statistical ϵ-almost periods** of Λ.

Prop 3.1. Let $\Lambda \in \mathbf{D_r}$.

i) $\mathbb{X}(\Lambda)$ is compact if and only if Λ has finite local complexity.
ii) $\mathbb{A}(\Lambda)$ is compact if and only if for all $\epsilon > 0$, P_ϵ is relatively dense.

Thus both $\mathbb{X}(\Lambda)$ and $\mathbb{A}(\Lambda)$ become dynamical systems. A dynamical system which is the closure of an orbit is minimal if and only if it is uniformly recurrent (for each point x and each open neighbourhood V of x the set of t in $\mathbb{R}^d$ for which $t.x$ is in V is relatively dense in $\mathbb{R}^d$. In the case of $\mathbb{X}(\Lambda)$ this condition on Λ is called **repetitivity**. In the case of $\mathbb{A}(\Lambda)$ is comes for free with the compactness, though one can see that in fact it amounts to the relative denseness of each of the sets of ϵ-almost periods P_ϵ.

4. Model sets

Fortunately there are large families of examples in which we can get a good feel for what $\mathbb{X}(\Lambda)$ and $\mathbb{A}(\Lambda)$ are like.

A **cut and project scheme** is a triple $(\mathbb{R}^d, H, \widetilde{L})$ of locally compact Abelian groups in which $\widetilde{L}$ is a lattice in $\mathbb{R}^d \times H$ and for which the natural projections π_1, π_2 satisfy $\pi_1|_{\widetilde{L}}$ is injective, and $\pi_2(\widetilde{L})$ is dense in H:

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\ & & \bigcup & & \\ & & \widetilde{L} & & \end{array} \quad . \tag{9}$$

We let $L := \pi_1(\widetilde{L})$ and $^* : L \longrightarrow H$ be the mapping $\pi_2 \circ (\pi_1|_{\widetilde{L}})^{-1}$. By hypothesis, $\widetilde{L}$ is a discrete group and the group $\mathbb{T} := (\mathbb{R}^d \times H)/\widetilde{L} = \{(t,t^*) | t \in L\}$ is compact. The obvious $\mathbb{R}^d$-action on $\mathbb{T}$ $(x + (t,t^*) + \widetilde{L} \mapsto (x+t, t^*) + \widetilde{L})$ makes it into a minimal dynamical system.

A **regular model set** (defined by the cut and project scheme (9)) is a non-empty set of the form $\Lambda = x + \{t \in L : t^* \in W\}$ where $W \subset H$ is compact and satisfies the conditions

$$W = \overline{W^\circ} \quad \text{and} \quad \theta_H(\partial W) = 0$$

where θ_H is Haar measure on H. It is possible to replace the cut and project scheme by one with a smaller H if necessary, so that for $u \in H$, $u + W = W$ if and only if $u = 0$ [16]. We will assume that this condition holds in what follows. The regular model set Λ is **generic** if $\partial W \cap L^* = \emptyset$. Generic model sets are repetitive. [c]

[c]Model sets were introduced by Y. Meyer [8] in his study of harmonious sets.

It is not hard to show that model sets are always Delone sets [10, 11] the uniform discreteness coming from the compactness of W and the relative denseness from its non-empty interior. In fact model sets are even Meyer sets, as can be seen by applying the previous sentence to the model set $\Lambda - \Lambda = \{t \in L : t^* \in W - W\}$.

Regular model sets have uniquely defined autocorrelations, [d] so we can consider the autocorrelation group $\mathbb{A}(\Lambda)$. A key point is that $\mathbb{A}(\Lambda)$ and $\mathbb{T}$ are isomorphic as topological groups [12], so in fact $\mathbb{T}(\Lambda)$ for a regular model set has a very natural interpretation – namely the completion of the orbit of Λ under the autocorrelation topology.

Prop 4.1. [12] Let Λ be a regular model set of the cut and project scheme (9). Then $\mathbb{A}(\Lambda) \simeq \mathbb{T}(\Lambda)$ and the isomorphism is also a G-mapping.

Now using the torus parameterization of Schlottmann [17] we have a fundamental result connecting our two topologies.

Prop 4.2. [17] Let Λ be a regular repetitive model set. Then the mapping β of (7) provides a continuous surjective $\mathbb{R}^d$-mapping

$$\beta : \mathbb{X}(\Lambda) \longrightarrow \mathbb{A}(\Lambda) \tag{10}$$

This mapping β is $1-1$ almost everywhere, in the sense that the set of elements of $\mathbb{A}(\Lambda)$ over which there lie more than one point of $\mathbb{X}(\Lambda)$ is of Haar measure 0.

