Honors Complex Variables (Math 411)

Volker Runde

January 16, 2009

Contents

1 The Complex Numbers

Definition 1.1. The *complex numbers*—denoted by $\mathbb{C}$—are $\mathbb{R}^2$ equipped with the operations

$$\begin{aligned}(x, y) + (u, v) &:= (x + u, y + v),\\ (x, y)(u, v) &:= (xu - yv, xv + yu)\end{aligned}$$

for $x, y, u, v \in \mathbb{R}$.

Theorem 1.2. *The complex numbers are a field. More specifically, we have:*

- $(0, 0)$ *is the neutral element of addition;*
- $-(x, y) = (-x, -y)$ *for* $x, y \in \mathbb{R}$*;*
- $(1, 0)$ *is the neutral element of multiplication;*
- $(x, y)^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$ *for* $x, y \in \mathbb{R}$ *with* $(x, y) \neq (0, 0)$.

Proof (of the last claim only). Let $x, y \in \mathbb{R}$ be such that $(x, y) \neq (0, 0)$, and note that

$$\begin{aligned}(x, y)\left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right) &= \left(\frac{x^2}{x^2 + y^2} - \frac{-y^2}{x^2 + y^2}, \frac{-xy}{x^2 + y^2} + \frac{xy}{x^2 + y^2}\right)\\ &= \left(\frac{x^2 + y^2}{x^2 + y^2}, \frac{-xy + xy}{x^2 + y^2}\right)\\ &= (1, 0).\end{aligned}$$

□

Proposition 1.3. *The set* $\{(x, 0) : x \in \mathbb{R}\}$ *is a subfield of* $\mathbb{C}$, *and the map*

$$\theta \colon \mathbb{R} \to \mathbb{C}, \quad x \mapsto (x, 0)$$

is an isomorphism onto its image.

Proposition 1.3 is often worded as:

$$\mathbb{R} \text{ “is” a subfield of } \mathbb{C}.$$

Set $1 := (1, 0)$ and $i := (0, 1)$. Then, for any $z = (x, y) \in \mathbb{C}$, we have

$$z = (x, 0) + (0, y) = (1, 0)(x, 0) + (0, 1)(y, 0) = x + iy.$$

We write

$$\operatorname{Re} z := x = \text{“the } \textit{real part} \text{ of } z\text{”}$$

and

$$\operatorname{Im} z := y = \text{“the } \textit{imaginary part} \text{ of } z\text{”}.$$

The complex number i is called the *imaginary unit* and satisfies

$$i^2 = (0,1)^2 = (-1,0) = -1.$$

Definition 1.4. For $z = x + iy \in \mathbb{C}$, its *complex conjugate* is defined as $\bar{z} = x - iy$.

Proposition 1.5. *For $z, w \in \mathbb{C}$, the following hold true:*

(i) $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$ *and* $\operatorname{Im} z = \frac{1}{2i}(z - \bar{z})$*;*

(ii) $\overline{z + w} = \bar{z} + \bar{w}$*;*

(iii) $\overline{zw} = \bar{z}\bar{w}$*;*

(iv) $\overline{z^{-1}} = \bar{z}^{-1}$ *if* $z \neq 0$.

Proof. (i): If $z = x + iy$, then $\bar{z} = x - iy$, so that $2x = z + \bar{z}$; this yields the claim for $\operatorname{Re} z$. The assertion for $\operatorname{Im} z$ is proven similarly.

(ii) is obvious.

(iii): Let $z = x + iy$ and $w = u + iv$, so that

$$zw = (xu - yv) + i(xv + yu)$$

and thus

$$\overline{zw} = (xu - yv) - i(xv + yu).$$

On the other hand, we have $\bar{z} = x - iy$ and $\bar{w} = u - iv$, which yields

$$\begin{aligned} \bar{z}\bar{w} &= (xu - (-y)(-v)) + i(x(-v) + (-y)u) \\ &= (xu - yv) - i(xv + yu) \\ &= \overline{zw}, \end{aligned}$$

as claimed.

(iv): By (iii), we have

$$\overline{z^{-1}}\bar{z} = \overline{z^{-1}z} = \bar{1} = 1,$$

which yields the claim. □

For any $z = x^2 + y^2 \in \mathbb{C}$, we have $z\bar{z} = x^2 + y^2 \geq 0$. Hence, the following makes sense:

Definition 1.6. For $z \in \mathbb{C}$, set $|z| := \sqrt{z\bar{z}}$.

Proposition 1.7. $|\cdot|$ *is the Euclidean norm on* $\mathbb{R}^2$*. In particular, the following hold:*

(i) $|z| \geq 0$ *with* $|z| = 0$ *if and only if* $z = 0$*;*

(ii) $|z + w| \leq |x| + |w|$ *for* $z, w \in \mathbb{C}$*.*

Moreover, we have $|zw| = |z||w|$ *for* $z, w \in \mathbb{C}$ *and* $z^{-1} = \frac{\bar{z}}{|z|^2}$ *for* $z \in \mathbb{C} \setminus \{0\}$*.*

Proof. In view of the remark immediately preceding Definition 1.6, it is clear that $|\cdot|$ is the Euclidean norm, which entails (i) and (ii).

For the moreover part, let $z, w \in \mathbb{C}$, and note that

$$|zw|^2 = zw\overline{zw} = (z\bar{z})(w\bar{w}) = |z|^2|w|^2.$$

Also, since $|z|^2 = z\bar{z}$, we have $1 = z\frac{\bar{z}}{|z|^2}$ for $z \neq 0$ and thus $z^{-1} = \frac{\bar{z}}{|z|^2}$. □

2 Complex Differentiation

Definition 2.1. Let $D \subset \mathbb{C}$, and let z_0 be an interior point of D, i.e., there is $\epsilon > 0$ such that $B_\epsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \epsilon\} \subset D$. A function $f \colon D \to \mathbb{C}$ is called *complex differentiable* at z_0 if

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Proposition 2.2. *Let $D \subset \mathbb{C}$, let $z_0 \in \operatorname{int} D$, and let $f \colon D \to \mathbb{C}$ be complex differentiable at z_0. Then f is continuous at z_0.*

Proof. Since $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists, we have

$$0 = \lim_{z \to z_0} (z - z_0) \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} (z - z_0) \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} (f(z) - f(z_0)),$$

so that $f(z_0) = \lim_{z \to z_0} f(z)$. □

Proposition 2.3. *Let $D \subset \mathbb{C}$, and let $f, g : D \to \mathbb{C}$ be complex differentiable at $z_0 \in \operatorname{int} D$. Then the following functions are complex differentiable at z_0: $f + g$, fg, and—if $g(z_0) \neq 0$—$\frac{f}{g}$. Moreover, we have:*

$$(f + g)'(z_0) = f'(z_0) + g'(z_0),$$
$$(fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0),$$

and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) + f(z_0)g'(z_0)}{g(z_0)^2}.$$

Proof. As over $\mathbb{R}$. □

Proposition 2.4. *Let $D, E \subset \mathbb{C}$, let $g \colon D \to \mathbb{C}$ and $f : E \to \mathbb{C}$ be such that $g(D) \subset E$, and let $z_0 \in \operatorname{int} D$ be such that $w_0 := g(z_0) \in \operatorname{int} E$. Further, suppose that g is complex differentiable at z_0 and f is complex differentiable at w_0. Then $f \circ g$ is complex differentiable at z_0 with*

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0).$$

Proof. As over $\mathbb{R}$. □

Examples. 1. All constant functions are (on all of $\mathbb{C}$) complex differentiable, as is $\mathbb{C} \ni z \mapsto z$. Consequently, all complex polynomials are complex differentiable on all of $\mathbb{C}$, and rational functions are complex differentiable wherever they are defined.

2. Let

$$f\colon \mathbb{C} \to \mathbb{C}, \quad z \mapsto \bar{z},$$

and let $z_0 = x_0 + iy_0 \in \mathbb{C}$. Assume that f is complex differentiable at z_0. Then we have

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0} \\ &= \lim_{y \to y_0} \frac{(x_0 - iy) - (x_0 - iy_0)}{(x_0 + iy) - (x_0 + iy_0)} \\ &= \lim_{y \to y_0} \frac{i(y_0 - y)}{i(y - y_0)} \\ &= -1 \end{aligned}$$

as well as

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0} \\ &= \lim_{x \to x_0} \frac{(x - iy_0) - (x_0 - iy_0)}{(x + iy_0) - (x_0 + iy_0)} \\ &= \lim_{x \to x_0} \frac{x - x_0}{x - x_0} \\ &= 1, \end{aligned}$$

which is impossible. Hence, f is *not* complex differentiable at *any* $z_0 \in \mathbb{C}$. (On the other hand, f is continuously partially differentiable—as a function of two real variables—on all of $\mathbb{C}$.)

Lemma 2.5. *The following are equivalent for a $\mathbb{R}$-linear map $T\colon \mathbb{C} \to \mathbb{C}$:*

(i) *there is $c \in \mathbb{C}$ such that $T(z) = cz$ for all $z \in \mathbb{C}$;*

(ii) *T is $\mathbb{C}$-linear;*

(iii) *$T(i) = iT(1)$;*

(iv) *if A is the real 2×2 matrix representing T with respect to the standard basis of $\mathbb{R}^2$, then there are $a, b \in \mathbb{R}$ such that*

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Proof. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\implies$ (i): Set $c := T(1)$. For $z = x + iy \in \mathbb{C}$, this means that

$$\begin{aligned} T(x+iy) &= T(x) + T(iy) \\ &= xT(1) + yT(i) \\ &= xT(1) + iyT(1) \\ &= zT(1) \\ &= cz. \end{aligned}$$

(iv) $\iff$ (iii): Let $a, b, c, d \in \mathbb{R}$ be such that

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

represents T with respect to the standard basis of $\mathbb{R}^2$. Note that

$$T(i) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} = c + id$$

and

$$T(1) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} = a + ib,$$

so that

$$iT(1) = -b + ia.$$

It follows that

$$T(i) = iT(1) \quad \iff \quad c = -b \text{ and } d = a. \qquad \square$$

Theorem 2.6 (Cauchy–Riemann Differential Equations)**.** *Let $D \subset \mathbb{C}$ be open, and let $z_0 \in D$. Then the following are equivalent for $f \colon D \to \mathbb{C}$:*

(i) *f is complex differentiable at z_0;*

(ii) *f is totally differentiable at z_0 (in the sense of several variable calculus), and the* Cauchy–Riemann differential equations *hold, i.e., for $g := \operatorname{Re} f$ and $h := \operatorname{Im} f$, we have*

$$\frac{\partial g}{\partial x}(z_0) = \frac{\partial h}{\partial y}(z_0) \qquad \textit{and} \qquad \frac{\partial g}{\partial y}(z_0) = -\frac{\partial h}{\partial x}(z_0).$$

Proof. (i) $\implies$ (ii): Define

$$T \colon \mathbb{C} \to \mathbb{C}, \quad z \mapsto f'(z_0)z,$$

and note that

$$\frac{|f(z) - f(z_0) - T(z - z_0)|}{|z - z_0|} = \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| \to 0$$

as $z_0 \to z_0$. Therefore, f is totally differentiable at z_0. From Math 217, it follows that the matrix representation of T with respect to the standard basis of $\mathbb{R}$ is the Jacobian of f, i.e.,

$$J_f(z_0) = \begin{bmatrix} \frac{\partial g}{\partial x}(z_0) & \frac{\partial g}{\partial y}(z_0) \\ \frac{\partial h}{\partial x}(z_0) & \frac{\partial h}{\partial y}(z_0) \end{bmatrix}.$$

Since T is $\mathbb{C}$-linear, Lemma 2.5 yields that

$$\frac{\partial g}{\partial x}(z_0) = \frac{\partial h}{\partial y}(z_0) \qquad \text{and} \qquad \frac{\partial h}{\partial x}(z_0) = -\frac{\partial g}{\partial y}(z_0).$$

(ii) $\Longrightarrow$ (i): Since f is totally differentiable at z_0, we have a unique $\mathbb{R}$-linear map $T\colon \mathbb{C} \to \mathbb{C}$ such that

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0) - T(z - z_0)|}{|z - z_0|} = 0.$$

As we know from Math 217, T is represented by $J_f(z_0)$ with respect to the standard basis of $\mathbb{R}^2$. Since the Cauchy–Riemann differential equations are supposed to hold, $J_f(z_0)$ is of the form described in Lemma 2.5(iv). By Lemma 2.5, there is thus $c \in \mathbb{C}$ such that $T(z) = cz$ for all $z \in \mathbb{C}$. It follows that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - c \right| = \frac{|f(z) - f(z_0) - c(z - z_0)|}{|z - z_0|} \to 0$$

as $z \to z_0$. Hence, f is complex differentiable at z_0. □

Remark. In the situation of Theorem 2.6, we have

$$f'(z_0)1 = \begin{bmatrix} \frac{\partial g}{\partial x}(z_0) & \frac{\partial g}{\partial y}(z_0) \\ \frac{\partial h}{\partial x}(z_0) & \frac{\partial h}{\partial y}(z_0) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\partial g}{\partial x}(z_0) + i\frac{\partial h}{\partial x}(z_0)$$

as well as

$$f'(z_0)i = \begin{bmatrix} \frac{\partial g}{\partial x}(z_0) & \frac{\partial g}{\partial y}(z_0) \\ \frac{\partial h}{\partial x}(z_0) & \frac{\partial h}{\partial y}(z_0) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\partial g}{\partial y}(z_0) + i\frac{\partial h}{\partial y}(z_0),$$

so that

$$f'(z_0) = \frac{\partial g}{\partial x}(z_0) + i\frac{\partial h}{\partial x}(z_0) = \frac{\partial h}{\partial y}(z_0) - i\frac{\partial g}{\partial y}(z_0).$$

Example. Let

$$f\colon \mathbb{C} \to \mathbb{C}, \quad z \mapsto |z|^2.$$

Then f is totally differentiable with $h = 0$. Moreover,

$$\frac{\partial g}{\partial x} = 2x \qquad \text{and} \qquad \frac{\partial g}{\partial y} = 2y$$

holds. Hence, we have that

$$\frac{\partial g}{\partial x}(z_0) = \frac{\partial h}{\partial y}(z_0) \qquad \text{and} \qquad \frac{\partial g}{\partial y}(z_0) = -\frac{\partial h}{\partial x}(z_0)$$

if and only if $z_0 = 0$. By Theorem 2.6, this means that f is complex differentiable at z_0 if and only if $z_0 = 0$.

Corollary 2.7. *Let $D \subset \mathbb{C}$ be open and connected, and let $f : D \to \mathbb{C}$ be complex differentiable. Then f is constant on D if and only if $f' \equiv 0$.*

Proof. Suppose that $f' \equiv 0$. From the remark after Theorem 2.6, it follows that

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial x} = \frac{\partial g}{\partial y} = \frac{\partial h}{\partial y} \equiv 0.$$

Several variable calculus then yields that f is constant. □

3 Power Series

Definition 3.1. A (complex) *power series* is an infinite series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ with $z, z_0, a_0, a_1, a_2, \ldots \in \mathbb{C}$. The point z_0 is called the *point of expansion* of the series.

Examples. 1. For $m \in \mathbb{N}$, we have

$$\sum_{n=0}^{m} z^n = \frac{1 - z^{m+1}}{1 - z}$$

if $z \neq 1$. For $|z| < 1$, we obtain (letting $m \to \infty$)

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

2. For $z \in \mathbb{C}$, define

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Let $z \neq 0$, and note that

$$\left|\frac{z^{n+1}}{(n+1)!}\right| \Bigg/ \left|\frac{z^n}{n!}\right| = \frac{|z|}{n+1} \to 0$$

as $n \to \infty$. As the ratio text holds for series with summands in $\mathbb{C}$ as well as for series over $\mathbb{R}$, we conclude that $\exp(z)$ converges absolutely.

Let $z, w \in \mathbb{C}$, and note that the Cauchy product formula for series over $\mathbb{R}$ also holds over $\mathbb{C}$. We obtain:

$$\begin{aligned}
\exp(z)\exp(w) &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n-k}}{(n-k)!} \frac{w^k}{k!}, \qquad \text{by the Cauchy product formula,} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^{n-k} w^k \\
&= \sum_{n=0}^{n} \frac{(z+w)^n}{n!} \\
&= \exp(z+w).
\end{aligned}$$

We call $\exp\colon \mathbb{C} \to \mathbb{C}$ the *exponential function*; we also write e^z instead of $\exp(z)$.

3. The *sine* and *cosine* on $\mathbb{C}$ are defined as

$$\sin(z) := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

and

$$\cos(z) := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

for $z \in \mathbb{C}$. As for $\exp(z)$, we see that both $\sin(z)$ and $\cos(z)$ converge absolutely for all $z \in \mathbb{C}$. Moroever, we have for $z \in \mathbb{C}$:

$$\begin{aligned} e^{iz} &= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(iz)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(iz)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \\ &= \cos(z) + i \sin(z). \end{aligned}$$

Theorem 3.2. *Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a complex power series. Then there is a unique $R \in [0, \infty]$ with the following properties:*

- *$\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely for each $z \in B_R(z_0)$;*
- *for each $r \in [0, R)$, the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges uniformly on $B_r[z_0] := \{z \in \mathbb{C} : |z - z_0| \leq r\}$;*
- *$\sum_{n=0}^{\infty} a_n (z - z_0)^n$ diverges for each $z \notin B_R[z_0]$.*

Moreover, R is computed via the Cauchy–Hadamard formula*:*

$$R = \frac{1}{\limsup_{n \to \infty} \sqrt[n]{|a_n|}}.$$

It is called the radius of convergence *for $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.*

Proof. The uniqueness of R is obvious.

Let $R \in [0, \infty]$ be *defined* by the Cauchy–Hadamard formula (we set $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$).

Let $r \in [0, R)$, and choose $r' \in (r, R)$. It follows that

$$\frac{1}{r'} > \frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{|a_n|},$$

so that there is $n_0 \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} < \frac{1}{r'}$ for $n \geq n_0$, i.e.,

$$|a_n| < \left(\frac{1}{r'}\right)^n$$

for $n \geq n_0$. For $n \geq n_0$ and $z \in B_r[z_0]$, we then have

$$|a_n(z - z_0)^n| < \left(\frac{r}{r'}\right)^n.$$

Since $\frac{r}{r'} < 1$, we have $\sum_{n=0}^{\infty} \left(\frac{r}{r'}\right)^n < \infty$. The Weierstraß M-test thus yields that $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges absolutely and uniformly on $B_r[z_0]$.

Since every $z \in B_R(z_0)$ is contained in $B_r[z_0]$ for some $r \in [0, R)$, it follows that $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges absolutely for each such z.

Let $z \notin B_R[z_0]$, i.e., $|z - z_0| > R$, so that

$$\frac{1}{|z - z_0|} < \frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

and thus

$$\frac{1}{|z - z_0|} < \sqrt[n]{|a_n|}$$

or, equivalently,

$$1 < |a_n(z - z_0)^n|$$

for infinitely many $n \in \mathbb{N}$. It follows that $(a_n(z - z_0)^n)_{n=1}^{\infty}$ does not converges to zero. Consequently, $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ diverges. □

Remark. The uniqueness of R follows already from the first and the last one of the three properties listed.

Examples.

1. $\sum_{n=0}^{\infty} z^n$: $R = 1$.
2. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$: $R = \infty$.
3. $\sum_{n=0}^{\infty} n! z^n$: $R = 0$.
4. $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ and $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$: $R = \infty$.

Theorem 3.3. *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a complex power series with radius of convergence R. Then*

$$f\colon B_R(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

is complex differentiable at each point $z \in B_R(z_0)$ with

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

Proof. Without loss of generality, suppose that $z_0 = 0$.

We first show that $\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges absolutely for each $z \in B_R(0)$.

Let $z \in B_R(0)$, and choose r such that $|z| < r < R$. Since $\frac{1}{r} > \limsup_{n\to\infty} \sqrt[n]{|a_n|}$, there is $n_0 \in \mathbb{N}$ such that $|a_n| < \left(\frac{1}{r}\right)^n$ for $n \geq n_0$ and thus

$$|na_n z^{n-1}| < \frac{n}{r}\left(\frac{|z|}{r}\right)^{n-1}$$

for $n \geq n_0$. Since $\frac{|z|}{r} < 1$, we have $\sum_{n=1}^{\infty} \frac{n}{r}\left(\frac{|z|}{r}\right)^{n-1} < \infty$; the comparison test therefore yields that $\sum_{n=1}^{\infty} na_n z^{n-1}$ converges absolutely.

In view of the foregoing, we may define

$$g\colon B_R(0) \to \mathbb{C}, \quad z \mapsto \sum_{n=1}^{\infty} na_n (z-z_0)^{n-1}.$$

We shall devote the rest of the proof to showing that f is complex differentiable on $B_R(0)$ with $f' = g$.

To this end, fix $\epsilon > 0$, and define, for $z \in B_R(0)$ and $n \in \mathbb{N}$,

$$s_n(z) := \sum_{k=0}^{n} a_k z^k \qquad \text{and} \qquad R_n(z) := \sum_{k=n+1}^{\infty} a_k z^k.$$

Fix $z \in B_R(0)$ and let $r \in (0, R)$ be such that $z \in B_r(0)$; note that

$$\frac{f(w)-f(z)}{w-z} - g(z) = \left(\frac{s_n(w)-s_n(z)}{w-z} - s_n'(z)\right) + (s_n'(z) - g(z)) + \frac{R_n(w)-R_n(z)}{w-z}$$

for all $w \in B_R(0)$. We shall see that each of the three summands on the right hand side of this equation has modulus less than $\frac{\epsilon}{3}$, provided that n is sufficiently large and w is sufficiently close to z.

We start with the last summand. First, note that

$$\frac{R_n(w)-R_n(z)}{w-z} = \sum_{k=n+1}^{\infty} a_k \frac{w^k - z^k}{w-z}$$

for all $w \in B_R(0)$ and also that

$$\left|\frac{w^k - z^k}{w-z}\right| = \left|\sum_{j=1}^{k} w^{k-j} z^{j-1}\right| \leq \sum_{j=1}^{k} |w|^{k-j}|z|^{j-1} \leq kr^{k-1}$$

for all $w \in B_r(0)$. Since $r < R$, we have $\sum_{k=1}^{\infty} k|a_k|r^{k-1} < \infty$. Consequently, there is $n_1 \in \mathbb{N}$ such that $\sum_{k=n+1}^{\infty} k|a_k|r^{k-1} < \frac{\epsilon}{3}$ for all $n \geq n_1$ and therefore

$$\left|\frac{R_n(w)-R_n(z)}{w-z}\right| = \left|\sum_{k=n+1}^{\infty} a_k \frac{w^k - z^k}{w-z}\right| \leq \sum_{k=n+1}^{\infty} k|a_k|r^{k-1} < \frac{\epsilon}{3}$$

for all $n \geq n_1$ and all $w \in B_r(0)$.

For the second summand, just note that $g(z) = \lim_{n\to\infty} s_n'(z)$; consequently, there is $n_2 \in \mathbb{N}$ such that $|g(z) - s_n'(z)| < \frac{\epsilon}{3}$ for all $n \geq n_2$.

For the first summand, fix $n \geq \max\{n_1, n_2\}$. Since

$$\lim_{w\to z} \frac{s_n(w) - s_n(z)}{w - z} = s_n'(z),$$

there is $\delta > 0$ such that

$$\left| \frac{s_n(w) - s_n(z)}{w - z} - s_n'(z) \right| < \frac{\epsilon}{3}$$

for all $w \in B_\delta(z) \cap B_R(0)$. Making $\delta > 0$ smaller, if necessary, we can achieve that $B_\delta(z) \subset B_r(0)$. Consequently, we obtain for all $w \in B_\delta(z)$ that

$$\left| \frac{R_n(w) - R_n(z)}{w - z} \right| < \frac{\epsilon}{3}$$

and

$$\left| \frac{s_n(w) - s_n(z)}{w - z} - s_n'(z) \right| < \frac{\epsilon}{3}.$$

These estimates, combined with $|g(z) - s_n'(z)| < \frac{\epsilon}{3}$, yield that

$$\left| \frac{f(w) - f(z)}{w - z} - g(z) \right| < \epsilon$$

for all $w \in B_\delta(z)$. Since $\epsilon > 0$ was arbitrary, we obtain that $f'(z)$ exists and equals $g(z)$. □

Examples. 1. $\exp'(z) = \exp(z)$.

2. $\sin'(z) = \cos(z)$.

3. $\cos'(z) = -\sin(z)$.

Corollary 3.4. *Let $\sum_{n=0}^\infty a_n(z-z_0)^n$ be a complex power series with radius of convergence R. Then*

$$f\colon B_R(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^\infty a_n(z - z_0)^n$$

is infinitely often complex differentiable on $B_R(z_0)$ with

$$f^{(k)}(z) = \sum_{n=k}^\infty n(n-1)\cdots(n-k+1)a_n(z-z_0)^{n-k}.$$

for $z \in B_R(z_0)$ and $k \in \mathbb{N}$. In particular,

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

holds for each $n \in \mathbb{N}_0$.

Corollary 3.5. *Let* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *be a complex power series with radius of convergence* R*. Then*

$$F\colon B_R(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}$$

is complex differentiable on $B_R(z_0)$ *with*

$$F'(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

for $z \in B_R(z_0)$*.*

4 Complex Line Integrals

We call a function $f\colon [a,b] \to \mathbb{C}$ *integrable* if $\operatorname{Re} f, \operatorname{Im} f\colon [a,b] \to \mathbb{R}$ are integrable in the sense of real variables. (The Riemann integral will do.) In this case, we define

$$\int_a^b f(t)\,dt := \int_a^b \operatorname{Re} f(t)\,dt + i \int_a^b \operatorname{Im} f(t)\,dt.$$

Definition 4.1. A *curve* (or *path*) in $\mathbb{C}$ is a continuous map $\gamma\colon [a,b] \to \mathbb{C}$. We call

- $\gamma(a)$ the *initial point* of γ,
- $\gamma(b)$ the *endpoint* (or *terminal point*) of γ, and
- $\{\gamma\} := \gamma([a,b])$ the *trajectory* of γ.

Examples. 1. Let $z, w \in \mathbb{C}$. Then

$$\gamma\colon [0,1] \to \mathbb{C}, \quad t \mapsto z + t(w-z)$$

has the initial point z and the endpoint w, and $\{\gamma\}$ is the line segment connecting z with w.

2. For $k \in \mathbb{Z}$, let

$$\gamma_k\colon [0,2\pi] \to \mathbb{C}, \quad t \mapsto e^{ikt}$$

Then $\gamma_k(0) = 1 = \gamma_k(1)$ holds, and for $k \neq 0$, we have $\{\gamma_k\} = \{z \in \mathbb{C} : |z| = 1\}$.

Definition 4.2. A curve $\gamma\colon [a,b] \to \mathbb{C}$ is called *piecewise smooth* if there is a partition $a = a_0 < a_1 < \cdots < a_n = b$ such that $\gamma|_{[a_{j-1},a_j]}$ is continuously differentiable for $j = 1, \ldots, n$.

Definition 4.3. The *lenght* of a piecewise smooth curve $\gamma\colon [a,b] \to \mathbb{C}$ is defined as

$$\ell(\gamma) := \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} |\gamma'(t)|\,dt,$$

where $a = a_0 < a_1 < \cdots < a_n = b$ is a partition such that $\gamma|_{[a_{j-1},a_j]}$ is continuously differentiable for $j = 1, \ldots, n$.

Definition 4.4. Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise smooth curve, let $a = a_0 < a_1 < \cdots < a_n = b$ be a partition such that $\gamma|_{[a_{j-1},a_j]}$ is continuously differentiable for $j = 1, \ldots, n$, and let $f\colon \{\gamma\} \to \mathbb{C}$ be continuous. Then the *line integral* (or *contour integral*) of f along γ is defined as

$$\int_\gamma f := \int_\gamma f(\zeta)\,d\zeta = \sum_{j=1}^{n} \int_{a_{j1}}^{a_j} f(\gamma(t))\gamma'(t)\,dt.$$

Properties of the Line Integral. 1. Let γ be a piecewise smooth curve, let $\lambda, \mu \in \mathbb{C}$, and let $f, g \colon \{\gamma\} \to \mathbb{C}$ be continuous. Then we have

$$\int_\gamma \lambda f + \mu g = \lambda \int_\gamma f + \mu \int_\gamma g.$$

2. Let γ be a piecewise smooth curve, let $f \colon \{\gamma\} \to \mathbb{C}$ be continuous, and let $C \geq 0$ be such that $|f(\zeta)| \leq C$ for $\zeta \in \{\gamma\}$. Then

$$\left| \int_\gamma f \right| \leq C\, \ell(\gamma)$$

holds.

3. Let $\gamma \colon [c, d] \to \mathbb{C}$ be a piecewise smooth curtve, let $\phi \colon [a, b] \to [c, d]$ be an increasing, continuously differentiable function with $\phi(a) = c$ and $\phi(b) = d$, and let $f \colon \{\gamma\} \to \mathbb{C}$ be continuous. Then we have

$$\int_\gamma f = \int_{\gamma \circ \phi} f.$$

4. Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be continuous with an anti-derivative, i.e., there is $F : D \to \mathbb{C}$ such that F is complex differentiable at each $z \in D$ with $F'(z) = f(z)$. Then

$$\int_\gamma f = F(\gamma(b)) - F(\gamma(a))$$

holds for every piecewise smooth curve $\gamma \colon [a, b] \to D$.

Definition 4.5. A curve $\gamma \colon [a, b] \to \mathbb{C}$ is called *closed* if $\gamma(a) = \gamma(b)$.

Proposition 4.6. *Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be continuous with an anti-derivative. Then $\int_\gamma f = 0$ holds for each closed, piecewise smooth curve γ in D.*

Example. Let $z_0 \in \mathbb{C}$, let $r > 0$, and let

$$\gamma \colon [0, 2\pi] \to \mathbb{C}, \quad t \mapsto re^{it} + z_0,$$

i.e., γ is the circle centered at z_0 with radius r in counterclockwise orientation.

Let $n \in \mathbb{Z}$, and consider $\int_\gamma (\zeta - z_0)^n \, d\zeta$.

For $n \neq -1$, let

$$F \colon \mathbb{C} \setminus \{z_0\} \to \mathbb{C}, \quad z \mapsto \frac{(z - z_0)^{n+1}}{n+1},$$

so that $F'(z) = (z - z_0)^n$ for all $z \in \mathbb{C} \setminus \{z_0\}$. It follows that $\int_\gamma (\zeta - z_0)^n \, d\zeta = 0$.

On the other hand, we have

$$\int_\gamma (\zeta - z_0)^{-1} \, d\zeta = \int_0^{2\pi} \frac{rie^{it}}{re^{it}} \, dt = \int_0^{2\pi} i \, dt = 2\pi i.$$

Consequently,

$$\mathbb{C} \setminus \{z_0\} \to \mathbb{C}, \quad z \mapsto \frac{1}{z - z_0}$$

has *no* anti-derivative.

Recall the following definition from Math 217:

Definition 4.7. A subset $D \subset \mathbb{C}$ is called *connected* if there are *no* open sets $U, V \subset \mathbb{C}$ with

- $U \cap D \neq \varnothing \neq V \cap D$,
- $U \cup V \supset D$, and
- $U \cap V \subset \mathbb{C} \setminus D$.

Definition 4.8. Let $D \subset \mathbb{C}$ be open. A function $f \colon D \to \mathbb{C}$ is called *locally constant* if, for each $z_0 \in D$, there is $\epsilon > 0$ such that $B_\epsilon(z_0) \subset D$ and f is constant on $B_\epsilon(z_0)$.

