Honors Complex Variables (Math 411)

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1 The Complex Numbers

Definition 1.1. The complex numbers—denoted by \( \mathbb{C} \)—are \( \mathbb{R}^2 \) equipped with the operations

\[
(x, y) + (u, v) := (x + u, y + v), \quad (x, y)(u, v) := (xu - yv, xv + yu)
\]

for \( x, y, u, v \in \mathbb{R} \).

Theorem 1.2. The complex numbers are a field. More specifically, we have:

• \((0, 0)\) is the neutral element of addition;
• \(-(x, y) = (-x, -y)\) for \( x, y \in \mathbb{R} \);
• \((1, 0)\) is the neutral element of multiplication;
• \((x, y)^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right)\) for \( x, y \in \mathbb{R} \) with \( (x, y) \neq (0, 0) \).

Proof (of the last claim only). Let \( x, y \in \mathbb{R} \) be such that \((x, y) \neq (0, 0)\), and note that

\[
(x, y)\left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right) = \left(\frac{x^2}{x^2 + y^2} - \frac{-y^2}{x^2 + y^2}, \frac{-xy}{x^2 + y^2} + \frac{xy}{x^2 + y^2}\right) = \left(\frac{x^2 + y^2}{x^2 + y^2}, \frac{-xy + xy}{x^2 + y^2}\right) = (1, 0).
\]

Proposition 1.3. The set \( \{(x, 0) : x \in \mathbb{R}\} \) is a subfield of \( \mathbb{C} \), and the map

\[
\theta : \mathbb{R} \to \mathbb{C}, \quad x \mapsto (x, 0)
\]

is an isomorphism onto its image.

Proposition 1.3 is often worded as:

\( \mathbb{R} \) “is” a subfield of \( \mathbb{C} \).

Set \( 1 := (1, 0) \) and \( i := (0, 1) \). Then, for any \( z = (x, y) \in \mathbb{C} \), we have

\[
z = (x, 0) + (0, y) = (1, 0)(x, 0) + (0, 1)(y, 0) = x + iy.
\]

We write

\[
\text{Re } z := x = \text{“the real part of } z\text{”}
\]
Im \( z := y = \) “the imaginary part of \( z \)”.

The complex number \( i \) is called the imaginary unit and satisfies \( i^2 = (0,1)^2 = (-1,0) = -1 \).

**Definition 1.4.** For \( z = x + iy \in \mathbb{C} \), its complex conjugate is defined as \( \bar{z} = x - iy \).

**Proposition 1.5.** For \( z, w \in \mathbb{C} \), the following hold true:

(i) \( \operatorname{Re} z = \frac{1}{2} (z + \bar{z}) \) and \( \operatorname{Im} z = \frac{1}{2i} (z - \bar{z}) \);

(ii) \( z + w = \bar{z} + \bar{w} \);

(iii) \( \bar{z} \bar{w} = \bar{z} \bar{w} \);

(iv) \( \frac{1}{z} = \frac{1}{\bar{z}} \) if \( z \neq 0 \).

**Proof.**

(i): If \( z = x + iy \), then \( \bar{z} = x - iy \), so that \( 2x = z + \bar{z} \); this yields the claim for \( \operatorname{Re} z \). The assertion for \( \operatorname{Im} z \) is proven similarly.

(ii) is obvious.

(iii): Let \( z = x + iy \) and \( w = u + iv \), so that

\[ zw = (xu - yv) + i(xv + yu) \]

and thus

\[ \bar{z} \bar{w} = (xu - yv) - i(xv + yu). \]

On the other hand, we have \( \bar{z} = x - iy \) and \( \bar{w} = u - iv \), which yields

\[ \bar{z} \bar{w} = (xu - (-y)(-v)) + i(x(-v) + (-y)u) \]

\[ = (xu - yv) - i(xv + yu) \]

\[ = \bar{z} \bar{w}, \]

as claimed.

(iv): By (iii), we have

\[ \frac{1}{z} \bar{z} = \frac{1}{\bar{z}} \bar{z} = 1 = 1, \]

which yields the claim.

For any \( z = x^2 + y^2 \in \mathbb{C} \), we have \( z \bar{z} = x^2 + y^2 \geq 0 \). Hence, the following makes sense:

**Definition 1.6.** For \( z \in \mathbb{C} \), set \( |z| := \sqrt{zz^*} \).

**Proposition 1.7.** \( |\cdot| \) is the Euclidean norm on \( \mathbb{R}^2 \). In particular, the following hold:
(i) \(|z| \geq 0\) with \(|z| = 0\) if and only if \(z = 0\);

(ii) \(|z + w| \leq |x| + |w|\) for \(z, w \in \mathbb{C}\).

Moreover, we have \(|zw| = |z||w|\) for \(z, w \in \mathbb{C}\) and \(z^{-1} = \frac{\bar{z}}{|z|^2}\) for \(z \in \mathbb{C} \setminus \{0\}\).

**Proof.** In view of the remark immediately preceding Definition 1.6, it is clear that \(|\cdot|\) is the Euclidean norm, which entails (i) and (ii).

For the moreover part, let \(z, w \in \mathbb{C}\), and note that

\[
|zw|^2 = zw\overline{zw} = (z\overline{z})(w\overline{w}) = |z|^2|w|^2.
\]

Also, since \(|z|^2 = z\overline{z}\), we have \(1 = z\frac{\bar{z}}{|z|^2}\) for \(z \neq 0\) and thus \(z^{-1} = \frac{\bar{z}}{|z|^2}\). \(\square\)


2 Complex Differentiation

Definition 2.1. Let $D \subset \mathbb{C}$, and let $z_0$ be an interior point of $D$, i.e., there is $\epsilon > 0$ such that $B_{\epsilon}(z_0) := \{ z \in \mathbb{C} : |z - z_0| < \epsilon \} \subset D$. A function $f : D \to \mathbb{C}$ is called complex differentiable at $z_0$ if

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Proposition 2.2. Let $D \subset \mathbb{C}$, let $z_0 \in \text{int} \, D$, and let $f : D \to \mathbb{C}$ be complex differentiable at $z_0$. Then $f$ is continuous at $z_0$.

Proof. Since $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists, we have

$$0 = \lim_{z \to z_0} (z - z_0) \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} (z - z_0) \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} (f(z) - f(z_0)),$$

so that $f(z_0) = \lim_{z \to z_0} f(z)$. □

Proposition 2.3. Let $D \subset \mathbb{C}$, and let $f, g : D \to \mathbb{C}$ be complex differentiable at $z_0 \in \text{int} \, D$. Then the following functions are complex differentiable at $z_0$: $f + g$, $fg$, and—if $g(z_0) \neq 0 - \frac{f'}{g}$—$f \cdot g$. Moreover, we have:

$$(f + g)'(z_0) = f'(z_0) + g'(z_0),$$

$$(fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0),$$

and

$$\left( \frac{f}{g} \right)'(z_0) = \frac{f'(z_0)g(z_0) + f(z_0)g'(z_0)}{g(z_0)^2}.$$ 

Proof. As over $\mathbb{R}$. □

Proposition 2.4. Let $D, E \subset \mathbb{C}$, let $g : D \to \mathbb{C}$ and $f : E \to \mathbb{C}$ be such that $g(D) \subset E$, and let $z_0 \in \text{int} \, D$ be such that $w_0 := g(z_0) \in \text{int} \, E$. Further, suppose that $g$ is complex differentiable at $z_0$ and $f$ is complex differentiable at $w_0$. Then $f \circ g$ is complex differentiable at $z_0$ with

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0).$$

Proof. As over $\mathbb{R}$. □

Examples. 1. All constant functions are (on all of $\mathbb{C}$) complex differentiable, as is $\mathbb{C} \ni z \mapsto z$. Consequently, all complex polynomials are complex differentiable on all of $\mathbb{C}$, and rational functions are complex differentiable wherever they are defined.
2. Let 

\[ f: \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto \bar{z}, \]

and let \( z_0 = x_0 + iy_0 \in \mathbb{C} \). Assume that \( f \) is complex differentiable at \( z_0 \). Then we have

\[
f'(z_0) = \lim_{{z \to z_0}} \frac{\bar{z} - \bar{z}_0}{z - z_0} \\
= \lim_{{y \to y_0}} \frac{(x - iy) - (x_0 - iy_0)}{(x + iy) - (x_0 + iy_0)} \\
= \lim_{{y \to y_0}} \frac{i(y_0 - y)}{i(y - y_0)} \\
= -1
\]

as well as

\[
f'(z_0) = \lim_{{z \to z_0}} \frac{\bar{z} - \bar{z}_0}{z - z_0} \\
= \lim_{{x \to x_0}} \frac{(x - iy) - (x_0 - iy_0)}{(x + iy) - (x_0 + iy_0)} \\
= \lim_{{x \to x_0}} \frac{x - x_0}{x - x_0} \\
= 1,
\]

which is impossible. Hence, \( f \) is not complex differentiable at any \( z_0 \in \mathbb{C} \). (On the other hand, \( f \) is continuously partially differentiable—as a function of two real variables—on all of \( \mathbb{C} \).)

