WHITE EXAM

Multiple Choice Questions

Please enter your answer as a CAPITAL LETTER in the BOX provided.

1. If \( z = f(x, y) \) is given implicitly by \( 2x \ln y + 4xz^2 + yz^3 = 3 \), then \( \frac{\partial f}{\partial x} \) evaluated at the point \((1, 1, -1)\) equals:
   \[(A) 0; (B) -\frac{4}{5}; (C) \frac{4}{5}; (D) \frac{1}{5}; (E) -\frac{1}{5} \.
   \]

2. The function \( f(x, y) = \begin{cases} \frac{x^4 - y^4}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0.1, & (x, y) = (0, 0) \end{cases} \) is continuous:
   \[(A) \text{ everywhere}; (B) \text{ only at } (0, 0); (C) \text{ everywhere except } (0, 0); (D) \text{ everywhere except } (0, 0.1); (E) \text{ nowhere}.
   \]

3. Use differentials to approximate \( \sqrt{23} + \sqrt{5} + e^{1/10} \).
   \[(A) 7.85 \quad (B) 7.95 \quad (C) 8.05 \quad (D) 8.15 \quad (E) 8.40 \]

4. Let \( f \) be an arbitrary, twice differentiable function for which \( f'' \neq 0 \). The function \( u(x, y) = f(x^2 +axy + y^2) \) satisfies the equation \( u_{xx} - u_{yy} = 0 \) if the constant “\( a \)” is:
   \[(A) 0; (B) 1; (C) 2; (D) 3; (E) 4. \]
Long Answer Problems. Show all your work.

1. Find the point(s) on the sphere \( x^2 + y^2 + z^2 = 2y - 2z + 22 \) at which the tangent plane is parallel to the plane \( x + 2y - z = 2 \).

   **Solution**
   The equation of the sphere can be written as
   \[ f(x, y, z) = x^2 + (y - 1)^2 + (z + 1)^2 = 24 \]
   Let \((x_0, y_0, z_0)\) be a point on the sphere. A normal direction of the tangent plane at \((x_0, y_0, z_0)\) is
   \[ \nabla f(x_0, y_0, z_0) = (2x_0, 2(y_0 - 1), 2(z_0 + 1)) \]
   or just \(\langle x_0, y_0 - 1, z_0 + 1 \rangle\). A normal direction for the plane \(x + 2y - z = 2\) is \(\langle 1, 2, -1 \rangle\). The tangent plane is parallel to the given plane if the vectors \(\langle x_0, y_0 - 1, z_0 + 1 \rangle\) and \(\langle 1, 2, -1 \rangle\) are parallel which means
   \[ \langle x_0, y_0 - 1, z_0 + 1 \rangle = \lambda \langle 1, 2, -1 \rangle \]
   where \(\lambda\) is a real number. Hence, on components,
   \[ x_0 = \lambda, \quad y_0 - 1 = 2\lambda, \quad z_0 + 1 = -\lambda \]
   Since \((x_0, y_0, z_0)\) is a point on the sphere \(x^2 + (y - 1)^2 + (z + 1)^2 = 24\) we must have:
   \[ x_0^2 + (y_0 - 1)^2 + (z_0 + 1)^2 = 24 \]
   that is,
   \[ \lambda^2 + 4\lambda^2 + \lambda^2 = 24 \iff \lambda = \pm 2 \]
   Therefore, the required points are: \((2, 5, -3)\) and \((-2, -3, 1)\).

2. The rate of change of a differentiable function \(f(x, y)\) at a point \((a, b)\) in the direction of \(\vec{v} = \vec{i} + \vec{j}\) is \(3\sqrt{2}\) and rate of change in the direction of \(\vec{w} = 3\vec{i} - 4\vec{j}\) is 5. Find \(\nabla f(a, b)\).

   **Solution**
   Let \(P = (a, b)\).
   Unit vector in the direction of \(\vec{v}\) is \(\vec{v}_0 = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle\). Hence:
   \[ \nabla f(P) \cdot \vec{v}_0 = 3\sqrt{2} \iff f_x(P) + f_y(P) = 6. \]
   Unit vector in the direction of \(\vec{w}\) is \(\vec{w}_0 = \langle 3/5, -4/5 \rangle\). Hence:
   \[ \nabla f(P) \cdot \vec{w}_0 = 5 \iff 3f_x(P) - 4f_y(P) = 25. \]
   Therefore:
   \[ f_x(P) = 7, \quad f_y(P) = -1, \]
   that is, \(\nabla f(a, b) = \langle 7, -1 \rangle\).
3. Find all critical points of the function \( f(x, y) = (x^2 + y^2)e^{-2y} \) and classify them as local minima, local maxima or saddle points.

