1. Evaluate the improper integral
\[ \int_0^\infty e^{-\alpha x} \cos(\beta x) \, dx \]
for \( \alpha > 0 \).

**Solution:** Let \( R > 0 \). Integration by parts yields
\[ \int_0^R e^{-\alpha x} \cos(\beta x) \, dx = -\frac{1}{\alpha} e^{-\alpha x} \cos(\beta x) \bigg|_0^R - \frac{\beta}{\alpha} \int_0^R e^{-\alpha x} \sin(\beta x) \, dx \]
\[ = -\frac{1}{\alpha} e^{-\alpha R} \cos(\beta R) + \frac{1}{\alpha} - \frac{\beta}{\alpha} \int_0^R e^{-\alpha x} \sin(\beta x) \, dx. \]

Using integration by parts again, we obtain
\[ \int_0^R e^{-\alpha x} \sin(\beta x) \, dx = -\frac{1}{\alpha} e^{-\alpha x} \sin(\beta x) \bigg|_0^R + \frac{\beta}{\alpha} \int_0^R e^{-\alpha x} \cos(\beta x) \, dx \]
\[ = -\frac{1}{\alpha} e^{-\alpha R} \sin(\beta R) + \frac{\beta}{\alpha} \int_0^R e^{-\alpha x} \cos(\beta x) \, dx. \]

Plugging the last equation into the right hand side of the first one, we get
\[ \int_0^R e^{-\alpha x} \cos(\beta x) \, dx \]
\[ = -\frac{1}{\alpha} e^{-\alpha R} \cos(\beta R) + \frac{1}{\alpha} + \frac{\beta}{\alpha} e^{-\alpha R} \sin(\beta R) - \frac{\beta^2}{\alpha^2} \int_0^R e^{-\alpha x} \cos(\beta x) \, dx \]
and thus
\[ \left( 1 + \frac{\beta^2}{\alpha^2} \right) \int_0^R e^{-\alpha x} \cos(\beta x) \, dx = -\frac{1}{\alpha} e^{-\alpha R} \cos(\beta R) + \frac{1}{\alpha} + \frac{\beta}{\alpha^2} e^{-\alpha R} \sin(\beta R). \]

Division by \( 1 + \frac{\beta^2}{\alpha^2} \), then yields
\[ \int_0^R e^{-\alpha x} \cos(\beta x) \, dx = \left( \frac{\alpha^2}{\alpha^2 + \beta^2} \right) \left( -\frac{1}{\alpha} e^{-\alpha R} \cos(\beta R) + \frac{1}{\alpha} + \frac{\beta}{\alpha^2} e^{-\alpha R} \sin(\beta R) \right). \]

Finally, letting \( R \to \infty \), we obtain
\[ \int_0^\infty e^{-\alpha x} \cos(\beta x) \, dx = \frac{\alpha}{\alpha^2 + \beta^2}. \]

2. Determine whether or not the following improper integrals exist:
(a) \[ \int_{0}^{\infty} \frac{x}{\sqrt{1+x^2}} \, dx; \]
(b) \[ \int_{0}^{1} \frac{dx}{\sqrt{\sin x}}; \]
(c) \[ \int_{0}^{\infty} \sin(x^2) \, dx. \]

(Hint for (c): Substitute \( x = \sqrt{u} \).)

Solution:

(a) As 
\[ \lim_{x \to \infty} \sqrt{\frac{x^3}{1+x^3}} = 1, \]
there is \( R_0 > 0 \) such that 
\[ \sqrt{\frac{x^3}{1+x^3}} = \frac{x}{\sqrt{1+x^3}} \sqrt{x} \geq \frac{1}{2} \]
for \( x \geq R_0 \) and thus 
\[ \frac{x}{\sqrt{1+x^3}} \geq \frac{1}{2\sqrt{x}} \]
for \( x \geq R_0 \). Since \( \int_{R_0}^{\infty} \frac{1}{2\sqrt{x}} \, dx \) does not exist, \( \int_{R_0}^{\infty} \frac{x}{\sqrt{1+x^3}} \, dx \) does not exist either by the Comparison Test. Consequently, \( \int_{0}^{\infty} \frac{x}{\sqrt{1+x^3}} \, dx \) does not exist.

(b) As 
\[ \lim_{x \to 0} \sqrt{\frac{x}{\sin x}} = 1, \]
there is \( C \geq 0 \) such that 
\[ \frac{1}{\sqrt{\sin x}} \leq \frac{C}{\sqrt{x}} \]
for \( x \in [0,1] \). Since \( \int_{0}^{1} \frac{1}{\sqrt{x}} \, dx \) easily seen to exist, the existence of \( \int_{0}^{1} \frac{dx}{\sqrt{\sin x}} \) follows from (an obvious modification of) the comparison test.

(c) It is sufficient to decide whether or not \( \int_{1}^{\infty} \sin(x^2) \, dx \) exists.