**Lemma 2.5.** The following are equivalent for a \( \mathbb{R} \)-linear map \( T: \mathbb{C} \rightarrow \mathbb{C} \):

(i) there is \( c \in \mathbb{C} \) such that \( T(z) = cz \) for all \( z \in \mathbb{C} \);

(ii) \( T \) is \( \mathbb{C} \)-linear;

(iii) \( T(i) = iT(1) \);

(iv) if \( A \) is the real \( 2 \times 2 \) matrix representing \( T \) with respect to the standard basis of \( \mathbb{R}^2 \), then there are \( a, b \in \mathbb{R} \) such that

\[
A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.
\]

**Proof.** (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) is obvious.
(iii) $\implies$ (i): Set $c := T(1)$. For $z = x + iy \in \mathbb{C}$, this means that
\[ T(x + iy) = T(x) + T(iy) = xT(1) + yT(i) = xT(1) + iyT(1) = zT(1) = cz. \]

(iv) $\iff$ (iii): Let $a, b, c, d \in \mathbb{R}$ be such that
\[ A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \]
represents $T$ with respect to the standard basis of $\mathbb{R}^2$. Note that
\[ T(i) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} = c + id \]
and
\[ T(1) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} = a + ib, \]
so that
\[ iT(1) = -b + ia. \]

It follows that
\[ T(i) = iT(1) \iff c = -b \text{ and } d = a. \quad \square \]

**Theorem 2.6** (Cauchy–Riemann Differential Equations). Let $D \subset \mathbb{C}$ be open, and let $z_0 \in D$. Then the following are equivalent for $f : D \to \mathbb{C}$:

(i) $f$ is complex differentiable at $z_0$;

(ii) $f$ is totally differentiable at $z_0$ (in the sense of several variable calculus), and the Cauchy–Riemann differential equations hold, i.e., for $g := \text{Re } f$ and $h := \text{Im } f$, we have
\[ \frac{\partial g}{\partial x}(z_0) = \frac{\partial h}{\partial y}(z_0) \quad \text{and} \quad \frac{\partial g}{\partial y}(z_0) = -\frac{\partial h}{\partial x}(z_0). \]

Proof. (i) $\implies$ (ii): Define
\[ T : \mathbb{C} \to \mathbb{C}, \quad z \mapsto f'(z_0)z, \]
and note that
\[ \frac{|f(z) - f(z_0) - T(z - z_0)|}{|z - z_0|} = \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| \to 0 \]
as $z_0 \to z_0$. Therefore, $f$ is totally differentiable at $z_0$. From Math 217, it follows that the matrix representation of $T$ with respect to the standard basis of $\mathbb{R}$ is the Jacobian of $f$, i.e.,

$$J_f(z_0) = \begin{bmatrix} \frac{\partial g}{\partial x}(z_0) & \frac{\partial g}{\partial y}(z_0) \\ \frac{\partial h}{\partial x}(z_0) & \frac{\partial h}{\partial y}(z_0) \end{bmatrix}.$$ 

Since $T$ is $\mathbb{C}$-linear, Lemma 2.5 yields that

$$\frac{\partial g}{\partial x}(z_0) = \frac{\partial h}{\partial y}(z_0) \quad \text{and} \quad \frac{\partial h}{\partial x}(z_0) = -\frac{\partial g}{\partial y}(z_0).$$

(ii) $\implies$ (i): Since $f$ is totally differentiable at $z_0$, we have a unique $\mathbb{R}$-linear map $T: \mathbb{C} \to \mathbb{C}$ such that

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0) - T(z - z_0)|}{|z - z_0|} = 0.$$

As we know from Math 217, $T$ is represented by $J_f(z_0)$ with respect to the standard basis of $\mathbb{R}^2$. Since the Cauchy–Riemann differential equations are supposed to hold, $J_f(z_0)$ is of the form described in Lemma 2.5(iv). By Lemma 2.5, there is thus $c \in \mathbb{C}$ such that $T(z) = cz$ for all $z \in \mathbb{C}$. It follows that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - c \right| = \frac{|f(z) - f(z_0) - c(z - z_0)|}{|z - z_0|} \to 0$$

as $z \to z_0$. Hence, $f$ is complex differentiable at $z_0$. \hfill \Box

Remark. In the situation of Theorem 2.6, we have

$$f'(z_0)1 = \begin{bmatrix} \frac{\partial g}{\partial x}(z_0) & \frac{\partial g}{\partial y}(z_0) \\ \frac{\partial h}{\partial x}(z_0) & \frac{\partial h}{\partial y}(z_0) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\partial g}{\partial x}(z_0) + i \frac{\partial h}{\partial x}(z_0)$$

as well as

$$f'(z_0)i = \begin{bmatrix} \frac{\partial g}{\partial x}(z_0) & \frac{\partial g}{\partial y}(z_0) \\ \frac{\partial h}{\partial x}(z_0) & \frac{\partial h}{\partial y}(z_0) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\partial g}{\partial y}(z_0) + i \frac{\partial h}{\partial y}(z_0),$$

so that

$$f'(z_0) = \frac{\partial g}{\partial x}(z_0) + i \frac{\partial h}{\partial x}(z_0) = \frac{\partial h}{\partial y}(z_0) - i \frac{\partial g}{\partial y}(z_0).$$

Example. Let

$$f: \mathbb{C} \to \mathbb{C}, \quad z \mapsto |z|^2.$$ 

Then $f$ is totally differentiable with $h = 0$. Moreover,

$$\frac{\partial g}{\partial x} = 2x \quad \text{and} \quad \frac{\partial g}{\partial y} = 2y$$

holds. Hence, we have that

$$\frac{\partial g}{\partial x}(z_0) = \frac{\partial h}{\partial y}(z_0) \quad \text{and} \quad \frac{\partial g}{\partial y}(z_0) = -\frac{\partial h}{\partial x}(z_0)$$

if and only if $z_0 = 0$. By Theorem 2.6, this means that $f$ is complex differentiable at $z_0$ if and only if $z_0 = 0$. 

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Corollary 2.7. Let $D \subset \mathbb{C}$ be open and connected, and let $f : D \to \mathbb{C}$ be complex differentiable. Then $f$ is constant on $D$ if and only if $f' \equiv 0$.

Proof. Suppose that $f' \equiv 0$. From the remark after Theorem 2.6, it follows that

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial x} = \frac{\partial g}{\partial y} = \frac{\partial h}{\partial y} \equiv 0.$$ 

Several variable calculus then yields that $f$ is constant. \hfill \Box
3 Power Series

Definition 3.1. A (complex) power series is an infinite series of the form \( \sum_{n=0}^{\infty} a_n(z-z_0)^n \) with \( z, z_0, a_0, a_1, a_2, \ldots \in \mathbb{C} \). The point \( z_0 \) is called the point of expansion of the series.

Examples. 1. For \( m \in \mathbb{N} \), we have
\[
\sum_{n=0}^{m} z^n = \frac{1 - z^{m+1}}{1 - z}
\]
if \( z \neq 1 \). For \( |z| < 1 \), we obtain (letting \( m \to \infty \))
\[
\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}.
\]

2. For \( z \in \mathbb{C} \), define
\[
\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.
\]
Let \( z \neq 0 \), and note that
\[
\left| \frac{z^{n+1}}{(n+1)!} \right| / \left| \frac{z^n}{n!} \right| = \frac{|z|}{n+1} \to 0
\]
as \( n \to \infty \). As the ratio test holds for series with summands in \( \mathbb{C} \) as well as for series over \( \mathbb{R} \), we conclude that \( \exp(z) \) converges absolutely.

Let \( z, w \in \mathbb{C} \), and note that the Cauchy product formula for series over \( \mathbb{R} \) also holds over \( \mathbb{C} \). We obtain:
\[
\exp(z) \exp(w) = \left( \sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \left( \sum_{n=0}^{\infty} \frac{w^n}{n!} \right)
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n-k} w^k}{(n-k)! k!},
\]
by the Cauchy product formula,
\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^{n-k} w^k
= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}
= \exp(z+w).
\]
We call \( \exp : \mathbb{C} \to \mathbb{C} \) the exponential function; we also write \( e^z \) instead of \( \exp(z) \).

3. The sine and cosine on \( \mathbb{C} \) are defined as
\[
\sin(z) := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}
\]
and
\[
\cos(z) := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}
\]
for \(z \in \mathbb{C}\). As for \(\exp(z)\), we see that both \(\sin(z)\) and \(\cos(z)\) converge absolutely for all \(z \in \mathbb{C}\). Moreover, we have for \(z \in \mathbb{C}\):
\[
e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{n=0}^{\infty} \frac{(iz)^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(iz)^{2n+1}}{(2n+1)!} = \cos(z) + i \sin(z).
\]

**Theorem 3.2.** Let \(\sum_{n=0}^{\infty} a_n(z - z_0)^n\) be a complex power series. Then there is a unique \(R \in [0, \infty]\) with the following properties:

- \(\sum_{n=0}^{\infty} a_n(z - z_0)^n\) converges absolutely for each \(z \in B_R(z_0)\);
- for each \(r \in [0, R)\), the series \(\sum_{n=0}^{\infty} a_n(z - z_0)^n\) converges uniformly on \(B_r[z_0] := \{z \in \mathbb{C} : |z - z_0| \leq r\}\);
- \(\sum_{n=0}^{\infty} a_n(z - z_0)^n\) diverges for each \(z \not\in B_R[z_0]\).

Moreover, \(R\) is computed via the Cauchy–Hadamard formula:
\[
R = \frac{1}{\limsup_{n \to \infty} \sqrt[n]{|a_n|}}.
\]
It is called the radius of convergence for \(\sum_{n=0}^{\infty} a_n(z - z_0)^n\).

**Proof.** The uniqueness of \(R\) is obvious.

Let \(R \in [0, \infty]\) be defined by the Cauchy–Hadamard formula (we set \(\frac{1}{0} = \infty\) and \(\frac{1}{\infty} = 0\)).

Let \(r \in [0, R)\), and choose \(r' \in (r, R)\). It follows that
\[
\frac{1}{r'} > \frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{|a_n|},
\]
so that there is \(n_0 \in \mathbb{N}\) such that \(\sqrt[n]{|a_n|} < \frac{1}{r'}\) for \(n \geq n_0\), i.e.,
\[
|a_n| < \left(\frac{1}{r'}\right)^n
\]
for \( n \geq n_0 \). For \( n \geq n_0 \) and \( z \in B_r(z_0) \), we then have

\[
|a_n(z - z_0)^n| < \left( \frac{r}{r'} \right)^n.
\]

Since \( \frac{r}{r'} < 1 \), we have \( \sum_{n=0}^{\infty} \left( \frac{r}{r'} \right)^n < \infty \). The Weierstraß M-test thus yields that \( \sum_{n=0}^{\infty} a_n(z - z_0)^n \) converges absolutely and uniformly on \( B_r(z_0) \).

Since every \( z \in B_R(z_0) \) is contained in \( B_r(z_0) \) for some \( r \in [0, R) \), it follows that \( \sum_{n=0}^{\infty} a_n(z - z_0)^n \) converges absolutely for each such \( z \).