Proposition 4.9. *Let $D \subset \mathbb{C}$ be open. Then the following are equivalent:*

(i) *D is connected;*

(ii) *every locally constant function $f \colon D \to \mathbb{C}$ is constant;*

(iii) *for any $z, w \in D$, there is a piecewise smooth curve $\gamma \colon [a, b] \to D$ such that $\gamma(a) = z$ and $\gamma(b) = w$.*

For the proof of Proposition 4.9 (and of the theorem following it), we require two constructions on curves:

1. Given $a < b < c$ and two curves $\gamma_1 \colon [a, b] \to \mathbb{C}$ and $\gamma_2 \colon [b, c] \to \mathbb{C}$ with $\gamma_1(b) = \gamma_2(b)$, the *concatenation* of γ_1 and γ_2 is the curve

$$\gamma_1 \oplus \gamma_2 \colon [a, c] \to \mathbb{C}, \quad t \mapsto \begin{cases} \gamma_1(t), & t \in [a, b], \\ \gamma_2(t), & t \in [b, c]. \end{cases}$$

   If γ_1 and γ_2 are piecewise smooth, then so is $\gamma_1 \oplus \gamma_2$, and we have

$$\int_{\gamma_1 \oplus \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

   for each continuous $f \colon \{\gamma_1\} \cup \{\gamma_2\} \to \mathbb{C}$.

2. For any curve $\gamma \colon [a, b] \to \mathbb{C}$, the *reversed curve* is defined as

$$\gamma^- \colon [a, b] \to \mathbb{C}, \quad t \mapsto \gamma(a + b - t).$$

   If γ is piecewise smooth, then so is γ^-, and we have

$$\int_{\gamma^-} f = -\int_\gamma f$$

   for each continuous $f \colon \{\gamma\} \to \mathbb{C}$.

Proof of Proposition 4.9. (i) $\Longrightarrow$ (iii): Let $z \in D$, and set

$$U := \{w \in D : \text{there is a piecewise smooth curve } \gamma\colon [a,b] \to D \text{ with } \gamma(a) = z \text{ and } \gamma(b) = w\}.$$

Obviously, $U \neq \varnothing$ (because $z \in U$).

We claim that U is open. To see this, let $w_0 \in U$, so that there is a piecewise smooth curve $\gamma\colon [a,b] \to D$ with $\gamma(a) = z$ and $\gamma(b) = w_0$. Choose $\epsilon > 0$ such that $B_\epsilon(w_0) \subset D$. For $w \in B_\epsilon(w_0)$, define

$$[w_0, w]\colon [0,1] \to \mathbb{C}, \quad t \mapsto w_0 + t(w - w_0).$$

Then $[w_0, w]$ is a curve in $B_\epsilon(w_0) \subset D$ with initial point w_0 and endpoint w. Consequently, $\gamma \oplus [w_0, w]$ is a piecewise smooth curve in D with initial point z and endpoint w. It follows that $w \in U$, and since $w \in B_\epsilon(w_0)$ was arbitrary, we have $B_\epsilon(w_0) \subset U$. This proves the openness of U.

Next, we claim that $D \setminus U$ is also open. To see this, let $w_0 \in D \setminus U$, and let $\epsilon > 0$ be so small that $B_\epsilon(w_0) \subset D$. Assume towards a contradiction that there is $w \in B_\epsilon(w_0) \cap U$. Let $\gamma : [a,b] \to D$ be a piecewise smooth curve with $\gamma(a) = z$ and $\gamma(b) = w$. Then $\gamma \oplus [w, w_0]$ is a piecewise smooth curve in D with initial point z and endpoint w_0, so that $w_0 \in U$. This contradicts the choice of $w_0 \in D \setminus U$. It follows that $B_\epsilon(w_0) \cap U = \varnothing$, i.e., $B_\epsilon(w_0) \subset D$.

Since U and $D \setminus U$ are both open with $U \cup (D \setminus U) = D$, the connectedness of D yields that $D \setminus U = \varnothing$, i.e., $D = U$.

(iii) $\Longrightarrow$ (ii): Let $f : D \to \mathbb{C}$ be a locally constant function, and let $z, w \in D$. Let $\gamma\colon [a,b] \to D$ be a piecewise smooth curve with $\gamma(a) = z$ and $\gamma(b) = w$. Since f is locally constant, the function

$$[a,b] \to \mathbb{C}, \quad t \mapsto f(\gamma(t))$$

is differentiable with zero derivative and therefore constant. It follows that $f(z) = f(\gamma(a)) = f(\gamma(b)) = f(w)$.

(ii) $\Longrightarrow$ (i): Suppose that D is *not* connected. Then there are non-empty open sets $U, V \subset \mathbb{C}$ with $U \cap V = \varnothing$ and $U \cup V = D$. Define

$$f\colon D \to \mathbb{C}, \quad z \mapsto \begin{cases} 0, & z \in U, \\ 1, & z \in V. \end{cases}$$

Then f is locally constant, but not constant. $\square$

Theorem 4.10. *Let $D \subset \mathbb{C}$ be open and connected, i.e., a* region*, and let $f\colon D \to \mathbb{C}$ be continuous. Then the following are equivalent:*

(i) *f has an anti-derivative;*

(ii) *$\int_\gamma f(\zeta)\, d\zeta = 0$ for any closed, piecewise smooth curve γ in D;*

(iii) *for any piecewise smooth curve γ in D, the value of $\int_\gamma f$ depends only on the inital point and the endpoint of γ.*

Proof. (i) $\Longrightarrow$ (ii) follows from Proposition 4.6.

(ii) $\Longrightarrow$ (iii): Let $\gamma, \gamma' \colon [a,b] \to D$ be piecewise smooth curves with $\gamma(a) = \gamma'(a)$ and $\gamma(b) = \gamma'(b)$. Then $\gamma \oplus (\gamma')^-$ is a closed, piecewise smooth curve, so that

$$0 = \int_{\gamma \oplus (\gamma')^-} f = \int_\gamma f + \int_{(\gamma')^-} f = \int_\gamma f - \int_{\gamma'} f.$$

(iii) $\Longrightarrow$ (i): Fix $z_0 \in D$. For any $z \in D$ choose a piecewise smooth curve $\gamma_z \colon [a,b] \to \mathbb{C}$ with $\gamma_z(a) = a_0$ and $\gamma_z(b) = z$. Define

$$F \colon D \to \mathbb{C}, \quad z \mapsto \int_{\gamma_z} f(\zeta)\, d\zeta.$$

We claim that F is an anti-derivative of f. Let $z \in D$, and choose $\delta > 0$ such that $B_\delta(z) \subset D$. For $w \in B_\delta(z)$, thus have

$$\begin{aligned} F(w) &= \int_{\gamma_w} f(\zeta)\, d\zeta \\ &= \int_{\gamma_z \oplus [z,w]} f(\zeta)\, d\zeta, \qquad \text{by (iii)}, \\ &= \int_{\gamma_z} f(\zeta)\, d\zeta + \int_{[z,w]} f(\zeta)\, d\zeta \\ &= F(z) + \int_{[z,w]} f(\zeta)\, d\zeta \end{aligned},$$

which entails

$$\begin{aligned} \left| \frac{F(w) - F(z)}{w - z} - f(z) \right| &= \frac{1}{|w-z|} \left| \int_{[z,w]} f - \int_{[z,w]} f(z) \right| \\ &= \frac{1}{|w-z|} \left| \int_{[z,w]} (f - f(z)) \right| \\ &\leq \frac{|w-z|}{|w-z|} \sup\{|f(\zeta) - f(z)| : \zeta \in \{[z,w]\}\} \\ &= \sup\{|f(\zeta) - f(z)| : \zeta \in \{[z,w]\}\}. \end{aligned}$$

Let $\epsilon > 0$ and choose $\delta > 0$ so small that $|f(\zeta) - f(z)| < \epsilon$ for all $\zeta \in D$ with $|\zeta - z| < \delta$. It follows from the foregoing that

$$\left| \frac{F(w) - F(z)}{w - z} - f(z) \right| \leq \epsilon$$

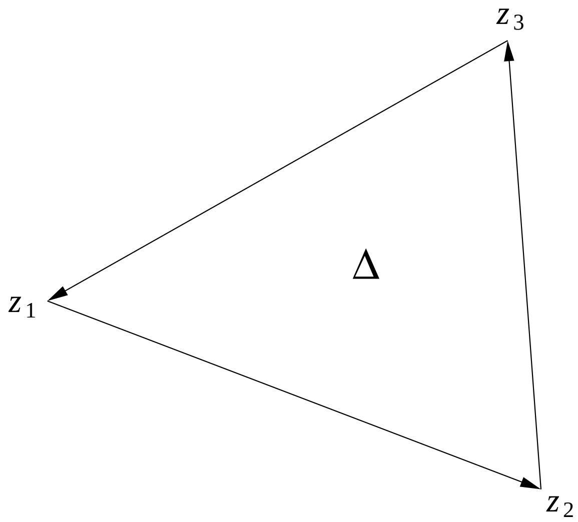
for all $w \in B_\delta(z)$. This, of course, means that $F'(z) = f(z)$. $\square$

From now on, we shall use the word *curve* as shorthand for *piecewise smooth curve.*

5 Cauchy's Integral Theorem and Formula

Definition 5.1. Let $D \subset \mathbb{C}$ be open. If $f : D \to \mathbb{C}$ is complex differentiable at each $z \in D$, then we call f *holomorphic* (or *analytic*) on D.

Let z_1, z_2, and z_3 be three different points in $\mathbb{C}$. They span a triangle Δ. Its boundary can be parametrized as a curve in counterclockwise orientation.

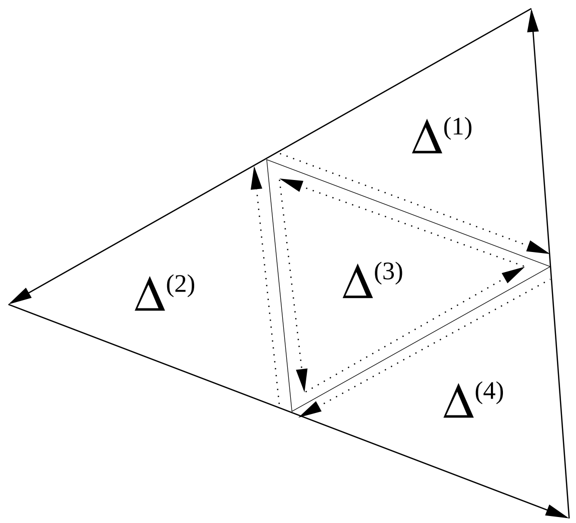


We denote this curve by $\partial\Delta$.

Theorem 5.2 (Goursat's "Lemma"). *Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $\Delta \subset D$ be a triangle. Then we have*

$$\int_{\partial\Delta} f(\zeta)\, d\zeta = 0.$$

Proof. We split up Δ into four subtriangles $\Delta^{(1)}$, $\Delta^{(2)}$, $\Delta^{(3)}$, and$\Delta^{(4)}$ as shown:



As the line sugments in the interior of Δ also occur as their reversed paths, we have

$$\int_{\partial\Delta} f = \sum_{j=1}^{4} \int_{\partial\Delta^{(j)}} f,$$

so that

$$\left|\int_{\partial\Delta} f\right| \leq \sum_{j=1}^{4} \left|\int_{\partial\Delta^{(j)}} f\right|.$$

Choose $j \in \{1,2,3,4\}$ such that $\left|\int_{\partial\Delta^{(j)}} f\right|$ is largest, and set $\Delta_1 := \Delta^{(j)}$. It follows that

$$\left|\int_{\partial\Delta} f\right| \leq 4\left|\int_{\partial\Delta_1} f\right|;$$

also, note that $\ell(\partial\Delta_1) = \frac{1}{2}\ell(\partial\Delta)$.

Repeat this argument with Δ_1 in place of Δ, and obtain a triangle $\Delta_2 \subset \Delta_1$ with

$$\ell(\partial\Delta_2) = \frac{1}{2}\ell(\partial\Delta_1) = \frac{1}{4}\ell(\partial\Delta)$$

and

$$\left|\int_{\partial\Delta_1} f\right| \leq 4\left|\int_{\partial\Delta_2} f\right|,$$

so that

$$\left|\int_{\partial\Delta} f\right| \leq 4\left|\int_{\partial\Delta_1} f\right| \leq 16\left|\int_{\partial\Delta_2} f\right|.$$

Inductively, we obtain triangles

$$\Delta \supset \Delta_1 \supset \Delta_2 \supset \cdots$$

with

$$\ell(\partial\Delta_n) = \frac{1}{2^n}\ell(\partial\Delta)$$

and

$$\left|\int_{\partial\Delta} f\right| \leq 4^n \left|\int_{\partial\Delta_n} f\right|$$

for $n \in \mathbb{N}$.

Let $z_0 \in \bigcap_{n=1}^\infty \Delta_n$, and define

$$r \colon D \to \mathbb{C}, \quad z \mapsto f(z) - f(z_0) - f'(z_0)(z - z_0),$$

so that $\lim_{z\to z_0} \frac{|r(z)|}{|z-z_0|} = 0$ and $\int_\gamma r = \int_\gamma f$ for each closed curve γ in D; hence, we have

$$\left|\int_{\partial\Delta} f\right| \leq 4^n \left|\int_{\partial\Delta_n} r\right|$$

for $n \in \mathbb{N}$. Let $\epsilon > 0$ and choose $\delta > 0$ such that

$$\left|\frac{r(z)}{z - z_0}\right| \leq \frac{\epsilon}{\ell(\partial\Delta)^2}$$

for all $z \in D$ with $|z - z_0| < \delta$. Choose $n \in \mathbb{N}$ such that $\Delta_n \subset B_\delta(z_0)$. For $z \in \Delta_n$, this means that

$$|z - z_0| \leq \ell(\partial\Delta_n) = \frac{1}{2^n}\ell(\partial\Delta).$$

We thus obtain:

$$\begin{aligned}
\left|\int_{\partial\Delta} f\right| &\leq 4^n \left|\int_{\partial\Delta_n} r\right| \\
&\leq 4^n \ell(\partial\Delta_n) \sup_{\zeta\in\partial\Delta_n} |r(\zeta)| \\
&= 2^n \ell(\partial\Delta) \sup_{\zeta\in\partial\Delta_n} |r(\zeta)| \\
&\leq 2^n \ell(\partial\Delta) \sup_{\zeta\in\partial\Delta_n} \frac{\epsilon}{\ell(\partial\Delta)^2} \underbrace{|\zeta - z_0|}_{\leq \frac{1}{2^n}\ell(\partial\Delta)} \\
&\leq \epsilon.
\end{aligned}$$

As $\epsilon > 0$ was arbitrary, this proves the claim. □

Definition 5.3. A set $D \subset \mathbb{C}$ is called *star shaped* if there is $z_0 \in D$ such that $\{z_0 + t(z - z_0) : t \in [0,1]\} \subset D$ for each $z \in D$. The point z_0 is called a *center* for D.

Corollary 5.4. *Let $D \subset \mathbb{C}$ be open and star shaped, and let $f \colon D \to \mathbb{C}$ be holomorphic. Then f has an anti-derivative.*

Proof. Let $z_0 \in D$ be a center for D. Define

$$F\colon D \to \mathbb{C}, \quad z \mapsto \int_{[z_0,z]} f.$$

Let $z \in D$, and let $\delta < 0$ be such that $B_\delta(z) \subset D$. Let $w \in D$. Since $[z_0, z] \oplus [z, w] \oplus [w, z_0]$ is the boundary of a triangle $\Delta \subset D$, Goursat's lemma yields

$$0 = \int_{[z_0,z]\oplus[z,w]\oplus[w,z_0]} f,$$

so that

$$F(w) = \int_{[z_0,w]} f = -\int_{[w,z_0]} f = \int_{[z_0,z]} f + \int_{[z,w]} f = F(z) + \int_{[z,w]} f.$$

As in the proof of Theorem 4.10, we see then see that F is an anti-derivative for f. □

Corollary 5.5. *Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be holomorphic. Then, for each $z_0 \in D$, there is an open neighborhood $U \subset D$ of z_0 such that $f|_U$ has an anti-derivative.*

Corollary 5.6 (Cauchy's Integral Theorem for Star Shaped Domains)**.** *Let $D \subset \mathbb{C}$ be open and star shaped, and let $f : D \to \mathbb{C}$ be holomorphic. Then $\int_\gamma f(\zeta)\, d\zeta = 0$ holds for each closed curve γ in D.*

Example. The *sliced plane* is defined as

$$\mathbb{C}_- := \{z \in \mathbb{C} : z \notin (-\infty, 0]\}.$$

Then $\mathbb{C}_-$ is star shaped (1 is a center, for instance). As seen in the proof of Corollary 5.4, the function

$$\operatorname{Log}\colon \mathbb{C}_- \to \mathbb{C}, \quad z \mapsto \int_{[1,z]} \frac{1}{\zeta}\, d\zeta$$

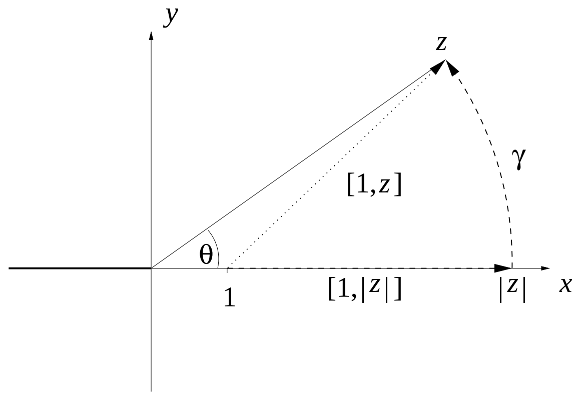
is an anti-derivative of $\mathbb{C}_- \ni z \mapsto \frac{1}{z}$; it is called the *principal branch of the logarithm.*

Let $z \in \mathbb{C}_-$, and let γ_z be any curve in $\mathbb{C}_-$ with initial point 1 and endpoint z. From Theorem 4.10, we conclude that $\operatorname{Log} z = \int_{\gamma_z} \frac{1}{\zeta}\, d\zeta$.

For any $z \in \mathbb{C}_-$, there is a unique $\theta \in (-\pi, \pi)$—the *argument* $\arg z$ of z—such that $z = |z|e^{i\theta}$. For $z \in \mathbb{C}_-$ with $\theta \geq 0$, the curve

$$\gamma\colon [0, \theta] \to \mathbb{C}, \quad t \mapsto |z|e^{it}.$$

has the initial point $|z|$ and the endpoint z. It follows that $[1, |z|] \oplus \gamma$ is curve with initial point 1 and endpoint z as shown:



It follows that

$$\operatorname{Log} z = \int_{[1,|z|]} \frac{1}{\zeta}\, d\zeta + \int_\gamma \frac{1}{\zeta}\, d\zeta = \log|z| + i\int_0^\theta \frac{e^{it}}{e^{it}}\, dt = \log|z| + i \arg z.$$

Similarly, we obtain the same equation for $\arg z < 0$.

Lemma 5.7. *Let $D \subset \mathbb{C}$ be open and star shaped with center z_0, and let $f : D \to \mathbb{C}$ be continuous such that*

$$\int_{\partial\Delta} f(\zeta)\, d\zeta = 0$$

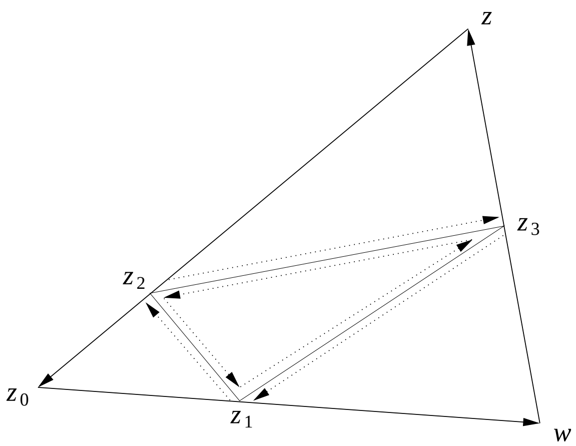
for each triangle $\Delta \subset D$ with z_0 as a corner. Then f has an anti-derivative.

Proof. This is proven just like Corollary 5.4. □

Lemma 5.8. *Let $D \subset \mathbb{C}$ be open and star shaped with center z_0, and let $f : D \to \mathbb{C}$ be continuous such that $f|_{D\setminus\{z_0\}}$ is holomorphic. Then f has an anti-derivative on D.*

Proof. We shall apply Lemma 5.7.

Let Δ be a triangle in D with z_0 as a corner:



Let z_1 be a interior point on the line segment connecting z_0 and w, let z_2 be an interior point on the line segment connecting z_0 and z, and let z_3 be an interior point on the line segment connecting w and z. Use these points to split Δ into four subtriangles—as shown above—, which we denote by $\Delta(z_0, z_1, z_2)$, $\Delta(z_1, z_3, z_2)$, $\Delta(z_1, w, z_3)$, and $\Delta(z_2, z_3, z)$. As in the proof of Goursat's Lemma, we have

$$\int_{\partial\Delta} f = \int_{\partial\Delta(z_0,z_1,z_2)} f + \int_{\partial\Delta(z_1,z_3,z_2)} f + \int_{\partial\Delta(z_1,w,z_3)} f + \int_{\partial\Delta(z_2,z_3,z)} f.$$

Since $\Delta(z_1, z_3, z_2), \Delta(z_1, w, z_3), \Delta(z_2, z_3, z) \subset D \setminus \{z_0\}$, and since f is holomorphic on $D \setminus \{z_0\}$, Goursat's Lemma yields

$$\int_{\partial\Delta(z_1,z_3,z_2)} f = \int_{\partial\Delta(z_1,w,z_3)} f = \int_{\partial\Delta(z_2,z_3,z)} f = 0,$$

so that

$$\int_{\partial\Delta} f = \int_{\partial\Delta(z_0,z_1,z_2)} f.$$

It follows that

$$\left|\int_{\partial\Delta} f\right| = \left|\int_{\partial\Delta(z_0,z_1,z_2)} f\right| \leq \ell(\partial\Delta(z_0, z_1, z_2)) \sup_{\zeta\in\partial\Delta(z_0,z_1,z_2)} |f(\zeta)|.$$

Placing z_1 and z_2 sufficiently close to z_0, $\ell(\partial\Delta(z_0, z_1, z_2))$ can be made smaller than every $\epsilon > 0$, so that $\left|\int_{\partial\Delta} f\right| = 0$. □

Let $z_0 \in \mathbb{C}$, and let $r > 0$. Slightly abusing notation, we use $\partial B_r(z_0)$ to denote the boundary of $B_r(z_0)$ in counterclockwise orientation.

Lemma 5.9. *Let $D \subset \mathbb{C}$ be open, let $z_0 \in D$, and let $r > 0$ be such that $B_r[z_0] \subset D$. Then*

$$\int_{\partial B_r(z_0)} \frac{1}{\zeta - z}\, d\zeta = 2\pi i$$

holds for all $z \in B_r(z_0)$.

Proof. Through direct computation, we saw that

$$\int_{\partial B_r(z_0)} \frac{1}{\zeta - z_0}\, d\zeta = 2\pi i.$$

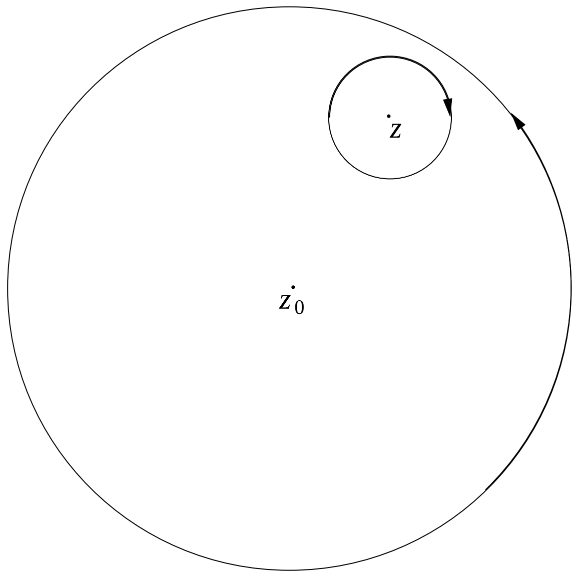
Let $z \in B_r(z_0)$, and choose $\epsilon > 0$ such that $B_\epsilon[z] \subset B_r(z_0)$, so that

$$\int_{\partial B_\epsilon(z)} \frac{1}{\zeta - z}\, d\zeta = 2\pi i.$$

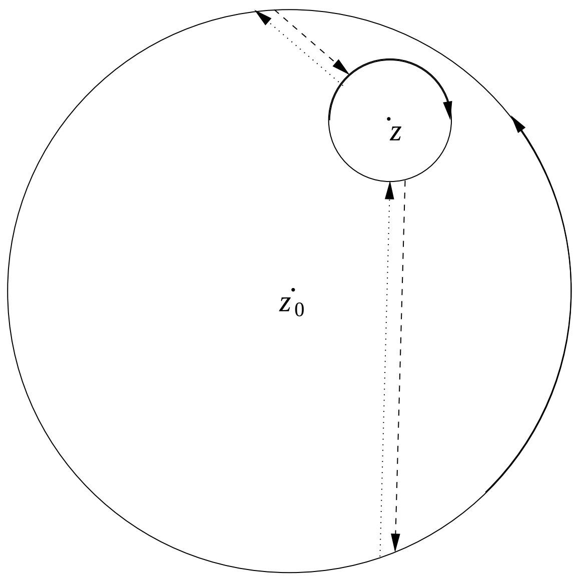
We need to show that

$$\int_{\partial B_r(z_0)} \frac{1}{\zeta - z}\, d\zeta + \int_{\partial B_\epsilon(z)^-} \frac{1}{\zeta - z}\, d\zeta = 0.$$

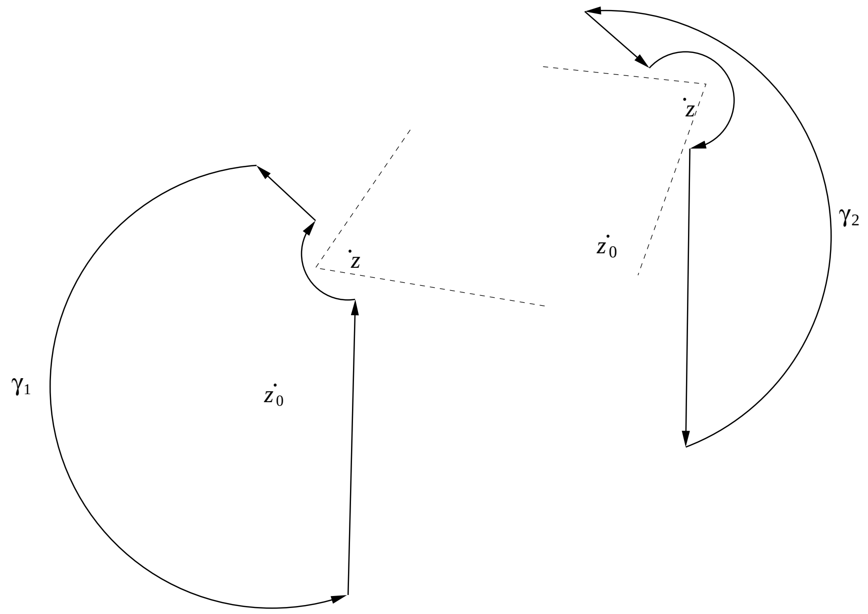
This is how the situation looks like:



We connect the boundaries of $B_r(z_0)$ and $B_\epsilon(z)$ through line segments:



Consider the following two curves γ_1 and γ_2:



Then it is clear that

$$\int_{\partial B_r(z_0)} \frac{1}{\zeta - z}\, d\zeta + \int_{\partial B_\epsilon(z)^-} \frac{1}{\zeta - z}\, d\zeta = \int_{\gamma_1} \frac{1}{\zeta - z}\, d\zeta + \int_{\gamma_2} \frac{1}{\zeta - z}\, d\zeta.$$

The sketches also show that there are star shaped open $D_j \subset \mathbb{C} \setminus \{z\}$ with $\{\gamma_j\} \subset D_j$ for $j = 1, 2$. Cauchy's integral theorem for star shaped domains thus yields that

$$\int_{\gamma_j} \frac{1}{\zeta - z}\, d\zeta = 0$$

for $j = 1, 2$, which proves the claim. □

Theorem 5.10 (Cauchy's Integral Formula for Circles)**.** *Let $D \subset \mathbb{C}$ be open, let $f\colon D \to \mathbb{C}$ be holomorphic, and let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$. Then we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for all $z \in B_r(z_0)$.

Remark. The values of f on all of $B_r[z_0]$ are already determined by those on $\partial B_r(z_0)$.

Proof of Theorem 5.10. Let $R > 0$ be such that $B_r[z_0] \subset B_R(z_0) \subset D$. Let $z \in B_r(z_0)$, and note that z is a center for the star shaped domain $B_R(z_0)$.

Define

$$g\colon D \to \mathbb{C}, \quad w \mapsto \begin{cases} \frac{f(w)-f(z)}{w-z}, & w \neq z, \\ f'(z), & w = z. \end{cases}$$

Then g is continuous an holomorphic on $D \setminus \{z\}$. By Lemma 5.8, g has an anti-derivative on $B_R(z_0)$. Since $\partial B_r(z_0)$ is closed, this means that

$$0 = \int_{\partial B_r(z_0)} g(\zeta)\, d\zeta = \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta + f(z) \underbrace{\int_{\partial B_r(z_0)} \frac{1}{\zeta - z}\, d\zeta}_{=2\pi i},$$

where the identity $\int_{\partial B_r(z_0)} \frac{1}{\zeta - z}\, d\zeta = 2\pi i$ holds by Lemma 5.9. Division by $2\pi i$ yields the claim. ☐

Corollary 5.11 (Mean Value Equation). *Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$. Then we have*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt.$$

Proof. Parametrize $\partial B_r(z_0)$ as

$$\gamma : [0, 2\pi] \to \mathbb{C}, \quad t \mapsto z_0 + re^{it}.$$

The Cauchy Integral Formula then yields

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z_0}\, d\zeta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it}\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt. \end{aligned}$$

☐

Lemma 5.12. *Let $D \subset \mathbb{R}^N$ be open, and let $f : [a, b] \times D \to \mathbb{R}$ be continuous such that $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N} : [a, b] \times D \to \mathbb{R}$ all exist and are continuous. Define*

$$g : D \to \mathbb{R}, \quad x \mapsto \int_a^b f(t, x)\, dt.$$

Then g is continuously partially differentiable with

$$\frac{\partial g}{\partial x_j}(x) = \int_a^b \frac{\partial f}{\partial x_j}(t, x)\, dt$$

for $x \in D$ and $j = 1, \ldots, N$.

Proof. There is no loss of generality supposing that $N = 1$.

Let $x_0, x \in D$ be such that every number between them is also in D. By the Mean Value Theorem of single variable calculus, there is, for each $t \in [a, b]$, an real number ξ_t between x_0 and x such that

$$\frac{f(t, x) - f(t, x_0)}{x - x_0} = \frac{\partial f}{\partial x}(t, \xi_t).$$

Let $\epsilon > 0$. By uniform continuity, there is $\delta > 0$ such that

$$\left|\frac{\partial f}{\partial x}(t, x_1) - \frac{\partial f}{\partial x}(t, x_2)\right| < \frac{\epsilon}{b-a}$$

for any $t \in [a, b]$ and all x_1, x_2 between x_0 and x with $|x_1 - x_2| < \delta$.