Let \( R > 1 \). The change of variables \( x = \sqrt{u} \) yields 
\[ \int_{1}^{R} \sin(x^2) \, dx = \frac{1}{2} \int_{1}^{R^2} \sin u \, du. \]
Integration by parts applied to the integral on the right hand side yields 
\[ \int_{1}^{R^2} \sin u \, du = -\frac{\cos u}{\sqrt{u}} \bigg|_{1}^{R^2} - \frac{1}{2} \int_{1}^{R^2} \frac{\cos u}{u^2} \, du. \]
By the Comparison Test, the improper integral \( \frac{1}{2} \int_{1}^{\infty} \frac{\cos u}{u^2} \, du \) converges absolutely, so that \( \lim_{R \to \infty} \int_{1}^{R^2} \frac{\cos u}{u^2} \, du \) exists. Since \( \lim_{R \to \infty} -\frac{\cos u}{\sqrt{u}} \bigg|_{1}^{R^2} = \cos 1 \), it follows that 
\[ \int_{1}^{\infty} \sin(x^2) \, dx = \lim_{R \to \infty} \int_{1}^{R} \sin(x^2) \, dx = \lim_{R \to \infty} \frac{1}{2} \int_{1}^{R^2} \sin u \, du \]
exists.
3. Determine those $p > 0$ for which the series $\sum_{n=10}^{\infty} \frac{1}{n^{(\log n)(\log(\log n))^{p}}$ converges.

**Solution:** The function

$$f : [10, \infty) \to \mathbb{R}, \quad x \mapsto \frac{1}{x(\log x)(\log(\log x))^{p}}$$

is non-negative and decreasing, so that the integral comparison test is applicable. Let $R > 10$. Changing variables twice yields

$$\int_{10}^{R} \frac{dx}{x(\log x)(\log(\log x))^{p}} = \int_{\log 10}^{\log R} \frac{du}{u(\log u)^{p}} = \int_{\log(\log 10)}^{\log(\log R)} \frac{dv}{v^{p}},$$

so that

$$\int_{10}^{R} \frac{dx}{x(\log x)(\log(\log x))^{p}} = \begin{cases} \frac{\log v}{1 \log(\log R)} - 1 \log(\log R), & \text{if } p = 1, \\ \frac{1}{1 - p} \frac{1}{v^{p}} - 1 \log(\log 10), & \text{if } p \neq 1, \end{cases}$$

As $\lim_{R \to \infty} \log(\log R) = \infty$, it follows that $\int_{10}^{R} \frac{dx}{x(\log x)(\log(\log x))^{p}}$ does exist if and only if $p > 1$. By the Integral Comparison Test, $\sum_{n=10}^{\infty} \frac{1}{n^{(\log n)(\log(\log n))^{p}}$ converges if and only if $p > 1$.

4. For $n \in \mathbb{N}$, let

$$f_{n} : [0, \infty) \to \mathbb{R}, \quad x \mapsto \frac{x}{n^{2}} e^{-\frac{x}{n}}.$$ 

Show that $f_{n} \to 0$ uniformly on $[0, \infty)$, but that

$$\lim_{n \to \infty} \int_{0}^{\infty} f_{n}(x) \, dx = 1.$$ 

Why doesn’t this contradict Corollary 8.1.4 from the notes?

**Solution:** First, note that

$$f_{n}'(x) = \frac{1}{n^{2}} e^{-\frac{x}{n}} - \frac{x}{n^{3}} e^{-\frac{x}{n}} = \left( \frac{1}{n^{2}} - \frac{x}{n^{3}} \right) e^{-\frac{x}{n}}$$

for $x \in [0, \infty)$. It follows that $f_{n}'(x) > 0$ for $x < n$, $f_{n}'(n) = 0$ and $f_{n}'(x) < 0$ for $x > n$. Hence, $f_{n}$ is increasing on $[0, n]$ and decreasing on $[n, \infty)$, so that

$$0 \leq f_{n}(x) \leq f_{n}(n) = \frac{1}{ne}$$

for $x \in [0, \infty)$. Let $\epsilon > 0$, and choose $n_{\epsilon} \in \mathbb{N}$ such that $\frac{1}{en} < \epsilon$ for $n \geq n_{\epsilon}$. It follows that $|f_{n}(x)| < \epsilon$ for $x \in [0, \infty)$ and $n \geq n_{\epsilon}$, so that $f_{n} \to 0$ uniformly on $[0, \infty)$.

Let $R > 0$ and note that

$$\int_{0}^{R} \frac{x}{n^{2}} e^{-\frac{x}{n}} \, dx = -\frac{x}{n} e^{-\frac{x}{n}} \bigg|_{0}^{R} + \frac{1}{n} \int_{0}^{R} e^{-\frac{x}{n}} \, dx$$

$$= -\frac{R}{n} e^{-\frac{R}{n}} - e^{-\frac{R}{n}} + 1$$

$$\to 1 \quad \text{as } R \to \infty.$$
Consequently,
\[ \int_0^\infty f_n(x) \, dx = 1 \]
holds for all \( n \in \mathbb{N} \).