Let \( z \notin B_{R}[z_0] \), i.e., \( |z - z_0| > R \), so that

\[
\frac{1}{|z - z_0|} < \frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{|a_n|}
\]

and thus

\[
\frac{1}{|z - z_0|} < \sqrt[n]{|a_n|}
\]

or, equivalently,

\[
1 < |a_n(z - z_0)^n|
\]

for infinitely many \( n \in \mathbb{N} \). It follows that \( (a_n(z - z_0)^n)_{n=1}^{\infty} \) does not converge to zero. Consequently, \( \sum_{n=0}^{\infty} a_n(z - z_0)^n \) diverges.

Remark. The uniqueness of \( R \) follows already from the first and the last one of the three properties listed.

Examples. 1. \( \sum_{n=0}^{\infty} z^n: R = 1. \)

2. \( \sum_{n=0}^{\infty} \frac{z^n}{n!}: R = \infty. \)

3. \( \sum_{n=0}^{\infty} n! z^n: R = 0. \)

4. \( \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \) and \( \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}: R = \infty. \)

Theorem 3.3. Let \( \sum_{n=0}^{\infty} a_n(z - z_0)^n \) be a complex power series with radius of convergence \( R \). Then

\[
f : B_R(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} a_n(z - z_0)^n
\]

is complex differentiable at each point \( z \in B_R(z_0) \) with

\[
f'(z) = \sum_{n=1}^{\infty} n a_n(z - z_0)^{n-1}.
\]

Proof. Without loss of generality, suppose that \( z_0 = 0. \)

We first show that \( \sum_{n=1}^{\infty} n a_n z^{n-1} \) converges absolutely for each \( z \in B_R(0) \).
Let \( z \in B_R(0) \), and choose \( r \) such that \(|z| < r < R\). Since \( \frac{1}{r} > \limsup_{n \to \infty} \sqrt[n]{|a_n|} \), there is \( n_0 \in \mathbb{N} \) such that \(|a_n| < \left( \frac{1}{r} \right)^n\) for \( n \geq n_0 \) and thus

\[
|na_n z^{n-1}| < \frac{n}{r} \left( \frac{|z|}{r} \right)^{n-1}
\]

for \( n \geq n_0 \). Since \( \frac{|z|}{r} < 1 \), we have \( \sum_{n=1}^{\infty} \frac{n}{r} \left( \frac{|z|}{r} \right)^{n-1} < \infty \); the comparison test therefore yields that \( \sum_{n=1}^{\infty} na_n z^{n-1} \) converges absolutely.

In view of the foregoing, we may define

\[ g: B_R(0) \to \mathbb{C}, \quad z \mapsto \sum_{n=1}^{\infty} na_n (z - z_0)^{n-1}. \]

We shall devote the rest of the proof to showing that \( f \) is complex differentiable on \( B_R(0) \) with \( f' = g \).

To this end, fix \( \epsilon > 0 \), and define, for \( z \in B_R(0) \) and \( n \in \mathbb{N} \),

\[ s_n(z) := \sum_{k=0}^{n} a_k z^k \quad \text{and} \quad R_n(z) := \sum_{k=n+1}^{\infty} a_k z^k. \]

Fix \( z \in B_R(0) \) and let \( r \in (0, R) \) be such that \( z \in B_r(0) \); note that

\[
\frac{f(w) - f(z)}{w - z} - g(z) = \left( s_n(w) - s_n(z) - s_n'(z) \right) + \left( s_n'(z) - g(z) \right) + \frac{R_n(w) - R_n(z)}{w - z}
\]

for all \( w \in B_R(0) \). We shall see that each of the three summands on the right hand side of this equation has modulus less than \( \frac{\epsilon}{3} \), provided that \( n \) is sufficiently large and \( w \) is sufficiently close to \( z \).

We start with the last summand. First, note that

\[
\frac{R_n(w) - R_n(z)}{w - z} = \sum_{k=n+1}^{\infty} a_k \frac{w^k - z^k}{w - z}
\]

for all \( w \in B_R(0) \) and also that

\[
\left| \frac{w^k - z^k}{w - z} \right| = \left| \sum_{j=1}^{k} w^{k-j} z^{j-1} \right| \leq \sum_{j=1}^{k} |w|^{k-j} |z|^{j-1} \leq kr^{k-1}
\]

for all \( w \in B_r(0) \). Since \( r < R \), we have \( \sum_{k=1}^{\infty} k |a_k| r^{k-1} < \infty \). Consequently, there is \( n_1 \in \mathbb{N} \) such that \( \sum_{k=n+1}^{\infty} k |a_k| r^{k-1} < \frac{\epsilon}{3} \) for all \( n \geq n_1 \) and therefore

\[
\left| \frac{R_n(w) - R_n(z)}{w - z} \right| = \left| \sum_{k=n+1}^{\infty} a_k \frac{w^k - z^k}{w - z} \right| \leq \sum_{k=n+1}^{\infty} k |a_k| r^{k-1} < \frac{\epsilon}{3}
\]

for all \( n \geq n_1 \) and all \( w \in B_r(0) \).
For the second summand, just note that $g(z) = \lim_{n \to \infty} s'_n(z)$; consequently, there is $n_2 \in \mathbb{N}$ such that $|g(z) - s'_n(z)| < \frac{\epsilon}{3}$ for all $n \geq n_2$.

For the first summand, fix $n \geq \max\{n_1, n_2\}$. Since

$$
\lim_{w \to z} \frac{s_n(w) - s_n(z)}{w - z} = s'_n(z),
$$

there is $\delta > 0$ such that

$$
\left| \frac{s_n(w) - s_n(z)}{w - z} - s'_n(z) \right| < \frac{\epsilon}{3}
$$

for all $w \in B_\delta(z) \cap B_R(0)$. Making $\delta > 0$ smaller, if necessary, we can achieve that $B_\delta(z) \subset B_r(0)$. Consequently, we obtain for all $w \in B_\delta(z)$ that

$$
\left| \frac{R_n(w) - R_n(z)}{w - z} \right| < \frac{\epsilon}{3}
$$

and

$$
\left| \frac{s_n(w) - s_n(z)}{w - z} - s'_n(z) \right| < \frac{\epsilon}{3}.
$$

These estimates, combined with $|g(z) - s'_n(z)| < \frac{\epsilon}{3}$, yield that

$$
\left| \frac{f(w) - f(z)}{w - z} - g(z) \right| < \epsilon
$$

for all $w \in B_\delta(z)$. Since $\epsilon > 0$ was arbitrary, we obtain that $f'(z)$ exists and equals $g(z)$. \hfill \square

**Examples.**

1. $\exp'(z) = \exp(z)$.

2. $\sin'(z) = \cos(z)$.

3. $\cos'(z) = -\sin(z)$.

**Corollary 3.4.** Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a complex power series with radius of convergence $R$. Then

$$
f: B_R(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} a_n(z - z_0)^n
$$

is infinitely often complex differentiable on $B_R(z_0)$ with

$$
f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n(z - z_0)^{n-k}.
$$

for $z \in B_R(z_0)$ and $k \in \mathbb{N}$. In particular,

$$
a_n = \frac{1}{n!} f^{(n)}(z_0)
$$

holds for each $n \in \mathbb{N}_0$. 

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Corollary 3.5. Let $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ be a complex power series with radius of convergence $R$. Then

$$F: B_R(z_0) \rightarrow \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-z_0)^{n+1}$$

is complex differentiable on $B_R(z_0)$ with

$$F'(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

for $z \in B_R(z_0)$. 


4 Complex Line Integrals

We call a function \( f : [a, b] \to \mathbb{C} \) integrable if \( \text{Re} \, f, \text{Im} \, f : [a, b] \to \mathbb{R} \) are integrable in the sense of real variables. (The Riemann integral will do.) In this case, we define

\[
\int_a^b f(t) \, dt := \int_a^b \text{Re} \, f(t) \, dt + i \int_a^b \text{Im} \, f(t) \, dt.
\]

**Definition 4.1.** A curve (or path) in \( \mathbb{C} \) is a continuous map \( \gamma : [a, b] \to \mathbb{C} \). We call

- \( \gamma(a) \) the initial point of \( \gamma \),
- \( \gamma(b) \) the endpoint (or terminal point) of \( \gamma \), and
- \( \{\gamma\} := \gamma([a, b]) \) the trajectory of \( \gamma \).

**Examples.**
1. Let \( z, w \in \mathbb{C} \). Then

\[
\gamma : [0, 1] \to \mathbb{C}, \quad t \mapsto z + t(w - z)
\]

has the initial point \( z \) and the endpoint \( w \), and \( \{\gamma\} \) is the line segment connecting \( z \) with \( w \).

2. For \( k \in \mathbb{Z} \), let

\[
\gamma_k : [0, 2\pi] \to \mathbb{C}, \quad t \mapsto e^{ikt}
\]

Then \( \gamma_k(0) = 1 = \gamma_k(1) \) holds, and for \( k \neq 0 \), we have \( \{\gamma_k\} = \{z \in \mathbb{C} : |z| = 1\} \).

**Definition 4.2.** A curve \( \gamma : [a, b] \to \mathbb{C} \) is called piecewise smooth if there is a partition

\[
a = a_0 < a_1 < \cdots < a_n = b
\]

such that \( \gamma|_{[a_{j-1}, a_j]} \) is continuously differentiable for \( j = 1, \ldots, n \).

**Definition 4.3.** The length of a piecewise smooth curve \( \gamma : [a, b] \to \mathbb{C} \) is defined as

\[
\ell(\gamma) := \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} |\gamma'(t)| \, dt,
\]

where \( a = a_0 < a_1 < \cdots < a_n = b \) is a partition such that \( \gamma|_{[a_{j-1}, a_j]} \) is continuously differentiable for \( j = 1, \ldots, n \).

**Definition 4.4.** Let \( \gamma : [a, b] \to \mathbb{C} \) be a piecewise smooth curve, let \( a = a_0 < a_1 < \cdots < a_n = b \) be a partition such that \( \gamma|_{[a_{j-1}, a_j]} \) is continuously differentiable for \( j = 1, \ldots, n \), and let \( f : \{\gamma\} \to \mathbb{C} \) be continuous. Then the line integral (or contour integral) of \( f \) along \( \gamma \) is defined as

\[
\int_{\gamma} f := \int_{\gamma} f(\zeta) \, d\zeta = \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} f(\gamma(t)) \gamma'(t) \, dt.
\]
Properties of the Line Integral. 1. Let \( \gamma \) be a piecewise smooth curve, let \( \lambda, \mu \in \mathbb{C} \), and let \( f, g : \{ \gamma \} \rightarrow \mathbb{C} \) be continuous. Then we have
\[
\int_{\gamma} \lambda f + \mu g = \lambda \int_{\gamma} f + \mu \int_{\gamma} g.
\]

2. Let \( \gamma \) be a piecewise smooth curve, let \( f : \{ \gamma \} \rightarrow \mathbb{C} \) be continuous, and let \( C \geq 0 \) be such that \( |f(\zeta)| \leq C \) for \( \zeta \in \{ \gamma \} \). Then
\[
\left| \int_{\gamma} f \right| \leq C \ell(\gamma)
\]
holds.