Suppose that $|x_0 - x| < \delta$. Then we have:

$$\begin{aligned}\left|\frac{g(x) - g(x_0)}{x - x_0} - \int_a^b \frac{\partial f}{\partial x}(t, x_0)\,dt\right| &= \left|\int_a^b \left(\frac{f(t,x) - f(t,x_0)}{x - x_0} - \frac{\partial f}{\partial x}(t, x_0)\right) dt\right| \\ &\leq \int_a^b \left|\frac{f(t,x) - f(t,x_0)}{x - x_0} - \frac{\partial f}{\partial x}(t, x_0)\right| dt \\ &= \int_a^b \left|\frac{\partial f}{\partial x}(t, \xi_t) - \frac{\partial f}{\partial x}(t, x_0)\right| dt \\ &\leq \int_a^b \frac{\epsilon}{b-a} dt, \qquad \text{because } |\xi_t - x_0| < \delta, \\ &\leq \epsilon.\end{aligned}$$

This proves that g is differentiable at x_0 with $g'(x_0) = \int_a^b \frac{\partial f}{\partial x}(t, x_0)\,dt$.

A similar (but easier) argument shows that g' is continuous. □

Corollary 5.13. *Let $D \subset \mathbb{C}$ be open, and let $f\colon [a, b] \times D \to \mathbb{C}$ be continuous such that $\frac{\partial f}{\partial z}\colon [a, b] \times D \to \mathbb{C}$ exists and is continuous. Define*

$$g\colon D \to \mathbb{R}, \quad z \mapsto \int_a^b f(t, z)\,dt.$$

Then g is holomorphic with

$$g'(z) = \int_a^b \frac{\partial f}{\partial z}(t, z)\,dt$$

for $z \in D$.

Proof. Apply Lemma 5.12 to $\operatorname{Re} f$ and $\operatorname{Im} f$ and check that the Cauchy Riemann differential equations are satisfied. □

Theorem 5.14. *Let $D \subset \mathbb{C}$ be open, let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$, and let $f\colon D \to \mathbb{C}$ be continuous such that*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

holds for all $z \in B_r(z_0)$. Then f is infinitely often complex differentiable on $B_r(z_0)$ and satisfies

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta \tag{$*$}$$

holds for all $z \in B_r(z_0)$ and $n \in \mathbb{N}_0$.

Proof. We prove by induction on $n \in \mathbb{N}_0$: f is n-times complex differentiable and $(*)$ holds.

For $n = 0$, the claim is clear, so suppose that it is true for some $n \in \mathbb{N}_0$. Define

$$F\colon [0, 2\pi] \times B_r(z_0) \to \mathbb{C}, \quad (t, z) \mapsto \frac{n!}{2\pi} \frac{f(z_0 + re^{it})re^{it}}{(z_0 + re^{it} - z)^{n+1}}.$$

Then F is continuous and, by the induction hypothesis, satisfies

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta = \int_0^{2\pi} F(t, z)\, dt$$

for all $z \in B_r(z_0)$. Furthermore,

$$\frac{\partial F}{\partial z}(t, z) = \frac{(n+1)!}{2\pi} \frac{f(z_0 + re^{it})re^{it}}{(z_0 + re^{it} - z)^{n+2}}$$

is continuous on $[0, 2\pi] \times B_r(z_0)$. From Corollary 5.13, we thus conclude that $f^{(n)}$ is holomorphic on D with

$$f^{(n+1)}(z) = \int_0^{2\pi} \frac{\partial F}{\partial z}(t, z)\, dt = \frac{(n+1)!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+2}}\, d\zeta. \qquad \square$$

Corollary 5.15 (Generalized Cauchy Integral Formula)**.** *Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be holomorphic. Then f is infinitely often complex differentiable on D. Moreover, for any $z_0 \in D$ and $r > 0$ such that $B_r[z_0] \subset D$, the* generalized Cauchy integral formula *holds, i.e.,*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

for all $z \in B_r(z_0)$ and $n \in \mathbb{N}_0$.

Example. We shall use Cauchy's integral theorem and formula to evaluate the line integral

$$\int_\gamma \frac{e^\zeta}{\zeta(\zeta - 1)}\, d\zeta$$

for various curves γ:

(a) $\gamma = \partial B_\pi(-3)$: The function

$$f\colon \mathbb{C} \setminus \{1\} \to \mathbb{C}, \quad z \mapsto \frac{e^z}{z - 1}$$

is holomorphic, and we have $B_\pi[-3] \subset \mathbb{C} \setminus \{1\}$. Cauchy's integral formula thus yields:

$$\int_{\partial B_\pi(-3)} \frac{e^\zeta}{\zeta(\zeta - 1)}\, d\zeta = \int_{\partial B_\pi(-3)} \frac{f(\zeta)}{\zeta}\, d\zeta = 2\pi i\, f(0) = -2\pi i.$$

(b) $\gamma = \partial B_{\frac{1}{2}}(i)$: As the integrand is holomorphic in on the star-shaped domain $B_{\frac{3}{4}}(i)$, Cauchy's integral theorem yields that

$$\int_{\partial B_{\frac{1}{2}}(i)} \frac{e^\zeta}{\zeta(\zeta-1)}\, d\zeta = 0.$$

(c) $\gamma = \partial B_2(0)$: Partial fractions yield

$$\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}.$$

Since $0, 1 \in B_2(0)$, we obtain with the help of Cauchy's integral formula:

$$\int_{\partial B_2(0)} \frac{e^\zeta}{\zeta(\zeta-1)}\, d\zeta = \int_{\partial B_2(0)} \frac{e^\zeta}{\zeta-1}\, d\zeta - \int_{\partial B_2(0)} \frac{e^\zeta}{\zeta}\, d\zeta = 2\pi i\,(e-1).$$

Characterizations of Holomorphic Functions. *Let $D \subset \mathbb{C}$ be open, and let $f\colon D \to \mathbb{C}$ be continuous. Then the following are equivalent:*

(i) *f is holomorphic;*

(ii) *the* Morera condition *holds, i.e., $\int_{\partial\Delta} f(\zeta)\, d\zeta = 0$ for each triangle $\Delta \subset D$;*

(iii) *for each $z_0 \in D$ and $r > 0$ with $B_r[z_0] \subset D$, we have*

$$f(z) = \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for $z \in B_r(z_0)$;

(iv) *for each $z_0 \in D$, there is $r > 0$ with $B_r[z_0] \subset D$ and*

$$f(z) = \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for $z \in B_r(z_0)$;

(v) *f is infinitely often complex differentiable on D;*

(vi) *for each $z_0 \in D$, there is an open neighborhood $U \subset D$ of z_0 such that f has an anti-derivative on U.*

Proof. (i) $\Longrightarrow$ (ii) is Goursat's Lemma.

(i) $\Longrightarrow$ (iii) is the Cauchy Integral Formula for circles, and (iii) $\Longrightarrow$ (iv) is trivial.

(iv) $\Longrightarrow$ (v) follows immediately from Theorem 5.14, and (v) $\Longrightarrow$ (i) is again trivial.

(ii) $\Longrightarrow$ (vi) follows from Lemma 5.7 because every $z_0 \in D$ has an open, star shaped neighborhood contained in D.

(vi) $\Longrightarrow$ (v): Let $z_0 \in D$, and let $U \subset D$ be an open neighborhood of z_0 such that f has an anti-derivative, say F, on U. Then F is holomorphic on U. Applying (i) $\Longrightarrow$ (v) to F, we see that F is infinitely often complex differentiable on U. Consequently, $f = F'$ is infinitely complex differentiable on U. Since $z_0 \in D$ was arbitrary, we conclude that f is infinitely often complex differentiable on D. □

We conclude this section with Liouville's Theorem and its application to the Fundamental Theorem of Algebra.

Definition 5.16. A holomorphic function defined on all of $\mathbb{C}$ is called *entire*.

Theorem 5.17 (Liouville's Theorem)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a bounded entire function. Then f is constant.*

Proof. We will show that $f' \equiv 0$.

Let $C \geq 0$ be such that $|f(z)| \leq C$ for all $z \in \mathbb{C}$. Let $z \in \mathbb{C}$ be arbitrary, and let $r > 0$. By the generalized Cauchy integral formula, we have

$$|f'(z)| = \frac{1}{2\pi} \left| \int_{\partial B_r(z)} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta \right| \leq \frac{1}{2\pi} \, \ell(\partial B_r(z)) \sup_{\zeta \in \partial B_r(z)} \frac{|f(\zeta)|}{|\zeta - z|^2} \leq \frac{1}{2\pi} 2\pi r \frac{C}{r^2} = \frac{C}{r}.$$

Letting $r \to \infty$, we obtain $f'(z) = 0$. This completes the proof. □

Corollary 5.18 (Fundamental Theorem of Algebra)**.** *Let p be a non-constant polynomial with complex coefficients. Then p has a zero.*

Proof. Assume that p has *no* zero. Then the function

$$f : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \frac{1}{p(z)}$$

is entire. Since p is a polynomial, we have $\lim_{|z| \to \infty} |p(z)| = \infty$ and thus $\lim_{|z| \to \infty} |f(z)| = 0$. Let $R > 0$ be such that $|f(z)| \leq 1$ for $|z| > R$. Since f is continuous it is bounded on $B_R[0]$, and by the choice of R, it is bounded on $\mathbb{C} \setminus B_R[0]$, too, and thus bounded on all of $\mathbb{C}$. By Liouville's Theorem, f is thus constant, and so is therefore p, which is a contradiction. □

6 Convergence of Holomorphic Functions

Just as a reminder:

Definition 6.1. Let $D \subset \mathbb{C}$ be open. Then a sequence $(f_n)_{n=1}^\infty$ of $\mathbb{C}$-valued functions on D is said to converge *uniformly* on D to $f\colon D \to \mathbb{C}$ if, for each $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \epsilon$ for all $n \geq n_\epsilon$ and all $z \in D$.

We recall the following theorem, which we won't prove:

Theorem 6.2. *Let $D \subset \mathbb{C}$ be open, and let $(f_n)_{n=1}^\infty$ of continuous, $\mathbb{C}$-valued functions on D converging uniformly on D to $f\colon D \to \mathbb{C}$. Then f is continuous.*

We now "localize" the notion of uniform convergence:

Definition 6.3. Let $D \subset \mathbb{C}$ be open. Then a sequence $(f_n)_{n=1}^\infty$ of $\mathbb{C}$-valued functions on D is said to converge *locally uniformly* on D to $f\colon D \to \mathbb{C}$ if, for each $z_0 \in D$, there is an open neighborhood $U \subset D$ of z_0 such that $(f_n|_U)_{n=1}^\infty$ converges to $f|_U$ uniformly on U.

Proposition 6.4. *Let $D \subset \mathbb{C}$ be open, and let $(f_n)_{n=1}^\infty$ of continuous, $\mathbb{C}$-valued functions on D converging locally uniformly on D to $f\colon D \to \mathbb{C}$. Then f is continuous.*

Proof. Let $z_0 \in D$, and let $U \subset D$ be an open neighborhood of z_0 such that $f_n|_U \to f|_U$ uniformly on U. By Theorem 6.2, $f|_U$ is continuous. Hence, f is continuous at z_0. □

Proposition 6.5. *Let $D \subset \mathbb{C}$ be open, and let $f, f_1, f_2, \ldots\colon D \to \mathbb{C}$ be functions. Then the following are equivalent:*

(i) *$(f_n)_{n=1}^\infty$ converges to f locally uniformly on D;*

(ii) *for each compact $K \subset D$, the sequence $(f_n|_K)_{n=1}^\infty$ converges to $f|_K$ uniformly on K.*

Proof. (i) $\Longrightarrow$ (ii): Let $K \subset D$ be compact. For each $z \in K$, there is an open neighborhood $U_z \subset D$ of z such that $f_n|_{U_z} \to f|_{U_z}$ unformly on U_z. Since K is compact, there are $z_1, \ldots, z_m \in K$ such that

$$K \subset U_{z_1} \cup \cdots \cup U_{z_m}.$$

Let $\epsilon > 0$. For each $j = 1, \ldots, m$, there is $n_j \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \epsilon$ for all $n \geq n_j$ and all $z \in U_{z_j}$. Set $n_\epsilon := \max\{n_1, \ldots, n_m\}$. Then $|f_n(z) - f(z)| < \epsilon$ holds for all $n \geq n_\epsilon$ and $z \in K$.

(ii) $\Longrightarrow$ (i): Let $z_0 \in D$, and let $r > 0$ be such that $B_r[z_0] \subset D$. Since $B_r[z_0]$ is compact, $(f_n|_{B_r[z_0]})_{n=1}^\infty$ converges uniformly on $B_r[z_0]$ to $f|_{B_r[z_0]}$. Trivially, $(f_n|_{B_r(z_0)})_{n=1}^\infty$ thus converges uniformly on $B_r(z_0)$ to $f|_{B_r(z_0)}$. □

Instead of locally uniform convergence, we therefore often speak of *compact convergence.*

Lemma 6.6. *Let $D \subset \mathbb{C}$ be open, let γ be a curve in D, and let $f, f_1, f_2, \ldots \colon D \to \mathbb{C}$ be continuous functions such that $(f_n|_{\{\gamma\}})_{n=1}^{\infty}$ converges to $f|_{\{\gamma\}}$ uniformly on $\{\gamma\}$. Then we have*

$$\int_\gamma f(\zeta)\, d\zeta = \lim_{n\to\infty} \int_\gamma f_n(\zeta)\, d\zeta.$$

Proof. Let $\epsilon > 0$, and choose $n_\epsilon \in \mathbb{N}$ such that

$$|f_n(\zeta) - f(\zeta)| < \frac{\epsilon}{\ell(\gamma) + 1}$$

for all $n \geq n_\epsilon$ and $z \in \{\gamma\}$. For $n \geq n_\epsilon$, we thus obtain:

$$\begin{aligned} \left| \int_\gamma f_n - \int_\gamma f \right| &= \left| \int_\gamma (f_n - f) \right| \\ &\leq \ell(\gamma) \sup\{|f_n(\zeta) - f(\zeta)| : \zeta \in \{\gamma\}\} \\ &\leq \frac{\epsilon\, \ell(\gamma)}{\ell(\gamma) + 1} \\ &< \epsilon. \end{aligned}$$

□

Corollary 6.7. *Let $D \subset \mathbb{C}$ be open, let γ be a curve in D, and let $f_1, f_2, \ldots \colon D \to \mathbb{C}$ be continuous functions converging compactly to $f \colon D \to \mathbb{C}$. Then f is continuous, and we have*

$$\int_\gamma f(\zeta)\, d\zeta = \lim_{n\to\infty} \int_\gamma f_n(\zeta)\, d\zeta.$$

Proof. If $\gamma \colon [a, b] \to \mathbb{C}$ is a curve—and thus continuous—, then $\{\gamma\} = \gamma([a, b])$ is compact. Hence, Lemma 6.6 applies. □

Theorem 6.8 (Weierstraß' Theorem)**.** *Let $D \subset \mathbb{C}$ be open, let $f_1, f_2, \ldots \colon D \to \mathbb{C}$ be holomorphic such that $(f_n)_{n=1}^{\infty}$ converges to $f \colon D \to \mathbb{C}$ compactly. Then f is holomorphic, and $(f_n^{(k)})_{n=1}^{\infty}$ converges compactly to $f^{(k)}$ for each $k \in \mathbb{N}$.*

Proof. Clearly, f is continuous.

To see that f is holomorphic, let $\Delta \subset D$ be a triangle. By Goursat's Lemma, $\int_{\partial\Delta} f_n(\zeta)\, d\zeta = 0$ holds for all $n \in \mathbb{N}$. From Corollary 6.7, we conclude that

$$\int_{\partial\Delta} f(\zeta)\, d\zeta = \lim_{n\to\infty} \int_{\partial\Delta} f_n(\zeta)\, d\zeta = 0,$$

i.e., f satisfies the Morera condition and thus is holomorphic.

Let $z_0 \in D$, and let $0 < r < R$ be such that $B_r[z_0] \subset B_R(z_0) \subset B_R[z_0] \subset D$. For any

$z \in B_r(z_0)$, we have

$$\begin{aligned}
|f_n'(z) - f'(z)| &= \frac{1}{2\pi} \left| \int_{\partial B_R(z_0)} \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2} \, d\zeta \right| \\
&\leq \frac{1}{2\pi} \ell(\partial B_R(z_0)) \sup_{\zeta \in \partial B_R(z_0)} \left| \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2} \right| \\
&\leq \frac{R}{(R-r)^2} \sup_{\zeta \in \partial B_R(z_0)} |f_n(\zeta) - f(\zeta)|.
\end{aligned}$$

Let $\epsilon > 0$, and choose $n_\epsilon \in \mathbb{N}$ such that

$$|f_n(\zeta) - f(\zeta)| < \epsilon \frac{(R-r)^2}{R}$$

for all $n \geq n_\epsilon$ and $\zeta \in \partial B_R(z_0)$. Then it follows from the above estimates that $|f_n'(z) - f'(z)| \leq \epsilon$ for all $n \geq n_\epsilon$ and $z \in B_r(z_0)$. Consequently, $(f_n'|_{B_r(z_0)})_{n=1}^\infty$ converges to $f'|_{B_r(z_0)}$ uniformly on $B_r(z_0)$. As $z_0 \in D$ is arbitrary, this means that $(f_n')_{n=1}^\infty$ converges to f locally uniformly, i.e., compactly, on D.

For higher derivatives, the claim now follows by induction. □

Lemma 6.9. *Let $z_0 \in \mathbb{C}$, let $r > 0$, and let $z \in B_r(z_0)$. Then*

$$\frac{1}{\zeta - z} = \sum_{n=0}^\infty \frac{1}{(\zeta - z_0)^{n+1}} (z - z_0)^n$$

holds for all $\zeta \in \partial B_r(z_0)$ with absolute and uniform converges on $\partial B_r(z_0)$.

Proof. Let $\zeta \in \partial B_r(z_0)$, and note that

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}.$$

Since $|z - z_0| < r$ and $|\zeta - z_0| = r$, we have $\left| \frac{z - z_0}{\zeta - z_0} \right| < 1$, so that

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0)} \sum_{n=0}^\infty \left(\frac{z - z_0}{\zeta - z_0} \right)^n = \sum_{n=1}^\infty \frac{1}{(\zeta - z_0)^{n+1}} (z - z_0)^n.$$

Since $\left| \frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}} \right| = \frac{|z - z_0|^n}{r^{n+1}}$ and $\sum_{n=0}^\infty \frac{|z-z_0|^n}{r^{n+1}} < \infty$, the Weierstraß M-test yields absolute and uniform convergence on $\partial B_r(z_0)$. □

Theorem 6.10. *Let $D \subset \mathbb{C}$ be open. Then the following are equivalent for $f \colon D \to \mathbb{C}$:*

(i) *f is holomorphic;*

(ii) *for each $z_0 \in D$, there are $r > 0$ with $B_r(z_0) \subset D$ and $a_0, a_1, a_2, \ldots \in \mathbb{C}$ such that $f(z) = \sum_{n=0}^\infty a_n (z - z_0)^n$ for all $z \in B_r(z_0)$;*

(iii) *for each* $z_0 \in D$ *and* $r > 0$ *with* $B_r(z_0) \subset D$, *we have*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

for all $z \in B_r(z_0)$.

Proof. (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are clear.

(i) $\Longrightarrow$ (iii): Let $z_0 \in D$, and let $r > 0$ be such that $B_r(z_0) \subset D$. Let $z \in B_r(z_0)$ and choose $\rho \in (0, r)$ such that $z \in B_\rho(z)$. Then we have:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} (z - z_0)^n\, d\zeta, \qquad \text{by Lemma 6.9,} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \right) (z - z_0)^n, \qquad \text{by Lemma 6.6,} \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n. \end{aligned}$$

□

7 Elementary Properties of Holomorphic Functions

Theorem 7.1 (Identity Theorem)**.** *Let $D \subset \mathbb{C}$ be open and connected, and let $f, g \colon D \to \mathbb{C}$ be holomorphic. Then the following are equivalent:*

(i) $f = g$;

(ii) *the set* $\{z \in D : f(z) = g(g)\}$ *has a cluster point in* D;

(iii) *there is* $z_0 \in D$ *such that* $f^{(n)}(z_0) = g^{(n)}(z_0)$.

Proof. Without loss of generality, let $g \equiv 0$.

(i) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (ii): Let $z_0 \in D$ be as in (iii), and let $r > 0$ be such that $B_r(z_0) \subset D$. Then we have by Theorem 6.10 that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = 0$$

for all $z \in B_r(z_0)$, so that

$$B_r(z_0) \subset \mathbf{Z}(f) := \{z \in D : f(z) = 0\}.$$

Every point in $B_r(z_0)$ is a cluster point of $\mathbf{Z}(f)$.

(ii) $\Longrightarrow$ (i): Let

$$V := \{z \in D : z \text{ is a cluster point of } \mathbf{Z}(f)\},$$

so that $V \neq \varnothing$ by (ii).

We claim that V is open. Let $z_0 \in V$, and let $r > 0$ be such that $B_r(z_0) \subset D$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

holds for some $a_0, a_1, a_2, \ldots \in \mathbb{C}$ and all $z \in B_r(z_0)$. We claim that $B_r(z_0) \subset \mathbf{Z}(f)$, so that $B_r(z_0) \subset V$. Assume that $B_r(z_0) \not\subset \mathbf{Z}(f)$. Then there must be $m \in \mathbb{N}_0$ such that $a_m \neq 0$; let $m \in \mathbb{N}_0$ be minimal with this property. As z_0 is a cluster point of $\mathbf{Z}(f)$, there is a sequence $(z_k)_{k=1}^{\infty}$ in $\mathbf{Z}(f) \setminus \{z_0\}$ such that $z_0 = \lim_{k\to\infty} z_k$. We obtain:

$$\begin{aligned} a_m &= \lim_{k\to\infty} \sum_{n=m}^{\infty} a_n (z_k - z_0)^{n-m} \\ &= \lim_{k\to\infty} \frac{1}{(z_k - z_0)^m} \sum_{n=0}^{\infty} a_n (z_k - z_0)^n \\ &= \lim_{k\to\infty} \frac{f(z_k)}{(z_k - z_0)^m} \\ &= 0, \end{aligned}$$

which is a contradiction. We conclude that $a_0 = a_1 = a_2 = \cdots = 0$, so that $B_r(z_0) \subset \mathbf{Z}(f)$.

We now claim that $D \setminus V$ is also open. Let $z_0 \in D \setminus V$, and $\epsilon > 0$ be such that $B_\epsilon(z_0) \subset D$ and $(B_\epsilon(z_0) \setminus \{z_0\}) \cap \mathbf{Z}(f) = \varnothing$. For any $z \in B_\epsilon(z_0) \setminus \{z_0\}$ and $\delta > 0$ such that $B_\delta(z) \subset B_\epsilon(z_0) \setminus \{z_0\}$, we thus have $(B_\delta(z) \setminus \{z\}) \cap \mathbf{Z}(f) = \varnothing$. It follows that $z \notin V$. Consequently, $B_\epsilon(z_0) \subset D \setminus V$.

All in all, V and $D \setminus V$ are both open and clearly satisfy $D = V \cup (D \setminus V)$ and $V \cap (D \setminus V) = \varnothing$. The connectedness of D yields $D = V$ and thus $\mathbf{Z}(f) = D$. $\square$

Examples.

1. There is non non-zero entire function $f\colon \mathbb{C} \to \mathbb{C}$ such that $f\left(\frac{1}{n}\right) = 0$ for $n \in \mathbb{N}$. For any entire function f with this property, we have $f(0) = 0$ by continuity, so that 0 is a clsuter point of $\mathbf{Z}(f)$. By the Identity Theorem, this means $f \equiv 0$.

2. Since $\mathbb{R}$ has cluster points in $\mathbb{C}$, the holomorphic extendsions of exp, cos, and sin from $\mathbb{R}$ to $\mathbb{C}$ are unique. For analogous reasons, Log is the only holomorphic extension of log to $\mathbb{C}_-$.

3. There is no entire function f such that

$$f\left(\frac{1}{n}\right) = \frac{1}{n} \qquad \text{and} \qquad f\left(-\frac{1}{n}\right) = \frac{1}{n^2}$$

for $n \in \mathbb{N}$. In the view of the identity theorem, the first condition necessitates that $f(z) = z$ whereas the second one implies that $f(z) = (-z)^2$ for all $z \in \mathbb{C}$.

Lemma 7.2. *Let $D \subset \mathbb{C}$ be open, let $f\colon D \to \mathbb{C}$ be holomorphic, and let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$. Suppose that*

$$|f(z_0)| < \inf_{z \in \partial B_r(z_0)} |f(z)|.$$

Then f has a zero in $B_r(z_0)$.

Proof. Assume otherwise, i.e., f has no zero in $B_r(z_0)$. The hypothesis implies that f has no zero on $\partial B_r(z_0)$, so that f thus has no zero in $B_r[z_0]$. Assume that, for each $R > 0$ such that $B_r[z_0] \subset B_R(z_0) \subset D$, there is a zero of f in $B_R(z_0)$. Then we have a sequence $(R_n)_{n=1}^\infty$ in (r, ∞) with $r = \lim_{n\to\infty} R_n$ such that $B_r[z_0] \subset B_{R_n}(z_0) \subset D$ and $\mathbf{Z}(f) \cap B_{R_n}(z_0) \neq \varnothing$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, pick $z_n \in \mathbf{Z}(f) \cap B_{R_n}(z_0)$. Then $(z_n)_{n=1}^\infty$ is bounded, and thus has a convergent subsequence $(z_{n_k})_{k=1}^\infty$ with limit z'. Clearly, $z' \in \mathbf{Z}(f)$, and since $\lim_{k\to\infty} R_{n_k} = r$, we have $z' \in B_r[z_0]$, which is impossible. Consequently, f has no zero on some $B_R(z_0)$ with $B_r[z_0] \subset B_R(z_0) \subset D$. Replacing D with $B_R(z_0)$, we can thus suppose that f has no zeros.

From the Cauchy Integral Formula, we obtain

$$\begin{aligned}\frac{1}{|f(z_0)|} &= \left| \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{1}{f(\zeta)} \frac{1}{\zeta - z_0}\, d\zeta \right| \\ &\leq \frac{1}{2\pi} 2\pi r \sup_{\zeta \in \partial B_r(z_0)} \frac{1}{|f(\zeta)| r} \\ &= \frac{1}{\inf_{\zeta \in \partial B_r(z_0)} |f(\zeta)|}\end{aligned}$$

and thus

$$|f(z_0)| \geq \inf_{\zeta \in \partial B_r(z_0)} |f(\zeta)|,$$

which is a contradiction. $\square$

Theorem 7.3 (Open Mapping Theorem)**.** *Let $D \subset \mathbb{C}$ be open and connected, and let $f \colon D \to \mathbb{C}$ be holomorphic and not constant. Then $f(D) \subset \mathbb{C}$ is open and connected.*

Proof. By the continuity of f, it is clear that $f(D)$ is connected.

Let $w_0 \in f(D)$, and let $z_0 \in D$ be such that $w_0 = f(z_0)$. Choose $r > 0$ such that $B_r[z_0] \subset D$ and such that $\{z \in B_r[z_0] : f(z) = w_0\} = \{w_0\}$. (This can be accomplished with the help of the Identity Theorem.) Choose $\epsilon > 0$ such that $|f(z) - w_0| \geq 3\epsilon$ for all $z \in \partial B_r(z_0)$. We claim that $B_\epsilon(w_0) \subset f(D)$. Let $w \in B_\epsilon(w_0)$. For $z \in \partial B_r(z_0)$, we have

$$|f(z) - w| \geq |f(z) - w_0| - |w - w_0| \geq 3\epsilon - \epsilon = 2\epsilon.$$

It follows that

$$|f(z_0) - w| < \epsilon < 2\epsilon \leq \inf_{z \in \partial B_r(z_0)} |f(z) - w|.$$

By Lemma 7.2, this means that

$$D \to \mathbb{C}, \quad z \mapsto f(z) - w$$

has a zero in $B_r(z_0)$. It follows that $w \in f(D)$. $\square$

Theorem 7.4 (Maximum Modulus Principle)**.** *Let $D \subset \mathbb{C}$ be open and connected, and let $f \colon D \to \mathbb{C}$ be holomorphic such that the function*

$$|f| \colon D \to \mathbb{C}, \quad z \mapsto |f(z)|$$

attains a local maximum on D. Then f is constant.

Proof. Let $z_0 \in D$ be such that $|f|$ attains a local maximum at z_0, i.e., there is $\epsilon > 0$ such that $B_\epsilon(z_0) \subset D$ and $|f(z_0)| \geq |f(z)|$ for all $z \in B_\epsilon(z_0)$. Then $f(z_0)$ is not an interior point of $f(B_\epsilon(z_0))$, so that $f|_{B_\epsilon(z_0)}$ is constant by the Open Mapping Theorem. The Identity Theorem then yields that f is constant. $\square$

Corollary 7.5. *Let $D \subset \mathbb{C}$ be open and connected, and let $f : D \to \mathbb{C}$ be holomorphic such that $|f|$ attains a local minimum on D. Then f is constant or f has a zero.*

Proof. Suppose that f has no zero. Applyint the Maximum Modulus Principle to $\frac{1}{f}$ then yields that f is constant. ☐

Corollary 7.6 (Maximum Modulus Principle for Bounded Domains)**.** *Let $D \subset \mathbb{C}$ be open, connected, and bouned, and let $f : \overline{D} \to \mathbb{C}$ be continuous such that $f|_D$ is holomorphic. Then $|f|$ attains its maximum on $\overline{D}$ on ∂D.*

Proof. The claim is trivial if f is constant, so suppose that f is not constant.

Since f is continuous and $\overline{D}$ is compact, there is a point $z_0 \in \overline{D}$ with $|f(z_0)| = \max\left\{|f(z)| : z \in \overline{D}\right\}$. If $z_0 \in D$, then $|f|$ would attain a local maximum at z_0, which is impossible by the Maximum Modulus Principle. Therefore $z_0 \in \partial D$ must hold. ☐

From now on, we shall use $\mathbb{D}$ to denote the open unit disc $B_1(0)$.

Theorem 7.7 (Schwarz' "Lemma")**.** *Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic such that $f(\mathbb{D}) \subset \overline{\mathbb{D}}$ and $f(0) = 0$. Then one has*

$$|f(z)| \leq |z| \quad \text{for } z \in \mathbb{D} \qquad \text{and} \qquad |f'(0)| \leq 1.$$

Moreover, if there is $z_0 \in \mathbb{D} \setminus \{0\}$ such that $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$, then the there is $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = cz$ for $z \in \mathbb{D}$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of f. Since $f(0) = 0$, we have $a_0 = 0$. Define

$$g : \mathbb{D} \to \mathbb{C}, \quad z \mapsto \sum_{n=1}^{\infty} a_n z^{n-1}.$$

Then g is holomorphic with $g(0) = a_1 = f'(0)$ and $f(z) = zg(z)$ for $z \in \mathbb{D}$. Let $r \in (0,1)$. Then we have

$$|g(z)| = \frac{|f(z)|}{r} \leq \frac{1}{r}$$

for $z \in \partial B_r(0)$ and thus for all $z \in B_r[0]$ by the Maximum Modulus Principle. Letting $r \to 1$, we obtain that $|g(z)| \leq 1$ for all $z \in \mathbb{D}$ and thus $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ as well as $|f'(0)| = |g(0)| =\leq 1$.

Suppose that there is $z_0 \in \mathbb{D} \setminus \{0\}$ such that $|f(z_0)| = |z_0|$ or $|f'(0)| = |g(0)| = 1$. Then $|g|$ has a maximum at z_0 or 0, respectively, so that g is constant. Hence, there is $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = zg(z) = cz$ for $z \in \mathbb{D}$. ☐

Definition 7.8. Let $D_1, D_2 \subset \mathbb{C}$ be open. Then $f : D_1 \to D_2$ is called *biholomorphic* (or *conformal*) if

(a) f is bijective and

(b) both f and f^{-1} are holomorphic.

