1. Use differentials to approximate $\sqrt{22} + \sqrt{5} + e^{1/10}$.
   (A) 7.85  (B) 7.95  (C) 8.05  (D) 8.15  (E) 8.40

2. Let $f$ be an arbitrary, twice differentiable function for which $f'' \neq 0$. The function $u(x, y) = f(x^2 + 2axy + y^2)$ satisfies the equation $u_{xx} - u_{yy} = 0$ if the constant “$a$” is:
   (A) 0;  (B) 1;  (C) 2;  (D) 3;  (E) 4.

3. If $z = f(x, y)$ is given implicitly by $2x \ln y + 4xz^2 + yz^3 = 3$, then $\frac{\partial f}{\partial x}$ evaluated at the point $(1, 1, -1)$ equals:
   (A) 0;  (B) $-\frac{4}{5}$;  (C) $\frac{4}{5}$;  (D) $\frac{1}{5}$;  (E) $-\frac{1}{5}$.

4. The function $f(x, y) = \begin{cases} 
\frac{x^4 - y^4}{x^2 + y^2}, & (x, y) \neq (0, 0) \\
0.1, & (x, y) = (0, 0) 
\end{cases}$ is continuous:
   (A) everywhere;
   (B) only at $(0,0)$;
   (C) everywhere except $(0,0)$;
   (D) everywhere except $(0,0,1)$;
   (E) nowhere.
Long Answer Problems. Show all your work.

1. Find all critical points of the function $f(x, y) = (x^2 + y^2)e^{-2x}$ and classify them as local minima, local maxima or saddle points.

**Solution**

Taking derivatives of $f(x, y) = (x^2 + y^2)e^{-2x}$ yields:

\[
\begin{align*}
  f_x &= e^{-2x}(2x - 2x^2 - 2y^2); \\
  f_{xx} &= e^{-2x}(-8x + 4x^2 + 4y^2 + 2); \\
  f_y &= 2ye^{-2x}; \\
  f_{xy} &= -4ye^{-2x}; \\
  f_{yy} &= 2e^{-2x}.
\end{align*}
\]

Solving for critical points:

\[
\begin{align*}
  f_y &= 0 \implies y = 0, \\
  f_x &= 0 \implies 2x^2 = 2x \implies x = 0, 1.
\end{align*}
\]

So the critical points are $(0, 0)$ and $(1, 0)$. Using the second derivative test leads to:

\[
\begin{align*}
  f_{xx}f_{yy} - f_{xy}^2 \bigg|_{(0,0)} &> 0, \quad f_{xx}(0,0) > 0 \implies f \text{ has a minimum at } (0,0); \\
  f_{xx}f_{yy} - f_{xy}^2 \bigg|_{(1,0)} &< 0, \quad \implies f \text{ has a saddle at } (1,0).
\end{align*}
\]
2. Find the maximum and minimum value of the function \( f(x, y) = x^4 + y^4 \) subject to the constraint \( x^2 + y^2 = 1 \).

**Solution**

Let \( g(x, y) = x^2 + y^2 - 1 \). Then we have

\[
\vec{\nabla} f = \langle 4x^3, 4y^3 \rangle, \quad \text{and} \quad \vec{\nabla} g = \langle 2x, 2y \rangle.
\]

Apply the Lagrange multiplier equations:

\[
\left\{ \begin{array}{c}
4x^3 = 2\lambda x \\
4y^3 = 2\lambda y \\
x^2 + y^2 = 1
\end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{c}
x(2x^2 - \lambda) = 0 \quad (1) \\
y(2y^2 - \lambda) = 0 \quad (2) \\
x^2 + y^2 = 1 \quad (3)
\end{array} \right.
\]

Notice that \( \lambda = 0 \) implies (using Eqs (1),(2)) that \( x = y = 0 \) which violates Eq. (3). Thus, we must conclude that \( \lambda \neq 0 \).

Eq. (1) implies \( x = 0 \) or \( x^2 = \frac{\lambda}{2} \). If \( x = 0 \) then from Eq. (3) we get \( y = \pm 1 \). Thus, two critical points are \((0, \pm 1)\).