3. Let \( \gamma : [c, d] \rightarrow \mathbb{C} \) be a piecewise smooth curve, let \( \phi : [a, b] \rightarrow [c, d] \) be an increasing, continuously differentiable function with \( \phi(a) = c \) and \( \phi(b) = d \), and let \( f : \{ \gamma \} \rightarrow \mathbb{C} \) be continuous. Then we have
\[
\int_{\gamma} f = \int_{\gamma \circ \phi} f.
\]

4. Let \( D \subset \mathbb{C} \) be open, and let \( f : D \rightarrow \mathbb{C} \) be continuous with an anti-derivative, i.e., there is \( F : D \rightarrow \mathbb{C} \) such that \( F \) is complex differentiable at each \( z \in D \) with \( F'(z) = f(z) \). Then
\[
\int_{\gamma} f = F(\gamma(b)) - F(\gamma(a))
\]
holds for every piecewise smooth curve \( \gamma : [a, b] \rightarrow D \).

Definition 4.5. A curve \( \gamma : [a, b] \rightarrow \mathbb{C} \) is called closed if \( \gamma(a) = \gamma(b) \).

Proposition 4.6. Let \( D \subset \mathbb{C} \) be open, and let \( f : D \rightarrow \mathbb{C} \) be continuous with an anti-derivative. Then \( \int_{\gamma} f = 0 \) holds for each closed, piecewise smooth curve \( \gamma \) in \( D \).

Example. Let \( z_0 \in \mathbb{C} \), let \( r > 0 \), and let
\[
\gamma : [0, 2\pi] \rightarrow \mathbb{C}, \quad t \mapsto re^{it} + z_0,
\]
i.e., \( \gamma \) is the circle centered at \( z_0 \) with radius \( r \) in counterclockwise orientation.

Let \( n \in \mathbb{Z} \), and consider \( \int_{\gamma} (\zeta - z_0)^n d\zeta \).

For \( n \neq -1 \), let
\[
F : \mathbb{C} \setminus \{z_0\} \rightarrow \mathbb{C}, \quad z \mapsto \frac{(z - z_0)^{n+1}}{n+1},
\]
so that \( F'(z) = (z - z_0)^n \) for all \( z \in \mathbb{C} \setminus \{z_0\} \). It follows that \( \int_{\gamma} (\zeta - z_0)^n d\zeta = 0 \).

On the other hand, we have
\[
\int_{\gamma} (\zeta - z_0)^{-1} d\zeta = \int_{0}^{2\pi} \frac{rie^{it}}{re^{it}} dt = \int_{0}^{2\pi} i dt = 2\pi i.
\]
Consequently,
\[ \mathbb{C} \setminus \{z_0\} \to \mathbb{C}, \quad z \mapsto \frac{1}{z - z_0} \]
has no anti-derivative.

Recall the following definition from Math 217:

**Definition 4.7.** A subset \( D \subset \mathbb{C} \) is called *connected* if there are no open sets \( U, V \subset \mathbb{C} \) with
- \( U \cap D \neq \emptyset \neq V \cap D \),
- \( U \cup V \supset D \), and
- \( U \cap V \subset \mathbb{C} \setminus D \).

**Definition 4.8.** Let \( D \subset \mathbb{C} \) be open. A function \( f: D \to \mathbb{C} \) is called *locally constant* if, for each \( z_0 \in D \), there is \( \epsilon > 0 \) such that \( B_\epsilon(z_0) \subset D \) and \( f \) is constant on \( B_\epsilon(z_0) \).

**Proposition 4.9.** Let \( D \subset \mathbb{C} \) be open. Then the following are equivalent:

(i) \( D \) is connected;

(ii) every locally constant function \( f: D \to \mathbb{C} \) is constant;

(iii) for any \( z, w \in D \), there is a piecewise smooth curve \( \gamma: [a, b] \to D \) such that \( \gamma(a) = z \) and \( \gamma(b) = w \).

For the proof of Proposition 4.9 (and of the theorem following it), we require two constructions on curves:

1. Given \( a < b < c \) and two curves \( \gamma_1: [a, b] \to \mathbb{C} \) and \( \gamma_2: [b, c] \to \mathbb{C} \) with \( \gamma_1(b) = \gamma_2(b) \), the *concatenation* of \( \gamma_1 \) and \( \gamma_2 \) is the curve
\[
\gamma_1 \oplus \gamma_2: [a, c] \to \mathbb{C}, \quad t \mapsto \begin{cases} 
\gamma_1(t), & t \in [a, b], \\
\gamma_2(t), & t \in [b, c]. 
\end{cases}
\]
If \( \gamma_1 \) and \( \gamma_2 \) are piecewise smooth, then so is \( \gamma_1 \oplus \gamma_2 \), and we have
\[
\int_{\gamma_1 \oplus \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f
\]
for each continuous \( f: \{\gamma_1\} \cup \{\gamma_2\} \to \mathbb{C} \).

2. For any curve \( \gamma: [a, b] \to \mathbb{C} \), the *reversed curve* is defined as
\[
\gamma^-: [a, b] \to \mathbb{C}, \quad t \mapsto \gamma(a + b - t).
\]
If \( \gamma \) is piecewise smooth, then so is \( \gamma^- \), and we have
\[
\int_{\gamma^-} f = -\int_{\gamma} f
\]
for each continuous \( f: \{\gamma\} \to \mathbb{C} \).
Proof of Proposition 4.9. (i) \implies (iii): Let \( z \in D \), and set

\[ U := \{ w \in D : \]

there is a piecewise smooth curve \( \gamma : [a, b] \to D \) with \( \gamma(a) = z \) and \( \gamma(b) = w \).

Obviously, \( U \neq \emptyset \) (because \( z \in U \)).

We claim that \( U \) is open. To see this, let \( w_0 \in U \), so that there is a piecewise smooth curve \( \gamma : [a, b] \to D \) with \( \gamma(a) = z \) and \( \gamma(b) = w_0 \). Choose \( \epsilon > 0 \) such that \( B_{\epsilon}(w_0) \subset D \). For \( w \in B_{\epsilon}(w_0) \), define

\[ [w_0, w] : [0, 1] \to \mathbb{C}, \quad t \mapsto w_0 + t(w - w_0).\]

Then \([w_0, w]\) is a curve in \( B_{\epsilon}(w_0) \subset D \) with initial point \( w_0 \) and endpoint \( w \). Consequently, \( \gamma + [w_0, w] \) is a piecewise smooth curve in \( D \) with initial point \( z \) and endpoint \( w \). It follows that \( w \in U \), and since \( w \in B_{\epsilon}(w_0) \) was arbitrary, we have \( B_{\epsilon}(w_0) \subset U \). This proves the openness of \( U \).

Next, we claim that \( D \setminus U \) is also open. To see this, let \( w_0 \in D \setminus U \), and let \( \epsilon > 0 \) be so small that \( B_{\epsilon}(w_0) \subset D \). Assume towards a contradiction that there is \( w \in B_{\epsilon}(w_0) \cap U \). Let \( \gamma : [a, b] \to D \) be a piecewise smooth curve with \( \gamma(a) = z \) and \( \gamma(b) = w \). Then \( \gamma + [w, w_0] \) is a piecewise smooth curve in \( D \) with initial point \( z \) and endpoint \( w_0 \), so that \( w_0 \in U \). This contradicts the choice of \( w_0 \in D \setminus U \). It follows that \( B_{\epsilon}(w_0) \cap U = \emptyset \), i.e., \( B_{\epsilon}(w_0) \subset D \).

Since \( U \) and \( D \setminus U \) are both open with \( U \cup (D \setminus U) = D \), the connectedness of \( D \) yields that \( D \setminus U = \emptyset \), i.e., \( D = U \).

(iii) \implies (ii): Let \( f : D \to \mathbb{C} \) be a locally constant function, and let \( z, w \in D \). Let \( \gamma : [a, b] \to D \) be a piecewise smooth curve with \( \gamma(a) = z \) and \( \gamma(b) = w \). Since \( f \) is locally constant, the function

\[ [a, b] \to \mathbb{C}, \quad t \mapsto f(\gamma(t)) \]

is differentiable with zero derivative and therefore constant. It follows that \( f(z) = f(\gamma(a)) = f(\gamma(b)) = f(w) \).

(ii) \implies (i): Suppose that \( D \) is not connected. Then there are non-empty open sets \( U, V \subset \mathbb{C} \) with \( U \cap V = \emptyset \) and \( U \cup V = D \). Define

\[ f : D \to \mathbb{C}, \quad z \mapsto \begin{cases} 0, & z \in U, \\ 1, & z \in V. \end{cases} \]

Then \( f \) is locally constant, but not constant. \( \square \)

Theorem 4.10. Let \( D \subset \mathbb{C} \) be open and connected, i.e., a region, and let \( f : D \to \mathbb{C} \) be continuous. Then the following are equivalent:
(i) $f$ has an anti-derivative;

(ii) $\int_{\gamma} f(\zeta) \, d\zeta = 0$ for any closed, piecewise smooth curve $\gamma$ in $D$;

(iii) for any piecewise smooth curve $\gamma$ in $D$, the value of $\int_{\gamma} f$ depends only on the initial point and the endpoint of $\gamma$.

Proof. (i) $\implies$ (ii) follows from Proposition 4.6.