Corollary 7.9. *Let $f : \mathbb{D} \to \mathbb{D}$ be biholomorphic such that $f(0) = 0$. Then there is $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = cz$ for $z \in \mathbb{D}$.*

Proof. Let $z \in \mathbb{D}$. Then $|f(z)| \leq |z|$ holds by Schwarz' Lemma as does

$$|z| = |f^{-1}(f(z))| \leq |f(z)|.$$

The claim then follows from Schwarz' Lemma. □

Lemma 7.10. *Let $w \in \mathbb{D}$, and define*

$$\phi_w : \mathbb{D} \to \mathbb{C}, \quad z \mapsto \frac{z - w}{\bar{w}z - 1}.$$

Then:

(i) *ϕ_w maps $\mathbb{D}$ bijectively onto $\mathbb{D}$;*

(ii) *$\phi_w(w) = 0$;*

(iii) *$\phi_w(0) = w$;*

(iv) *$\phi_w^{-1} = \phi_w$.*

Proof. Obviously, ϕ_w is holomorphic and extends continuously to $\overline{\mathbb{D}}$.

Let $z \in \partial\mathbb{D}$. Then we have

$$|\phi_w(z)| = \left|\frac{z - w}{\bar{w}z - 1}\right| = \left|\frac{z - w}{-(\overline{z - w})}\right| = 1.$$

By the Maximum Modulus Principle, thus $\phi_w(\mathbb{D}) \subset \overline{\mathbb{D}}$ holds. Since ϕ_w is not constant, $\phi_w(\mathbb{D})$ is open and thus contained in the interior of $\overline{\mathbb{D}}$, i.e., in $\mathbb{D}$.

It is obvious that $\phi_w(w) = 0$ and $\phi_w(0) = w$.

Moreover, we have for $z \in \mathbb{D}$:

$$\begin{aligned}
(\phi_w \circ \phi_w)(z) &= \frac{\frac{z-w}{\bar{w}z-1} - w}{\bar{w}\frac{z-w}{\bar{w}z-1} - 1} \\
&= \frac{(z - w) - w(\bar{w}z - 1)}{\bar{w}(z - w) - (\bar{w}z - 1)} \\
&= \frac{z - w - w\bar{w}z + w}{\bar{w}z - w\bar{w} - \bar{w}z + 1} \\
&= \frac{z(1 - |w|^2)}{1 - |w|^2} \\
&= z.
\end{aligned}$$

Hence, ϕ_w is bijective with $\phi_w^{-1} = \phi_w$. □

Theorem 7.11. *Let $\phi\colon \mathbb{D} \to \mathbb{D}$ be biholomorphic. Then there are $w \in \mathbb{D}$ and $c \in \partial\mathbb{D}$ with*

$$\phi(z) = c\frac{z - w}{\bar{w}z - 1}$$

for $z \in \mathbb{D}$.

Proof. Set $w := \phi^{-1}(0)$. Then $\phi \circ \phi_w : \mathbb{D} \to \mathbb{D}$ is biholomorphic with $(\phi \circ \phi_w)(0) = 0$. By Corollary 7.9, there is $c \in \mathbb{C}$ with $|c| = 1$ such that $\phi(\phi_w(z)) = cz$ for $z \in \mathbb{D}$, so that

$$\phi(z) = \phi(\phi_w(\phi_w(z))) = c\,\phi_w(z)$$

for $z \in \mathbb{D}$. □

Definition 7.12. Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be holomorphic. We call $z_0 \in \mathbb{C} \setminus D$ an *isolated singularity* for f if there is $\epsilon > 0$ such that $B_\epsilon(z_0) \setminus \{z_0\} \subset D$. We say that the isolated singularity z_0 is *removable* if there is a holomorphic function $g\colon D \cup \{z_0\} \to \mathbb{C}$ such that $g|_D = f$.

Theorem 7.13 (Riemann's Removability Condition). *Let $D \subset \mathbb{C}$ be open, let $f\colon D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be an isolated singularity for f. Then the following are equivalent:*

(i) *z_0 is removable;*

(ii) *there is a continuous function $g\colon D \cup \{z_0\} \to \mathbb{C}$ such that $g|_D = f$;*

(iii) *there is $\epsilon > 0$ with $B_\epsilon(z_0) \setminus \{z_0\} \subset D$ such that f is bounded on $B_\epsilon(z_0) \setminus \{z_0\}$.*

Proof. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is clear.

(iii) $\Longrightarrow$ (i): Let $C \geq 0$ be such that $|f(z)| \leq C$ for $z \in B_\epsilon(z_0) \setminus \{z_0\}$. Define

$$h\colon D \cup \{z_0\} \to \mathbb{C}, \quad z \mapsto \begin{cases} (z - z_0)^2 f(z), & z \neq z_0, \\ 0, & z = z_0. \end{cases}$$

Then we have for $z \in B_\epsilon(z_0) \setminus \{z_0\}$ that

$$\left|\frac{h(z) - h(z_0)}{z - z_0}\right| = |(z - z_0)f(z)| \leq C|z - z_0|.$$

Hence, h is holomorphic with $h'(z_0) = h(z_0) = 0$. Let $h(z) = \sum_{n=0}^\infty a_n(z - z_0)^n$ be the power series representation of h on $B_\epsilon(z_0)$. Then $h'(z_0) = h(z_0) = 0$ means that $a_0 = a_1 = 0$, so that $h(z) = \sum_{n=2}^\infty a_n(z - z_0)^n$ for $z \in B_\epsilon(z_0)$ and thus $f(z) = \sum_{n=0}^\infty a_{n+2}(z - z_0)^n$ for $z \in B_\epsilon(z_0) \setminus \{z_0\}$.

Define

$$g\colon D \cup \{z_0\} \to \mathbb{C}, \quad z \mapsto \begin{cases} \sum_{n=0}^\infty a_{n+2}(z - z_0)^n, & z \in B_\epsilon(z_0), \\ f(z), & z \in D. \end{cases}$$

Then g is a holomorphic function extending f. □

8 Analytic Continuation along a Curve

Example. Let

$$D_1 := \{z \in \mathbb{C} : \operatorname{Re} z > 0\},$$
$$D_2 := \{z \in \mathbb{C} : \operatorname{Im} z > \operatorname{Re} z\},$$

and

$$D_3 := \{z \in \mathbb{C} : \operatorname{Im} z < -\operatorname{Re} z\},$$

so that

$$D_1 \cup D_2 \cup D_3 = \mathbb{C} \setminus \{0\}.$$

Let

$$g \colon \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad z \mapsto \frac{1}{z},$$

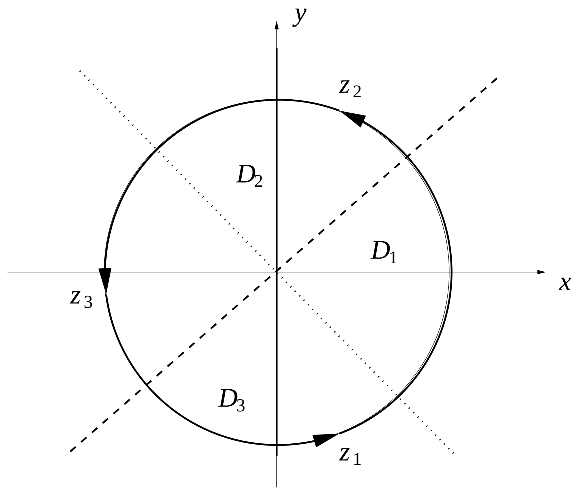
and let $f_1 = \operatorname{Log}|_{D_1}$, so that f_1 is an anti-derivative of g on D_1. Since D_2 is star shaped, g also has an anti-derivative on D_2; since $f_1' - f_2'|_{D_1 \cap D_2} = g - g_{D_1 \cap D_2} \equiv 0$, it follows that $f_1 - f_2|_{D_1 \cap D_2}$ is constant, and by altering f_2 by an additive constant, we can achieve that $f_1|_{D_1 \cap D_2} = f_2|_{D_1 \cap D_2}$. In the same fashion, we can find an anti-derivative f_3 of g on D_3 such that $f_2|_{D_2 \cap D_3} = f_3|_{D_2 \cap D_3}$. However, $f_1|_{D_1 \cap D_3} \neq f_3|_{D_1 \cap D_3}$ because otherwise, we would have an anti-derivative of g on all of $\mathbb{C} \setminus \{0\}$, which we know to be impossible.

Since $f_1' - f_3'|_{D_1 \cap D_3} = g - g_{D_1 \cap D_3} \equiv 0$, however, there is $c \in \mathbb{C}$ such that $f_3(z) = f_1(z) + c$ for $z \in D_1 \cap D_2$. We claim that $c = 2\pi i$. To see this, let $z_1, z_2, z_3 \in \partial\mathbb{D}$ be such that $z_1 \in D_1 \cap D_3$, $z_2 \in D_2 \cap D_1$, and $z_3 \in D_3 \cap D_2$. Let γ_{z_1,z_2}, γ_{z_2,z_3}, and γ_{z_3,z_1} be the arc segments of $\partial\mathbb{D}$ from z_1 to z_2, from z_2 to z_3, and from z_3 to z_1, respectively. Since f_j is an anti-derivative of g on D_j for $j = 1, 2, 3$, we obtain

$$\int_{\gamma_{z_1,z_2}} g = f_1(z_2) - f_1(z_1), \qquad \int_{\gamma_{z_2,z_3}} g = f_2(z_3) - f_2(z_2),$$
$$\text{and} \qquad \int_{\gamma_{z_3,z_1}} g = f_3(z_1) - f_3(z_3).$$

It follows that

$$\begin{aligned} c &= f_3(z_1) - f_1(z_1) \\ &= f_3(z_1) - f_3(z_3) + f_2(z_3) - f_2(z_2) + f_1(z_2) - f_1(z_1) \\ &= \int_{\gamma_{z_3,z_1}} g + \int_{\gamma_{z_2,z_3}} g + \int_{\gamma_{z_1,z_2}} g \\ &= \int_{\gamma_{z_1,z_2} \oplus \gamma_{z_2,z_3} \oplus \gamma_{z_3,z_1}} g \\ &= \int_{\partial\mathbb{D}} \frac{1}{\zeta}\, d\zeta \\ &= 2\pi i. \end{aligned}$$



Definition 8.1. A *function element* is a pair (D, f), where $D \subset \mathbb{C}$ is open and connected, and $f\colon D \to \mathbb{C}$ is a holomorphic function. For a given function element (D, f) and $z_0 \in D$, the *germ of f at z_0*—denoted by $\langle f \rangle_{z_0}$—is the collection of all function elements (E, g) such that $z_0 \in E$ and there is a an open neighborhood $U \subset D \cap E$ of z_0 such that $f(z) = g(z)$ for all $z \in U$.

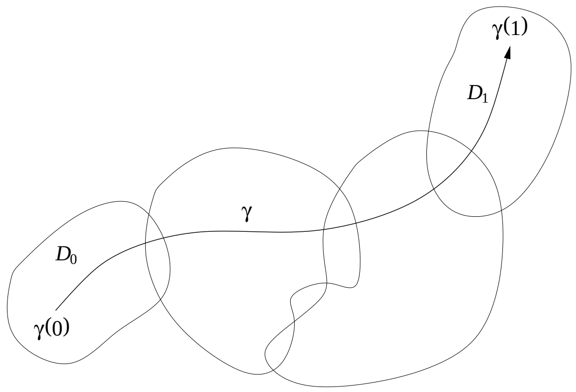
Definition 8.2. Let $\gamma\colon [0,1] \to \mathbb{C}$ be a path, and suppose that, for each $t \in [0,1]$, there is a function element (D_t, f_t) such that:

(a) $\gamma(t) \in D_t$ for $t \in [0,1]$;

(b) for each $t \in [0,1]$, there is $\delta > 0$ such that, whehenver $s \in [0,1]$ is such that $|s-t| < \delta$, then $\gamma(s) \in D_t$ and $\langle f_s \rangle_{\gamma(s)} = \langle f_t \rangle_{\gamma(s)}$.

Then we call $\{(D_t, f_t) : t \in [0,1]\}$ an *analytic continuation* along γ and say that (D_1, f_1) is obtained by analytic continuation of (D_0, f_0) along γ.

Remark. Since γ is continuous and D_t is open for each $t \in D_t$, it is clear that there is $\delta > 0$ such that $\gamma(s) \in D_t$ for all $s \in [0,1]$ such that $|s - t| < \delta$. What is important about Definition 8.2(b) is that $\langle f_s \rangle_{\gamma(s)} = \langle f_t \rangle_{\gamma(s)}$, i.e., there is an open neighborhood $U_s \subset D_s \cap D_t$ of $\gamma(s)$ such that $f_s(z) = f_t(z)$ for $z \in U_s$.



Theorem 8.3. *Let $\gamma\colon [0,1] \to \mathbb{C}$ be a path, and let $\{(D_t, f_t) : t \in [0,1]\}$ and $\{(E_t, g_t) : t \in [0,1]\}$ be analytic continuations along γ such that $\langle f_0\rangle_{\gamma(0)} = \langle g_0\rangle_{\gamma(0)}$. Then we have $\langle f_1\rangle_{\gamma(1)} = \langle g_1\rangle_{\gamma(1)}$.*

Proof. Let

$$I = \{t \in [0,1] : \langle f_t\rangle_{\gamma(t)} = \langle g_t\rangle_{\gamma(t)}\},$$

so that $0 \in I$.

We first claim that I is closed. Let $t \in \overline{I}$, and let $\delta > 0$ be such that $\gamma(s) \in D_t \cap E_t$ and

$$\langle f_s\rangle_{\gamma(s)} = \langle f_t\rangle_{\gamma(s)} \qquad \text{and} \qquad \langle g_s\rangle_{\gamma(s)} = \langle g_t\rangle_{\gamma(s)}$$

for all $s \in [0,1]$ with $|s-t| < \delta$. Since $t \in \overline{I}$, there is $s \in I$ with $|s-t| < \delta$. There is thus an open neighborhood $U \subset D_t \cap D_s \cap E_t \cap E_s$ of $\gamma(s)$ such that $f_s(z) = g_s(z)$ for all $z \in U$ by the definition of I. From the choice of δ, we also have—after possibly making U smaller—that $f_s(z) = f_t(z)$ and $g_s(z) = g_t(z)$ for $z \in U$. It follows that $f_t(z) = g_t(z)$ for $z \in U$, so that $t \in I$.

Let $t_0 := \sup I$. Let $\delta > 0$ be such that $\gamma(s) \in D_{t_0} \cap E_{t_0}$ and

$$\langle f_s\rangle_{\gamma(s)} = \langle f_{t_0}\rangle_{\gamma(s)} \qquad \text{and} \qquad \langle g_s\rangle_{\gamma(s)} = \langle g_{t_0}\rangle_{\gamma(s)}$$

for all $s \in [0,1]$ with $|s-t| < \delta$. Since I is closed, we have $t_0 \in I$ and thus $f_{t_0}(z) = g_{t_0}(z)$ for all z in some neighborhood V of $\gamma(t_0)$ contained in $D_{t_0} \cap E_{t_0}$. It follows that $\langle f_{t_0}\rangle_{\gamma(s)} = \langle g_{t_0}\rangle_{\gamma(s)}$ for all $s \in [0,1]$ such that $\gamma(s) \in V$. For $\delta > 0$ sufficiently small, we thus have $\langle f_s\rangle_{\gamma(s)} = \langle g_s\rangle_{\gamma(s)}$ for any $s \in [0,1]$ with $|s-t| < \delta$. It follows that $[0,1] \cap (t_0 - \delta, t_0 + \delta) \subset I$. Since $t_0 = \sup I$, this means that $t_0 = 1$, so that $I = [0,1]$. ☐

9 Harmonic Functions

Definition 9.1. Let $D \subset \mathbb{R}^N$ be open, and let $u \colon D \to \mathbb{R}$ be twice continuously partially differentiable. Then f is called *harmonic* if

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_N^2} \equiv 0.$$

In this course, we will only be concerned with harmonic functions on $\mathbb{R}^2$, i.e., on $\mathbb{C}$.

Proposition 9.2. *Let $D \subset \mathbb{C}$ be open, and let $f \colon D \to \mathbb{C}$ be holomorphic. Then* $\operatorname{Re} f$ *and* $\operatorname{Im} f$ *are harmonic.*

Proof. Clearly, $\operatorname{Re} f$ and $\operatorname{Im} f$ are twice continuously differentiable.

We have

$$\begin{aligned}
\frac{\partial^2 (\operatorname{Re} f)}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial}{\partial x}\operatorname{Re} f \\
&= \frac{\partial}{\partial x}\frac{\partial}{\partial y}\operatorname{Im} f, \qquad \text{by Cauchy–Riemann,} \\
&= \frac{\partial}{\partial y}\frac{\partial}{\partial x}\operatorname{Im} f \\
&= -\frac{\partial^2 (\operatorname{Re} f)}{\partial y^2}, \qquad \text{by Cauchy–Riemann again,}
\end{aligned}$$

so that $\Delta \operatorname{Re} f \equiv 0$, i.e., $\operatorname{Re} f$ is harmonic.

Similarly, one sees that $\operatorname{Im} f$ is harmonic. ☐

Example. Consider

$$u \colon \mathbb{C} \setminus \{0\} \to \mathbb{R}, \quad z \mapsto \log|z|,$$

so that

$$u(x,y) = \log \sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2)$$

for $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then the first partial derivatives of u are computed as

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \qquad \text{and} \qquad \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2},$$

so that

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

Consequently, u is harmonic.

Assume that there is a holomorphic function $f\colon \mathbb{C}\setminus\{0\}\to\mathbb{C}$ such that $\operatorname{Re} f = u$. On $\mathbb{C}_-$, we then have that $\operatorname{Re} f = \operatorname{Re}\operatorname{Log}$. The Cauchy–Riemann Equations thus yield

$$\frac{\partial(\operatorname{Im} f)}{\partial x}(z) = -\frac{\partial u}{\partial y}(z) = \frac{\partial(\operatorname{Im}\operatorname{Log})}{\partial x}(z),$$

so that

$$f'(z) = \frac{\partial u}{\partial x}(z) + i\frac{\partial(\operatorname{Im} f)}{\partial x}(z) = \frac{\partial u}{\partial x}(z) + i\frac{\partial(\operatorname{Im}\operatorname{Log})}{\partial x}(z) = \operatorname{Log}' z = \frac{1}{z}$$

for $z\in\mathbb{C}_-$. By continuity, it follows that $f'(z)=\frac{1}{z}$ for all $z\in\mathbb{C}\setminus\{0\}$, so that f is an anti-derivative of $\mathbb{C}\setminus\{0\}\ni z\mapsto\frac{1}{z}$. This is impossible.

Definition 9.3. Let $D\subset\mathbb{C}$ be open, and let $u\colon D\to\mathbb{R}$ be harmonic. We call a harmonic function $v\colon D\to\mathbb{C}$ a *harmonic conjugate* of u if $u+i\,v$ is holomorphic.

Theorem 9.4. *Let $D\subset\mathbb{C}$ be open and suppose that there is $(x_0,y_0)\in D$ with the following property: for each $(x,y)\in D$, we have*

- *$(x,t)\in D$ for each t between y and y_0 and*
- *$(s,y_0)\in D$ for each s between x and x_0.*

Then every harmonic function on D has a harmonic conjugate.

Proof. Let $u\colon D\to\mathbb{R}$ be harmonic. We will find a harmonic $v\colon D\to\mathbb{R}$ such that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\text{and}\qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{$*$}$$

For $(x,y)\in D$, define

$$v(x,y) = \int_{y_0}^{y}\frac{\partial u}{\partial x}(x,t)\,dt + \phi(x),$$

where ϕ will be specified later. First, note that

$$\begin{aligned}\frac{\partial v}{\partial x}(x,y) &= \int_{y_0}^{y}\frac{\partial^2 u}{\partial x^2}(x,t)\,dt + \phi'(x), \qquad \text{by Lemma 5.12,}\\ &= -\int_{y_0}^{y}\frac{\partial^2 u}{\partial y^2}(x,t)\,dt + \phi'(x)\\ &= -\frac{\partial u}{\partial y}(x,y) + \frac{\partial u}{\partial y}(x,y_0) + \phi'(x).\end{aligned}$$

Hence, if we want the Cauchy–Riemann Differential Equations to hold for $u+iv$, we need that $\phi'(x) = -\frac{\partial u}{\partial y}(x,y_0)$. We thus set

$$v(x,y) = \int_{y_0}^{y}\frac{\partial u}{\partial x}(x,t)\,dt - \int_{x_0}^{x}\frac{\partial u}{\partial y}(s,y_0)\,ds.$$

Then $(*)$ holds, so that

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = -\frac{\partial^2 v}{\partial y^2},$$

i.e., v is harmonic. $\square$

Example. Let

$$u \colon \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto xy.$$

Then u is harmonic and

$$v(x, y) = \int_0^y t\,dt - \int_0^x s\,ds = \frac{y^2}{2} - \frac{x^2}{2}$$

is a harmonic conjugate for u.

Corollary 9.5. *Let $D \subset \mathbb{C}$ be open, and let $u : D \to \mathbb{R}$ be harmonic. Then, for each $z_0 \in D$, there is an open neighborhood $U \subset D$ of z_0 such that $u|_U$ has a harmonic conjugate.*

Corollary 9.6. *Let $D \subset \mathbb{C}$ be open, and let $u \colon D \to \mathbb{R}$ be harmonic. Then u is infinitely often partially differentiable.*

Corollary 9.7. *Let $D \subset \mathbb{C}$ be open and connected, and let $u \colon D \to \mathbb{R}$ be harmonic. Then the following are equivalent:*

(i) $u \equiv 0$;

(ii) *there is $\varnothing \neq U \subset D$ open with $u|_U \equiv 0$.*

Proof. Of course, only (ii) $\Longrightarrow$ (i) needs proof. Let

$$V := \{z \in D : u \equiv 0 \text{ on an open neighborhood of } z\}.$$

Clearly, V is open and not empty.

We will show that $D \setminus V$ is open, too. As D is connected, this establishes $D = V$.

Let $z_0 \in D \setminus V$ and $\epsilon > 0$ be such that $B_\epsilon(z_0) \subset D$. Assume that $B_\epsilon(z_0) \cap V \neq \varnothing$. Let $z \in B_\epsilon(z_0) \cap V$, and let $\delta > 0$ be such that $B_\delta(z) \subset B_\epsilon(z_0) \cap V$. By Theorem 9.4, u has a harmonic conjugate v on $B_\epsilon(z_0)$. Set $f := u + i\,v$, so that f is holomorphic with

$$f' = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \equiv 0$$

on $B_\delta(z)$. Consequently, f is constant on $B_\delta(z)$ and thus—by the Identity Theorem—on all of $B_\epsilon(z_0)$. Hence, $u = \operatorname{Re} f$ is constant on $B_\epsilon(z_0)$, and since $u \equiv 0$ on $B_\delta(z)$, this means $u \equiv 0$ on $B_\epsilon(z_0)$, which contradicts $z_0 \notin V$. $\square$

Corollary 9.8. *Let $D \subset \mathbb{C}$ be open, let $u\colon D \to \mathbb{R}$ be harmonic, and let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$. Then we have*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})\, dt.$$

Corollary 9.9. *Let $D \subset \mathbb{C}$ be open and connected, and let $u\colon D \to \mathbb{R}$ be harmonic with a local maximum or minimum on D. Then u is constant.*

Proof. It is enough to consider the case of a local maximum: otherwise, replace u by $-u$.

Let $z_0 \in D$ be a point where u attains a local maximum. Let $\epsilon > 0$ be such that $B_\epsilon(z_0) \subset D$ and $u(z) \leq u(z_0)$ for all $z \in B_\epsilon(z_0)$. Let v be a harmonic conjugate of u on $B_\epsilon(z_0)$. Hence, $f := u + i\,v\colon B_\epsilon(z_0) \to \mathbb{C}$ is holomorphic such that $\operatorname{Re} f$ has a local maximum at z_0. By a midterm practice problem, this means that f is constant, i.e., there is $c \in \mathbb{C}$ such that $u - c|_{B_\epsilon(z_0)} \equiv 0$. By Corollary 9.7, this means that $u \equiv c$ on D. $\square$

Corollary 9.10. *Let $D \subset \mathbb{C}$ be open, connected, and bounded, and let $u : \overline{D} \to \mathbb{R}$ be continuous such that $u|_D$ is harmonic. Then u attains its maximum and minimum on $\overline{D}$ on ∂D.*

The Dirichlet Problem. Let $D \subset \mathbb{C}$ be open, connected, and bounded, and let $f : \partial D \to \mathbb{R}$ be continuous. Is there a continuous $g\colon \overline{D} \to \mathbb{R}$ such that $g|_{\partial D} = f$ and $f|_D$ is harmonic?

Remark. If the Dirichlet problem has a solution, then it is unique. To see this, let $g_1, g_2 : \overline{D} \to \mathbb{R}$ be such that $g_j|_{\partial D} = f$ and $g_j|_D$ is harmonic for $j = 1, 2$. Then $g_1 - g_2$ vanishes on ∂D. Since $g_1 - g_2$ attains both its maximum and minimum on ∂D, it follows that $g_1 - g_2 \equiv 0$ on $\overline{D}$.

Definition 9.11. Let $r > 0$. The *Poisson kernel* for $B_r(0)$ is defined as

$$P_r(\zeta, z) := \frac{1}{2\pi} \frac{r^2 - |z|^2}{|\zeta - z|^2}$$

for $z \in B_r(0)$ and $\zeta \in \partial B_r(0)$.

Lemma 9.12. *Let $D \subset \mathbb{C}$ be open, let $r > 0$ be such that $B_r[0] \subset D$, and let $f\colon D \to \mathbb{C}$ be holomorphic. Then we have*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) \frac{r^2 - |z|^2}{|re^{it} - z|^2}\, dt = \int_0^{2\pi} f(re^{it}) P_r(re^{it}, z)\, dt.$$

for $z \in B_r(0)$.

Proof. The Cauchy Integral Formula yields

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(re^{it}) i re^{it}}{re^{it} - z}\, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{it}) r^2}{r^2 - \overline{re^{it}} z}\, dt \qquad (**)$$

for $z \in B_r(0)$.

Fix $z \in B_r(0)$, and define $g(w) := \frac{f(w)}{r^2 - w\bar{z}}$ for w in a neighborhood of $B_r[0]$. Then g is holomorphic, and from $(**)$—with f replaced by g—we obtain

$$\begin{aligned} \frac{f(z)}{r^2 - |z|^2} &= g(z) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{g(re^{it}) r^2}{r^2 - \overline{re^{it}} z}\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{it}) r^2}{(r^2 - re^{it}\bar{z})(r^2 - \overline{re^{it}} z)}\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{it}) r^2}{|r^2 - \overline{re^{it}} z|^2}\, dt, \end{aligned}$$

so that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) \frac{r^2 (r^2 - |z|^2)}{|r^2 - \overline{re^{it}} z|^2}\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) \frac{r^2 - |z|^2}{|re^{it} - z|^2}\, dt. \qquad \square$$

Theorem 9.13 (Poisson's Integral Formula)**.** *Let $r > 0$, and let $u \colon B_r[0] \to \mathbb{R}$ be continuous such that $u|_{B_r(0)}$ is harmonic. Then*

$$u(z) = \int_0^{2\pi} u(re^{it}) P_r(re^{it}, z)\, dt$$

holds for all $z \in B_r(0)$.

Proof. Suppose first that u extends to $B_R(0)$ for some $R > r$ as a harmonic function. Then u has a harmonic conjugate v on $B_R(0)$, so that $f := u + i\,v$ is holomorphic. By Lemma 9.12, we have, for $z \in B_r(0)$, that

$$\begin{aligned} u(z) + i\,v(z) &= f(z) \\ &= \int_0^{2\pi} f(re^{it}) P_r(re^{it}, z)\, dt = \int_0^{2\pi} u(re^{it}) P_r(re^{it}, z)\, dt + i \int_0^{2\pi} v(re^{it}) P_r(re^{it}, z)\, dt, \end{aligned}$$

so that

$$u(z) = \int_0^{2\pi} u(re^{it}) P_r(re^{it}, z)\, dt.$$

Suppose now that u is arbitrary. For $\theta \in (0,1)$, define

$$u_\theta \colon B_{\frac{r}{\theta}}(0) \to \mathbb{R}, \quad z \mapsto u(\theta z).$$

Then u_θ is harmonic, and by the foregoing we have

$$u_\theta(z) = \int_0^{2\pi} u_\theta(re^{it}) P_r(re^{it}, z)\, dt$$

for $z \in B_r(0)$. Letting $\theta \to 1$ (compare a related homework problem on the Cauchy Integral Formula), we obtain for $z \in B_r(0)$ that

$$u(z) = \lim_{\theta \to 1} u_\theta(z) = \lim_{\theta \to 1} \int_0^{2\pi} u_\theta(re^{it}) P_r(re^{it}, z)\, dt = \int_0^{2\pi} u(re^{it}) P_r(re^{it}, z)\, dt. \qquad \square$$

Theorem 9.14. *Let $r > 0$, and let $f \colon \partial B_r(0) \to \mathbb{R}$ be continuous. Define*

$$g \colon B_r[0] \to \mathbb{C}, \quad z \mapsto \begin{cases} f(z), & z \in \partial B_r(0), \\ \int_0^{2\pi} f(re^{it}) P_r(re^{it}, z)\, dt, & z \in B_r(0). \end{cases}$$

Then g is continuous and harmonic on $B_r(0)$.

Proof. There is no loss of generality to suppose that $r = 1$.

For $z \in \mathbb{D}$ and $\zeta \in \partial\mathbb{D}$, note that

$$\operatorname{Re} \frac{\zeta + z}{\zeta - z} = \operatorname{Re} \frac{(\zeta + z)(\bar{\zeta} - \bar{z})}{|\zeta - z|^2} = \frac{1}{|\zeta - z|^2} \operatorname{Re}(|\zeta|^2 - |z|^2 + z\bar{\zeta} - \zeta\bar{z}) = \frac{1 - |z|^2}{|\zeta - z|^2}.$$

As the real part of a holomorphic function,

$$\mathbb{D} \to \mathbb{R}, \quad z \mapsto P_1(\zeta, z)$$

is therefore harmonic for each $\zeta \in \partial\mathbb{D}$. We thus obtain for $z \in \mathbb{D}$:

$$(\Delta g)(z) = \frac{\partial^2 g}{\partial x^2}(z) + \frac{\partial^2 g}{\partial y^2}(z) = \int_0^{2\pi} f(e^{it}) \left(\frac{\partial^2}{\partial x^2} P_1(e^{it}, z) + \frac{\partial^2}{\partial y^2} P_1(e^{it}, z) \right) dt = 0.$$

Consequently, g is harmonic on $B_r(0)$.

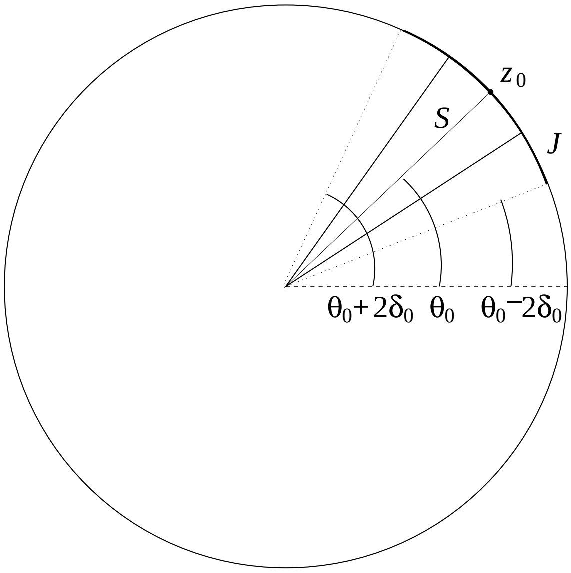
What remains to be shown is that g is continuous at any point $z_0 \in \partial\mathbb{D}$.

Let $z_0 = e^{i\theta_0}$, and supose without loss of generality that $\theta_0 \in (0, 2\pi)$. Let $\epsilon > 0$. We need to find $\delta > 0$ such that $|g(z_0) - g(z)| < \epsilon$ for all $z \in \mathbb{D}$ with $|z_0 - z| < \delta$.