Eq. (2) implies \( y = 0 \) or \( y^2 = \frac{\lambda}{2} \). If \( y = 0 \) then from Eq. (3) we get \( x = \pm 1 \). Thus, two critical points are \((\pm 1, 0)\).

The only other alternative is \( x^2 = y^2 = \frac{\lambda}{2} \). Inserting this into Eq. (3) gives

\[
\frac{\lambda}{2} + \frac{\lambda}{2} = 1 \quad \Rightarrow \quad \lambda = 1.
\]

This gives

\[
x^2 = y^2 = \frac{1}{2} \quad \Rightarrow \quad x = \pm \frac{1}{\sqrt{2}}, \quad y = \pm \frac{1}{\sqrt{2}}.
\]

Thus, we now have 4 more critical points:

\[
\left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \quad \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right), \quad \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \quad \text{and} \quad \left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).
\]

To determine the maximum and minimum values, we evaluate \( f(x, y) \) at each critical point \((x_c, y_c)\):

\[
\begin{array}{cccccccc}
(x_c, y_c) & (0, 1) & (0, -1) & (1, 0) & (-1, 0) & \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) & \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) & \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) & \left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \\
 f(x_c, y_c) & 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{array}
\]

Therefore the maximum value is 1 and the minimum value is \( \frac{1}{2} \).
3. Find the point(s) on the sphere  \( x^2 + y^2 + z^2 = 2y - 2z + 22 \) at which the tangent plane is parallel to the plane  \( x + 2y - z = 2 \).

**Solution**

The equation of the sphere can be written as

\[
f(x, y, z) = x^2 + (y - 1)^2 + (z + 1)^2 = 24
\]

Let \((x_0, y_0, z_0)\) be a point on the sphere. A normal direction of the tangent plane at \((x_0, y_0, z_0)\) is

\[
\nabla f(x_0, y_0, z_0) = \langle 2x_0, 2(y_0 - 1), 2(z_0 + 1) \rangle
\]

or just \(\langle x_0, y_0 - 1, z_0 + 1 \rangle\). A normal direction for the plane \(x + 2y - z = 2\) is \(\langle 1, 2, -1 \rangle\). The tangent plane is parallel to the given plane if the vectors \(\langle x_0, y_0 - 1, z_0 + 1 \rangle\) and \(\langle 1, 2, -1 \rangle\) are parallel which means

\[
(x_0, y_0 - 1, z_0 + 1) = \lambda(1, 2, -1)
\]

where \(\lambda\) is a real number. Hence, on components,

\[
x_0 = \lambda, \quad y_0 - 1 = 2\lambda, \quad z_0 + 1 = -\lambda
\]

Since \((x_0, y_0, z_0)\) is a point on the sphere \(x^2 + (y - 1)^2 + (z + 1)^2 = 24\) we must have:

\[
x_0^2 + (y_0 - 1)^2 + (z_0 + 1)^2 = 24
\]

that is,

\[
\lambda^2 + 4\lambda^2 + \lambda^2 = 24 \iff \lambda = \pm 2
\]

Therefore, the required points are: \((2, 5, -3)\) and \((-2, -3, 1)\).

4. The rate of change of a differentiable function \(f(x, y)\) at a point \((a, b)\) in the direction of \(\vec{v} = \vec{i} + \vec{j}\) is \(3\sqrt{2}\) and rate of change in the direction of \(\vec{w} = 3\vec{i} - 4\vec{j}\) is 5. Find \(\nabla f(a, b)\).

**Solution**

Let \(P = (a, b)\).

Unit vector in the direction of \(\vec{v}\) is \(\vec{v}_0 = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle\). Hence:

\[
\nabla f(P) \cdot \vec{v}_0 = 3\sqrt{2} \iff f_x(P) + f_y(P) = 6.
\]

Unit vector in the direction of \(\vec{w}\) is \(\vec{w}_0 = \langle 3/5, -4/5 \rangle\). Hence:

\[
\nabla f(P) \cdot \vec{w}_0 = 5 \iff 3f_x(P) - 4f_y(P) = 25.
\]

Therefore:

\[
f_x(P) = 7, \quad f_y(P) = -1,
\]

that is, \(\nabla f(a, b) = \langle 7, -1 \rangle\).