(ii) $\implies$ (iii): Let $\gamma, \gamma': [a, b] \to D$ be piecewise smooth curves with $\gamma(a) = \gamma'(a)$ and $\gamma(b) = \gamma'(b)$. Then $\gamma \oplus (\gamma')^-$ is a closed, piecewise smooth curve, so that

$$0 = \int_{\gamma \oplus (\gamma')^-} f = \int_{\gamma} f + \int_{(\gamma')^-} f = \int_{\gamma} f - \int_{\gamma'},$$

(iii) $\implies$ (i): Fix $z_0 \in D$. For any $z \in D$ choose a piecewise smooth curve $\gamma_z: [a, b] \to D$ with $\gamma_z(a) = a_0$ and $\gamma_z(b) = z$. Define

$$F: D \to \mathbb{C}, \quad z \mapsto \int_{\gamma_z} f(\zeta) \, d\zeta.$$ We claim that $F$ is an anti-derivative of $f$. Let $z \in D$, and choose $\delta > 0$ such that $B_\delta(z) \subset D$. For $w \in B_\delta(z)$, thus have

$$F(w) = \int_{\gamma_w} f(\zeta) \, d\zeta$$

$$= \int_{\gamma_z \oplus [z, w]} f(\zeta) \, d\zeta, \quad \text{by (iii)},$$

$$= \int_{\gamma_z} f(\zeta) \, d\zeta + \int_{[z, w]} f(\zeta) \, d\zeta,$$

$$= F(z) + \int_{[z, w]} f(\zeta) \, d\zeta$$

which entails

$$\left| \frac{F(w) - F(z)}{w - z} - f(z) \right| = \frac{1}{|w - z|} \left| \int_{[z, w]} f - \int_{[z, w]} f(z) \right|$$

$$= \frac{1}{|w - z|} \left| \int_{[z, w]} (f - f(z)) \right|$$

$$\leq \frac{|w - z|}{|w - z|} \sup\{|f(\zeta) - f(z)| : \zeta \in ([z, w])\}$$

$$= \sup\{|f(\zeta) - f(z)| : \zeta \in ([z, w])\}.$$

Let $\epsilon > 0$ and choose $\delta > 0$ so small that $|f(\zeta) - f(z)| < \epsilon$ for all $\zeta \in D$ with $|\zeta - z| < \delta$. It follows from the foregoing that

$$\left| \frac{F(w) - F(z)}{w - z} - f(z) \right| \leq \epsilon$$

for all $w \in B_\delta(z)$. This, of course, means that $F'(z) = f(z)$. \qed
From now on, we shall use the word *curve* as shorthand for *piecewise smooth curve*.
5 Cauchy’s Integral Theorem and Formula

**Definition 5.1.** Let $D \subset \mathbb{C}$ be open. If $f : D \to \mathbb{C}$ is complex differentiable at each $z \in D$, then we call $f$ holomorphic (or analytic) on $D$.

Let $z_1$, $z_2$, and $z_3$ be three different points in $\mathbb{C}$. They span a triangle $\Delta$. Its boundary can be parametrized as a curve in counterclockwise orientation.

We denote this curve by $\partial \Delta$.

**Theorem 5.2** (Goursat’s “Lemma”). Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $\Delta \subset D$ be a triangle. Then we have

$$\int_{\partial \Delta} f(\zeta) \, d\zeta = 0.$$

**Proof.** We split up $\Delta$ into four subtriangles $\Delta^{(1)}$, $\Delta^{(2)}$, $\Delta^{(3)}$, and $\Delta^{(4)}$ as shown:
As the line segments in the interior of $\Delta$ also occur as their reversed paths, we have

$$\int_{\partial \Delta} f = \sum_{j=1}^{4} \int_{\partial \Delta(j)} f,$$

so that

$$\left| \int_{\partial \Delta} f \right| \leq \sum_{j=1}^{4} \left| \int_{\partial \Delta(j)} f \right|.$$

Choose $j \in \{1, 2, 3, 4\}$ such that $\left| \int_{\partial \Delta(j)} f \right|$ is largest, and set $\Delta_1 := \Delta(j)$. It follows that

$$\left| \int_{\partial \Delta} f \right| \leq 4 \left| \int_{\partial \Delta_1} f \right|;$$

also, note that $\ell(\partial \Delta_1) = \frac{1}{2} \ell(\partial \Delta)$.

Repeat this argument with $\Delta_1$ in place of $\Delta$, and obtain a triangle $\Delta_2 \subset \Delta_1$ with

$$\ell(\partial \Delta_2) = \frac{1}{2} \ell(\partial \Delta_1) = \frac{1}{4} \ell(\partial \Delta)$$

and

$$\left| \int_{\partial \Delta_1} f \right| \leq 4 \left| \int_{\partial \Delta_2} f \right|,$$

so that

$$\left| \int_{\partial \Delta} f \right| \leq 4 \left| \int_{\partial \Delta_1} f \right| \leq 16 \left| \int_{\partial \Delta_2} f \right|.$$

Inductively, we obtain triangles

$$\Delta \supset \Delta_1 \supset \Delta_2 \supset \cdots$$
with

$$\ell(\partial \Delta_n) = \frac{1}{2^n} \ell(\partial \Delta)$$

and

$$\left| \int_{\partial \Delta} f \right| \leq 4^n \left| \int_{\partial \Delta_n} f \right|$$

for \( n \in \mathbb{N} \).

Let \( z_0 \in \bigcap_{n=1}^{\infty} \Delta_n \), and define

$$r : D \to \mathbb{C}, \quad z \mapsto f(z) - f(z_0) - f'(z_0)(z - z_0),$$

so that \( \lim_{z \to z_0} \frac{|r(z)|}{|z - z_0|} = 0 \) and \( \int_{\gamma} r = \int_{\gamma} f \) for each closed curve \( \gamma \) in \( D \); hence, we have

$$\left| \int_{\partial \Delta} f \right| \leq 4^n \left| \int_{\partial \Delta_n} r \right|$$

for \( n \in \mathbb{N} \). Let \( \epsilon > 0 \) and choose \( \delta > 0 \) such that

$$\left| \frac{r(z)}{z - z_0} \right| \leq \frac{\epsilon}{\ell(\partial \Delta)^2}$$

for all \( z \in D \) with \( |z - z_0| < \delta \). Choose \( n \in \mathbb{N} \) such that \( \Delta_n \subset B_\delta(z_0) \). For \( z \in \Delta_n \), this means that

$$|z - z_0| \leq \ell(\partial \Delta_n) = \frac{1}{2^n} \ell(\partial \Delta).$$

We thus obtain:

$$\left| \int_{\partial \Delta} f \right| \leq 4^n \left| \int_{\partial \Delta_n} r \right| \leq 4^n \ell(\partial \Delta_n) \sup_{\zeta \in \partial \Delta_n} |r(\zeta)| = 2^n \ell(\partial \Delta) \sup_{\zeta \in \partial \Delta_n} |r(\zeta)| \leq 2^n \ell(\partial \Delta) \sup_{\zeta \in \partial \Delta_n} \frac{\epsilon}{\ell(\partial \Delta)^2} \frac{|\zeta - z_0|}{\ell(\partial \Delta)} \leq \epsilon.$$  

As \( \epsilon > 0 \) was arbitrary, this proves the claim. \( \Box \)

**Definition 5.3.** A set \( D \subset \mathbb{C} \) is called *star shaped* if there is \( z_0 \in D \) such that \( \{z_0 + t(z - z_0) : t \in [0, 1]\} \subset D \) for each \( z \in D \). The point \( z_0 \) is called a *center* for \( D \).

**Corollary 5.4.** Let \( D \subset \mathbb{C} \) be open and star shaped, and let \( f : D \to \mathbb{C} \) be holomorphic. Then \( f \) has an anti-derivative.
Proof. Let \( z_0 \in D \) be a center for \( D \). Define

\[
F: D \to \mathbb{C}, \quad z \mapsto \int_{[z_0,z]} f.
\]

Let \( z \in D \), and let \( \delta < 0 \) be such that \( B_\delta(z) \subset D \). Let \( w \in D \). Since \([z_0, z] \oplus [z, w] \oplus [w, z_0]\) is the boundary of a triangle \( \Delta \subset D \), Goursat’s lemma yields

\[
0 = \int_{[z_0,z] \oplus [z,w] \oplus [w,z_0]} f,
\]

so that

\[
F(w) = \int_{[z_0,w]} f = - \int_{[w,z_0]} f = \int_{[z_0,z]} f + \int_{[z,w]} f = F(z) + \int_{[z,w]} f.
\]

As in the proof of Theorem 4.10, we see then see that \( F \) is an anti-derivative for \( f \). \( \Box \)

Corollary 5.5. Let \( D \subset \mathbb{C} \) be open, and let \( f: D \to \mathbb{C} \) be holomorphic. Then, for each \( z_0 \in D \), there is an open neighborhood \( U \subset D \) of \( z_0 \) such that \( f|_U \) has an anti-derivative.

Corollary 5.6 (Cauchy’s Integral Theorem for Star Shaped Domains). Let \( D \subset \mathbb{C} \) be open and star shaped, and let \( f: D \to \mathbb{C} \) be holomorphic. Then \( \int_\gamma f(\zeta) \, d\zeta = 0 \) holds for each closed curve \( \gamma \) in \( D \).

Example. The sliced plane is defined as

\[
\mathbb{C}_- := \{ z \in \mathbb{C} : z \notin (-\infty, 0] \}.
\]

Then \( \mathbb{C}_- \) is star shaped (1 is a center, for instance). As seen in the proof of Corollary 5.4, the function

\[
\text{Log}: \mathbb{C}_- \to \mathbb{C}, \quad z \mapsto \int_{[1,z]} \frac{1}{\zeta} \, d\zeta
\]

is an anti-derivative of \( \mathbb{C}_- \ni z \mapsto \frac{1}{z} \); it is called the principal branch of the logarithm.

Let \( z \in \mathbb{C}_- \), and let \( \gamma_z \) be any curve in \( \mathbb{C}_- \) with initial point 1 and endpoint \( z \). From Theorem 4.10, we conclude that \( \text{Log} \, z = \int_{\gamma_z} \frac{1}{\zeta} \, d\zeta \).

