

Math 518 (Winter 2012)

Functional Analysis

Solution #1

1. Let V be a vector space, let p be a seminorm on V , and let $C := \{x \in V : p(x) < 1\}$. Show that

$$p(x) = \inf\{t \geq 0 : x \in tC\} \quad (x \in V).$$

Solution: Set

$$q(x) := \inf\{t \geq 0 : x \in tC\} \quad (x \in V).$$

Let $x \in V$, and let $\epsilon > 0$. As $\frac{1}{p(x)+\epsilon}x \in C$, i.e., $x \in (p(x) + \epsilon)C$, it is immediate that $q(x) \leq p(x) + \epsilon$; since $\epsilon > 0$ is arbitrary, we have $q(x) \leq p(x)$. For the reversed inequality, choose $t \geq 0$ such that $x \in tC$ —and thus $p(x) \leq t$ —and $q(x) \geq t - \epsilon$. It follows that $q(x) \geq p(x) - \epsilon$ and thus $q(x) \geq p(x)$.

2. Let (V, \mathcal{P}) be a finite-dimensional LCS. Show that there is a norm $\|\cdot\|$ on V that induces the same topology on V as \mathcal{P} .

Solution: We inductively determine $p_1, \dots, p_n \in \mathcal{P}$ and subspaces $V = V_1 \supseteq \dots \supseteq V_n$ such that p_n is a norm on V_n and that

$$V_{k+1} := \{x \in V_k : p_k(x) = 0\} \quad (k = 1, \dots, n-1).$$

Set $V_1 := V$, and let $p_1 \in \mathcal{P}$ be arbitrary. If p_1 is a norm on V_1 , we stop. Otherwise, set $V_2 := \{x \in V_1 : p_1(x) = 0\}$.

Suppose p_1, \dots, p_k and $V_1, \dots, V_k \neq \{0\}$ have already been constructed. Choose $p_{k+1} \in \mathcal{P}$ such that $p_{k+1}|_{V_k} \neq 0$; if p_{k+1} is a norm on V_k , we stop. Otherwise, set $V_{k+1} := \{x \in V_k : p_k(x) = 0\}$, so that $V_{k+1} \subsetneq V_k$.

Since $\dim V < \infty$, this process breaks off after a finite number of steps.

Set

$$\|x\| := \max_{j=1, \dots, n} p_j(x) \quad (x \in V).$$

Then $\|\cdot\|$ is a continuous norm on V , i.e., the topology induced by $\|\cdot\|$ is coarser than the one induced by \mathcal{P} .

Let $p \in \mathcal{P}$ be arbitrary. Define a norm $|||\cdot|||$ on V by letting

$$|||x||| := \max\{\|x\|, p(x)\} \quad (x \in V).$$

Since norms on finite-dimensional spaces are equivalent, there is $C \geq 0$ such that

$$p(x) \leq |||x||| \leq C\|x\| \quad (x \in V).$$

Consequently, p is a continuous seminorm on $(V, \|\cdot\|)$. Since $p \in \mathcal{P}$ was arbitrary, this means that the topology induced by \mathcal{P} is coarser than the norm topology.

3. Let V be a vector space, and let $\phi, \phi_1, \dots, \phi_n: V \rightarrow \mathbb{F}$ be linear. Show that ϕ is a linear combination of ϕ_1, \dots, ϕ_n if and only if $\bigcap_{j=1}^n \ker \phi_j \subset \ker \phi$.

Solution: Of course, if $\phi = \lambda_1\phi_1 + \dots + \lambda_n\phi_n$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{F}$, then $\bigcap_{j=1}^n \ker \phi_j \subset \ker \phi$ holds.