For $\delta_0 > 0$, let $J := [\theta_0 - 2\delta_0, \theta_0 + 2\delta_0]$. Making $\delta_0 > 0$ sufficiently small, we can achieve that $J \subset [0, 2\pi]$ and $|f(z_0) - f(e^{i\theta})| < \frac{\epsilon}{2}$ for $\theta \in J$. Set

$$S := \{se^{i\theta} : s \in [0, 1),\, \theta \in [\theta_0 - \delta_0, \theta_0 + \delta_0]\},$$

and note that $C := \inf\{|e^{i\theta} - z| : \theta \in [0, 2\pi] \setminus J,\, z \in S\} > 0$.



Since $\int_0^{2\pi} P_1(e^{it},z)\,dt = 1$ for all $z \in \mathbb{D}$, we have

$$\begin{aligned} g(z) - g(z_0) &= \int_0^{2\pi} (f(e^{it}) - f(z_0)) P_1(e^{it},z)\,dt \\ &= \underbrace{\int_J (f(e^{it}) - f(z_0)) P_1(e^{it},z)\,dt}_{=:I_1} + \underbrace{\int_{[0,2\pi]\setminus J} (f(e^{it}) - f(z_0)) P_1(e^{it},z)\,dt}_{=:I_2}. \end{aligned}$$

Note that

$$|I_1| \le \int_J \underbrace{|f(e^{it}) - f(z_0)|}_{<\frac{\epsilon}{2}} P_1(e^{it},z)\,dt \le \frac{\epsilon}{2}\int_0^{2\pi} P_1(e^{it},z)\,dt = \frac{\epsilon}{2}.$$

Set $K := \sup_{\zeta\in\partial\mathbb{D}} |f(\zeta)|$. For $z \in S$, we then have

$$\begin{aligned} |I_2| &\le \int_{[0,2\pi]\setminus J} (|f(e^{it})| + |f(z_0)|) P_1(e^{it},z)\,dt \\ &= \frac{1}{2\pi}\int_{[0,2\pi]\setminus J} (|f(e^{it})| + |f(z_0)|) \frac{1-|z|^2}{|e^{it}-z|^2}\,dt \\ &\le \frac{K}{\pi}\int_{[0,2\pi]\setminus J} \frac{1-|z|^2}{|e^{it}-z|^2}\,dt \\ &\le \frac{K}{\pi C^2}\int_{[0,2\pi]\setminus J} 1-|z|^2\,dt, \qquad \text{because } z \in S, \\ &\le \frac{2K}{C^2}(1-|z|^2) \end{aligned}$$

Choose $\delta \in (0,\delta_0)$ so small that $|z_0 - z| < \delta$ for $z \in \mathbb{D}$ implies $z \in S$ and

$$1 - |z|^2 < \frac{C^2}{2K}\frac{\epsilon}{2}.$$

For $z \in \mathbb{D}$ with $|z_0 - z| < \delta$, we then have $|I_2| < \frac{\epsilon}{2}$ and therefore, all in all, $|g(z_0) - g(z)| < \epsilon$. $\square$

Definition 9.15. Let $D \subset \mathbb{C}$ be open, and let $f\colon D \to \mathbb{C}$ be continuous. We say that f has the *mean value property* if, for every $z_0 \in D$, there is $R > 0$ with $B_R[z_0] \subset D$ such that

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt =: \mu_r(f, z_0)$$

for all $r \in [0, R]$.

Theorem 9.16. *Let $D \subset \mathbb{C}$ be open, and let $f\colon D \to \mathbb{C}$ have the the mean value property such that $|f|$ attains a local maximum at $z_0 \in D$. Then f is constant on a neighborhood of z_0.*

Proof. Withoug loss of generality, suppose that $f(z_0) > 0$.

Choose $R > 0$ with $B_R[z_0] \subset D$ such that $f(z_0) \geq |f(z)|$ for all $z \in B_R[z_0]$ and $f(z_0) = \mu_r(f, z_0)$ for all $r \in [0, R]$. Set $g := \operatorname{Re} f - f(z_0)$. Then g has the mean value property and satisfies

$$g(z) \leq |f(z)| - f(z_0) \leq 0$$

for $z \in B_R[z_0]$. It follows that

$$0 = g(z_0) = \mu_r(g, z_0) = \int_0^{2\pi} \underbrace{g(z_0 + re^{it})}_{\leq 0}\, dt$$

for all $r \in [0, R]$. As the integrand is continuous, we conclude that $g(z_0 + re^{it}) = 0$ for all $r \in [0, R]$ and $\theta \in [0, 2\pi]$, i.e., $g \equiv 0$ on $B_R[z_0]$. This means that, for $z \in B_R[z_0]$, we have $f(z_0) = \operatorname{Re} f(z)$ as well as

$$|f(z)| \leq f(z_0) = \operatorname{Re} f(z) \leq |f(z)|,$$

so that $|f(z)| = \operatorname{Re} f(z)$ for $z \in B_R[z_0]$ and thus $f(z) = f(z_0)$. $\square$

Corollary 9.17. *Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{R}$ have the mean value property, and suppose that f has a local maximum or minimum at $z_0 \in D$. Then f is constant in a neighborhood of z_0.*

Proof. We only consider the case of a local maximum (for a local minimum, replace f by $-f$).

Let $R > 0$ be such that $B_R[z_0] \subset D$ and $f(z) \leq f(z_0)$ for all $z \in B_R[z_0]$. Choose C such that $f(z) + C \geq 0$ for all $z \in B_R[z_0]$. It follows that $|f + C|$ has a local maximum at z_0. Hence, $f + C$ is constant in a neighborhood of z_0, as is f. $\square$

Corollary 9.18. *Let $D \subset \mathbb{C}$ be open, connected, and bounded, and let $f : \overline{D} \to \mathbb{R}$ be continuous such that $f|_D$ has the mean value property. Then f attains its maximum and mimimum on ∂D.*

Proof. Without loss of generality, suppose that f is not constant. Let $z_0 \in \overline{D}$ be such that $f(z_0)$ is maximal. Set

$$V := \{z \in D : f(z) < f(z_0)\}.$$

Then V is open and not empty. Let $z \in D \setminus V$, i.e., $f(z) = f(z_0)$. Then f has a local maximum at z, so that, by Corollary 9.17, $f(w) = f(z) = f(z_0)$ for w in an open neighborhood, say $W \subset D$, of z. Consequently, $W \subset D \setminus V$ holds, so that z is an interior point of $D \setminus V$. Since $z \in D \setminus V$ is arbitrary, this shows that $D \setminus V$ is open. Since D is connected, and $V \neq \varnothing$, we must have $D \setminus V = \varnothing$, i.e., $V = D$.

The case of the minimum is treated analogously. $\square$

Corollary 9.19. *Let $D \subset \mathbb{C}$ be open, and let $f\colon D \to \mathbb{R}$ be continuous. Then the following are equivalent:*

(i) *f is harmonic;*

(ii) *f has the mean value property.*

Proof. Only (ii) $\Longrightarrow$ (i) needs proof.

Let $z_0 \in D$, and let $R > 0$ be such that $B_R[z_0] \subset D$. By Theorem 9.14, there is a continuous function $g : B_R[z_0] \to \mathbb{R}$ such that $g|_{\partial B_R[z_0]} = f|_{\partial B_R[z_0]}$ and $g|_{B_r(z_0)}$ is harmonic. Consequently, $f - g|_{B_R(z_0)}$ has the mean value property. By Corollary 9.18, this means that $f - g$ attains its maximum on $B_R[z_0]$ on $\partial B_R[z_0]$, so that $g = f|_{B_R[z_0]}$. Hence, $f|_{B_R(z_0)}$ is harmonic, i.e., $\Delta f \equiv 0$ on $B_R[z_0]$. Since $z_0 \in D$ was arbitrary, this means that $\Delta f \equiv 0$. $\square$

10 The Singularities of a Holomorphic Function

Definition 10.1. Let $D \subset \mathbb{C}$ be open, let $f\colon D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be an isolated singularity for f. Then z_0 is called a *pole* for f if $\lim_{z \to z_0} |f(z)| = \infty$.

Example. For $n \in \mathbb{N}$, the function

$$\mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad z \mapsto \frac{1}{z^n}$$

has a pole at 0.

Theorem 10.2. *Let $D \subset \mathbb{C}$ be open, let $f\colon D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be an isolated singularity for f. Then the following are equivalent:*

(i) *z_0 is a pole for f;*

(ii) *there are unique $k \in \mathbb{N}$ and holomorphic $g\colon D \cup \{z_0\} \to \mathbb{C}$ such that $g(z_0) \neq 0$ and*

$$f(z) = \frac{g(z)}{(z - z_0)^k}$$

for $z \in D$.

Proof. (ii) $\Longrightarrow$ (i) is obvious.

(i) $\Longrightarrow$ (ii): We prove the uniqueness first.

Suppose that there are $k_1, k_2 \in \mathbb{N}$ and holomorphic $g_1, g_2 : D \cup \{z_0\} \to \mathbb{C}$ such that $g_j(z_0) \neq 0$ and

$$f(z) = \frac{g_j(z)}{(z - z_0)^{k_j}}$$

for $z \in D$ and $j = 1, 2$. Assume that $k_1 \neq k_2$ and without loss of generality that $k_1 < k_2$. For all $z \in D$, we then have

$$g_1(z) = \frac{g_2(z)}{(z - z_0)^{\overbrace{k_2 - k_1}^{}}_{>0}}.$$

It follows that

$$|g_1(z_0)| = \lim_{z \to z_0} \frac{|g_2(z)|}{|z - z_0|^{k_2 - k_1}} = \infty,$$

which is impossible. Hence, $k_1 = k_2$ must hold and thus $g_1 = g_2$ on D and—by continuity—on all of $D \cup \{z_0\}$.

To establish the existence of k and g, choose $r > 0$ such that $B_r(z_0) \setminus \{z_0\} \subset D$ and $|f(z)| \geq 1$ for all $z \in B_r(z_0) \setminus \{z_0\}$. Then

$$B_r(z_0) \setminus \{z_0\} \to \mathbb{C}, \quad z \mapsto \frac{1}{f(z)}$$

is holomorphic and bounded and thus—by Riemann's Removability Criterion—, has a holomorphic extension $h\colon B_r(z_0) \to \mathbb{C}$ with $h(z_0) = \lim_{z\to z_0} \frac{1}{f(z)} = 0$. (Note that z_0 is the *only* zero of h.) Let

$$h(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for $z \in B_r(z_0)$ be the power series representation of h. Set $k := \min\{n \in \mathbb{N}_0 : a_n \neq 0\}$. Since $a_0 = h(z_0) = 0$, we have $k \geq 1$. Define

$$\tilde{h}\colon B_r(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=k}^{\infty} a_n (z - z_0)^{n-k}.$$

Then $\tilde{h}$ is holomorphic, has no zeros, and satisfies $h(z) = (z - z_0)^k \tilde{h}(z)$ for $z \in B_r(z_0)$. For $z \in B_r(z_0) \setminus \{z_0\}$, we thus have

$$f(z) = \frac{1}{\tilde{h}(z)(z - z_0)^k}.$$

Define

$$g\colon D \cup \{z_0\} \quad \mathbb{C}, \quad z \mapsto \begin{cases} (z - z_0)^k f(z), & z \neq z_0, \\ \frac{1}{\tilde{h}(z)}, & z \in B_r(z_0). \end{cases}$$

□

Definition 10.3. Let $D \subset \mathbb{C}$ be open, let $f\colon D \to \mathbb{C}$ be holomoprhic, and let $z_0 \in \mathbb{C} \setminus D$ be a pole for f. Then the positive integer k in Theorem 10.2(ii) is called the *order* of z_0 and denoted by $\operatorname{ord}(f, z_0)$. If $\operatorname{ord}(f, z_0) = 1$, we call z_0 a *simple pole* of f.

Example. For $m \in \mathbb{N}$, consider

$$f_m\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad z \mapsto \frac{\sin z}{z^m}.$$

We claim that f_1 has a removable singularity at 0 whereas f_n has a pole of order $m-1$ at 0 for $m \geq 2$.

Recall that

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

for $z \in \mathbb{C}$. For $z \neq 0$, we thus have

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}.$$

Define

$$g\colon \mathbb{C} \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}.$$

Then g is holomorphic with $g(0) \neq 0$ and clearly extends f_1. Hence, f_1 has a removable singularity at 0.

For $m \geq 2$ and $z \neq 0$, note that $f_m(z) = \frac{g(z)}{z^{m-1}}$. Hence, f_m has a pole of order $m - 1$ at 0.

Example. Consider

$$f\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}, \quad z \mapsto e^{\frac{1}{z}}.$$

Then 0 is not removable because $\lim_{n\to\infty} f\left(\frac{1}{n}\right) = \lim_{n\to\infty} e^n = \infty$. But 0 is not a pole for f either: for $n \in \mathbb{N}$, we have

$$\left| f\left(\frac{i}{n}\right)\right| = |e^{-in}| = 1.$$

Definition 10.4. Let $D \subset \mathbb{C}$ be open, let $f\colon D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C}\setminus D$ be an isolated singularity for f. Then z_0 is called *essential* if it is neither removable nor a pole.

Theorem 10.5 (Casorati–Weierstraß Theorem)**.** *Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C}\setminus D$ be an isolated singularity for f. Then the following are equivalent:*

(i) *z_0 is essential;*

(ii) *$\overline{f(B_\epsilon(z_0)\cap D)} = \mathbb{C}$ for each $\epsilon > 0$.*

Proof. (ii) $\Longrightarrow$ (i): For each $n \in \mathbb{N}$ choose $z_n \in B_{\frac{1}{n}}(z_0)\cap D$ such that $|f(z_n) - n| < \frac{1}{n}$. It follows that $\lim_{n\to\infty} |f(z_n)| = \infty$. Hence, z_0 cannot be removable.

For each $n \in \mathbb{N}$, choose $z_n' \in B_{\frac{1}{n}}(z_0)\cap D$ such that $|f(z_n')| < \frac{1}{n}$. This means that $\lim_{n\to\infty} f(z_n') = 0$, so that z_0 is not a pole either.

(i) $\Longrightarrow$ (ii): Assume otherwise, i.e., there is $\epsilon_0 > 0$ such that $\overline{f(B_{\epsilon_0}(z_0)\cap D)} \neq \mathbb{C}$. Without loss of generality, suppose that $B_{\epsilon_0}(z_0)\setminus\{z_0\} \subset D$. Let $w_0 \in \mathbb{C}$ and $\delta > 0$ be such that $B_\delta(w_0) \subset \mathbb{C}\setminus f(B_{\epsilon_0}(z_0)\setminus\{z_0\})$. Consider

$$g\colon B_{\epsilon_0}(z_0)\setminus\{z_0\} \to \mathbb{C}, \quad z \mapsto \frac{1}{f(z) - w_0}.$$

Then g is holomorphic with

$$|g(z)| = \frac{1}{|f(z) - w_0|} \leq \frac{1}{\delta}$$

for $z \in B_{\epsilon_0}(z_0)\setminus\{z_0\}$. Hence, z_0 is a removable singularity for g. Let $\tilde{g}\colon B_{\epsilon_0}(z_0) \to \mathbb{C}$ be a holomorphic extension of g.

Case 1: $g(z_0) \neq 0$. Since $f(z) = \frac{1}{g(z)} + w_0$ for $z \in B_{\epsilon_0}(z_0)\setminus\{z_0\}$, this means that z_0 is a removable singularity for f, which contradicts (i).

Case 2: $g(z_0) = 0$. For $z \neq z_0$, we have

$$|f(z)| \geq \frac{1}{|g(z)|} - |w_0| \overset{z\to z_0}{\to} \infty.$$

Hence, z_0 is a pole for f, again contradicting (i). □

11 Holomorphic Functions on Annuli

Definition 11.1. Let $z_0 \in \mathbb{C}$, and let $r, R \in [0, \infty]$ be such that $r < R$. Then the *annulus* centered at z_0 with inner radius r and outer radius R is defined as

$$A_{r,R}(z_0) := \{z \in \mathbb{C} : r < |z - z_0| < R\}.$$

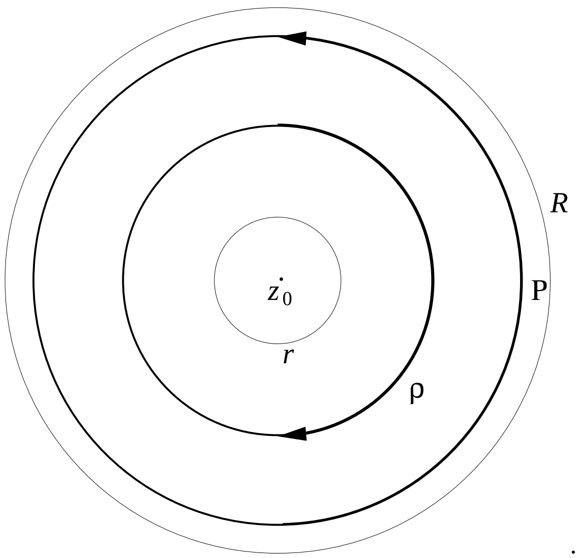
Theorem 11.2 (Cauchy's Integral Theorem for Annuli). *Let $z_0 \in \mathbb{C}$, let $r, \rho, \mathrm{P}, R \in [0, \infty]$ be such that $r < \rho < \mathrm{P} < R$, and let $f : A_{r,R}(z_0) \to \mathbb{C}$ be holomorphic. Then we have*

$$\int_{\partial B_\rho(z_0)} f(\zeta)\, d\zeta = \int_{\partial B_{\mathrm{P}}(z_0)} f(\zeta)\, d\zeta.$$

Proof. The claim is equivalent to

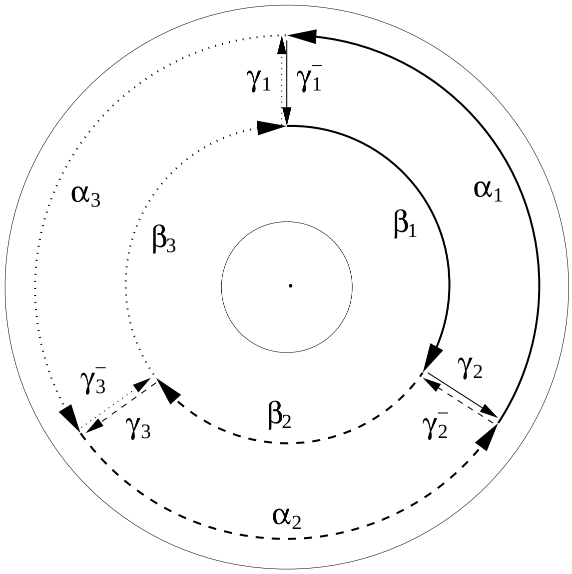
$$\int_{\partial B_{\mathrm{P}}(z_0)} f(\zeta)\, d\zeta + \int_{\partial B_\rho(z_0)^-} f(\zeta)\, d\zeta = 0.$$

Consider



.

Split $\partial B_{\mathrm{P}}(z_0)$ and $\partial B_\rho(z_0)^-$ into finitely many arc segments—say $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$—, and connect them with line segments $\gamma_1, \ldots, \gamma_n$ as shown (for $n = 3$):

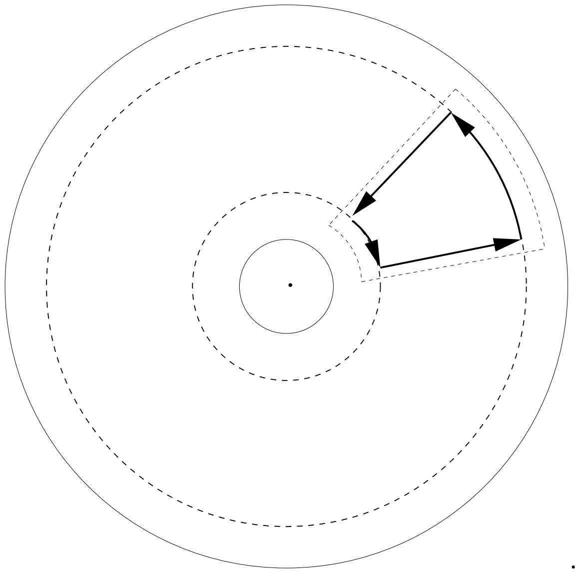


.

We thus obtain

$$\begin{aligned}\int_{\partial B_{\mathrm{P}}(z_0)} f(\zeta)\,d\zeta + \int_{\partial B_\rho(z_0)^-} f(\zeta)\,d\zeta &= \sum_{j=1}^{n}\int_{\alpha_j} f(\zeta)\,d\zeta + \sum_{j=1}^{n}\int_{\beta_j} f(\zeta)\,d\zeta \\ &= \sum_{j=1}^{n-1}\int_{\alpha_j\oplus\gamma_j^-\oplus\beta_j\oplus\gamma_{j+1}} f(\zeta)\,d\zeta + \int_{\alpha_n\oplus\gamma_n^-\oplus\beta_n\oplus\gamma_1} f(\zeta)\,d\zeta\end{aligned}$$

Making the arc segments $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ sufficiently, we can achieve that each of the closed curves $\alpha_1 \oplus \gamma_1^- \oplus \beta_1 \oplus \gamma_2, \ldots, \alpha_{n-1} \oplus \gamma_{n-1}^- \oplus \beta_{n-1} \oplus \gamma_n, \alpha_n \oplus \gamma_n^- \oplus \beta_n \oplus \gamma_1$ lies inside a star shaped open subset of $A_{r,R}(z_0)$:



It follows that

$$\int_{\partial B_{\mathrm{P}}(z_0)} f(\zeta)\,d\zeta + \int_{\partial B_\rho(z_0)^-} f(\zeta)\,d\zeta$$
$$= \sum_{j=1}^{n-1} \int_{\alpha_j \oplus \gamma_j^- \oplus \beta_j \oplus \gamma_{j+1}} f(\zeta)\,d\zeta + \int_{\alpha_n \oplus \gamma_n^- \oplus \beta \oplus \gamma_1} f(\zeta)\,d\zeta = 0$$

as claimed. □

Theorem 11.3 (Laurent Decomposition)**.** *Let $z_0 \in \mathbb{C}$, let $r, R \in [0,\infty]$ be such that $r < R$, and let $f\colon A_{r,R}(z_0) \to \mathbb{C}$ be holomorphic. Then there are holomorphic*

$$g\colon B_R(z_0) \to \mathbb{C} \qquad \textit{and} \qquad h\colon \mathbb{C} \setminus B_r[z_0] \to \mathbb{C}$$

with $f = g + h$ on $A_{r,R}(z_0)$. Moreover, h can be chosen such that $\lim_{|z|\to\infty} |h(z)| = 0$, in which case g and h are uniquely determined.

Proof. We prove the uniqueness assertion first.

Let $g_1, g_2 : B_R(z_0) \to \mathbb{C}$ and $h_1, h_2 : \mathbb{C} \setminus B_r[z_0] \to \mathbb{C}$ be holomorphic such that $\lim_{|z|\to\infty} |h_j(z)| = 0$ for $j = 1, 2$ and

$$f = g_1 + h_1 = g_2 + h_2.$$

It follows that $g_1 - g_2 = h_2 - h_1$ on $A_{r,R}(z_0)$. Define

$$F\colon \mathbb{C} \to \mathbb{C}, \quad z \mapsto \begin{cases} g_1(z) - g_2(z), & z \in B_R(z_0), \\ h_2(z) - h_1(z), & z \in \mathbb{C} \setminus B_r[z_0]. \end{cases}$$

Then F is entire with $\lim_{|z|\to\infty}|F(z)| = \lim_{|z|\to\infty}|h_2(z)-h_1(z)| = 0$. Hence, F is bounded and entire and thus bounded by Liouville's theorem. Since $\lim_{|z|\to\infty}|F(z)| = 0$, this means that $F \equiv 0$, so that $g_1 = g_2$ and $h_1 = h_2$.

To show that g and h exists, let $\rho \in (r, R)$, and define

$$g_\rho \colon B_\rho(z_0) \to \mathbb{C}, \quad z \mapsto \frac{1}{2\pi i}\int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Then g_ρ is holomorphic. Let $\rho_1, \rho_2 \in (r, R)$ be such that $\rho_1 < \rho_2$. Let $z \in B_{\rho_1}(z_0)$, and choose $r_0 \in (|z - z_0|, \rho_1)$. Since

$$A_{r_0,R}(z_0) \to \mathbb{C}, \quad w \mapsto \frac{1}{2\pi i}\frac{f(w)}{w - z}$$

is holomorphic, Cauchy's Integral Theorem for Annuli yields that

$$g_{\rho_1}(z) = \frac{1}{2\pi i}\int_{\partial B_{\rho_1}(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\int_{\partial B_{\rho_2}(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = g_{\rho_2}(z).$$

Hence, we can define $g \colon B_R(z_0) \to \mathbb{C}$ as follows: for $z \in B_R(z_0)$, choose $\rho \in (r, R)$ with $z \in B_\rho(z_0)$, and set

$$g(z) := \frac{1}{2\pi i}\int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Similarly, define $h \colon \mathbb{C} \setminus B_r[z_0] \to \mathbb{C}$ as follows: for $z \in \mathbb{C} \setminus B_r[z_0]$, choose $\sigma \in (r, R)$ such that $z \in \mathbb{C} \setminus B_\sigma[z_0]$ and set

$$h(z) = \frac{-1}{2\pi i}\int_{\partial B_\sigma(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Then h is holomorphic and satisfies

$$|h(z)| \le \sigma \sup_{\zeta \in \partial B_\sigma(z_0)} \left|\frac{f(\zeta)}{\zeta - z}\right| \le \sigma \frac{\sup_{\zeta\in\partial B_\sigma(z_0)} |f(\zeta)|}{\operatorname{dist}(z, \partial B_\sigma(z_0))} \overset{|z|\to\infty}{\longrightarrow} 0.$$

Let $z \in A_{r,R}(z_0)$ and choose σ and ρ such that $r < \sigma < |z - z_0| < \rho < R$. Define

$$G \colon A_{r,R}(z_0) \to \mathbb{C}, \quad w \mapsto \begin{cases} \frac{f(w)-f(z)}{w-z}, & w \ne z, \\ f'(z), & z = w. \end{cases}$$

Then G is holomorphic on $A_{r,R}(z_0) \setminus \{z\}$ and continuous on $A_{r,R}(z_0)$. By Riemann's Removability Criterion, this means that G is holomorphic on all of $A_{r,R}(z_0)$. It follows that

$$\int_{\partial B_\sigma(z_0)} G(\zeta)\, d\zeta = \int_{\partial B_\rho(z_0)} G(\zeta)\, d\zeta,$$

i.e.,

$$\underbrace{\int_{\partial B_\sigma(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta}_{=-2\pi i\, h(z)} - f(z) \underbrace{\int_{\partial B_\sigma(z_0)} \frac{1}{\zeta - z}\, d\zeta}_{=0} = \underbrace{\int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta}_{=2\pi i\, g(z)} + f(z) \underbrace{\int_{\partial B_\rho(z_0)} \frac{1}{\zeta - z}\, d\zeta}_{=2\pi i},$$

so that

$$-2\pi i\, h(z) = 2\pi i\, g(z) - 2\pi i\, f(z)$$

and thus

$$f(z) = g(z) + h(z). \qquad \square$$

Terminology. The function h in Theorem 11.3 is called the *principal part* and g is called the *secondary part* of the *Laurent decomposition* $f = g + h$.

Theorem 11.4. *Let $z_0 \in \mathbb{C}$, let $r, R \in [0, \infty]$ be such that $r < R$, and let $f\colon A_{r,R}(z_0) \to \mathbb{C}$ be holomorphic. Then f has a representation*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for $z \in A_{r,R}(z_0)$ as a Laurent series, *which converges uniformly and absolutely on compact subsets of $A_{r,R}(z_0)$. Moreover, for any $n \in \mathbb{Z}$ and $\rho \in (r, R)$, we have*

$$a_n = \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta.$$

Proof. Let g and h be as in Theorem 11.3 (in particular, with $\lim_{|z|\to\infty} |h(z)| = 0$).

For $z \in B_R(z_0)$, we have the Taylor series

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with uniform and absolute convergence on the compact subsets of $B_R(z_0)$.

Define

$$\tilde{h}\colon A_{0,\frac{1}{r}}(0) \to \mathbb{C}, \quad z \mapsto h\left(z_0 + \frac{1}{z}\right),$$

so that $\tilde{h}$ is holomorphic with $\lim_{z\to 0} \tilde{h}(z) = 0$. Hence, $\tilde{h}$ has a removable singularity at 0 and thus extends to $B_{\frac{1}{r}}(0)$ as a holomorphic function (which we denote likewise by $\tilde{h}$). For $z \in B_{\frac{1}{r}}(0)$, we have the Taylor series

$$\tilde{h}(z) = \sum_{n=0}^{\infty} b_n z^n = \sum_{n=1}^{\infty} b_n z^n,$$

so that

$$h(z) = \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

for $z \in \mathbb{C} \setminus B_r[z_0]$ with uniform and absolute convergence on compact subsets.

Set $a_n := b_{-n}$ for $n < 0$. For $z \in A_{r,R}(z_0)$, we obtain

$$f(z) = g(z) + h(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n} = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Finally, pick $m \in \mathbb{Z}$ and $\rho \in (r, R)$. Note that

$$\frac{f(z)}{(z-z_0)^{m+1}} = \sum_{n=-\infty}^{-1} a_{n+m+1}(z-z_0)^n + \sum_{n=0}^{\infty} a_{n+m+1}(z-z_0)^n$$

converges uniformly on $\partial B_\rho(z_0)$. Hence, we obtain:

$$\begin{aligned}
&\int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{(\zeta-z_0)^{m+1}}\, d\zeta \\
&= \sum_{n=-\infty}^{-1} a_{n+m+1} \int_{\partial B_\rho(z_0)} (\zeta-z_0)^n\, d\zeta + \sum_{n=0}^{\infty} a_{n+m+1} \int_{\partial B_\rho(z_0)} (\zeta-z_0)^n\, d\zeta \\
&= a_m \int_{\partial B_\rho(z_0)} \frac{1}{\zeta-z_0}\, d\zeta \\
&= 2\pi i\, a_m,
\end{aligned}$$

i.e.,

$$a_m = \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{1}{\zeta-z_0}\, d\zeta \qquad \square$$

Corollary 11.5. *Let $z_0 \in \mathbb{C}$, let $r > 0$, and let $f \colon B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic with Laurent representation $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$. Then the singularity z_0 of f is*

(i) *removable if and only if $a_n = 0$ for $n < 0$;*

(ii) *a pole of order $k \in \mathbb{N}$ if and only if $a_{-k} \neq 0$ and $a_n = 0$ for all $n < -k$;*

(iii) *essential if and only if $a_n \neq 0$ for infinitely many $n < 0$.*

Proof. (i) The "if" part is clear.

Conversely, suppose that z_0 is a removable singularity, and let $\tilde{f} : B_r(z_0) \to \mathbb{C}$ be a holomorphic extension of f with Taylor expansion $f(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$ for $z \in B_r(z_0)$. Uniqueness of the Laurent representation yields $a_n = b_n$ for $n \in \mathbb{N}_0$ and $a_n = 0$ for $n < 0$.

(ii) For the "if" part, set

$$g(z) := (z-z_0)^k f(z) = \sum_{n=-k}^{\infty} a_n(z-z_0)^{n+k}$$

for $z \in B_r(z_0) \setminus \{z_0\}$. Then g extends holomorphically to $B_r(z_0)$ with $g(z_0) = a_{-k} \neq 0$. By definition, we have $f(z) = \frac{g(z)}{(z-z_0)^k}$ for $z \in B_r(z_0) \setminus \{z_0\}$. Hence, f has a pole of order k at z_0.