**Solution**

Taking derivatives of \( f(x, y) = (x^2 + y^2)e^{-2y} \) yields:

\[
\begin{align*}
f_x &= 2xe^{-2y}; \\
f_{xx} &= 2e^{-2y}; \\
f_{xy} &= -4xe^{-2y}; \\
f_y &= e^{-2y}(-2x^2 - 2y^2 + 2y); \\
f_{yy} &= e^{-2y}(4x^2 + 4y^2 - 8y + 2).
\end{align*}
\]

Solving for critical points:

\[
f_x = 0 \implies x = 0, \quad f_y = 0 \implies y^2 = y \implies y = 0, 1.
\]

So the critical points are \((0, 0)\) and \((0, 1)\). Using the second derivative test leads to:

\[
\begin{align*}
f_{xx}f_{yy} - f_{xy}^2 \bigg|_{(0,0)} &> 0, \quad f_{xx}(0,0) > 0 \implies f \text{ has a minimum at } (0, 0); \\
f_{xx}f_{yy} - f_{xy}^2 \bigg|_{(0,1)} &< 0, \quad \implies f \text{ has a saddle at } (0, 1).
\end{align*}
\]
4. Find the maximum and minimum value of the function \( f(x, y) = x^2 + y^2 \) subject to the constraint \( x^4 + y^4 = 1 \).

**Solution**

Let \( g(x, y) = x^4 + y^4 - 1 \). Then we have

\[ \vec{\nabla} f = \langle 2x, 2y \rangle, \quad \text{and} \quad \vec{\nabla} g = \langle 4x^3, 4y^3 \rangle. \]

Apply the Lagrange multiplier equations:

\[ \begin{align*}
\vec{\nabla} f &= \lambda \vec{\nabla} g \\
g &= 0
\end{align*} \]

\[ \begin{cases}
2x = 4\lambda x^3 \\
2y = 4\lambda y^3 \\
x^4 + y^4 = 1
\end{cases} \]

Notice that \( \lambda = 0 \) implies (using Eqs (1),(2)) that \( x = y = 0 \) which violates Eq. (3). Thus, we must conclude that \( \lambda \neq 0 \).

Eq. (1) implies \( x = 0 \) or \( x^2 = \frac{1}{2\lambda} \). If \( x = 0 \) then from Eq. (3) we get \( y = \pm 1 \). Thus, two critical points are \((0, \pm 1)\).

Eq. (2) implies \( y = 0 \) or \( y^2 = \frac{1}{2\lambda} \). If \( y = 0 \) then from Eq. (3) we get \( x = \pm 1 \). Thus, two critical points are \((\pm 1, 0)\).

The only other alternative is \( x^2 = y^2 = \frac{1}{2\lambda} \). Inserting this into Eq. (3) gives

\[ \left( \frac{1}{2\lambda} \right)^2 + \left( \frac{1}{2\lambda} \right)^2 = 1 \quad \Rightarrow \quad \lambda = \pm \frac{1}{\sqrt{2}}. \]

But \( x^2 = \frac{1}{2\lambda} \) can not be negative, so we must have \( \lambda = \frac{1}{\sqrt{2}} \). This gives

\[ x^2 = y^2 = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad x = \pm \frac{1}{2^{1/4}}, \quad y = \pm \frac{1}{2^{1/4}}. \]

Thus, we now have 4 more critical points:

\[ \left( \frac{1}{2^{1/4}}, \frac{1}{2^{1/4}} \right), \quad \left( \frac{1}{2^{1/4}}, -\frac{1}{2^{1/4}} \right), \quad \left( -\frac{1}{2^{1/4}}, \frac{1}{2^{1/4}} \right), \quad \text{and} \quad \left( -\frac{1}{2^{1/4}}, -\frac{1}{2^{1/4}} \right). \]

To determine the maximum and minimum values, we evaluate \( f(x, y) \) at each critical point \((x_c, y_c)\):

\[
\begin{array}{cccccccc}
(x_c, y_c) & (0, 1) & (0, -1) & (1, 0) & (-1, 0) & \left( \frac{1}{2^{1/4}}, \frac{1}{2^{1/4}} \right) & \left( \frac{1}{2^{1/4}}, -\frac{1}{2^{1/4}} \right) & \left( -\frac{1}{2^{1/4}}, \frac{1}{2^{1/4}} \right) & \left( -\frac{1}{2^{1/4}}, -\frac{1}{2^{1/4}} \right) \\
\text{f}(x_c, y_c) & 1 & 1 & 1 & 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2}
\end{array}
\]

Therefore the maximum value is \( \sqrt{2} \) and the minimum value is 1. □

*Note on grading:* The point of a Lagrange multiplier problem is to find ALL critical points. Half the marks for this question are associated with each group of four critical points. Missing one group of four critical points entirely resulted in an immediate loss of half the marks. More errors resulted further marks being deducted.