For any \( z \in \mathbb{C}_- \), there is a unique \( \theta \in (-\pi, \pi) \)—the argument \( \arg z \) of \( z \)—such that \( z = |z|e^{i\theta} \). For \( z \in \mathbb{C}_- \) with \( \theta \geq 0 \), the curve

\[
\gamma: [0, \theta] \to \mathbb{C}, \quad t \mapsto |z|e^{it}.
\]

has the initial point \( |z| \) and the endpoint \( z \). It follows that \([1,|z|] \oplus \gamma\) is curve with initial point 1 and endpoint \( z \) as shown.
It follows that

\[
\log z = \int_{|1, z|} \frac{1}{\zeta} d\zeta + \int_{\gamma} \frac{1}{\zeta} d\zeta = \log |z| + i \int_0^\theta \frac{e^{it}}{e^t} dt = \log |z| + i \arg z.
\]

Similarly, we obtain the same equation for \( \arg z < 0 \).

**Lemma 5.7.** Let \( D \subset \mathbb{C} \) be open and star shaped with center \( z_0 \), and let \( f : D \to \mathbb{C} \) be continuous such that

\[
\int_{\partial \Delta} f(\zeta) d\zeta = 0
\]

for each triangle \( \Delta \subset D \) with \( z_0 \) as a corner. Then \( f \) has an anti-derivative.

**Proof.** This is proven just like Corollary 5.4. \( \square \)

**Lemma 5.8.** Let \( D \subset \mathbb{C} \) be open and star shaped with center \( z_0 \), and let \( f : D \to \mathbb{C} \) be continuous such that \( f|D\setminus\{z_0\} \) is holomorphic. Then \( f \) has an anti-derivative on \( D \).

**Proof.** We shall apply Lemma 5.7.

Let \( \Delta \) be a triangle in \( D \) with \( z_0 \) as a corner:
Let $z_1$ be an interior point on the line segment connecting $z_0$ and $w$, let $z_2$ be an interior point on the line segment connecting $z_0$ and $z$, and let $z_3$ be an interior point on the line segment connecting $w$ and $z$. Use these points to split $\Delta$ into four subtriangles—as shown above—, which we denote by $\Delta(z_0, z_1, z_2)$, $\Delta(z_1, z_3, z_2)$, $\Delta(z_1, w, z_3)$, and $\Delta(z_2, z_3, z)$. As in the proof of Goursat’s Lemma, we have

$$\int_{\partial \Delta} f = \int_{\partial \Delta(z_0, z_1, z_2)} f + \int_{\partial \Delta(z_1, z_3, z_2)} f + \int_{\partial \Delta(z_1, w, z_3)} f + \int_{\partial \Delta(z_2, z_3, z)} f.$$ 

Since $\Delta(z_1, z_3, z_2), \Delta(z_1, w, z_3), \Delta(z_2, z_3, z) \subset D \setminus \{z_0\}$, and since $f$ is holomorphic on $D \setminus \{z_0\}$, Goursat’s Lemma yields

$$\int_{\partial \Delta(z_1, z_3, z_2)} f = \int_{\partial \Delta(z_1, w, z_3)} f = \int_{\partial \Delta(z_2, z_3, z)} f = 0,$$

so that

$$\int_{\partial \Delta} f = \int_{\partial \Delta(z_0, z_1, z_2)} f.$$

It follows that

$$\left| \int_{\partial \Delta} f \right| = \left| \int_{\partial \Delta(z_0, z_1, z_2)} f \right| \leq \ell(\partial \Delta(z_0, z_1, z_2)) \sup_{\zeta \in \partial \Delta(z_0, z_1, z_2)} |f(\zeta)|.$$

Placing $z_1$ and $z_2$ sufficiently close to $z_0$, $\ell(\partial \Delta(z_0, z_1, z_2))$ can be made smaller than every $\epsilon > 0$, so that $\left| \int_{\partial \Delta} f \right| = 0$. \hfill $\square$

Let $z_0 \in \mathbb{C}$, and let $r > 0$. Slightly abusing notation, we use $\partial B_r(z_0)$ to denote the boundary of $B_r(z_0)$ in counterclockwise orientation.

**Lemma 5.9.** Let $D \subset \mathbb{C}$ be open, let $z_0 \in D$, and let $r > 0$ be such that $B_r[z_0] \subset D$. Then

$$\int_{\partial B_r(z_0)} \frac{1}{\zeta - z} \, d\zeta = 2\pi i$$

holds for all $z \in B_r(z_0)$.

**Proof.** Through direct computation, we saw that

$$\int_{\partial B_r(z_0)} \frac{1}{\zeta - z_0} \, d\zeta = 2\pi i.$$ 

Let $z \in B_r(z_0)$, and choose $\epsilon > 0$ such that $B_\epsilon[z] \subset B_r(z_0)$, so that

$$\int_{\partial B_\epsilon(z)} \frac{1}{\zeta - z} \, d\zeta = 2\pi i.$$ 

We need to show that

$$\int_{\partial B_r(z_0)} \frac{1}{\zeta - z} \, d\zeta + \int_{\partial B_\epsilon(z)} \frac{1}{\zeta - z} \, d\zeta = 0.$$

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This is how the situation looks like:

We connect the boundaries of $B_r(z_0)$ and $B_\epsilon(z)$ through line segments:

Consider the following two curves $\gamma_1$ and $\gamma_2$: 
Then it is clear that
\[
\int_{\partial B_r(z_0)} \frac{1}{\zeta - z} \, d\zeta + \int_{\partial B_r(z)} \frac{1}{\zeta - z} \, d\zeta = \int_{\gamma_1} \frac{1}{\zeta - z} \, d\zeta + \int_{\gamma_2} \frac{1}{\zeta - z} \, d\zeta.
\]

The sketches also show that there are star shaped open \( D_j \subset \mathbb{C} \setminus \{z\} \) with \( \{\gamma_j\} \subset D_j \) for \( j = 1, 2 \). Cauchy’s integral theorem for star shaped domains thus yields that
\[
\int_{\gamma_j} \frac{1}{\zeta - z} \, d\zeta = 0
\]
for \( j = 1, 2 \), which proves the claim. \( \square \)

**Theorem 5.10** (Cauchy’s Integral Formula for Circles). Let \( D \subset \mathbb{C} \) be open, let \( f: D \to \mathbb{C} \) be holomorphic, and let \( z_0 \in D \) and \( r > 0 \) be such that \( B_r(z_0) \subset D \). Then we have
\[
f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta
\]
for all \( z \in B_r(z_0) \).

**Remark.** The values of \( f \) on all of \( B_r(z_0) \) are already determined by those on \( \partial B_r(z_0) \).

**Proof of Theorem 5.10.** Let \( R > 0 \) be such that \( B_r(z_0) \subset B_R(z_0) \subset D \). Let \( z \in B_r(z_0) \), and note that \( z \) is a center for the star shaped domain \( B_R(z_0) \).

Define
\[
g: D \to \mathbb{C}, \quad w \mapsto \begin{cases} 
\frac{f(w) - f(z)}{w - z}, & w \neq z, \\
f'(z), & w = z.
\end{cases}
\]
Then $g$ is continuous and holomorphic on $D \setminus \{z\}$. By Lemma 5.8, $g$ has an anti-derivative on $B_r(z_0)$. Since $\partial B_r(z_0)$ is closed, this means that

$$0 = \int_{\partial B_r(z_0)} g(\zeta) \, d\zeta = \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta + f(z) \int_{\partial B_r(z_0)} \frac{1}{\zeta - z} \, d\zeta,$$

where the identity $\int_{\partial B_r(z_0)} \frac{1}{\zeta - z} \, d\zeta = 2\pi i$ holds by Lemma 5.9. Division by $2\pi i$ yields the claim.

**Corollary 5.11** (Mean Value Equation). Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $z_0 \in D$ and $r > 0$ be such that $B_r(z_0) \subset D$. Then we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) \, dt.$$

**Proof.** Parametrize $\partial B_r(z_0)$ as

$$\gamma : [0, 2\pi] \to \mathbb{C}, \quad t \mapsto z_0 + re^{it}.$$

The Cauchy Integral Formula then yields

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z_0} \, d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} \, ire^{it} \, dt = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) \, dt. \quad \square$$

**Lemma 5.12.** Let $D \subset \mathbb{R}^N$ be open, and let $f : [a, b] \times D \to \mathbb{R}$ be continuous such that $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N} : [a, b] \times D \to \mathbb{R}$ all exist and are continuous. Define

$$g : D \to \mathbb{R}, \quad x \mapsto \int_a^b f(t, x) \, dt.$$

Then $g$ is continuously partially differentiable with

$$\frac{\partial g}{\partial x_j}(x) = \int_a^b \frac{\partial f}{\partial x_j}(t, x) \, dt$$

for $x \in D$ and $j = 1, \ldots, N$.

**Proof.** There is no loss of generality supposing that $N = 1$.

Let $x_0, x \in D$ be such that every number between them is also in $D$. By the Mean Value Theorem of single variable calculus, there is, for each $t \in [a, b]$, an real number $\xi_t$ between $x_0$ and $x$ such that

$$f(t, x) - f(t, x_0) = \frac{\partial f}{\partial x}(t, \xi_t).$$
Let $\epsilon > 0$. By uniform continuity, there is $\delta > 0$ such that

$$\left| \frac{\partial f}{\partial x}(t, x_1) - \frac{\partial f}{\partial x}(t, x_2) \right| < \frac{\epsilon}{b-a}$$

for any $t \in [a, b]$ and all $x_1, x_2$ between $x_0$ and $x$ with $|x_1 - x_2| < \delta$.

Suppose that $|x_0 - x| < \delta$. Then we have:

$$\left| \frac{g(x) - g(x_0)}{x - x_0} - \int_a^b \frac{\partial f}{\partial x}(t, x_0) \, dt \right| = \left| \int_a^b \left( \frac{f(t, x) - f(t, x_0)}{x - x_0} - \frac{\partial f}{\partial x}(t, x_0) \right) \, dt \right|
\leq \int_a^b \left| \frac{f(t, x) - f(t, x_0)}{x - x_0} - \frac{\partial f}{\partial x}(t, x_0) \right| \, dt
= \int_a^b \left| \frac{\partial f}{\partial x}(t, \xi_t) - \frac{\partial f}{\partial x}(t, x_0) \right| \, dt
\leq \int_a^b \frac{\epsilon}{b-a} \, dt,
$$

because $|\xi_t - x_0| < \delta$.

This proves that $g$ is differentiable at $x_0$ with $g'(x_0) = \int_a^b \frac{\partial f}{\partial x}(t, x_0) \, dt$.