For the converse, consider

$$\Phi: V \rightarrow \mathbb{F}^n, \quad x \mapsto (\phi_1(x), \dots, \phi_n(x)).$$

Then $V/\ker \Phi$ is a finite-dimensional vector space, and ϕ_1, \dots, ϕ_n as well as ϕ drop to functionals on $V/\ker \Phi$, which we denote by $\tilde{\phi}_1, \dots, \tilde{\phi}_n$ and $\tilde{\phi}$. Let $x_1, \dots, x_m \in V/\ker \Phi$ be a basis for $V/\ker \Phi$. The matrix

$$\begin{bmatrix} \tilde{\phi}_1(x_1) & \dots & \tilde{\phi}_n(x_1) \\ \vdots & & \vdots \\ \tilde{\phi}_1(x_m) & \dots & \tilde{\phi}_n(x_m) \end{bmatrix}$$

then has rank m . Consequently, there are $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ that solve the system

$$\begin{bmatrix} \tilde{\phi}_1(x_1) & \dots & \tilde{\phi}_n(x_1) \\ \vdots & & \vdots \\ \tilde{\phi}_1(x_m) & \dots & \tilde{\phi}_n(x_m) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} \tilde{\phi}(x_1) \\ \vdots \\ \tilde{\phi}(x_m) \end{bmatrix}$$

of linear equations or, equivalently, satisfy

$$\tilde{\phi} = \lambda_1\tilde{\phi}_1 + \dots + \lambda_n\tilde{\phi}_n.$$

This, of course, means that $\phi = \lambda_1\phi_1 + \dots + \lambda_n\phi_n$ holds.

4. Let E be a normed space. Show that the following are equivalent:

- (i) $\dim E < \infty$;
- (ii) $\sigma(E, E^*)$ and the norm topology coincide;
- (iii) $\sigma(E, E^*)$ is metrizable.

(*Hint:* For the implication from (c) to (a), use Problem 3 and the fact that no Banach space can have a countably infinite Hamel basis.)

Solution: (a) \implies (b) is clear by Problem 2.

[Even though we will prove that (c) \implies (a), we will give a direct proof of (b) \implies (a) because it makes the idea of (c) \implies (a) somewhat more transparent: If the open

unit ball in E is $\sigma(E, E^*)$ -open, there are $\phi_1, \dots, \phi_n \in E^*$ and $\epsilon_1, \dots, \epsilon_n > 0$ such that

$$\{x \in E : |\phi_j(x)| < \epsilon_j \text{ for } j = 1, \dots, n\} \subset \{x \in E : \|x\| < 1\}$$

and thus

$$\bigcap_{j=1}^n \ker \phi_j \subset \{x \in E : \|x\| < 1\}.$$

This is possible only if $\bigcap_{j=1}^n \ker \phi_j = \{0\}$. From Problem 3, we conclude that every $\phi \in E^*$ is a linear combination of ϕ_1, \dots, ϕ_n .

(b) \implies (c) is trivial.

(c) \implies (a): Let d be a metric on E that induces $\sigma(E, E^*)$. For $n \in \mathbb{N}$, set $U_n := \{x \in E : d(x, 0) < \frac{1}{n}\}$, so that U_n is open.

By the definition of $\sigma(E, E^*)$, there are, for each $n \in \mathbb{N}$, functionals $\phi_1^{(n)}, \dots, \phi_{m_n}^{(n)} \in E^*$ as well as $\epsilon_1^{(n)}, \dots, \epsilon_{m_n}^{(n)} > 0$ such that

$$\{x \in E : |\phi_j^{(n)}(x)| < \epsilon_j^{(n)} \text{ for } j = 1, \dots, m_n\} \subset U_n$$

and, in particular,

$$\bigcap_{j=1}^{m_n} \ker \phi_j^{(n)} \subset U_n.$$

Let $\phi \in E^*$, so that ϕ is $\sigma(E, E^*)$ -continuous and thus continuous with respect to d . Let $n \in \mathbb{N}$ be so large that $|\phi(x)| < 1$ for all $x \in U_n$. It follows that

$$\bigcap_{j=1}^{m_n} \ker \phi_j^{(n)} \subset \{x \in E : |\phi(x)| < 1\},$$

which is possible only if $\bigcap_{j=1}^{m_n} \ker \phi_j^{(n)} \subset \ker \phi$. By the previous problem, this means that ϕ is a linear combination of $\phi_1^{(n)}, \dots, \phi_{m_n}^{(n)}$.

Since $\phi \in E^*$ is arbitrary, we have thus seen that $\{\phi_1^{(n)}, \dots, \phi_{m_n}^{(n)} : n \in \mathbb{N}\}$ spans E^* . Consequently, the Banach space E^* has a countable Hamel basis, which is possible only if $\dim E^* < \infty$, so that $\dim E < \infty$.