It is interesting to look at the nature of the $1-1$-ness condition. We have pointed out that $\mathbb{A}(\Lambda) \simeq \mathbb{T}$. Now $\mathbb{T}$ may be viewed as a parametrization of the different sets $\Lambda(x, y) := x + \{t \in L : t^* \in -y + W\}$ that are obtainable from W by changing varying x and y: $\Lambda(x, y) = \Lambda(x', y')$ if $(x, y) \equiv (x', y') \mod \widetilde{L}$. Each point set Λ' in $\mathbb{X}(\Lambda)$ is repetitive, since Λ is, and if $\beta(\Lambda') = (x, y) \mod \widetilde{L}$ then

$$x + \{t \in L : t^* \in -y + W^\circ\} \subset \Lambda' \subset x + \{t \in L : t^* \in -y + W\},$$

that is, it is determined by some set $-y + W'$ between the interior of $-y + W$ and $-y + W$ itself. The situation is easy if $(-y + \partial W) \cap L^* = \emptyset$ for then there are no points of Λ coming from the boundary and there is only one possibility for Λ'. But if $t^* \in (-y + \partial W) \cap L^*$ then there are at least two preimages under β, one of which has t and one which does not. As y varies over H there is a dense set of translates $y - W$ for which $-y + \partial W$ meets L^*, whence the singular cases are inextricably bound in with the non-singular cases. For the construction of the mapping $\mathbb{X}(\Lambda)$ see [17].

It was noted by Anderson and Putnam [1] that locally $\mathbb{X}(\Lambda)$ has the structure of an open ball in $\mathbb{R}^d$ times a Cantor set. To see this let us suppose that $0 \in \Lambda$. The set $B_r \times Z$ where Z is all the points $\Lambda' \in \mathbb{X}(\Lambda)$ for which $0 \in \Lambda'$ is an open set neighbourhood of Λ in $\mathbb{X}(\Lambda)$. Furthermore, for each finite patch S of points of Λ containing 0, the set of points Λ' of $\mathbb{X}(\Lambda)$ containing S is a both open and closed in the induced topology on Z, so Z is totally disconnected. Z itself is closed in $\mathbb{X}(\Lambda)$, hence compact, and it is also perfect, so a Cantor set.

[d] The autpcorrelation can be expressed, using a Weyl-type theorem for uniform distributiion [16, 11], as an integral over H which is dependent only on W.

The cut and project formalism is sometimes criticized because of its introduction of non-physical internal dimensions. In fact these internal dimensions are of completely physical origins (that is, determined by the geometry of the model set). We have just seen that $\mathbb{T}$ is none other than $\mathbb{A}(\Lambda)$ which is the completion of $\mathbb{R}^d$ under the topology that considers the long-range order of Λ. We also have a remarkably simple desription of the internal space H.

Consider the subgroup L of $\mathbb{R}^d$ generated by the set of differences $\Lambda - \Lambda$. Supply this with the topology of the (coarse) autocorrelation – that is the topology for which an open neighbourhood basis of each point $t \in L$ is given by the sets $t + P_\epsilon$, $\epsilon > 0$. Then H is the completion of L under this topology [3]. So the internal space is simply a reflection of the almost periodic structure of Λ.

5. Diffraction

The diffraction of a distribution of density in space is usually described as the square absolute value of some suitably normalized Fourier transform of this density. Alternatively it is the Fourier transform of the volume averaged autocorrelation. For finite sets of scatterers or for cyrstals, both lead to the same thing. With quasicrystals one has to be careful - because of issues of convergence, only the second one makes mathematical sense. Formally the definitions are as follows:

Let $\Lambda \in \mathbf{D_r}$. We represent this set as a non-zero regular positive Borel measure in the form of a *Dirac comb*

$$\omega \;=\; \sum_{x \in \Lambda} \delta_x$$

where δ_x is the unit point (or Dirac) measure located at x.[e]