A similar (but easier) argument shows that $g'$ is continuous. \qed

**Corollary 5.13.** Let $D \subset \mathbb{C}$ be open, and let $f : [a, b] \times D \to \mathbb{C}$ be continuous such that $\frac{\partial f}{\partial z} : [a, b] \times D \to \mathbb{C}$ exists and is continuous. Define

$$g : D \to \mathbb{R}, \quad z \mapsto \int_a^b f(t, z) \, dt.$$  

Then $g$ is holomorphic with

$$g'(z) = \int_a^b \frac{\partial f}{\partial z}(t, z) \, dt$$

for $z \in D$.

**Proof.** Apply Lemma 5.12 to $\text{Re} \, f$ and $\text{Im} \, f$ and check that the Cauchy Riemann differential equations are satisfied. \qed

**Theorem 5.14.** Let $D \subset \mathbb{C}$ be open, let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$, and let $f : D \to \mathbb{C}$ be continuous such that

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

holds for all $z \in B_r(z_0)$. Then $f$ is infinitely often complex differentiable on $B_r(z_0)$ and satisfies

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta \quad (*)$$

holds for all $z \in B_r(z_0)$ and $n \in \mathbb{N}_0$.  

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Proof. We prove by induction on $n \in \mathbb{N}_0$: $f$ is $n$-times complex differentiable and $(*)$ holds.

For $n = 0$, the claim is clear, so suppose that it is true for some $n \in \mathbb{N}_0$. Define

$$F : [0, 2\pi] \times B_r(z_0) \to \mathbb{C}, \quad (t, z) \mapsto \frac{n!}{2\pi} \frac{f(z_0 + re^{it})re^{it}}{(z_0 + re^{it} - z)^{n+1}}.$$  

Then $F$ is continuous and, by the induction hypothesis, satisfies

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta = \int_0^{2\pi} F(t, z) \, dt$$

for all $z \in B_r(z_0)$. Furthermore,

$$\frac{\partial F}{\partial z}(t, z) = \frac{(n + 1)!}{2\pi} \frac{f(z_0 + re^{it})re^{it}}{(z_0 + re^{it} - z)^{n+2}}$$

is continuous on $[0, 2\pi] \times B_r(z_0)$. From Corollary 5.13, we thus conclude that $f^{(n)}$ is holomorphic on $D$ with

$$f^{(n+1)}(z) = \int_0^{2\pi} \frac{\partial F}{\partial z}(t, z) \, dt = \frac{(n + 1)!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+2}} d\zeta. \quad \square$$

**Corollary 5.15** (Generalized Cauchy Integral Formula). Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be holomorphic. Then $f$ is infinitely often complex differentiable on $D$. Moreover, for any $z_0 \in D$ and $r > 0$ such that $B_r(z_0) \subset D$, the generalized Cauchy integral formula holds, i.e.,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

for all $z \in B_r(z_0)$ and $n \in \mathbb{N}_0$.

**Example.** We shall use Cauchy’s integral theorem and formula to evaluate the line integral

$$\int_\gamma \frac{e^\zeta}{\zeta(\zeta - 1)} \, d\zeta$$

for various curves $\gamma$:

(a) $\gamma = \partial B_\pi(-3)$: The function

$$f : \mathbb{C} \setminus \{1\} \to \mathbb{C}, \quad z \mapsto \frac{e^z}{z - 1}$$

is holomorphic, and we have $B_\pi(-3) \subset \mathbb{C} \setminus \{1\}$. Cauchy’s integral formula thus yields:

$$\int_{\partial B_\pi(-3)} \frac{e^\zeta}{\zeta(\zeta - 1)} \, d\zeta = \int_{\partial B_\pi(-3)} \frac{f(\zeta)}{\zeta} \, d\zeta = 2\pi i f(0) = -2\pi i.$$
(b) $\gamma = \partial B_{2}(i)$: As the integrand is holomorphic in on the star-shaped domain $B_{2}(i)$, Cauchy’s integral theorem yields that
\[
\int_{\partial B_{2}(i)} \frac{e^{\zeta}}{\zeta(\zeta - 1)} \, d\zeta = 0.
\]

(c) $\gamma = \partial B_{2}(0)$: Partial fractions yield
\[
\frac{1}{z(z - 1)} = \frac{1}{z - 1} - \frac{1}{z}.
\]
Since $0, 1 \in B_{2}(0)$, we obtain with the help of Cauchy’s integral formula:
\[
\int_{\partial B_{2}(0)} \frac{e^{\zeta}}{\zeta(\zeta - 1)} \, d\zeta = \int_{\partial B_{2}(0)} \frac{e^{\zeta}}{\zeta - 1} \, d\zeta - \int_{\partial B_{2}(0)} \frac{e^{\zeta}}{\zeta} \, d\zeta = 2\pi i (e - 1).
\]

**Characterizations of Holomorphic Functions.** Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be continuous. Then the following are equivalent:

(i) $f$ is holomorphic;

(ii) the Morera condition holds, i.e., $\int_{\partial \Delta} f(\zeta) \, d\zeta = 0$ for each triangle $\Delta \subset D$;

(iii) for each $z_{0} \in D$ and $r > 0$ with $B_{r}[z_{0}] \subset D$, we have
\[
f(z) = \frac{1}{2\pi i} \int_{\partial B_{r}(z_{0})} \frac{f(\zeta)}{\zeta - z} \, d\zeta
\]
for $z \in B_{r}(z_{0})$;

(iv) for each $z_{0} \in D$, there is $r > 0$ with $B_{r}[z_{0}] \subset D$ and
\[
f(z) = \frac{1}{2\pi i} \int_{\partial B_{r}(z_{0})} \frac{f(\zeta)}{\zeta - z} \, d\zeta
\]
for $z \in B_{r}(z_{0})$;

(v) $f$ is infinitely often complex differentiable on $D$;

(vi) for each $z_{0} \in D$, there is an open neighborhood $U \subset D$ of $z_{0}$ such that $f$ has an anti-derivative on $U$.

**Proof.** (i) $\implies$ (ii) is Goursat’s Lemma.

(i) $\implies$ (iii) is the Cauchy Integral Formula for circles, and (iii) $\implies$ (iv) is trivial.

(iv) $\implies$ (v) follows immediately from Theorem 5.14, and (v) $\implies$ (i) is again trivial.

(ii) $\implies$ (vi) follows from Lemma 5.7 because every $z_{0} \in D$ has an open, star shaped neighborhood contained in $D$. 

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(vi) $\implies$ (v): Let $z_0 \in D$, and let $U \subset D$ be an open neighborhood of $z_0$ such that $f$ has an anti-derivative, say $F$, on $U$. Then $F$ is holomorphic on $U$. Applying (i) $\implies$ (v) to $F$, we see that $F$ is infinitely often complex differentiable on $U$. Consequently, $f = F'$ is infinitely complex differentiable on $U$. Since $z_0 \in D$ was arbitrary, we conclude that $f$ is infinitely often complex differentiable on $D$.

We conclude this section with Liouville’s Theorem and its application to the Fundamental Theorem of Algebra.

**Definition 5.16.** A holomorphic function defined on all of $\mathbb{C}$ is called *entire*.

**Theorem 5.17** (Liouville’s Theorem). Let $f : \mathbb{C} \to \mathbb{C}$ be a bounded entire function. Then $f$ is constant.

**Proof.** We will show that $f' \equiv 0$.

Let $C \geq 0$ be such that $|f(z)| \leq C$ for all $z \in \mathbb{C}$. Let $z \in \mathbb{C}$ be arbitrary, and let $r > 0$. By the generalized Cauchy integral formula, we have

$$|f'(z)| = \frac{1}{2\pi} \left| \int_{\partial B_r(z)} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta \right| \leq \frac{1}{2\pi} \ell(\partial B_r(z)) \sup_{\zeta \in \partial B_r(z)} \left| \frac{f(\zeta)}{|\zeta - z|^2} \right| \leq \frac{1}{2\pi} \frac{2\pi r C}{r^2} = \frac{C}{r}.$$  

Letting $r \to \infty$, we obtain $f'(z) = 0$. This completes the proof.

**Corollary 5.18** (Fundamental Theorem of Algebra). Let $p$ be a non-constant polynomial with complex coefficients. Then $p$ has a zero.

**Proof.** Assume that $p$ has no zero. Then the function

$$f : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \frac{1}{p(z)}$$

is entire. Since $p$ is a polynomial, we have $\lim_{|z| \to \infty} |p(z)| = \infty$ and thus $\lim_{|z| \to \infty} |f(z)| = 0$. Let $R > 0$ be such that $|f(z)| \leq 1$ for $|z| > R$. Since $f$ is continuous it is bounded on $B_R[0]$, and by the choice of $R$, it is bounded on $\mathbb{C} \setminus B_R[0]$, too, and thus bounded on all of $\mathbb{C}$. By Liouville’s Theorem, $f$ is thus constant, and so is therefore $p$, which is a contradiction.
6 Convergence of Holomorphic Functions

Just as a reminder:

**Definition 6.1.** Let \( D \subset \mathbb{C} \) be open. Then a sequence \((f_n)_{n=1}^{\infty}\) of \( \mathbb{C} \)-valued functions on \( D \) is said to converge **uniformly** on \( D \) to \( f : D \to \mathbb{C} \) if, for each \( \epsilon > 0 \), there is \( n_\epsilon \in \mathbb{N} \) such that \(|f_n(z) - f(z)| < \epsilon\) for all \( n \geq n_\epsilon \) and all \( z \in D \).

We recall the following theorem, which we won’t prove:

**Theorem 6.2.** Let \( D \subset \mathbb{C} \) be open, and let \((f_n)_{n=1}^{\infty}\) of continuous, \( \mathbb{C} \)-valued functions on \( D \) converging uniformly on \( D \) to \( f : D \to \mathbb{C} \). Then \( f \) is continuous.

We now “localize” the notion of uniform convergence:

**Definition 6.3.** Let \( D \subset \mathbb{C} \) be open. Then a sequence \((f_n)_{n=1}^{\infty}\) of \( \mathbb{C} \)-valued functions on \( D \) is said to converge **locally uniformly** on \( D \) to \( f : D \to \mathbb{C} \) if, for each \( z_0 \in D \), there is an open neighborhood \( U \subset D \) of \( z_0 \) such that \((f_n|_U)_{n=1}^{\infty}\) converges to \( f|_U \) uniformly on \( U \).