5. Let (V, \mathcal{P}) and (W, \mathcal{Q}) be LCSs. Show that a linear map $T: V \rightarrow W$ is continuous if and only if, for each $q \in \mathcal{Q}$, there are $C \geq 0$ and $p_1, \dots, p_n \in \mathcal{P}$ such that

$$(*) \quad q(Tx) \leq C \max_{j=1, \dots, n} p_j(x) \quad (x \in V).$$

Solution: Suppose first that the given condition holds, and let $(x_\alpha)_\alpha$ be a net in V such that $x_\alpha \rightarrow 0$.

Let $q \in \mathcal{Q}$ be arbitrary. Then there are $C \geq 0$ and $p_1, \dots, p_n \in \mathcal{P}$ such that (*) holds, and we have

$$q(Tx_\alpha) \leq C \max_{j=1, \dots, n} p_j(x_\alpha) \rightarrow 0.$$

Since $q \in \mathcal{Q}$ was arbitrary, this means that $Tx_\alpha \rightarrow 0$.

Conversely, suppose that T is continuous, and let $q \in \mathcal{Q}$. Define

$$p: V \rightarrow [0, \infty), \quad x \mapsto q(Tx).$$

Then p is a continuous seminorm on V , and 0 is thus an interior point of $\{x \in V : p(x) < 1\}$. By the definition of the topology on (V, \mathcal{P}) , there are thus $p_1, \dots, p_n \in \mathcal{P}$ and $\epsilon > 0$ such that

$$\{x \in V : p_j(x) < \epsilon \text{ for } j = 1, \dots, n\} \subset \{x \in V : p(x) < 1\}.$$

Assume that (*) is false. Then, for each $k \in \mathbb{N}$, there is $x_k \in V$ such that $p(x_k) > k \max_{j=1, \dots, n} p_j(x_k)$ (and thus, in particular, that $p(x_k) > 0$). For $k \in \mathbb{N}$ set $y_k := p(x_k)^{-1}x_k$; it follows that

$$1 = p(y_k) > k \max_{j=1, \dots, n} p_j(y_k) \quad (k \in \mathbb{N}).$$

This is possible only if $\lim_{k \rightarrow \infty} \max_{j=1, \dots, n} p_j(y_k) = 0$. In particular, there is $k_0 \in \mathbb{N}$ such that $\max_{j=1, \dots, n} p_j(y_k) < \epsilon$ for all $k \geq k_0$. From the choice of $p_1, \dots, p_n \in \mathcal{P}$ and $\epsilon > 0$, however, this means that $p(y_k) < 1$ for $k \geq k_0$, which is a contradiction.

6. Let V be a TVS. Show that a closed subset $C \subset V$ is convex if and only if $\frac{1}{2}(x+y) \in C$ for all $x, y \in C$. Does this statement remain true if the demand that C be closed is dropped?

Solution: Clearly, only the “if” part needs proof.

Let $x, y \in C$, and let $t \in [0, 1]$. We need to show that $tx + (1-t)y \in C$.

We prove that this is true whenever $t = \frac{m}{2^n}$ with $n \in \mathbb{N}$ and $m \in \{1, \dots, 2^n\}$ by induction on n .

The claim is true if $n = 1$, so suppose that it is true for arbitrary, but fixed $n \in \mathbb{N}$. Let $m \in \{0, \dots, 2^{n+1}\}$, and let $m_1, m_2 \in \{0, \dots, 2^n\}$ be such that $m = m_1 + m_2$. Then we have:

$$\begin{aligned} \frac{m}{2^{n+1}}x + \left(1 - \frac{m}{2^{n+1}}\right)y &= \frac{1}{2} \left(\frac{m}{2^n}x + \left(2 - \frac{m}{2^n}\right)y \right) \\ &= \frac{1}{2} \left(\frac{m_1 + m_2}{2^n}x + \left(2 - \frac{m_2 + m_2}{2^n}\right)y \right) \\ &= \frac{1}{2} \left(\left(\frac{m_1}{2^n}x + \left(1 - \frac{m_1}{2^n}\right)y \right) + \left(\frac{m_2}{2^n}x + \left(1 - \frac{m_2}{2^n}\right)y \right) \right). \end{aligned}$$

By the induction hypothesis, we have $\frac{m_j}{2^n}x + (1 - \frac{m_j}{2^n})y \in C$ for $j = 1, 2$ and thus, by the foregoing, $\frac{m}{2^{n+1}}x + (1 - \frac{m}{2^{n+1}})y \in C$.