For the converse, let $g : B_r(z_0) \to \mathbb{C}$ be holomorphic such that $g(z_0) \neq 0$ and $f(z) = \frac{g(z)}{(z-z_0)^k}$ for $z \in B_r(z_0) \setminus \{z_0\}$. Let $g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$ for $z \in B_r(z_0)$ be the Taylor series of g, so that

$$f(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^{n-k}$$

for $z \in B_r(z_0)\setminus\{z_0\}$. Uniqueness of the Laurent representation yields that $a_n = b_{n+k}$ for $n \geq -k$ and $a_n = 0$ for $n < -k$.

(iii) This follows by default from (i) and (ii).

□

Examples. 1. Let

$$f : \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad z \mapsto e^{-\frac{1}{z^2}}.$$

Then f has the Laurent representation

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z^{2n}}$$

for $z \in \mathbb{C} \setminus \{0\}$ and thus an essential singularity at 0.

2. Let

$$f : \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad z \mapsto \frac{e^z - 1}{z^3}, .$$

so that

$$f(z) = \frac{1}{z^3} \sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-3}}{n!}$$

for $z \in \mathbb{C} \setminus \{0\}$. Hence, f has a pole of order two at 0.

The Laurent representation of a holomorphic function on an annulus $A_{r,R}(z_0)$ does not only depend on z_0, but also on r and R.

Example. Consider the function

$$f : \mathbb{C} \setminus \{1, 3\} \to \mathbb{C}, \quad z \mapsto \frac{2}{z^2 - 4z + 3},$$

and note that

$$f(z) = \frac{1}{1-z} - \frac{1}{3-z}.$$

Then f has the following Laurent representations:

(a) On $A_{0,1}(0)$: For $|z| < 1$, we have

$$\frac{1}{1-z} = \sum_{n=0}^{n} z^n$$

and for $|z| < 3$ that

$$\frac{1}{3-z} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n.$$

We thus have for $z \in A_{0,1}(0)$:

$$f(z) = \sum_{n=0}^{\infty}\left(1 - \frac{1}{3^{n+1}}\right) z^n.$$

(b) On $A_{1,3}(0)$: For $|z| > 1$, we have

$$-\frac{1}{1-z} = \frac{1}{z-1} = \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}},$$

so that, for $z \in A_{1,3}(0)$:

$$f(z) = -\left(\sum_{n=1}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}\right).$$

(c) On $A_{3,\infty}(0)$: For $|z| > 3$, we have

$$-\frac{1}{3-z} = \frac{1}{z-3} = \frac{1}{z\left(1-\frac{3}{z}\right)} = \sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}}$$

and thus, for $z \in A_{3,\infty}(0)$:

$$f(z) = \sum_{n=1}^{\infty}(3^{n-1} - 1)\frac{1}{z^n}.$$

12 The Winding Number of a Curve

Definition 12.1. Let γ be a closed curve in $\mathbb{C}$, and let $z \in \mathbb{C} \setminus \{\gamma\}$. Then the *winding number* of γ with resepct to z is defined as

$$\nu(\gamma, z) := \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z}\, d\zeta.$$

Remark. Geometrically, $\nu(\gamma, z)$ is the number of times γ winds around z in counterclockwise direction.

Lemma 12.2. *Let $\gamma\colon [0,1] \to \mathbb{C}$ be curve, and let $z \in \mathbb{C} \setminus \{\gamma\}$. Then there are a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and open discs $D_1, \ldots, D_n \subset \mathbb{C} \setminus \{z\}$ such that $\gamma([t_{j-1}, t_j]) \subset D_j$ for $j = 1, \ldots, n$.*

Proof. Let $\epsilon := \operatorname{dist}(z, \{\gamma\}) > 0$. Since γ is uniformly continuous, there is $\delta > 0$ such that $|\gamma(t) - \gamma(t')| < \epsilon$ for all $t, t' \in [0,1]$ such that $|t - t'| < \delta$. Choose $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $|t_{j-1} - t_j| < \delta$ for $j = 1, \ldots, n$, and set $D_j := B_\epsilon(\gamma(t_j))$ for $j = 1, \ldots, n$. By the choice of ϵ, it is clear that $D_1, \ldots, D_n \subset \mathbb{C} \setminus \{z\}$. For $j = 1, \ldots, n$, let $t \in [t_{j-1}, t_j]$, and note that $|t - t_j| \le |t_{j-1} - t_j| < \delta$, so that $|\gamma(t) - \gamma(t_j)| < \epsilon$, i.e., $\gamma(t) \in D_j$; consequently, $\gamma([t_{j-1}, t_j]) \subset D_j$ holds. □

Proposition 12.3. *Let γ be a closed curve in $\mathbb{C}$, and let $z \in \mathbb{C} \setminus \{\gamma\}$. Then $\nu(\gamma, z) \in \mathbb{Z}$.*

Proof. Choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and open discs $D_1, \ldots, D_n$ as Lemma 12.2.

Let $j \in \{1, \ldots, n\}$. Since $z \notin D_j$, there is a holomorphic function $L_j : D_j \to \mathbb{C}$ such that

$$e^{L_j(w)} = w - z$$

for $w \in D_j$. Differentiation yields $L_j'(w) e^{L_j(w)} = 1$ and thus

$$L_j'(w) = \frac{1}{w - z}$$

for $w \in D_j$. It follows that

$$\begin{aligned}
\int_\gamma \frac{1}{\zeta - z}\, d\zeta &= \sum_{j=1}^n \int_{\gamma|_{[t_{j-1}, t_j]}} \frac{1}{\zeta - z}\, d\zeta \\
&= \sum_{j=1}^n (L_j(\gamma(t_j)) - L_j(\gamma(t_{j-1}))) \\
&= L_n(\gamma(t_n)) - L_1(\gamma(t_0)) + \sum_{j=1}^{n-1} (L_j(\gamma(t_j)) - L_{j+1}(\gamma(t_j))).
\end{aligned}$$

Since

$$\exp(L_j(\gamma(t_j))) = \exp(L_{j+1}(\gamma(t_j))) = \gamma(t_j) - z,$$

we have

$$\exp(L_j(\gamma(t_j)) - L_{j+1}(\gamma(t_j))) = 1$$

and thus

$$L_j(\gamma(t_j)) - L_{j+1}(\gamma(t_j)) \in 2\pi i\, \mathbb{Z}$$

for $j = 1, \ldots, n-1$. With a similar argument—using that $\gamma(t_n) = \gamma(t_0)$—, we see that $L_n(\gamma(t_n)) - L_1(\gamma(t_0)) \in 2\pi i\, \mathbb{Z}$, too.

All in all, $\int_\gamma \frac{1}{\zeta - z}\, d\zeta \in 2\pi i\, \mathbb{Z}$ holds. □

Definition 12.4. Let γ be a closed curve in $\mathbb{C}$. We define the *interior* and *exterior* of γ to be

$$\operatorname{int}\gamma := \{z \in \mathbb{C} \setminus \{\gamma\} : \nu(\gamma, z) \neq 0\}$$

and

$$\operatorname{ext}\gamma := \{z \in \mathbb{C} \setminus \{\gamma\} : \nu(\gamma, z) = 0\}.$$

Proposition 12.5. *Let γ be a closed curve in $\mathbb{C}$. Then:*

(i) *the map*

$$\mathbb{C} \setminus \{\gamma\} \to \mathbb{C}, \quad z \mapsto \nu(\gamma, z)$$

is locally constant;

(ii) *there is $R > 0$ such that $\mathbb{C} \setminus B_R[z_0] \subset \operatorname{ext}\gamma$.*

Proof. (i): Let $z_0 \in \mathbb{C} \setminus \{\gamma\}$. Let $R > r > 0$ be such that $B_R(z_0) \subset \mathbb{C} \setminus \{\gamma\}$.

Consider the function

$$F \colon \{\gamma\} \times B_r[z_0] \to \mathbb{C}, \quad (\zeta, z) \mapsto \frac{1}{\zeta - z}.$$

Then F is continuous and thus uniformly continuous. Choose $\delta \in (0, r)$ such that

$$|F(\zeta', z') - F(\zeta, z)| < \frac{\pi}{\ell(\gamma) + 1}$$

for all $(\zeta', z'), (\zeta, z) \in \{\gamma\} \times B_r[z_0]$ such that $\|(\zeta' z') - (\zeta, z)\| < \delta$.

Let $z \in B_\delta(z_0)$ and note that

$$|F(\zeta, z) - F(\zeta, z_0)| < \frac{\pi}{\ell(\gamma) + 1}$$

holds for all $\zeta \in \{\gamma\}$. Consequently, we have

$$\begin{aligned}
|\nu(\gamma, z) - \nu(\gamma, z_0)| &= \left| \frac{1}{2\pi i} \int_\gamma \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0} \right) d\zeta \right| \\
&\leq \frac{\ell(\gamma)}{2\pi} \sup_{\zeta \in \{\gamma\}} |F(\zeta, z) - F(\zeta, z_0)| \\
&\leq \frac{\ell(\gamma)}{2\pi} \frac{\pi}{\ell(\gamma) + 1} \\
&< \frac{1}{2}.
\end{aligned}$$

Since $\nu(\gamma, z) - \nu(\gamma, z_0) \in \mathbb{Z}$, this means that $\nu(\gamma, z) = \nu(\gamma, z_0)$.

(ii): For any $z \in \mathbb{C} \setminus \{\gamma\}$, we have

$$|\nu(\gamma, z)| = \left| \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z} \, d\zeta \right| \leq \frac{\ell(\gamma)}{2\pi} \frac{1}{\operatorname{dist}(z, \{\gamma\})}.$$

Since $\lim_{|z| \to \infty} \operatorname{dist}(z, \{\gamma\}) = \infty$, there is $R > 0$ such that $|\nu(\gamma, z)| \leq \frac{\ell(\gamma)}{2\pi} \frac{1}{\operatorname{dist}(z, \{\gamma\})} < 1$ for all $z \in \mathbb{C}$ such that $|z| > R$. Since $\nu(\gamma, z) \in \mathbb{Z}$ for all $z \in \mathbb{C} \setminus \{\gamma\}$, this implies that $\nu(\gamma, z) = 0$ for all $z \in \mathbb{C}$ with $|z| > R$. $\square$

13 The Residue Theorem and Applications

Definition 13.1. Let $z_0 \in \mathbb{C}$, let $r > 0$, and let $f \colon B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic with Laurent series representation

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for $z \in B_r(z_0) \setminus \{z_0\}$. Then a_{-1} is called the *residue* of f at z_0 and denoted by $\operatorname{res}(f, z_0)$.

Remarks. 1. By Theorem 11.4, we have

$$\operatorname{res}(f, z_0) = \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} f(\zeta)\, d\zeta$$

for any $\rho \in (0, r)$.

2. If f has a removable singularity at z_0, then $\operatorname{res}(f, z_0) = 0$.

3. Suppose that f has a simple pole at z_0, i.e.,

$$f(z) = \sum_{n=-1}^{\infty} a_n (z - z_0)^n$$

with $a_{-1} \neq 0$, then

$$\operatorname{res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

4. Suppose that f has a pole of order k at z_0, and let g be as specified in Theorem 10.2(ii). Then we have

$$\operatorname{res}(f, z_0) = \frac{g^{(k-1)}(z_0)}{(k-1)!}.$$

Examples. 1. Let

$$f(z) = \frac{e^{iz}}{z^2 + 1},$$

so that f has a simple pole at $z_0 = i$. It follows that

$$\operatorname{res}(f, i) = \lim_{z \to i} (z - i) f(z) = \lim_{z \to i} \frac{e^{iz}}{z + i} = -\frac{i}{2e}.$$

2. Let

$$f(z) = \frac{\cos(\pi z)}{\sin(\pi z)},$$

so that f has a simple pole at each $n \in \mathbb{Z}$. For $n \in \mathbb{Z}$, we thus have:

$$\begin{aligned}
\operatorname{res}(f, n) &= \lim_{z \to n} (z - n) \frac{\cos(\pi z)}{\sin(\pi z)} \\
&= \lim_{z \to n} (z - n) \frac{\cos(\pi z)}{\sin(\pi z) - \sin(\pi n)} \\
&= \frac{1}{\pi} \lim_{z \to n} \frac{\pi z - \pi n}{\sin(\pi z) - \sin(\pi n)} \cos(\pi z) \\
&= \frac{1}{\pi}.
\end{aligned}$$

3. Let
$$f(z) = \frac{1}{(z^2+1)^3};$$
then f has a pole of order 3 at $z_0 = i$. With
$$g(z) = (z-i)^3 f(z) = \frac{1}{(z+i)^3},$$
we have
$$g'(z) = -\frac{3}{(z+i)^4} \qquad \text{and} \qquad g''(z) = \frac{12}{(z+i)^5},$$
so that
$$\operatorname{res}(f,i) = \frac{1}{2}\frac{12}{(2i)^5} = -\frac{3i}{16}.$$

Theorem 13.2 (Residue Theorem)**.** *Let $D \subset \mathbb{C}$ be open and star shaped, let $z_1, \ldots, z_n \in D$ be such that $z_j \neq z_k$ for $j \neq k$, let $f\colon D \setminus \{z_1, \ldots, z_n\} \to \mathbb{C}$ be holomorphic, and let γ be a closed curve in $D \setminus \{z_1, \ldots, z_n\}$. Then we have*
$$\int_\gamma f(\zeta)\,d\zeta = 2\pi i \sum_{j=1}^{n} \nu(\gamma, z_j) \operatorname{res}(f, z_j).$$

Proof. Let $\epsilon > 0$ be such that $B_\epsilon(z_j) \subset D$ for $j = 1, \ldots, n$. For $j = 1, \ldots, n$, we have Laurent representations
$$f(z) = \sum_{k=-\infty}^{\infty} a_k^{(j)} (z - z_j)^k$$
for $z \in B_\epsilon(z_j) \setminus \{z_j\}$, so that $\operatorname{res}(f, z_j) = a_{-1}^{(j)}$. For $j = 1, \ldots, n$, define
$$h_j\colon \mathbb{C} \setminus \{z_j\} \to \mathbb{C}, \quad z \mapsto \sum_{k=-\infty}^{-1} a_k^{(j)} (z - z_j)^k,$$
so that h_j is holomorphic on $\mathbb{C} \setminus \{z_j\}$. Define
$$g\colon D \setminus \{z_1, \ldots, z_n\} \to \mathbb{C}, \quad z \mapsto f(z) - \sum_{j=1}^{n} h_j(z),$$
and note that $z_1, \ldots, z_n$ are removable singularities for g.

Since D is star shaped, Cauchy's Integral Theorem yields:

$$\begin{aligned}
0 &= \int_\gamma g(\gamma)\, d\zeta \\
&= \int_\gamma f(\zeta)\, d\zeta - \sum_{j=1}^{n} \int_\gamma h_j(\zeta)\, d\zeta \\
&= \int_\gamma f(\zeta)\, d\zeta - \sum_{j=1}^{n} \int_\gamma \left(\sum_{k=-\infty}^{-1} a_k^{(j)} (\zeta - z_j)^k \right) d\zeta \\
&= \int_\gamma f(\zeta)\, d\zeta - \sum_{j=1}^{n} \sum_{k=-\infty}^{-1} a_k^{(j)} \int_\gamma (\zeta - z_j)^k\, d\zeta \\
&= \int_\gamma f(\zeta)\, d\zeta - \sum_{j=1}^{n} a_{-1}^{(j)} \int_\gamma \frac{1}{\zeta - z_j}\, d\zeta \\
&= \int_\gamma f(\zeta)\, d\zeta - \sum_{j=1}^{n} a_{-1}^{(j)}\, 2\pi i\, \nu(\gamma, z_j).
\end{aligned}$$

□

Corollary 13.3. *Let $D \subset \mathbb{C}$ be open, and star shaped, let $f\colon D \to \mathbb{C}$ be holomorphic, and let γ be a closed curve in D. Then we have*

$$\nu(\gamma, z)\, f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for $z \in D \setminus \{\gamma\}$.

Proof. Fix $z \in D \setminus \{\gamma\}$, and define

$$g\colon D \setminus \{z\} \to \mathbb{C}, \quad w \mapsto \frac{f(w)}{w - z}.$$

Then g is holomorphic with an isolated singularity at z. Let

$$f(w) = \sum_{n=0}^{\infty} a_n (w - z)^n$$

be the Taylor series expansion of f near z, so that

$$g(w) = \sum_{n=-1}^{\infty} a_{n+1} (w - z)^n,$$

and thus $\operatorname{res}(g, z) = a_0 = f(z)$. The Residue Theorem then yields:

$$2\pi i\, \nu(\gamma, z)\, f(z) = 2\pi i\, \nu(\gamma, z) \operatorname{res}(g, z) = \int_\gamma g(\zeta)\, d\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

□

Applications of the Residue Theorem to Real Integrals

Proposition 13.4. *Let p and q be polynomials of two real variables such that $q(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2$ with $x^2+y^2=1$. Then we have*

$$\int_0^{2\pi} \frac{p(\cos t, \sin t)}{q(\cos t, \sin t)}\, dt = 2\pi i \sum_{z\in\mathbb{D}} \operatorname{res}(f,z),$$

where

$$f(z) = \frac{1}{iz} \frac{p\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)}{q\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)}.$$

Proof. Just note that, by the Residue Theorem,

$$\begin{aligned} 2\pi i \sum_{z\in\mathbb{D}} \operatorname{res}(f,z) &= \int_{\partial\mathbb{D}} f(\zeta)\, d\zeta \\ &= \int_0^{2\pi} f(e^{it}) i\, e^{it}\, dt \\ &= \int_0^{2\pi} \frac{p(\cos t, \sin t)}{q(\cos t, \sin t)}\, dt. \end{aligned}$$

□

Examples. 1. Let $a > 1$. What is $\int_0^\pi \frac{dt}{a+\cos t}$?

First, note that

$$\int_0^\pi \frac{dt}{a+\cos t} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{dt}{a+\cos t} = \frac{1}{2}\int_0^{2\pi} \frac{dt}{a+\cos t}.$$

Let

$$p(x,y) = 1 \qquad \text{and} \qquad q(x,y) = a + x,$$

so that

$$\begin{aligned} f(z) &= \frac{1}{iz} \frac{1}{a + \frac{1}{2}\left(z+\frac{1}{z}\right)} \\ &= \frac{1}{iz} \frac{1}{a + \frac{1}{2}\left(\frac{z^2+1}{z}\right)} \\ &= \frac{1}{i} \frac{1}{az + \frac{z^2}{2} + \frac{1}{2}} \\ &= \frac{2}{i} \frac{1}{z^2 + 2az + 1} \\ &= \frac{2}{i} \frac{1}{(z-z_1)(z-z_2)}, \end{aligned}$$

where

$$z_1 = -a + \sqrt{a^2-1} \in \mathbb{D} \qquad \text{and} \qquad z_2 = -a - \sqrt{a^2-1} \notin \mathbb{D}.$$

By Proposition 13.4, we thus obtain:

$$\begin{aligned}
\int_0^{\pi} \frac{dt}{a+\cos t} &= \frac{1}{2}\int_0^{2\pi} \frac{dt}{a+\cos t} \\
&= \pi i \operatorname{res}(f, z_1) \\
&= \pi i \frac{2}{i} \frac{1}{(z_1 - z_2)} \\
&= 2\pi \frac{1}{2\sqrt{a^2-1}} \\
&= \frac{\pi}{\sqrt{a^2-1}}.
\end{aligned}$$

2. Let $a > b > 0$. What is $\int_0^{2\pi} \frac{dt}{(a+b\cos t)^2}$?

Let

$$p(x,y) = 1 \qquad \text{and} \qquad q(x,y) = (a+bx)^2,$$

so that

$$\begin{aligned}
f(z) &= \frac{1}{iz} \frac{1}{\left(a + \frac{b}{2}\left(z + \frac{1}{z}\right)\right)^2} \\
&= \frac{1}{iz} \frac{1}{\left(a + \frac{bz^2+b}{2z}\right)^2} \\
&= \frac{1}{i} \frac{z}{\left(az + \frac{bz^2}{2} + \frac{b}{2}\right)^2} \\
&= \frac{z}{i} \frac{1}{\frac{b^2}{4}(z-z_1)^2(z-z_2)^2},
\end{aligned}$$

where

$$z_1 = -\frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1} \in \mathbb{D} \qquad \text{and} \qquad z_2 = -\frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1} \notin \mathbb{D}.$$

At z_1, the function f has a pole of order two. In order to calculate $\operatorname{res}(f, z_1)$, set

$$g(z) := (z-z_1)^2 f(z) = \frac{4z}{b^2 i} \frac{1}{(z-z_2)^2},$$

so that

$$\begin{aligned}
g'(z) &= \frac{4}{b^2 i}\left(\frac{1}{(z-z_2)^2} - \frac{2z}{(z-z_2)^3}\right) \\
&= \frac{4}{b^2 i} \frac{1}{(z-z_2)^2}\left(1 - \frac{2z}{z-z_2}\right) \\
&= \frac{4}{b^2 i} \frac{1}{(z-z_2)^3}((z-z_2) - 2z) \\
&= -\frac{4}{b^2 i} \frac{z+z_2}{(z-z_2)^3};
\end{aligned}$$

it follows that

$$\begin{aligned}\operatorname{res}(f, z_1) &= g'(z_1)\\ &= -\frac{4}{b^2 i}\frac{-2\frac{a}{b}}{8\left(\sqrt{\frac{a^2}{b^2}-1}\right)^3}\\ &= \frac{a}{b^3 i}\frac{1}{\left(\sqrt{\frac{a^2}{b^2}-1}\right)^3}\\ &= \frac{1}{i}\frac{a}{\left(\sqrt{a^2-b^2}\right)^3}.\end{aligned}$$

From Proposition 13.4, we conclude that

$$\int_0^{2\pi} \frac{dt}{(a+b\cos t)^2} = 2\pi i \operatorname{res}(f, z_1) = \frac{2\pi a}{\left(\sqrt{a^2-b^2}\right)^3}.$$

Proposition 13.5. *Let p and q be polynomials of one real variable with* $\deg q \geq \deg p + 2$ *and such that $q(x) \neq 0$ for $x \in \mathbb{R}$. Then we have*

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx = 2\pi i \sum_{z\in\mathbb{H}} \operatorname{res}\left(\frac{p}{q}, z\right),$$

where

$$\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

Proof. Since $\deg q \geq \deg p + 2$, the comparison test yields that the indefinite integral exists.

Let $r > 0$, and define

$$\gamma_r \colon [0, \pi] \to \mathbb{C}, \quad t \mapsto re^{it}.$$

Let $\epsilon > 0$ be such that, for $D := \{z \in \mathbb{C} : \operatorname{Im} z > -\epsilon\}$, we have

$$\{z \in \mathbb{H} : q(z) = 0\} = \{z \in D : q(z) = 0\}.$$

Then D is star shaped, and $\frac{p}{q}$ is holomorphic on D except at the zeros of q in $\mathbb{H}$. For large enough r—such that all zeros of q in D lie in the interior of $[-r, r] \oplus \gamma_r$—, we have by the Residue Theorem:

$$\int_{[-r,r]\oplus\gamma_r} \frac{p(\zeta)}{q(\zeta)}\,d\zeta = 2\pi i \sum_{z\in D} \operatorname{res}\left(\frac{p}{q}, z\right) = 2\pi i \sum_{z\in\mathbb{H}} \operatorname{res}\left(\frac{p}{q}, z\right).$$

There are $R > 0$ and $C \geq 0$ such that

$$\left|\frac{p(z)}{q(z)}\right| \leq \frac{C}{|z|^2}$$

for all $z \in \mathbb{C}$ with $|z| \geq R$. It follows that

$$\left| \int_{\gamma_r} \frac{p(\zeta)}{q(\zeta)} \, d\zeta \right| \leq \pi r \sup_{\zeta \in \{\gamma_r\}} \frac{C}{|\zeta|^2} \leq \frac{\pi C}{r}$$

for $r \geq R$ and thus

$$\lim_{r \to \infty} \int_{\gamma_r} \frac{p(\zeta)}{q(\zeta)} \, d\zeta = 0.$$

All in all, we obtain:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \, dx &= \lim_{r \to \infty} \int_{[-r,r]} \frac{p(\zeta)}{q(\zeta)} \, d\zeta \\
&= \lim_{r \to \infty} \int_{[-r,r]} \frac{p(\zeta)}{q(\zeta)} \, d\zeta + \lim_{r \to \infty} \int_{\gamma_r} \frac{p(\zeta)}{q(\zeta)} \, d\zeta \\
&= \lim_{r \to \infty} \int_{[-r,r] \oplus \gamma_r} \frac{p(\zeta)}{q(\zeta)} \, d\zeta \\
&= 2\pi i \sum_{z \in \mathbb{H}} \operatorname{res}\left(\frac{p}{q}, z \right).
\end{aligned}$$

□

Examples. 1. What is $\int_0^\infty \frac{1}{1+x^6} \, dx$?

The zeros of $q(z) = 1 + z^6$ are of the form $e^{i\theta}$ where $\theta \in [0, 2\pi)$ is such that $e^{i6\theta} = -1 = e^{i\pi}$, i.e., $6\theta - \pi \in 2\pi\,\mathbb{Z}$, so that $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$. For $k = 1, \ldots, 6$, let

$$z_k = e^{i(2k-1)\frac{\pi}{6}}.$$

Then $\frac{1}{q}$ has a simple pole at z_k for $k = 1, \ldots, 6$.

By a homework problem, we have

$$\operatorname{res}\left(\frac{1}{q}, z_k \right) = \frac{1}{6z_k^5} = -\frac{z_k}{6},$$

so that by, Proposition 13.5,

$$\begin{aligned}
\int_0^\infty \frac{1}{1+x^6} \, dx &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1+x^6} \, dx \\
&= \pi i \sum_{k=1}^{3} \operatorname{res}\left(\frac{1}{q}, z_k \right) \\
&= -\frac{\pi i}{6} \left(e^{i\frac{\pi}{6}} + e^{i\frac{\pi}{2}} + e^{i\frac{5\pi}{6}} \right) \\
&= -\frac{\pi i}{6} \left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6} + i + \cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6} \right) \\
&= \frac{\pi}{6} \left(2 \sin\frac{\pi}{6} + 1 \right) \\
&= \frac{\pi}{3}.
\end{aligned}$$

2. What is $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n}\,dx$, where $n \in \mathbb{N}$?

   The polynomial $q(z) := (z^2+1)^n$ has zeros of order n at $\pm i$. Define

$$g(z) = (z-i)^n \frac{1}{q(z)} = (z+i)^{-n},$$

   so that

$$g^{(n-1)}(z) = (-n)\cdots(-2n+2)(z+i)^{-2n+1}$$

   and thus

$$\begin{aligned}\operatorname{res}\left(\frac{1}{q}, i\right) &= \frac{g^{(n-1)}(i)}{(n-1)!} \\ &= \frac{1}{(n-1)!}\frac{1}{2^{2n-1}}\frac{1}{i}n\cdots(2n-2) \\ &= \frac{1}{i}\frac{1}{2^{2n-1}}\frac{(2n-2)!}{((n-1)!)^2}.\end{aligned}$$

   It follows that

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n}\,dx = 2\pi i \operatorname{res}\left(\frac{1}{q}, i\right) = \frac{\pi}{2^{2n-2}}\frac{(2n-2)!}{((n-1)!)^2};$$

   in particular, we have

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \pi,$$
$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx = \frac{\pi}{2}, \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3}\,dx = \frac{3\pi}{8}.$$

14 Function Theoretic Consequences of the Residue Theorem

Definition 14.1. Let $D \subset \mathbb{C}$ be open. We call $S \subset D$ *discrete* in D if it has no cluster points in D.

Example. If D is connected, and $f\colon D \to \mathbb{C}$ is holomorphic, then $\mathbf{Z}(f)$ is discrete if $f \not\equiv 0$.

Remark. Let $S \subset D$ be discrete, and let $K \subset D$ be compact. Assume that $K \cap S$ is infinite. Then $K \cap S$ has cluster points, which must lie in K in thus in D. Hence, $K \cap S$ must be finite. In the homework, it was shown that there are compact $K_1 \subset K_2 \subset K_3 \subset \cdots \subset D$ with $D = \bigcup_{n=1}^{\infty} K_n$. It follows that

$$S = \bigcup_{n=1}^{\infty} K_n \cap S$$

is countable.

Proposition 14.2. *Let $D \subset \mathbb{C}$ be open, let γ be a closed curve in D, and let $S \subset D$ be discrete. Then $S \cap \operatorname{int}\gamma$ is finite.*

Proof. By Proposition 12.5(ii), there is $R > 0$ such that $\operatorname{int}\gamma \subset B_R[0]$. □

Definition 14.3. Let $D \subset \mathbb{C}$ be open. A *meromorphic function* on D is a holomorphic function $f\colon D \setminus \mathbf{P}(f) \to \mathbb{C}$, where $\mathbf{P}(f)$ is a discrete subset of D such that each point in $\mathbf{P}(f)$ is a pole of f.

Remark. If D is connected, and $f, g\colon D \to \mathbb{C}$ are holomorphic, then $\frac{f}{g}$ is meromorphic if $g \neq 0$.

Proposition 14.4. *Let $D \subset \mathbb{C}$ be open, and let f be meromorphic on D. Then, for each $z_0 \in D$, there are $\epsilon > 0$ with $B_\epsilon(z_0) \subset D$ as well as holomorphic $g, h\colon B_\epsilon(z_0) \to \mathbb{C}$ such that $f(z) = \frac{g(z)}{h(z)}$ for $z \in B_\epsilon(z_0) \setminus \{z_0\}$.*

Proof. If z_0 is not a pole of f, the claim is clear.

Otherwise, choose $\epsilon > 0$ so small that $B_\epsilon(z_0) \subset D$ and $\mathbf{P}(f) \cap B_\epsilon(z_0) = \{z_0\}$. We can then find holomorphic $g\colon B_\epsilon(z_0) \to \mathbb{C}$ with $g(z_0) \neq 0$ and $k \in \mathbb{N}$ such that

$$f(z) = \frac{g(z)}{(z - z_0)^k}$$

for $z \in B_\epsilon(z_0) \setminus \{z_0\}$. Setting $h(z) := (z - z_0)^n$ yields the claim. □

Lemma 14.5. *Let $D \subset \mathbb{C}$ be open and connected, and let $S \subset D$ be discrete. Then $D \setminus S$ is open and connected.*

Proof. Let $z \in D \setminus S$. Since S is discrete in D, there is $\epsilon_1 > 0$ such that $B_{\epsilon_1}(z) \cap S = \varnothing$. Also, since D is open, there is $\epsilon_2 > 0$ with $B_{\epsilon_2}(z) \subset D$. Setting $\epsilon := \min\{\epsilon_1, \epsilon_2\}$, we get $B_\epsilon(z)$. This proves the openness of $D \setminus S$.

Assume that $D \setminus S$ is not connected. Then there are open sets $U \neq \varnothing \neq V$ with $U \cap V = \varnothing$ and $U \cup V = D \setminus S$. Let $s \in S$, and choose $\epsilon > 0$ such that $B_\epsilon(s) \subset D$ and $B_\epsilon(s) \cap S = \{s\}$. Set $W := B_\epsilon(s) \setminus \{s\}$, and note that W is open and connected. Since $(U \cap W) \cap (V \cap W) = \varnothing$ and $(U \cap W) \cup (V \cup W) = W$, the connectedness of W yields that either $U \cap W = \varnothing$ or $V \cap W = \varnothing$ and thus $W \subset U$ or $W \subset V$.