Define $\omega_R = \omega|_{B_R}$ and $\tilde{\omega}_R = (\omega_R)\tilde{}$. Then, the measure

$$\gamma_\omega^{(R)} \;:=\; \frac{\omega_R * \tilde{\omega}_R}{\mathrm{vol}(B_R)} \;=\; \frac{1}{\mathrm{vol}(B_R)} \sum_{x,y \in S \cap B_R} \delta_{x-y}$$

is well defined, since it is the (volume averaged) convolution of two *finite* measures. The **autocorrelation** γ_ω of ω exists and is the limit of $\left(\gamma_\omega^{(R)}\right)_R$ in the vague topology as $R \to \infty$, if this limit exists [7]. It is then a positive definite measure with $\gamma_\omega = \sum_{t \in \Lambda - \Lambda} \eta(t)\delta_t$ where the coefficients are given by

$$\eta(t) \;=\; \lim_{n \to \infty} \frac{1}{\mathrm{vol}(B_R)} \sum_{\substack{x,y \in \Lambda \cap B_n \\ x-y=t}} 1 = \frac{1}{\mathrm{vol}(B_R)} \sharp(\Lambda \cap (t + \Lambda)) \cap B_R. \tag{11}$$

The second inequality uses the fact that as R becomes large the error caused by the mismatching of B_R and $t + B_R$ becomes negligible with respect to the volume of B_R. The function η can be extended from $\Lambda - \Lambda$ to all of $\mathbb{R}^d$ by setting $\eta(t) = 0$ for all other t. It is a positive definite function.

What is particularly relevant here is the fact that for all t,

$$d(\Lambda, t + \Lambda) = \frac{\sharp((\Lambda \,\triangle\, \Lambda') \cap B_r)}{\mathrm{vol}(B_r)} = 2(\eta(0) - \eta(t))\,, \tag{12}$$

[e]In a more realistic setting these points might be weighted to represent different scattering intensities or convolved with some atomic profiles.

as can be seen from $\varLambda \triangle (t+\varLambda) = (\varLambda \cup (t+\varLambda))\backslash \varLambda \cap (t+\varLambda)$. Thus the metric d and the autocorrelation coefficients $\eta(t)$ are directly related, whence our choice of name for the autocorrelation topology.

The autocorrelation measure γ_ω is a positive definite, translation bounded measure and, as such, has a well-defined Fourier transform $\widehat{\gamma}_\omega$ (also defined on $\mathbb{R}^d$), which is a positive and also translation bounded measure [4, 7].

Definition 5.1. $\widehat{\gamma}_\omega$ is the **diffraction** of ω. The pure point part of this measure is called the **Bragg spectrum**. The measure ω is **pure point diffractive** if $\widehat{\gamma}_\omega$ is a pure point (also called discrete or atomic) measure on $\mathbb{R}^d$.

Prop 5.1. [6] The measure ω is pure point diffractive if and only if the autocorrelation measure γ_ω is strongly almost periodic.

This key result of the theory introduces a new concept – strong almost periodicity – which by virtue of this result is clearly crucial for the fundamental understanding of pure point diffraction. However it is not particularly intuitive. Fortunately, under the assumption that $\varLambda - \varLambda$ is uniformly discrete (the *Meyer* condition), we have

Prop 5.2. [3] The autocorrelation measure γ_ω of ω is strongly almost periodic if and only if, for all $\epsilon > 0$, P_ϵ is relatively dense.

Thus, for our point set $\varLambda$, pure pointedness of the diffraction hinges around the relative denseness of the sets P_ϵ – in other words the compactness of $\mathbb{A}(\varLambda)$.

6. The main theorem

Prop 6.1. [2] Let $\varLambda \subset \mathbb{R}^d$ be a Meyer set with a well-defined autocorrelation. Suppose that $\beta : \mathbb{X}(\varLambda) \longrightarrow \mathbb{A}(\varLambda)$ is continuous and $1-1$ almost everywhere. Then $\varLambda$ differs in density 0 from a regular model set. Furthermore, if $\varLambda$ is repetitive then

i) $\mathbb{X}(\varLambda)$ is the hull of a regular model set;
ii) $\mathbb{X}(\varLambda)$ is uniquely ergodic and minimal (strictly ergodic);
iii) $\varLambda$ is pure point diffractive;
iv) $\mathbb{X}(\varLambda)$ has pure point dynamical spectrum;
v)) $\mathbb{X}(\varLambda)$ has continuous eigenfunctions.

β is a bijection if and only if $\varLambda$ is a crystal.

Thus model sets are characterized as Meyer sets for which a good map from $\mathbb{X}(\varLambda)$ to $\mathbb{A}(\varLambda)$ exists.

All of ii) through v) are consequences of being a regular model set. It is conjectured that, along with the Meyer condition, collectively they are equivalent to the existence of the existence of the continuous almost everywhere $1-1$ map β.