Let $t \in [0, 1]$ now be arbitrary. We construct a sequence $(t_k)_{k=1}^\infty$ in $[0, 1]$ as well as a sequence $(I_k)_{k=1}^\infty$ of closed subintervals as follows. Let $t_1 = 0$ and let $I_1 = [0, 1]$. Suppose that $t_1, \dots, t_k \in [0, 1]$ have already been constructed as have $I_1, \dots, I_{k-1}, I_k = [a, b]$. Let t_{k+1} be the midpoint of I_k . If $t \in [a, t_k]$, let $I_{k+1} = [a, t_k]$; otherwise, set $I_{k+1} := [t_k, b]$.

It follows from this construction that $t = \lim_{k \rightarrow \infty} t_k$ and that each t_k is of the form $\frac{m}{2^n}$ with $n \in \mathbb{N}$ and $m \in \{0, \dots, 2^n\}$. Since C is closed, we thus obtain that

$$tx + (1 - t)y = \lim_{k \rightarrow \infty} t_k x + (1 - t_k)y \in C.$$

Hence, C is convex.

Let $V = \mathbb{R}$ and $C = \mathbb{Q}$. Then $\frac{1}{2}(x + y) \in C$ for all $x, y \in C$, but C is not convex.

7. Let E and F be normed spaces, and let $\mathcal{B}(E, F)$ denote the space of all bounded linear operators from E to F . Define families $\{p_x : x \in E\}$ and $\{q_{x, \phi} : x \in E, \phi \in F^*\}$ on $\mathcal{B}(E, F)$ of seminorms by letting

$$p_x(T) := \|Tx\| \quad \text{and} \quad q_{x, \phi}(T) := |\phi(Tx)| \quad (x \in E, \phi \in F^*, T \in \mathcal{B}(E, F)).$$

The locally convex topologies on $\mathcal{B}(E, F)$ induced by $\{p_x : x \in E\}$ and $\{q_{x, \phi} : x \in E, \phi \in F^*\}$ are called the *strong* and the *weak operator topology*, respectively.

Show that the following are equivalent for $\phi: \mathcal{B}(E, F) \rightarrow \mathbb{F}$:

- (i) ϕ continuous with respect to the weak operator topology;
- (ii) ϕ is continuous with respect to the strong operator topology;
- (iii) there are $x_1, \dots, x_n \in E$ and $\phi_1, \dots, \phi_n \in F^*$ such that

$$\phi(T) = \sum_{j=1}^n \phi_j(Tx_j) \quad (T \in \mathcal{B}(E, F)).$$

Solution: Of course, only the implication (ii) \implies (iii) needs proof.

Let $\phi: \mathcal{B}(E, F) \rightarrow \mathbb{F}$ be continuous with respect to the strong operator topology. By Problem 5, there are thus $x_1, \dots, x_n \in E$ such that

$$(**) \quad |\phi(T)| \leq \max_{j=1, \dots, n} \|Tx_j\| \quad (T \in \mathcal{B}(E, F)).$$

Let F^n be equipped with the norm

$$\|(y_1, \dots, y_n)\|_\infty := \max_{j=1, \dots, n} \|y_j\| \quad ((y_1, \dots, y_n) \in F^n),$$

and let X be the the subspace $\{(Tx_1, \dots, Tx_n) : T \in \mathcal{B}(E, F)\}$ of F^n . By (**), the functional

$$\tilde{\phi}: X \rightarrow \mathbb{F}, \quad (Tx_1, \dots, Tx_n) \mapsto \phi(T)$$

is well defined and continuous. The Hahn–Banach theorem yields an extension of $\tilde{\phi}$ to all of F^n , which we also denote by $\tilde{\phi}$. Obviously, there are $\phi_1, \dots, \phi_n \in F^*$ such that

$$\tilde{\phi}(y_1, \dots, y_n) = \phi_1(y_1) + \dots + \phi_n(y_n) \quad ((y_1, \dots, y_n) \in F^n),$$

so that

$$\phi(T) = \tilde{\phi}(Tx_1, \dots, Tx_n) = \sum_{j=1}^n \phi_j(Tx_j).$$