Set

$$S_U := \{s \in S : \text{there is } \epsilon > 0 \text{ such that } B_\epsilon(s) \setminus \{s\} \subset U\}$$

and

$$S_V := \{s \in S : \text{there is } \epsilon > 0 \text{ such that } B_\epsilon(s) \setminus \{s\} \subset V\}.$$

By the foregoing, we have $S = S_U \cup S_V$, and trivially, $S_U \cap S_V = \varnothing$ holds. Set

$$\tilde{U} := U \cup S_U \qquad \text{and} \qquad \tilde{V} := V \cup S_V.$$

Then $\tilde{U} \neq \varnothing \neq \tilde{V}$ are easily seen to be open and clearly satisfy $\tilde{U} \cap \tilde{V} = \varnothing$ and $\tilde{U} \cup \tilde{V} = D$, which contradicts the connectedness of D. □

Theorem 14.6. *Let $D \subset \mathbb{C}$ be open and connected. Then the meromorphic functions on D—with pointwise addition and multiplication—form a field.*

Proof. It is routinely checked that the meromorphic functions do indeed form a commutative ring with identity.

Let f be a non-zero meromorphic function on D, and let $\mathbf{P}(f)$ and $\mathbf{Z}(f)$ denote the poles and zeros, respectively, of f in D. As $\mathbf{P}(f)$ is discrete, $D \setminus \mathbf{P}(f)$ is connected by Lemma 14.5. From the Identity Theorem, we conclude that $\mathbf{Z}(f)$ is discrete, too.

Define

$$\tilde{f} \colon D \setminus \mathbf{Z}(f) \to \mathbb{C}, \quad z \mapsto \begin{cases} \frac{1}{f(z)}, & z \notin \mathbf{P}(f), \\ 0, & z \in \mathbf{P}(f). \end{cases}$$

Then $\tilde{f}$ is holomorphic, and we have $f(z)\tilde{f}(z) = 1$ for $z \in D \setminus (\mathbf{P}(f) \cup \mathbf{Z}(f))$. □

Theorem 14.7 (Argument Principle)**.** *Let $D \subset \mathbb{C}$ be open and star shaped, let f be meromorphic on D, and let γ be a closed curve in $D \setminus (\mathbf{P}(f) \cup \mathbf{Z}(f))$. Then we have*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \sum_{z \in \mathbf{Z}(f)} \nu(\gamma, z) \operatorname{ord}(f, z) - \sum_{z \in \mathbf{P}(f)} \nu(\gamma, z) \operatorname{ord}(f, z).$$

(The order of a zero of a holomorphic function was defined on the homework.)

Proof. By the Residue Theorem, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\,d\zeta = \sum_{z\in\mathbf{Z}(f)} \nu(\gamma,z)\,\mathrm{res}\left(\frac{f'}{f},z\right) + \sum_{z\in\mathbf{P}(f)} \nu(\gamma,z)\,\mathrm{res}\left(\frac{f'}{f},z\right).$$

Let $z_0 \in \mathbf{Z}(f)$, and let $k := \mathrm{ord}(f,z)$. Then there is a holomorphic function g with $g(z_0) \neq 0$ such that $f(z) = (z-z_0)^k g(z_0)$ and thus

$$f'(z) = k\,(z-z_0)^{k-1}g(z) + (z-z_0)^k g'(z).$$

It follows that

$$\frac{f'(z)}{f(z)} = \frac{k}{z-z_0} + \frac{g'(z)}{g(z)}$$

for z near z_0, so that

$$\mathrm{res}\left(\frac{f'}{f},z\right) = k.$$

Let $z_0 \in \mathbf{P}(f)$, and let $k := \mathrm{ord}(f,z_0)$. Then $f(z) = \frac{g(z)}{(z-z_0)^k}$ holds with g holomorphic such that $g(z_0)$ and, consequently,

$$f'(z) = -k(z-z_0)^{-(k+1)}g(z) + (z-z_0)^{-k}g'(z).$$

It follows that

$$\frac{f'(z)}{f(z)} = \frac{-k}{z-z_0} + \frac{g'(z)}{g(z)}$$

for $z \neq z_0$ near z_0, so that

$$\mathrm{res}\left(\frac{f'}{f},z\right) = -k. \qquad \square$$

Definition 14.8. Let $D \subset \mathbb{C}$ be open, and let $f\colon D \to \mathbb{C}$ be holomorphic. We say that f *attains $w_0 \in \mathbb{C}$ with multiplicity $k \in \mathbb{N}$ at $z_0 \in D$* if the function

$$D \to \mathbb{C}, \quad z \mapsto f(z) - w_0$$

has a zero of order k at z_0.

Theorem 14.9. *Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and suppose that, at $z_0 \in D$, the function f attains w_0 with multiplicity $k \in \mathbb{N}$. Then there are open neighborhoods $V \subset D$ of z_0 and $W \subset f(V)$ of w_0 such that, for each $w \in W \setminus \{w_0\}$, there are pairwise distinct $z_1, \ldots, z_k \in V$ with $f(z_1) = \cdots = f(z_k) = w$, where f attains w at each z_j with multiplicity one.*

Proof. Without loss of generality, suppose that D is star shaped.

Choose $\epsilon > 0$ with $B_\epsilon[z_0] \subset D$ such that $f(z) \neq w_0$ and $f'(z) \neq 0$ for all $z \in B_\epsilon[z_0] \setminus \{z_0\}$.

Set $V := B_\epsilon(z_0)$ and $\gamma := \partial B_\epsilon(z_0)$. Choose $\delta > 0$ such that $B_\delta(w_0) \subset \mathbb{C} \setminus \{f \circ \gamma\}$, and set $W := B_\delta(w_0)$. Let $w \in W$. By the Argument Principle, the number of times w is attained in V (with multiplicity) is

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(\zeta)}{f(\zeta) - w}\, d\zeta = \frac{1}{2\pi i} \int_{f\circ\gamma} \frac{d\zeta}{\zeta - w} = \nu(f \circ \gamma, w).$$

As $\nu(f \circ \gamma, \cdot)$ is constant on W, the number of times w is attained in V is the same as the number of times w_0 is attained in V, i.e., k. Suppose that $w \in W \setminus \{w_0\}$. Since $f'(z) \neq 0$ for all $z \in V \setminus \{z_0\}$, there are pairwise distinct $z_1, \ldots, z_k \in V \setminus \{z_0\}$ such that $f(z_1) = \cdots = f(z_k) = w$; necessarily, f attains w at each z_j with multiplicity one. $\square$

Theorem 14.10 (Hurwitz' Theorem)**.** *Let $D \subset \mathbb{C}$ be open and connected, let $f, f_1, f_2, \ldots \colon D \to \mathbb{C}$ be holomorphic such that $(f_n)_{n=1}^\infty$ converges to f compactly on D, and suppose that $\mathbf{Z}(f_n) = \varnothing$ for $n \in \mathbb{N}$. Then $f \equiv 0$ or $\mathbf{Z}(f) = \varnothing$.*

Proof. Suppose that $f \not\equiv 0$, but that there is $z_0 \in \mathbf{Z}(f)$. Choose $\epsilon > 0$ such that $B_\epsilon[z_0] \subset D$ and $f(z) \neq 0$ for all $z \in B_\epsilon[z_0] \setminus \{z_0\}$, and note that

$$0 = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f_n'(\zeta)}{f_n(\zeta)}\, d\zeta = \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \operatorname{ord}(f, z_0),$$

which is a contradiction. $\square$

Corollary 14.11. *Let $D \subset \mathbb{C}$ be open and connected, let $f, f_1, f_2, \ldots \colon D \to \mathbb{C}$ be holomorphic such that $(f_n)_{n=1}^\infty$ converges to f compactly on D, and suppose that f_n is injective for $n \in \mathbb{N}$. Then f is constant or injective.*

Proof. Suppose that f is not constant. Let $z_0 \in D$ be arbitrary, and define

$$g_n \colon D \setminus \{z_0\} \to \mathbb{C}, \quad z \mapsto f_n(z) - f_n(z_0)$$

for $n \in \mathbb{N}$. Then $g_1, g_2, \ldots$ have no zeros. Since f is not constant, the function

$$D \setminus \{z_0\} \to \mathbb{C}, \quad z \mapsto f(z) - f(z_0)$$

is not zero, so that that is has no zeros by Hurwitz's theorem, i.e., $f(z) \neq f(z_0)$ for all $z \in D$, $z \neq z_0$. $\square$

Theorem 14.12 (Rouché's Theorem)**.** *Let $D \subset \mathbb{C}$ be open and star shaped, and let $f, g \colon D \to \mathbb{C}$ be holomorphic. Suppose that γ is a closed curve in D such that* $\operatorname{int}\gamma = \{z \in D \setminus \{\gamma\} : \nu(\gamma, z) = 1\}$ *and that*

$$|f(\zeta) - g(\zeta)| < |f(\zeta)|$$

for $\zeta \in \{\gamma\}$. Then f and g have the same number of zeros in $\operatorname{int}\gamma$ *(counted with multiplicity).*

Proof. For $t \in [0,1]$, define $h_t := f + t(g - f)$, so that $h_0 = f$ and $h_1 = g$. Also, since

$$|t(g - f)| \leq |g - f| < |f|$$

for any $t \in [0,1]$ on $\{\gamma\}$, the functions h_t have no zeros on $\{\gamma\}$. For $t \in [0,1]$, let $n_t \in \mathbb{N}_0$ denote the number of zeros of h_t in $\operatorname{int}\gamma$. From the Argument Principle, we obtain that

$$n_t = \frac{1}{2\pi i} \int_\gamma \frac{h_t'(\zeta)}{h_t(\zeta)}\, d\zeta = \frac{1}{2\pi i} \int_\gamma \frac{f'(\zeta) + t(g'(\zeta) - f'(\zeta))}{f(\zeta) - t(g(\zeta) - f(\zeta))}\, d\zeta.$$

Since the integral on the right hand side depends continuously on t, we conclude that $n_0 = n_1$. □

Example. How many zeros does $z^4 - 4z + 2$ have in $\mathbb{D}$?

Set

$$g(z) := z^4 - 4z + 2 \qquad \text{and} \qquad f(z) = -4z + 2.$$

For $\zeta \in \partial\mathbb{D}$, we have $|f(\zeta)| \geq 4 - 2 = 2$, so that

$$|f(\zeta) - g(\zeta)| = |\zeta^4| = 1 < 2 \leq |f(\zeta)|.$$

Since f has precisely one zero in $\mathbb{D}$, so has g.

Corollary 14.13. *Let p be a polynomial with $n := \deg p \geq 1$. Then p has n zeros (counted with multiplicity).*

Proof. Let

$$p(z) = a_n z^n + \cdots + a_1 z + a_0$$

with $a_n \neq 0$, and let $g(z) := a_n z^n$, so that $\lim_{|z| \to \infty} \left| \frac{p(z) - g(z)}{g(z)} \right| = 0$. Choose $R > 0$ such that

$$\left| \frac{p(z) - g(z)}{g(z)} \right| < 1$$

for $z \in \mathbb{C}$ with $|z| \geq R$. Consequently, if $\zeta \in \partial B_R(0)$, we have $|p(\zeta) - g(\zeta)| < |g(\zeta)|$. By Rouche's Theorem, p thus has as many zeros in $B_R(0)$ as g, namely n. Since p has at most n zeros, these are all the zeros of p. □

15 The General Cauchy Integral Theorem

Definition 15.1. Let $D \subset \mathbb{C}$ be open. We call a closed curve γ in D *homologous to zero* if $\nu(\gamma, z) = 0$ for each $z \in \mathbb{C} \setminus D$.

Theorem 15.2 (Cauchy's Integral Theorem)**.** *Let $D \subset \mathbb{C}$ be open, let $f\colon D \to \mathbb{C}$ be holomorphic, and let γ be a closed curve in D that is homologous to zero. Then $\int_\gamma f(\zeta)\, d\zeta = 0$ holds.*

Theorem 15.3 (Cauchy's Integral Formula)**.** *Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let γ be a closed curve in D that is homologous to zero. Then, for $n \in \mathbb{N}_0$ and $z \in D \setminus \{\gamma\}$, we have*

$$\nu(\gamma, z) f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

Proof. It is enough to prove the claim for $n = 0$: for general $n \in \mathbb{N}_0$, use induction and differentiation under the integral.

Define

$$g\colon D \times D \to \mathbb{C}, \quad (w, z) \mapsto \begin{cases} \frac{f(w)-f(z)}{w-z}, & w \neq z, \\ f'(z), & w = z. \end{cases}$$

We claim that g is continuous. To see this, let $(w_0, z_0) \in D \times D$. As g is clearly continuous at (w_0, z_0) if $w_0 \neq z_0$, we suppose without loss of generality that $w_0 = z_0$. Let $\delta > 0$ be such that $B_\delta[z_0] \subset D$. For $(w, z) \in B_\delta(z_0) \times B_\delta(z_0)$, we then have:

- *if* $w = z$:

$$g(w, z) - g(z_0, z_0) = f'(z) - f'(z_0);$$

- *if* $w \neq z$:

$$g(w, z) - g(z_0, z_0) = \frac{f(w) - f(z)}{w - z} - f'(z_0) = \frac{1}{w - z} \int_{[z,w]} (f'(\zeta) - f'(z_0))\, d\zeta.$$

Let $\epsilon > 0$, and choose $\delta > 0$ so small that $|f'(z) - f'(z_0)| < \epsilon$ for all $z \in B_\delta[z_0]$. Let $z, w \in B_\delta(z_0)$. If $w = z$, we have

$$|g(w, z) - g(z_0, z_0)| = |f'(z) - f'(z_0)| < \epsilon$$

and, if $w \neq z$,

$$|g(w, z) - g(z_0, z_0)| \leq \frac{1}{|w - z|} |z - w| \sup_{\zeta \in \{[z,w]\}} |f'(\zeta) - f'(z_0)| \leq \epsilon.$$

All in all, g is continous at (z_0, w_0).

Next, define

$$h_0\colon D \to \mathbb{C}, \quad z \mapsto \int_\gamma g(\zeta, z)\, d\zeta.$$

We claim that h_0 is holomorphic. It is easy to see that h_0 is continuous. To see that it is indeed holomorphic, we shall show that it satisfies the Morera condition. Let $\Delta \subset D$ be a triangle. For fixed $\zeta \in \{\gamma\}$, the function

$$D \to \mathbb{C}, \quad z \mapsto g(\zeta, z)$$

is holomorphic as a consequence of Riemann's Removability Condition. Goursat's Lemma thus yields that

$$\int_{\partial\Delta} g(\zeta, z)\, dz = 0$$

for each $\zeta \in \{\gamma\}$. As a consequence, we obtain that

$$\begin{aligned} 0 &= \int_\gamma \left(\int_{\partial\Delta} g(\zeta, z)\, dz \right) d\zeta \\ &= \int_{\partial\Delta} \left(\int_\gamma g(\zeta, z)\, d\zeta \right) dz \\ &= \int_{\partial D} h_0(z)\, dz, \end{aligned}$$

so that h_0 is holomorphic as claimed.

Define

$$h_1\colon \operatorname{ext}\gamma \to \mathbb{C}, \quad z \mapsto \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Then h_1 is holomorphic. For $z \in D \cap \operatorname{ext}\gamma$, we have

$$\begin{aligned} h_0(z) &= \int_\gamma g(\zeta, z)\, d\zeta \\ &= \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta \\ &= \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta + f(z) \underbrace{\int_\gamma \frac{1}{\zeta - z}\, d\zeta}_{=0} \\ &= \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta. \end{aligned}$$

Define

$$h\colon D \cup \operatorname{ext}\gamma, \quad z \mapsto \begin{cases} h_0(z), & z \in D, \\ h_1(z), & z \in \operatorname{ext}\gamma. \end{cases}$$

Then h is holomorphic. Since γ is homologous to zero, we have $\mathbb{C} \setminus D \subset \operatorname{ext}\gamma$. Hence, h is entire.

For any $z \in \operatorname{ext}\gamma$, we have the estimate

$$|h(z)| = |h_1(z)| \leq \frac{\ell(\gamma)}{\operatorname{dist}(z, \{\gamma\})} \sup_{\zeta \in \{\gamma\}} |f(\zeta)|. \tag{$*$}$$

Let $R > 0$ be such that $\mathbb{C} \setminus B_R(0) \subset \operatorname{ext}\gamma$. Then $(*)$ yields that h is bounded on $\mathbb{C} \setminus B_R(0)$; by continuity, h is trivially bounded on $B_R[0]$. All in all, h is bounded and thus constant by Liouville's Theorem. From $(*)$ again, we see that $\lim_{|z|\to\infty} |h(z)| = 0$. Hence, $h \equiv 0$ holds.

All in all, we have for $z \in D \setminus \{\gamma\}$ that

$$0 = h(z) = h_0(z) = \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta$$ □

Proof of Cauchy's Integral Theorem. Let $z_0 \in D \setminus \{\gamma\}$ be arbitrary, and define

$$g \colon D \to \mathbb{C}, \quad z \mapsto (z - z_0) f(z),$$

so that

$$0 = \nu(\gamma, z_0) g(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{g(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_\gamma f(\zeta)\, d\zeta$$ □

Definition 15.4. Let $D \subset \mathbb{C}$ be open. We call D *simply connected* if D is connected, and every closed curve in D is homologous to zero.

Corollary 15.5. *The following are equivalent for an open, connected set $D \subset \mathbb{C}$:*

(i) *D is simply connected;*

(ii) *every holomorphic function on D has an anti-derivative.*

Corollary 15.6 (Existence of Holomorphic Logarithms)**.** *Let $D \subset \mathbb{C}$ be simply connected, and let $f : D \to \mathbb{C}$ be holomorphic such that $\mathbf{Z}(f) = \varnothing$. Then there is a holomorphic $g \colon D \to \mathbb{C}$ with $f = \exp \circ g$.*

Proof. This was proven in the homework for star shaped D; the argument carries over verbatim. □

Corollary 15.7 (Existence of Holomorphic Roots)**.** *Let $D \subset \mathbb{C}$ be simply connected, and let $f : D \to \mathbb{C}$ be holomorphic such that $\mathbf{Z}(f) = \varnothing$. Then, for each $n \in \mathbb{N}$, there is a holomorphic $h_n \colon D \to \mathbb{C}$ such that $h_n(z)^n = f(z)$ for $z \in D$.*

Proof. Let g be as in the previous corollary, and set $h_n := \exp \circ \left(\frac{g}{n}\right)$ for $n \in \mathbb{N}$. □

16 Montel's Theorem

Definition 16.1. Let $S \subset \mathbb{R}^N$. A family $\mathcal{F}$ of functions on S into $\mathbb{R}^M$ is called *equicontinuous* if, for each $\epsilon > 0$, there is $\delta > 0$ such that $\|f(x) - f(y)\| < \epsilon$ for all $f \in \mathcal{F}$ and for all $x, y \in S$ such that $\|x - y\| < \delta$.

Lemma 16.2. *Let $S \subset \mathbb{R}^N$. Then S contains a countable dense subset.*

Proof. Let $\{x_1, x_2, x_3, \ldots\}$ be a dense, countable subset of $\mathbb{R}^N$, e.g., $\mathbb{Q}^N$. For $n, m \in \mathbb{N}$ with $S \cap B_{\frac{1}{m}}(x_n) \neq \varnothing$, choose $y_{n,m} \in S \cap B_{\frac{1}{m}}(x_n)$. Then

$$S_0 := \left\{ y_{n,m} : S \cap B_{\frac{1}{m}}(x_n) \neq \varnothing \right\} \subset S$$

is countable.

Let $\epsilon > 0$ and $x \in S$. Choose $m \in \mathbb{N}$ so large that $\frac{1}{m} < \frac{\epsilon}{2}$. Since $\{x_1, x_2, x_3, \ldots\}$ is dense in $\mathbb{R}^N$, there is $n \in \mathbb{N}$ such that $\|x_n - x\| < \frac{1}{m}$ and thus $x \in S \cap B_{\frac{1}{m}}(x_n) \neq \varnothing$. It follows that

$$\|y_{n,m} - x\| \leq \|y_{n,m} - x_n\| + \|x_n - x\| < \frac{2}{m} < \epsilon$$ □

Theorem 16.3 (Arzelà–Ascoli Theorem)**.** *Let $K \subset \mathbb{R}^N$ be compact, and let $\mathcal{F}$ be a bounded, equicontinuous family of functions from K to $\mathbb{R}^M$. Then every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on K.*

Proof. Let $(f_n)_{n=1}^\infty$ be a sequence in $\mathcal{F}$, and let $\{x_1, x_2, x_3, \ldots\}$ be a countable dense subset of K.

Since $(f_n(x_1)_{n=1}^\infty$ is a bounded sequence in $\mathbb{R}^M$, there is a subsequence $(f_{n,1})_{n=1}^\infty$ of $(f_n)_{n=1}^\infty$ such that $(f_{n,1}(x_1))_{n=1}^\infty$ converges.

Since $(f_{n,1}(x_2))_{n=1}^\infty$ is a bounded sequence in $\mathbb{R}^M$, there is a subsequence $(f_{n,2})_{n=1}^\infty$ of $(f_{n,1})_{n=1}^\infty$ such that $(f_{n,2}(x_2))_{n=1}^\infty$ converges.

Continuing inductively in this fashion, we obtain, for each $k \in \mathbb{N}$, a subsequence $(f_{n,k})_{n=1}^\infty$ of $(f_n)_{n=1}^\infty$ such that, for each $k \in \mathbb{N}$,

- $(f_{n,k+1})_{n=1}^\infty$ is a subsequence of $(f_{n,k})_{n=1}^\infty$, and
- $(f_{n,k}(x_k))_{n=1}^\infty$ converges.

For $n \in \mathbb{N}$, set $g_n := f_{n,n}$. Then $(g_n)_{n=1}^\infty$ is a subsequence of $(f_n)_{n=1}^\infty$, and $(g_n(x_k))_{n=1}^\infty$ converges for each $k \in \mathbb{N}$.

We claim that $(g_n)_{n=1}^\infty$ is a uniform Cauchy sequence on K (and thus convergent).

Let $\epsilon > 0$. Choose $\delta > 0$ such that $\|f(x) - f(y)\| < \frac{\epsilon}{3}$ for all $f \in \mathcal{F}$ and for all $x, y \in K$ with $\|x - y\| < \delta$. Since K is compact, there are $y_1, \ldots, y_\nu \in K$ such that $K \subset \bigcup_{j=1}^{\nu} B_{\frac{\delta}{2}}(y_j)$. Since $\{x_1, x_2, x_3, \ldots\}$ is dense in K, there are $k_1, \ldots, k_\nu \in \mathbb{N}$ such that $x_{k_j} \in B_{\frac{\delta}{2}}(y_j)$. It follows that $K \subset \bigcup_{j=1}^{\nu} B_\delta(x_{k_j})$.

By definition, $(g_n(x_k))_{n=1}^\infty$ is a Cauchy sequence for each $k \in \mathbb{N}$. Choose $n_\epsilon \in \mathbb{N}$ such that

$$\|g_n(x_{k_j}) - g_m(x_{k_j})\| < \frac{\epsilon}{3}$$

for $n, m \geq n_\epsilon$ and $j = 1, \ldots, \nu$. Let $x \in K$ be arbitrary, and let $n, m \geq n_\epsilon$. Choose $j \in \{1, \ldots, \nu\}$ such that $x \in B_\delta(x_{k_j})$, and note that

$$\|g_n(x) - g_m(x)\| \leq \underbrace{\|g_n(x) - g_n(x_{k_j})\|}_{<\frac{\epsilon}{3}} + \underbrace{\|g_n(x_{k_j}) - g_m(x_{k_j})\|}_{<\frac{\epsilon}{3}} + \underbrace{\|g_m(x_{k_j}) - g_m(x)\|}_{<\frac{\epsilon}{3}} < \epsilon.$$

Hence, $(g_n)_{n=1}^\infty$ is a uniform Cauchy sequence as claimed. $\square$

Definition 16.4. Let $D \subset \mathbb{R}^N$ be open. We call a family $\mathcal{F}$ of functions from D into $\mathbb{R}^M$

(a) *locally bounded* if, for each $x_0 \in D$, there is $\epsilon > 0$ with $B_\epsilon(x_0) \subset D$ such that $\{f|_{B_\epsilon(x_0)} : f \in \mathcal{F}\}$ is bounded and

(b) *locally equicontinuous* if, for each $x_0 \in D$, there is $\epsilon > 0$ with $B_\epsilon(x_0) \subset D$ such that $\{f|_{B_\epsilon(x_0)} : f \in \mathcal{F}\}$ is equicontinuous.

For any non-empty set $S \subset \mathbb{R}$, we define its *diameter* as

$$\operatorname{diam}(S) := \sup\{\|x - y\| : x, y \in S\}.$$

Proposition 16.5 (Lebesgue's Covering "Lemma")**.** *Let $K \subset \mathbb{R}^N$ be compact, and let $\mathcal{U}$ be an open cover for K. Then there is a number $\ell(\mathcal{U}) > 0$—the* Lebesgue number *of $\mathcal{U}$—such that, for each $\varnothing \neq S \subset K$ with* $\operatorname{diam}(S) < \ell(\mathcal{U})$*, there is $U \in \mathcal{U}$ such that $S \subset U$.*

Proof. Assume towards a contradiction that the assertion is false. Then, for each $n \in \mathbb{N}$, there is $\varnothing \neq S_n \subset K$ such that $\operatorname{diam}(S_n) < \frac{1}{n}$ and $S_n \not\subset U$ for each $U \in \mathcal{U}$. For $n \in \mathbb{N}$, choose $x_n \in S_n$. Since K is compact, $(x_n)_{n=1}^\infty$ has a convergent subsequence, say $(x_{n_k})_{k=1}^\infty$, with limit $x_0 \in K$. Let $U_0 \in \mathcal{U}$ be such that $x_0 \in U_0$. Let $\epsilon > 0$ be such that $B_\epsilon(x_0) \subset U_0$ (this is possible because U_0 is open). Let $k_0 \in \mathbb{N}$ be so large that $\frac{1}{n_{k_0}} < \frac{\epsilon}{2}$ and $\|x_{n_{k_0}} - x_0\| < \frac{\epsilon}{2}$. Let $x \in S_{n_{k_0}}$ be arbitrary, and note that

$$\|x - x_0\| \leq \underbrace{\|x - x_{n_{k_0}}\|}_{\leq \operatorname{diam}(S_{n_{k_0}})} + \underbrace{\|x_{n_{k_0}} - x_0\|}_{<\frac{\epsilon}{2}} < \epsilon.$$

It follows that $S_{n_{k_0}} \subset B_\epsilon(x_0) \subset U_0$, which contradicts the choices of $S_1, S_2, \ldots$. $\square$

Lemma 16.6. *Let $D \subset \mathbb{R}^N$ be open, and let $\mathcal{F}$ be a family of functions from D into $\mathbb{R}^M$. For $K \subset D$ compact, set*

$$\mathcal{F}|_K := \{f|_K : f \in \mathcal{F}\}.$$

Then we have:

(i) *if* $\mathcal{F}$ *is locally bounded, then* $\mathcal{F}|_K$ *is bounded;*

(ii) *if* $\mathcal{F}$ *is locally equicontinuous, then* $\mathcal{F}|_K$ *is equicontinuous.*

Proof. (i): For each $x \in K$, there is $\epsilon_x > 0$ such that $B_{\epsilon_x}(x) \subset D$ and $\{f|_{B_{\epsilon_x}(x)} : f \in \mathcal{F}\}$ is bounded, i.e., there is $C_x \geq 0$ such that

$$\sup\{\|f(y)\| : f \in \mathcal{F},\, y \in B_{\epsilon_x}(x)\} \leq C_x.$$

Since $\{B_{\epsilon_x}(x) : x \in K\}$ is an open cover for K, and since K is compact, there are $x_1, \ldots, x_n \in K$ such that

$$K \subset \bigcup_{j=1}^{n} B_{\epsilon_{x_j}}(x_j).$$

Set $C := \max\{C_{x_1}, \ldots, C_{x_n}\}$. Then

$$\sup\{\|f(y)\| : f \in \mathcal{F},\, y \in K\} \leq C$$

holds, i.e., $\mathcal{F}|_K$ is bounded.

(ii): For each $x \in K$, there is $\epsilon_x > 0$ with $B_{\epsilon_x}(x) \subset D$ such that $\{f|_{B_{\epsilon_x}(x)} : f \in \mathcal{F}\}$ is equicontinuous.

Let $\epsilon > 0$ be arbitrary. Then, for each $x \in K$, there is $\delta_x > 0$ such that $\|f(y) - f(z)\| < \epsilon$ for all $f \in \mathcal{F}$ and all $y, z \in B_{\epsilon_x}(x)$ with $\|y - z\| < \delta_x$. Since K is compact, there are $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup_{j=1}^{n} B_{\epsilon_{x_j}}(x_j)$. Let $\ell > 0$ be the Lebesgue number of the open cover $\left\{B_{\epsilon_{x_j}}(x_j) : j = 1, \ldots, n\right\}$ of K, and choose $\delta := \min\{\delta_{x_1}, \ldots, \delta_{x_n}, \ell\}$. Let $f \in \mathcal{F}$, and let $x, y \in K$ be such that $\|x - y\| < \delta$, so that $\operatorname{diam}(\{x, y\}) < \delta \leq \ell$. By the definition of ℓ, there is $j \in \{1, \ldots, n\}$ such that $\{x, y\} \subset B_{\epsilon_{x_j}}(x_j)$. Since $\|x - y\| < \delta_{x_j} \leq \delta$, this means that $\|f(x) - f(y)\| < \epsilon$. Hence, $\mathcal{F}|_K$ is indeed equicontinuous. □

Proposition 16.7. *Let* $D \subset \mathbb{R}^N$ *be open, and let* $\mathcal{F}$ *be a locally bounded and locally equicontinuous family of functions from* D *to* $\mathbb{R}^M$*. Then every sequence in* $\mathcal{F}$ *has a compactly convergent subsequence.*

Proof. Let $(f_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{F}$.

Define a sequence $K_1, K_2, \ldots$ of compact sets such that

- $\bigcup_{k=1}^{\infty} K_k = D$ and
- $K_k \subset \operatorname{int} K_{k+1}$ for $n \in \mathbb{N}$.

(As in the $N = 2$ case, this can be accomplished, for instance, by letting $K_k := B_k[0]$ if $D = \mathbb{C}$ and $K_k := \left\{x \in D : \operatorname{dist}(x, \partial D) \geq \frac{1}{k},\, \|x\| \leq k\right\}$ if $D \neq \mathbb{C}$.)

By Lemma 16.6, $\mathcal{F}|_{K_1}$ is bounded and equicontinuous. By the Arzelà–Ascoli Theorem, there is thus a subsequence $(f_{n,1})_{n=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ and a function $g_1 \colon K_1 \to \mathbb{R}^M$ such that $f_{n,1}|_{K_1} \to g_1$ uniformly on K_1.

Invoking Lemma 16.6 and the Arzelà–Ascoli Theorem again, we obtain a subsequence $(f_{n,2})_{n=1}^\infty$ of $(f_{n,1})_{n=1}^\infty$ and a function $g_2 \colon K_2 \to \mathbb{R}^M$ such that $f_{n,2}|_{K_2} \to g_2$ uniformly on K_2.

Inductively, we thus obtain, for each $k \in \mathbb{N}$, a subsequence $(f_{n,k})_{n=1}^\infty$ of $(f_n)_{n=1}^\infty$ and a function $g_k \colon K_k \to \mathbb{R}^M$ such that, for each $k \in \mathbb{N}$,

- $(f_{n,k+1})_{n=1}^\infty$ is a subsequence of $(f_{n,k})_{n=1}^\infty$, and
- $f_{n,k}|_{K_k} \to g_k$ uniformly on K_k.

Define $g : D \to \mathbb{R}^M$ as follows: for $x \in D$, let $k \in \mathbb{N}$ be such that $x \in K_k$, set $g(x) := g_k(x)$. It is easy to see that g well defined. From the definition of g, it is obvious that $f_{n,n}|_{K_k} \to g|_{K_k}$ uniformly on K_k.