Since the integrals in this problem are not Riemann integrals, but improper integrals, there is no contradiction.

5. Show that the series \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) does not uniformly converge to \( e^x \) on all of \( \mathbb{R} \).

\textit{Solution:} Assume that we have uniform convergence. Then there is \( N \in \mathbb{N} \) such that
\[ \left| e^x - \sum_{n=0}^{N} \frac{x^n}{n!} \right| < 1 \]
for all \( x \in \mathbb{R} \). Division by \( |x^{N+1}| \) yields
\[ \left| \frac{e^x}{x^{N+1}} - \sum_{n=0}^{N} \frac{x^{n-N-1}}{n!} \right| < \frac{1}{|x^{N+1}|} \]
for \( x \in \mathbb{R} \) and thus
\[ \left| \frac{e^x}{x^{N+1}} \right| < \frac{1}{|x^{N+1}|} + \sum_{n=0}^{N} \frac{|x^{n-N-1}|}{n!} \]
for \( x \in \mathbb{R} \). The right hand side of this inequality tends to zero if \( x \to \infty \), and so the same must be true for the left hand side. However, it is well known (from de l’Hospital’s Rule) that \( \lim_{x \to \infty} \frac{e^x}{x^{N+1}} = \infty \).

6*. Let \( \emptyset \neq D \subset \mathbb{R}^N \) have content, and let \( (f_n)_{n=1}^{\infty} \) be a sequence of Riemann-integrable functions on \( D \) that converges uniformly to a function \( f : D \to \mathbb{R} \). Show that \( f \) is Riemann-integrable as well such that
\[ \int_D f = \lim_{n \to \infty} \int_D f_n. \]

Give an example of a sequence of Riemann-integrable functions on \([0,1]\) that converges pointwise to a bounded, but not Riemann-integrable function.

\textit{Solution:} Without loss of generality, suppose that \( D \) is an \( N \)-dimensional compact interval \( I \) (with \( \mu(I) > 0 \)).

Let \( \epsilon > 0 \). Choose \( n_\epsilon \in \mathbb{N} \) such that \( |f_n(x) - f(x)| < \frac{\epsilon}{5\mu(I)} \) for all \( n \geq n_\epsilon \) and for all \( x \in I \).

By the Cauchy Criterion for the Riemann-integrability, there is a partition \( \mathcal{P}_\epsilon \) of \( I \) with the following property: Whenever \( \mathcal{P} \) is a partition finer than \( \mathcal{P}_\epsilon \), \((I_\nu)_\nu\) is the
subdivision of $I$ corresponding to $\mathcal{P}$, and $\xi_\nu$ and $\eta_\nu$ are arbitrary points in $I_\nu$, we have
\[
\left| \sum_\nu f_n(\xi_\nu)\mu(I_\nu) - \sum_\nu f_n(\eta_\nu)\mu(I_\nu) \right| < \frac{\epsilon}{3}
\]
For such $\mathcal{P}$, $(I_\nu)_\nu$, and $\xi_\nu$ and $\eta_\nu$, we obtain
\[
\left| \sum_\nu f(\xi_\nu)\mu(I_\nu) - \sum_\nu f(\eta_\nu)\mu(I_\nu) \right|
\leq \left| \sum_\nu f(\xi_\nu)\mu(I_\nu) - \sum_\nu f_n(\xi_\nu)\mu(I_\nu) \right| + \left| \sum_\nu f_n(\xi_\nu)\mu(I_\nu) - \sum_\nu f_n(\eta_\nu)\mu(I_\nu) \right|
+ \left| \sum_\nu f_n(\eta_\nu)\mu(I_\nu) - \sum_\nu f(\eta_\nu)\mu(I_\nu) \right|
\leq \sum_\nu |f(\xi_\nu) - f_n(\xi_\nu)|\mu(I_\nu) + \frac{\epsilon}{3} + \sum_\nu |f_n(\eta_\nu) - f(\eta_\nu)|\mu(I_\nu)
\leq \frac{2\epsilon}{3\mu(I)} \sum_\nu \mu(I_\nu) + \frac{\epsilon}{3}
= \frac{2\epsilon}{3} + \frac{\epsilon}{3}
= \epsilon.
\]
By the Cauchy criterion again, this means that $f$ is Riemann-integrable.

The fact that
\[
\int_D f = \lim_{n \to \infty} \int_D f_n.
\]
is proven exactly as for continuous functions.

Let $\{q_1, q_2, \ldots\}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$. For $n \in \mathbb{N}$, let $f_n := \chi_{\{q_1, \ldots, q_n\}}$. Then $(f_n)_{n=1}^\infty$ converges pointwise to $f := \chi_{\mathbb{Q} \cap [0,1]}$. Since each $f_n$ is discontinuous only at $\{q_1, \ldots, q_n\}$, it follows that each $f_n$ is Riemann-integrable. However, $f$ clearly isn’t.