In outline, the proof of Prop.6.1 for repetitive $\varLambda$ may be sketched as follows. The Meyer property implies finite local complexity (see equations (1) and (2)) and hence compacntess of $\mathbb{X}(\varLambda)$ (see Prop.3.1). Since β is continuous, $\mathbb{A}(\varLambda)$ is compact, whence by Prop.3.1 the sets P_ϵ of almost periods are relatively dense. This implies pure pointedness, but more importantly, it provides us with a way to create the cut and project scheme.

We let L be the subgroup of $\mathbb{R}^d$ generated by the set $\mathit{\Lambda}-\mathit{\Lambda}$ and give it the group topology for which the sets P_ϵ form a neighbourhood basis at 0. This is a uniform topology and its Hausforff completion $\phi : L \longrightarrow H$ provides us with the internal group H. The subgroup $\widetilde{L} := \{(t, \phi(t)) : t \in L\}$ of $\mathbb{R}^d \times H$ is a lattice and $\mathbb{R}^d, H, \widetilde{L})$ is the cut and project scheme. Its *-map is the mapping ϕ. One proves that $\mathbb{A}(\mathit{\Lambda})$ is isomorphic to the group $\mathbb{T}$ emerging from the cut and project scheme.

There is no harm is shifting $\mathit{\Lambda}$ so that it contains 0, whence it also lies in L. Define $W \subset H$ as the closure of $\mathit{\Lambda}^*$ in H. One proves that W is compact and $\mathit{\Lambda} \subset \{t \in L : t^* \in W\}$. The hard part is to show that this is equality. In fact it is not true in general. But it is true that if $(x, y) + \widetilde{L}$ is a point at which β is $1-1$ then over that point, the corresponding set $\mathit{\Lambda}' := x + \{t \in L : t^* \in -y + W\}$ is a model set. Finally the almost everywhere $1-1$-ness can be used to show that the boundary of W has measure 0. Hence $\mathbb{X}(\mathit{\Lambda}) = \mathbb{X}(\mathit{\Lambda}')$ is the local hull of a regular model set.

7. Additional thoughts

The point of the paper is to show the relationship of the nature of quasicrystals to the intertwining of topologies of long-range and local order. From a physical point of view such a connection seems to be a basic underlying assumption. If small samples of quasicrysalline materials are supposed to be representative of some materials that can be extended indefinitely in space, and if the long-range order is supposed to be a defining characteristic of these materials then there has to be some notion that of convergence in long-range structure on the basis of local convergence of structure.

The present version of this work was intended to shed light on the nature of model sets, and as such makes some assumptions that are physically unrealistic. The primary of these assumptions are the treatment of atoms as points of equal scattering intensity and the very strong assumption of the Meyer condition.

It might be as well to show what this innocent looking condition implies. Suppose that $\mathit{\Lambda}$ is a Meyer set. Then $\mathit{\Lambda}-\mathit{\Lambda}$ is uniformly discrete. A consequence of this (due to J. Lagarias) is that $\mathit{\Lambda} - \mathit{\Lambda} \subset \mathit{\Lambda} + F$ for some finite set F. A consequence of that is that $\mathit{\Lambda} \pm \cdots \pm \mathit{\Lambda}$ (any fixed number of terms) is also uniformly discrete. For a whole ciricle of equivalences one may consult [9] or the starting point for all these results [8]. Recently Strungaru [13] proved that for any Meyer set (even weighted Meyer sets) the support of Bragg spectrum is relatively dense in $\mathbb{R}^d$. In other words all Meyer sets are, in the general sense of a pervasive Bragg spectrum, quasicrystals.

The rigidity of Meyer sets, and even finite local complexity, limits the applicability of this work in understanding physical quasicrystals. Its value is more in pointing out the way in which the tension between long and short range order can be resolved in a nice mathematical setting

Future work will focus on removing the Meyer condition and relaxing the notion of the point sets under consideration to measures. Generalizing in this way forces one to generalize the local and autocorrelation topologies and will create new versions of $\mathbb{X}$ and $\mathbb{A}$. But one expects that some version of Prop. 6.1 will emerge.

Acknowledgments

Thanks to Michael Baake, Daniel Lenz, Nicolae Strungaru, all of whom have contributed substantially to this project. I would also like to thank the Natural Sciences and Engineering Research Council of Canada for its continuing support of my work over many years. I would also like to thank the Banff International Research Station for supporting my collaborators in the preparation of this research.

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