Let $K \subset D$ be compact. By the choices of $K_1, K_2, \ldots$, we have $K \subset D \subset \bigcup_{k=1}^\infty \operatorname{int} K_k$, so that $\{\operatorname{int} K_k : k \in \mathbb{N}\}$ is an open cover for K. Since K is compact, and since $\operatorname{int} K_k \subset \operatorname{int} K_{k+1}$ for $k \in \mathbb{N}$, there is $k_0 \in \mathbb{N}$ such that $K \subset \operatorname{int} K_{k_0} \subset K_{k_0}$. Since $f_{n,n}|_{K_{k_0}} \to g|_{K_{k_0}}$ uniformly on K_{k_0}, it follows that $f_{n,n}|_K \to g|_K$ uniformly on K. $\square$

Theorem 16.8 (Montel's Theorem)**.** *Let $D \subset \mathbb{C}$ be open, and let $\mathcal{F}$ be a locally bounded family of holomorphic functions on D. Then every sequence in $\mathcal{F}$ has a subsequence that converges compactly to a holomorphic function on D.*

Proof. In view of Proposition 16.7, we only need to show that $\mathcal{F}$ is locally equicontinuous.

Let $z_0 \in D$, and let $r > 0$ be such that $B_{2r}[z_0] \subset D$. By Lemma 16.6(ii), there is $C > 0$ such that $|f(\zeta)| \le C$ for all $f \in \mathcal{F}$ and for all $\zeta \in \partial B_{2r}(z_0)$.

Let $f \in \mathcal{F}$, and let $z, w \in B_r(z_0)$. Then we have:

$$\begin{aligned}
|f(z) - f(w)| &= \left| \frac{1}{2\pi i} \int_{\partial B_{2r}(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{\partial B_{2r}(z_0)} \frac{f(\zeta)}{\zeta - w}\, d\zeta \right| \\
&= \frac{1}{2\pi} \left| \int_{\partial B_{2r}(z_0)} \left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w} \right) d\zeta \right| \\
&= \frac{1}{2\pi} \left| \int_{\partial B_{2r}(z_0)} \frac{f(\zeta)(w - z)}{(\zeta - z)(\zeta - w)}\, d\zeta \right| \\
&= \frac{|z - w|}{2\pi} \left| \int_{\partial B_{2r}(z_0)} \frac{f(\zeta)}{(\zeta - z)(\zeta - w)}\, d\zeta \right| \\
&\le \frac{|z - w|}{2\pi} 4\pi r \frac{C}{r^2} \\
&= \frac{2C}{r} |z - w|.
\end{aligned}$$

For $\epsilon > 0$, thus choose $\delta := \frac{r\epsilon}{2C}$, so that $|f(z) - f(w)| < \epsilon$ for all $z, w \in B_r(z_0)$ with $|z - w| < \delta$. $\square$

17 The Riemann Mapping Theorem

Definition 17.1. Let $D_1, D_2 \subset \mathbb{C}$ be open and connected. We say that D_1 and D_2 are *biholomorphically equivalent* if there is a biholomorphic map from D_1 onto D_2.

Examples. 1. Let $z_1, z_2 \in \mathbb{C}$, and let $r_1, r_2 > 0$. Then $B_{r_1}(z_1)$ and $B_{r_2}(z_2)$ are biholomorphically equivalent because

$$B_{r_1}(z_1) \to B_{r_2}(z_2), \quad z \mapsto \frac{r_2}{r_1}(z - z_1) + z_2$$

is biholomorphic.

2. Consider the *Cayley transform*

$$f \colon \mathbb{H} \to \mathbb{C}, \quad z \mapsto \frac{z-i}{z+i}.$$

Let $x, y \in \mathbb{R}$ with $y > 0$, and let $z = x + iy$. Then

$$\begin{aligned} |z-i|^2 &= |x + i(y-1)|^2 \\ &= x^2 + y^2 - 2y + 1 \\ &< x^2 + y^2 + 2y + 1 \\ &= |x + i(y+1)|^2 \\ &= |z+i|^2 \end{aligned}$$

holds, so that $|f(z)| < 1$. Consequently, we have $f(\mathbb{H}) \subset \mathbb{D}$. Consider

$$g \colon \mathbb{D} \to \mathbb{C}, \quad z \mapsto i\frac{1+z}{1-z},$$

and note that

$$\begin{aligned} g(f(z)) &= i\frac{1 + \frac{z-i}{z+i}}{1 - \frac{z-i}{z+i}} \\ &= i\frac{z+i+z-i}{z+i-z+i} \\ &= i\frac{2z}{2i} \\ &= z \end{aligned}$$

for $z \in \mathbb{H}$. Hence, f is injective. Let $x^2 + y^2 < 1$, and note that

$$\begin{aligned} g(x+iy) &= i\frac{(1+x)+iy}{(1-x)-iy} \\ &= i\frac{((1+x)+iy)((1-x)+iy)}{(1-x)^2+y^2} \\ &= -\frac{2y}{(1-x)^2+y^2} + i\underbrace{\frac{1-(x^2+y^2)}{(1-x)^2+y^2}}_{>0} \in \mathbb{H}. \end{aligned}$$

For $z \in \mathbb{D}$, we can thus evaluate

$$\begin{aligned} f(g(z)) &= \frac{i\frac{1+z}{1-z} - i}{i\frac{1+z}{1-z} + i} \\ &= \frac{1+z-1+z}{1+z+1-z} \\ &= \frac{2z}{2} \\ &= z. \end{aligned}$$

Hence, f is also surjective and thus bijective with inverse g. Since f and g are obviously holomorphic, this means that $\mathbb{D}$ and $\mathbb{H}$ are biholomorphically equivalent.

3. There is no biholomorphic map $f : \mathbb{C} \to \mathbb{D}$ because any holomorphic map from $\mathbb{C}$ to $\mathbb{D}$ is bounded and thus constant by Liouville's theorem. Hence, $\mathbb{C}$ and $\mathbb{D}$ are not biholomorphically equivalent.

Lemma 17.2. *Let $D_1, D_2 \subset \mathbb{C}$ be open, let $f : D_1 \to D_2$ be holomorphic, let $g : D_2 \to \mathbb{C}$ be continuous, and let γ be a curve in D_1. Then we have*

$$\int_{f\circ\gamma} g(\zeta)\, d\zeta = \int_\gamma g(f(\zeta))f'(\zeta)\, d\zeta.$$

Proof. Let $a = t_0 < t_1 < \cdots < t_n = b$ be such that $\gamma|_{[t_{j-1},t_j]}$ is continuously differentiable for $j = 1, \ldots, n$, and note that

$$\int_{f\circ\gamma} g(\zeta)\, d\zeta = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} g(f(\gamma(t))f'(\gamma(t))\gamma'(t)\, dt = \int_\gamma g(f(\zeta))f'(\zeta)\, d\zeta. \qquad \square$$

Proposition 17.3. *Let $D_1, D_2 \subset \mathbb{C}$ be open and connected such that D_1 is simply connected, and suppose that D_1 and D_2 are biholomorphically equivalent. Then D_2 is simply connected.*

Proof. Let $f : D_1 \to D_2$ be biholomorphic. Let $g : D_2 \to \mathbb{C}$ be holomorphic, and let γ be a closed curve in D_2. We shall see that $\int_\gamma g(\zeta)\, d\zeta = 0$. Note that $f^{-1} \circ \gamma$ is a closed curve in D_1, so that by Lemma 17.2 and the simple connectedness of D_1,

$$\int_\gamma g(\zeta)\, d\zeta = \int_{f\circ f^{-1}\circ\gamma} g(\zeta)\, d\zeta = \int_{f^{-1}\circ\gamma} g(f(\zeta))f'(\zeta)\, d\zeta = 0 \qquad \square$$

Example. Let $r, R \in [0, \infty]$ be such that $r < R$. Then $A_{r,R}(0)$ is not biholomorphically equivalent to $\mathbb{D}$ or $\mathbb{C}$.

Biholomorphic maps have a very interesting geometric property.

Given and open set $D \subset \mathbb{R}^N$ and curves $\gamma_1, \gamma_2 : [0, 1] \to D$, suppose there are $t_1, t_2 \in (0, 1)$ such that $\gamma_1(t_1) = \gamma_2(t_2) = x_0$. In order to define the *angle between γ_1 and γ_2 at x_0*, we further suppose that there is $\epsilon > 0$ such that γ_j is differentiable on $(t_j - \epsilon, t_j + \epsilon)$ for $j = 1, 2$ with $\gamma_j'(t_j) \neq 0$. The angle is then defined to be the unique $\theta \in [0, \pi]$ such that

$$\cos\theta = \frac{\gamma_1'(t_1) \cdot \gamma_2'(t_2)}{\|\gamma_1'(t_1)\|\|\gamma_2'(t_2)\|}.$$

Given two open sets $D_1, D_2 \subset \mathbb{R}^N$, a differentiable map $f : D_1 \to D_2$ is called *angle preserving at $x_0 \in D_1$* if, for any two curves γ_1 and γ_2 in D_1, the angle between $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(x_0)$ is the same between γ_1 and γ_2 at x_0.

Recall that a real $N \times N$ matrix A is called *orthogonal* if it is invertible with $A^{-1} = A^t$.

Lemma 17.4. *Let $D_1, D_2 \subset \mathbb{R}^N$ be open, let $x_0 \in D_1$, and let $f : D_1 \to D_2$ be differentiable such that $J_f(x_0)$ is orthogonal. Then f is angle preserving at x_0.*

Proof. Let γ_1 and γ_2 be two curves in D_1 satisfying the necessary requirements, and note that

$$\begin{aligned}
&\text{cosine of the angle between } f \circ \gamma_1 \text{ and } f \circ \gamma_2 \text{ at } f(x_0) \\
&= \frac{(f \circ \gamma_1)'(t_1) \cdot (f \circ \gamma_2)'(t_2)}{\|(f \circ \gamma_1)'(t_1)\|\|(f \circ \gamma_2)'(t_2)\|} \\
&= \frac{J_f(x_0)\gamma_1'(t_1) \cdot J_f(x_0)\gamma_2'(t_2)}{\|J_f(x_0)\gamma_1'(t_1)\|\|J_f(x_0)\gamma_2'(t_2)\|}, \qquad \text{by the chain rule,} \\
&= \frac{J_f(x_0)^t J_f(x_0)\gamma_1'(t_1) \cdot \gamma_2'(t_2)}{\|J_f(x_0)\gamma_1'(t_1)\|\|J_f(x_0)\gamma_2'(t_2)\|} \\
&= \frac{\gamma_1'(t_1) \cdot \gamma_2'(t_2)}{\|\gamma_1'(t_1)\|\|\gamma_2'(t_2)\|} \\
&= \text{cosine of the angle between } \gamma_1 \text{ and } \gamma_2 \text{ at } x_0.
\end{aligned}$$

□

Example. Let z be a complex number. Then multiplication by z is a $\mathbb{R}$-linear map from $\mathbb{C} = \mathbb{R}^2$ into itself and thus uniquely represented by a real 2×2 matrix A of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

where $a = \operatorname{Re} z$ and $b = \operatorname{Im} z$. It follows that A^t is the matrix representing $\bar{z}$. Hence, A is orthogonal if and only if $|z| = 1$.

Proposition 17.5. *Let $D_1, D_2 \subset \mathbb{C}$ be open, and let $f : D_1 \to D_2$ be holomorphic. Then f is angle preserving at $z_0 \in D_1$ whenever $f'(z_0) \neq 0$.*

Proof. Let $z_0 \in D_1$ be such that $f'(z_0) \neq 0$. In view of Lemma 17.4 and the example following it, the claim is clear if $|f'(z_0)| = 1$.

For the general case, let

$$\frac{1}{|f'(z_0)|} D_2 := \left\{ \frac{z}{|f'(z_0)|} : z \in D_2 \right\},$$

and define

$$g\colon D_1 \to \frac{1}{|f'(z_0)|}D_2, \quad z \mapsto \frac{f(z)}{|f'(z_0)|}$$

and

$$h\colon \frac{1}{|f'(z_0)|}D_2 \to D_2, \quad z \mapsto |f'(z_0)|z.$$

Then g is angle preserving at z_0 because $|g'(z_0)| = 1$, and it is easily seen that h is angle preserving at $g(z_0)$. Consequently, $f = h \circ g$ is angle preserving at z_0. □

Corollary 17.6. *Let $D_1, D_2 \subset \mathbb{C}$ be open and connected, and let $f\colon D_1 \to D_2$ be biholomorphic. Then f is angle preserving at every point of D_1.*

Proposition 17.7. *Let $D_1, D_2 \subset \mathbb{C}$ be open and connected, and let $f : D_1 \to D_2$ be holomorphic and bijective. Then f is biholomorphic.*

Proof. We first show that f^{-1} is continuous.

Let $w_0 \in D_2$, and let $\epsilon > 0$. Without loss of generality suppose that $B_\epsilon(f^{-1}(w_0)) \subset D_1$. By the Open Mapping Theorem, $f(B_\epsilon(f^{-1}(w_0)))$ is open. Hence, there is $\delta > 0$ such that $B_\delta(w_0) \subset f(B_\epsilon(f^{-1}(w_0)))$. Hence, if $w \in D_2$ is such that $|w - w_0| < \delta$, i.e., $w \in B_\delta(w_0)$, we have $f^{-1}(w) \in B_\epsilon(f^{-1}(w_0))$, i.e., $|f^{-1}(w) - f^{-1}(w_0)| < \epsilon$. Hence, f^{-1} is continuous at w_0.

Since f is not constant, we have $f' \neq 0$, so that $\mathbf{Z}(f')$ is discrete. We claim that $f(\mathbf{Z}(f'))$ is also discrete. Assume that $f(\mathbf{Z}(f'))$ is not discrete. Then there are $w_0 \in D_2$ and a sequence $(z_n)_{n=1}^\infty$ in $\mathbf{Z}(f')$ such that $w_0 \neq f(z_n)$ for $n \in \mathbb{N}$, but $w_0 = \lim_{n\to\infty} f(z_n)$. By the bijectivity and continuity of f^{-1}, we have $f^{-1}(w_0) \neq z_n$ for $n \in \mathbb{N}$ and $f^{-1}(w_0) = \lim_{n\to\infty} z_n$. Hence, $f^{-1}(w_0)$ is a cluster point of $\mathbf{Z}(f')$, which is impossible.

Let $z_0 \in D_1$ be such that $f'(z_0) \neq 0$, and let $w_0 := f(z_0)$. For $w \in D_2 \setminus \{w_0\}$, we then have

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{f^{-1}(w) - f^{-1}(w_0)}{f(f^{-1}(w)) - f(f^{-1}(w_0))} \overset{w\to w_0}{\longrightarrow} \frac{1}{f'(z_0)}.$$

Hence, f^{-1} is holomorphic on $D_2 \setminus f(\mathbf{Z}(f'))$. Since f^{-1} is continuous and $f(\mathbf{Z}(f'))$ is discrete, Riemann's Removability Criterion yields the holomorphy of f^{-1} on all of D_2. □

Corollary 17.8. *Let $D \subset \mathbb{C}$ be open and connected, and let $f : D \to \mathbb{C}$ be holomorphic and injective. Then $f'(z) \neq 0$ for all $z \in D$.*

Proof. If f is injective, it is not constant. By the Open Mapping Theorem, $f(D)$ is therefore open and connected, and by the previous proposition, $f\colon D \to f(D)$ is biholomorphic, so that

$$1 = (f^{-1} \circ f)'(z) = (f^{-1})'(f(z))f'(z)$$

for all $z \in D$, which is possible only if $f'(z) \neq 0$ for all $z \in D$. □

Definition 17.9. Let $D \subset \mathbb{C}$ be open and connected. We say that D admits

(a) *holomorphic logarithms* if, for every holomorphic $f \colon D \to \mathbb{C}$ with $\mathbf{Z}(f) = \varnothing$, there is a holomorphic $g \colon D \to \mathbb{C}$ with $f = \exp \circ g$;

(b) *holomorphic roots* if, for every holomorphic $f \colon D \to \mathbb{C}$ with $\mathbf{Z}(f) = \varnothing$ and each $n \in \mathbb{N}$, there is a holomorphic $h_n \colon D \to \mathbb{C}$ with $f(z) = h_n(z)^n$ for $z \in D$;

(c) *holomorphic square roots* if, for every holomorphic $f \colon D \to \mathbb{C}$ with $\mathbf{Z}(f) = \varnothing$, there is a holomorphic $h \colon D \to \mathbb{C}$ with $f(z) = h(z)^2$ for $z \in D$.

Remark. By Corollary 15.6, every simply connected D admits holomorphic logarithms, and the proof of Corollary 15.7 shows that every D that admits holomorphic logarithms also admits holomorphic roots.

Theorem 17.10 (Riemann's Mapping Theorem)**.** *Let $D \subsetneq \mathbb{C}$ be open and connected and admit holomorphic square roots, and let $z_0 \in D$. Then there is a unique biholomorphic $f \colon D \to \mathbb{D}$ with $f(z_0) = 0$ and $f'(z_0) > 0$.*

Proof. Uniqueness: Let $g : D \to \mathbb{D}$ be another such function. Then $f \circ g^{-1} : \mathbb{D} \to \mathbb{D}$ is biholomorphic with $(f \circ g^{-1})(0) = f(z_0) = 0$. By Corollary 7.9, there is thus $c \in \mathbb{C}$ with $|c| = 1$ such that

$$f(g^{-1}(z)) = cz$$

for $z \in \mathbb{D}$ and thus

$$f(z) = f(g^{-1}(g(z))) = c\, g(z)$$

for $z \in D$. Differentiation yields $f'(z) = c\, g'(z)$ for $z \in D$. Since $|c| = 1$ and $f'(z_0), g'(z_0) > 0$, we conclude that $c = 1$.

Existence: Let

$$\mathcal{F} := \{f \colon D \to \mathbb{D} : f \text{ is injective and holomorphic with } f(z_0) = 0 \text{ and } f'(z_0) > 0\}.$$

Claim 1 $\mathcal{F} \neq \varnothing$.

Since $D \neq \mathbb{C}$, there is $w \in \mathbb{C} \setminus D$. Since D admits holomorphic square roots, there is a holomorphic $g \colon D \to \mathbb{C}$ such that $g(z)^2 = z - w$ for $z \in D$. Let $z_1, z_2 \in D$ be such that $g(z_1) = \pm g(z_2)$. Then $z_1 = z_2$ holds, and, in particular, g is injective and thus not constant.

By the Open Mapping Theorem, there is $r > 0$ with $B_r(g(z_0)) \subset g(D)$.

Assume that there is $z \in D$ with $g(z) \in B_r(-g(z_0))$. This means that

$$r > |g(z) + g(z_0)| = |-g(z) - g(z_0)|,$$

so that $-g(z) \in B_r(g(z_0)) \subset g(D)$. Hence, there is $\tilde{z} \in D$ with $g(\tilde{z}) = -g(z)$ and thus $\tilde{z} = z$, which, in turn, yields that $g(z) = 0$, so that

$$0 = g(z)^2 = z - w.$$

This contradicts $w \notin D$. Hence, $g(D) \cap B_r(-g(z_0)) = \varnothing$ must hold.

Define

$$T\colon \mathbb{C} \setminus B_r[-g(z_0)] \to \mathbb{C}, \quad z \mapsto \frac{r}{z + g(z_0)},$$

and set $\tilde{g} := T \circ g$. Then $\tilde{g}\colon D \to \mathbb{D}$ is holomorphic and injective. Let $\tilde{w} := \tilde{g}(z_0)$. Then $\phi_{\tilde{w}} \circ \tilde{g}\colon D \to \mathbb{D}$ is holomorphic and injective with $(\phi_{\tilde{w}} \circ \tilde{g})(z_0) = 0$ and $(\phi_{\tilde{w}} \circ \tilde{g})'(z_0) \neq 0$ by Corollary 17.8. Let $c \in \mathbb{C}$ with $|c| = 1$ be such that $c\,(\phi_{\tilde{w}} \circ \tilde{g})'(z_0) > 0$. Then $c\,(\phi_{\tilde{w}} \circ \tilde{g}) \in \mathcal{F}$ holds, so that indeed $\mathcal{F} \neq \varnothing$.

Claim 2 Let $(f_n)_{n=1}^\infty$ be a sequence in $\mathcal{F}$ converging compactly to $f : D \to \mathbb{C}$. Then either $f \equiv 0$ or $f \in \mathcal{F}$.

It is straightforward that $f(z_0) = 0$, $f'(z_0) \geq 0$, and $f(D) \subset \overline{\mathbb{D}}$. By Corollary 14.11, $f \equiv 0$ or f is injective. If f is injective, then $f'(z_0) \neq 0$ must hold by Corollary 17.8, i.e., $f'(z_0) > 0$. Also, since $f(D) \subset \overline{\mathbb{D}}$ is open, we have $f(D) \subset \mathbb{D}$, so that $f \in \mathcal{F}$.

Claim 3 There is $f \in \mathcal{F}$ such that $f(D) = \mathbb{D}$.

Choose a sequence $(f_n)_{n=1}^\infty$ such that

$$\lim_{n\to\infty} f_n'(z_0) = \sup\{(\tilde{f})'(z_0) : \tilde{f} \in \mathcal{F}\} \in (0, \infty].$$

By Montel's Theorem, we can suppose, by passing to a subsequence, that $(f_n)_{n=1}^\infty$ converges compactly to some $f\colon D \to \mathbb{C}$. In particular,

$$f'(z_0) = \sup\{(\tilde{f})'(z_0) : \tilde{f} \in \mathcal{F}\} > 0 \tag{$*$}$$

holds, so that $f \in \mathcal{F}$ by Claim 2.

Assume that there is $w \in \mathbb{D} \setminus f(D)$. Since D admits holomorphic square roots, there is a holomorphic $h\colon D \to \mathbb{C}$ such that

$$h(z)^2 = \frac{f(z) - w}{1 - \bar{w} f(z)} = (\phi_w \circ f)(z) \tag{$**$}$$

for $z \in D$. In particular, $h(D) \subset \mathbb{D}$, and h is injective. Define

$$f_0\colon D \to \mathbb{C}, \quad z \mapsto \frac{|h'(z_0)|}{h'(z_0)} \frac{h(z) - h(z_0)}{1 - \overline{h(z_0)} h(z)} = \frac{|h'(z_0)|}{h'(z_0)} (\phi_{h(z_0)} \circ h)(z).$$

Then f_0 is injective with $f_0(D) \subset \mathbb{D}$ and $f_0(z_0) = 0$. Differentiation yields

$$f_0'(z) = \frac{|h'(z_0)|}{h'(z_0)} \phi_{h(z_0)}'(h(z)) h'(z) = \frac{|h'(z_0)|}{h'(z_0)} \frac{1 - |h(z_0)|^2}{(1 - \overline{h(z_0)} h(z))^2} h'(z)$$

for $z \in D$ and thus, in particular,

$$f_0'(z_0) = \frac{|h'(z_0)|}{1 - |h(z_0)|^2} > 0,$$

so that $f_0 \in \mathcal{F}$.

Differentiating $(**)$, we obtain

$$2h(z)h'(z) = \frac{1 - |w|^2}{(1 - \bar{w}f(z))^2} f'(z)$$

for $z \in D$ and thus, letting $z = z_0$,

$$2h(z_0)h'(z_0) = (1 - |w|^2) f'(z_0).$$

Since $|h(z_0)|^2 = |-w| = |w|$, we obtain

$$\begin{aligned} f_0'(z_0) &= \frac{|h'(z_0)|}{1 - |h(z_0)|^2} \\ &= \frac{2|h(z_0)h'(z_0)|}{2|h(z_0)|(1 - |h(z_0)|^2)} \\ &= \frac{(1 - |w|^2) f'(z_0)}{2\sqrt{|w|}(1 - |w|)} \\ &= f'(z_0) \frac{1 + |w|}{2\sqrt{|w|}} \\ &> f'(z_0), \end{aligned}$$

which contradicts $(*)$. □

Characterizations of Simply Connected Domains. *The following are equivalent for an open and connected set $D \subset \mathbb{C}$:*

(i) *D is simply connected;*

(ii) *D admits holomorphic logarithms;*

(iii) *D admits holomorphic roots;*

(iv) *D admits holomorphic square roots;*

(v) *D is all of $\mathbb{C}$ or biholomorphically equivalent to $\mathbb{D}$;*

(vi) *every holomorphic function $f \colon D \to \mathbb{C}$ has an anti-derivative;*

(vii) *$\int_\gamma f(\zeta)\, d\zeta = 0$ for each holomorphic $f \colon D \to \mathbb{C}$ and each closed curve γ in D;*

(viii) *for every holomorphic function $f\colon D \to \mathbb{C}$, we have*

$$\nu(f, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

for each closed curve γ in D and all $z \in D \setminus \{\gamma\}$;

(ix) *every harmonic function $u\colon D \to \mathbb{R}$ has a harmonic conjugate.*

Proof. (i) $\Longrightarrow$ (ii) is Corollary 15.6, (ii) $\Longrightarrow$ (iii) is shown in the proof of Corollary 15.7, (iii) $\Longrightarrow$ (iv) is trivial, (iv) $\Longrightarrow$ (v) follows from Theorem 17.10, and (v) $\Longrightarrow$ (i) is implied by Proposition 17.3.

(i) $\Longleftrightarrow$ (vi) is Corollary 15.5, and (vi) $\Longleftrightarrow$ (vii) follows from Theorem 4.10.

(i) $\Longrightarrow$ (viii) follows from Theorem 15.3, and (viii) $\Longrightarrow$ (vii) is seen as in the proof of Theorem 15.2.

(v) $\Longrightarrow$ (ix): Let $u : D \to \mathbb{R}$ be harmonic. If $D = \mathbb{C}$, the existence of a harmonic conjugate is immediate by Theorem 9.4. So suppose that $D \neq \mathbb{C}$. Hence, there is a biholomorphic map $f\colon D \to \mathbb{D}$. It is easily seen that $\tilde{u} := u \circ f^{-1}\colon \mathbb{D} \to \mathbb{R}$ is harmonic and thus has a harmonic conjugate $\tilde{v}\colon \mathbb{D} \to \mathbb{R}$ by Theorem 9.4. Then $v := \tilde{v} \circ f\colon D \to \mathbb{R}$ is a harmonic conjugate of u.

(ix) $\Longrightarrow$ (ii): Let $f : D \to \mathbb{R}$ be holomorphic such that $\mathbf{Z}(f) = \varnothing$. Then $u := \log|f|$ is harmonic and thus has a harmonic conjugate $v\colon D \to \mathbb{R}$, i.e., v is harmonic such that $g := u + i\,v$ is holomorphic. For $z \in D$, we have

$$|\exp(g(z))| = |\exp(u(z) + i\,v(z))| = |\exp(u(z))||\exp(i\,v(z))| = \exp(u(z)) = |f(z)|.$$

Thus,

$$D \to \mathbb{C}, \quad z \mapsto \frac{f(z)}{\exp(g(z))}$$

is a holomorphic function whose range lies in $\partial\mathbb{D}$ and therefore isn't open. By the Open Mapping Theorem, this means that there is $c \in \mathbb{C}$—necessarily of modulus one—such that $f(z) = c\exp(g(z))$ for $z \in D$. Choose $\theta \in \mathbb{R}$ with $\exp(i\theta) = c$, and note that $f(z) = \exp(g(z) + i\theta)$ for $z \in D$. $\square$

Further Characterizations of Simply Connected Domains. There are more conditions that characterize the simply connected domains. We will only mention them without giving proofs.

(x) *for every holomorphic function $f\colon D \to \mathbb{C}$, there is a sequence of polynomials converging to f compactly on D;*

It was shown in the homework that, if every holomorphic function on D can be approximated by polynomials, then D is indeed simply connected. The proof of the converse relies on *Runge's Approximation Theorem*, which we didn't prove in this course.

Two (not necessarily piecewise smooth) curves $\gamma_1, \gamma_2 \colon [0,1] \to D$ are with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$ are called *path homotopic* if there is a continuous function $\Gamma \colon [0,1] \times [0,1] \to D$ such that,

$$\Gamma(0,t) = \gamma_1(t) \qquad \text{and} \qquad \Gamma(1,t) = \gamma_2(t)$$

for $t \in [0,1]$ and

$$\Gamma(s,0) = \gamma_1(0) \qquad \text{and} \qquad \Gamma(s,1) = \gamma_1(1)$$

for $s \in [0,1]$. A closed curve γ with starting point/endpoint z_0 is called *homotopic to zero* if γ and the constant curve z_0 are path homotopic.

Simple connectedness is equivalent to:

(xi) *every (not necessarily smooth) curve in D is homotopic to zero*;

This conditions makes no reference to holomorphic functions and is entirely topological in nature, like:

(xii) *D is homeomorphic to $\mathbb{D}$*;

This means that there is a bijective, continuous map $f : D \to \mathbb{C}$ with continuous inverse. Since (xi) is preserved under homeomorphisms, it is clear that D satisfies (xi) if it is homeomorphic to $\mathbb{D}$. For the converse, it is sufficient to show that $\mathbb{C}$ is homeomorphic to $\mathbb{D}$ (for $D \neq \mathbb{C}$, this is clear by Theorem 17.10). Since

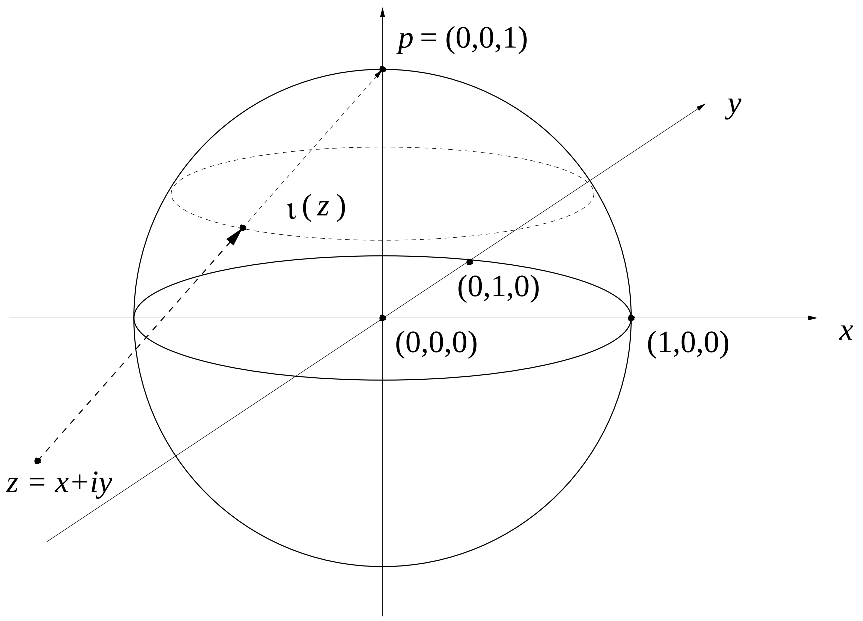
$$\mathbb{C} \to \mathbb{D}, \quad z \mapsto \frac{z}{1+|z|}$$

and

$$\mathbb{D} \to \mathbb{C}, \quad z \mapsto \frac{z}{1-|z|}$$

are continuous and inverse to each other, this is indeed the case.

Let $\mathbb{C}_\infty$ denote the unit sphere in $\mathbb{R}^3$. We can connect every point z in xy-plane, which we identify with $\mathbb{C}$, with $(0,0,1)$ through a straight line that intersects $\mathbb{C}_\infty$ in a point $\iota(z)$. Thus, we obtain an injective map $\iota \colon \mathbb{C} \to \mathbb{C}_\infty$.



One can show that ι is continous, and that the inverse $\iota^{-1}\colon \iota(\mathbb{C}) \to \mathbb{C}$ is also continuous. Hence, we may identify $\mathbb{C}$ with $\iota(\mathbb{C})$.

With regards to simple connectedness, we have another equivalent conditions involving $\mathbb{C}_\infty$:

(xiii) $\mathbb{C}_\infty \setminus D$ *is connected.*