**Proposition 6.4.** Let \( D \subset \mathbb{C} \) be open, and let \((f_n)_{n=1}^{\infty}\) of continuous, \( \mathbb{C} \)-valued functions on \( D \) converging locally uniformly on \( D \) to \( f : D \to \mathbb{C} \). Then \( f \) is continuous.

**Proof.** Let \( z_0 \in D \), and let \( U \subset D \) be an open neighborhood of \( z_0 \) such that \( f_n|_U \to f|_U \) uniformly on \( U \). By Theorem 6.2, \( f|_U \) is continuous. Hence, \( f \) is continuous at \( z_0 \). \( \Box \)

**Proposition 6.5.** Let \( D \subset \mathbb{C} \) be open, and let \( f, f_1, f_2, \ldots : D \to \mathbb{C} \) be functions. Then the following are equivalent:

(i) \((f_n)_{n=1}^{\infty}\) converges to \( f \) locally uniformly on \( D \);

(ii) for each compact \( K \subset D \), the sequence \((f_n|_K)_{n=1}^{\infty}\) converges to \( f|_K \) uniformly on \( K \).

**Proof.** (i) \(\implies\) (ii): Let \( K \subset D \) be compact. For each \( z \in K \), there is an open neighborhood \( U_z \subset D \) of \( z \) such that \( f_n|_{U_z} \to f|_{U_z} \) uniformly on \( U_z \). Since \( K \) is compact, there are \( z_1, \ldots, z_m \in K \) such that

\[
K \subset U_{z_1} \cup \cdots \cup U_{z_m}.
\]

Let \( \epsilon > 0 \). For each \( j = 1, \ldots, m \), there is \( n_j \in \mathbb{N} \) such that \(|f_n(z) - f(z)| < \epsilon\) for all \( n \geq n_j \) and all \( z \in U_{z_j} \). Set \( n_\epsilon := \max\{n_1, \ldots, n_m\} \). Then \(|f_n(z) - f(z)| < \epsilon\) holds for all \( n \geq n_\epsilon \) and \( z \in K \).

(ii) \(\implies\) (i): Let \( z_0 \in D \), and let \( r > 0 \) be such that \( B_r[z_0] \subset D \). Since \( B_r[z_0] \) is compact, \((f_n|_{B_r[z_0]})_{n=1}^{\infty}\) converges uniformly on \( B_r[z_0] \) to \( f|_{B_r[z_0]} \). Trivially, \((f_n|_{B_r[z_0]})_{n=1}^{\infty}\) thus converges uniformly on \( B_r(z_0) \) to \( f|_{B_r(z_0)} \). \( \Box \)

Instead of locally uniform convergence, we therefore often speak of **compact convergence**.
Lemma 6.6. Let $D \subset \mathbb{C}$ be open, let $\gamma$ be a curve in $D$, and let $f,f_1,f_2,\ldots : D \to \mathbb{C}$ be continuous functions such that $(f_n|_{\gamma})_{n=1}^{\infty}$ converges to $f|_{\gamma}$ uniformly on $\{\gamma\}$. Then we have

$$\int_{\gamma} f(\zeta) \, d\zeta = \lim_{n \to \infty} \int_{\gamma} f_n(\zeta) \, d\zeta.$$ 

Proof. Let $\epsilon > 0$, and choose $n_{\epsilon} \in \mathbb{N}$ such that

$$|f_n(\zeta) - f(\zeta)| < \frac{\epsilon}{\ell(\gamma) + 1}$$

for all $n \geq n_{\epsilon}$ and $z \in \{\gamma\}$. For $n \geq n_{\epsilon}$, we thus obtain:

$$\left|\int_{\gamma} f_n - \int_{\gamma} f\right| \leq \ell(\gamma) \sup \{|f_n(\zeta) - f(\zeta)| : \zeta \in \{\gamma\}\} \leq \frac{\epsilon \ell(\gamma)}{\ell(\gamma) + 1} < \epsilon.$$

Corollary 6.7. Let $D \subset \mathbb{C}$ be open, let $\gamma$ be a curve in $D$, and let $f_1,f_2,\ldots : D \to \mathbb{C}$ be continuous functions converging compactly to $f : D \to \mathbb{C}$. Then $f$ is continuous, and we have

$$\int_{\gamma} f(\zeta) \, d\zeta = \lim_{n \to \infty} \int_{\gamma} f_n(\zeta) \, d\zeta.$$ 

Proof. If $\gamma : [a,b] \to \mathbb{C}$ is a curve—and thus continuous—, then $\{\gamma\} = \gamma([a,b])$ is compact. Hence, Lemma 6.6 applies.

Theorem 6.8 (Weierstraß' Theorem). Let $D \subset \mathbb{C}$ be open, let $f_1,f_2,\ldots : D \to \mathbb{C}$ be holomorphic such that $(f_n)_{n=1}^{\infty}$ converges to $f : D \to \mathbb{C}$ compactly. Then $f$ is holomorphic, and $(f_n^{(k)})_{n=1}^{\infty}$ converges compactly to $f^{(k)}$ for each $k \in \mathbb{N}$.

Proof. Clearly, $f$ is continuous.

To see that $f$ is holomorphic, let $\Delta \subset D$ be a triangle. By Goursat’s Lemma, $\int_{\partial \Delta} f_n(\zeta) \, d\zeta = 0$ holds for all $n \in \mathbb{N}$. From Corollary 6.7, we conclude that

$$\int_{\partial \Delta} f(\zeta) \, d\zeta = \lim_{n \to \infty} \int_{\partial \Delta} f_n(\zeta) \, d\zeta = 0,$$

i.e., $f$ satisfies the Morera condition and thus is holomorphic.

Let $z_0 \in D$, and let $0 < r < R$ be such that $B_r[z_0] \subset B_R(z_0) \subset B_R[z_0] \subset D$. For any
Let \( z \in B_r(z_0) \), we have

\[
|f'_n(z) - f'(z)| = \frac{1}{2\pi} \left| \int_{\partial B_R(z_0)} \frac{f_n(\zeta) - f(\zeta)}{\zeta - z} \, d\zeta \right|
\]

\[
\leq \frac{1}{2\pi} \left( \int_{\partial B_R(z_0)} |f(\zeta)| \, d\zeta \right) \left( \int_{\partial B_R(z_0)} \frac{1}{|\zeta - z|^2} \, d\zeta \right)
\]

\[
\leq \frac{R}{(R - r)^2} \sup_{\zeta \in \partial B_R(z_0)} |f_n(\zeta) - f(\zeta)|.
\]

Let \( \epsilon > 0 \), and choose \( n_\epsilon \in \mathbb{N} \) such that

\[
|f_n(\zeta) - f(\zeta)| < \epsilon \frac{(R - r)^2}{R}
\]

for all \( n \geq n_\epsilon \) and \( \zeta \in \partial B_R(z_0) \). Then it follows from the above estimates that \( |f'_n(z) - f'(z)| \leq \epsilon \) for all \( n \geq n_\epsilon \) and \( z \in B_r(z_0) \). Consequently, \( (f'_n|_{B_r(z_0)})_{n=1}^\infty \) converges to \( f'|_{B_r(z_0)} \) uniformly on \( B_r(z_0) \). As \( z_0 \in D \) is arbitrary, this means that \( (f'_n)_{n=1}^\infty \) converges to \( f \) locally uniformly, i.e., compactly, on \( D \).

For higher derivatives, the claim now follows by induction. \( \square \)

**Lemma 6.9.** Let \( z_0 \in \mathbb{C} \), let \( r > 0 \), and let \( z \in B_r(z_0) \). Then

\[
\frac{1}{\zeta - z} = \sum_{n=0}^\infty \frac{1}{(\zeta - z_0)^{n+1}} (z - z_0)^n
\]

holds for all \( \zeta \in \partial B_r(z_0) \) with absolute and uniform converges on \( \partial B_r(z_0) \).

**Proof.** Let \( \zeta \in \partial B_r(z_0) \), and note that

\[
\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}
\]

Since \( |z - z_0| < r \) and \( |\zeta - z_0| = r \), we have \( \frac{z - z_0}{\zeta - z_0} \) < 1, so that

\[
\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0)} \sum_{n=0}^\infty \left( \frac{z - z_0}{\zeta - z_0} \right)^n = \sum_{n=1}^\infty \frac{1}{(z - z_0)^{n+1}} (z - z_0)^n.
\]

Since \( \frac{(z-z_0)^n}{(\zeta-z_0)^{n+1}} = \frac{|z-z_0|^n}{|\zeta-z_0|^{n+1}} \) and \( \sum_{n=0}^\infty \frac{|z-z_0|^n}{|\zeta-z_0|^{n+1}} < \infty \), the Weierstraß M-test yields absolute and uniform convergence on \( \partial B_r(z_0) \). \( \square \)

**Theorem 6.10.** Let \( D \subset \mathbb{C} \) be open. Then the following are equivalent for \( f : D \to \mathbb{C} \):

(i) \( f \) is holomorphic;

(ii) for each \( z_0 \in D \), there are \( r > 0 \) with \( B_r(z_0) \subset D \) and \( a_0, a_1, a_2, \ldots \in \mathbb{C} \) such that

\[
f(z) = \sum_{n=0}^\infty a_n(z - z_0)^n \text{ for all } z \in B_r(z_0);
\]

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(iii) for each \( z_0 \in D \) and \( r > 0 \) with \( B_r(z_0) \subset D \), we have

\[
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n
\]

for all \( z \in B_r(z_0) \).

Proof. (iii) \( \Rightarrow \) (ii) \( \Rightarrow \) (i) are clear.

(i) \( \Rightarrow \) (iii): Let \( z_0 \in D \), and let \( r > 0 \) be such that \( B_r(z_0) \subset D \). Let \( z \in B_r(z_0) \) and choose \( \rho \in (0, r) \) such that \( z \in B_\rho(z) \). Then we have:

\[
f(z) = \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta
\]

\[
= \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} (z - z_0)^n \, d\zeta, \quad \text{by Lemma 6.9,}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \right) (z - z_0)^n, \quad \text{by Lemma 6.6,}
\]

\[
= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n. \quad \square
\]
7 Elementary Properties of Holomorphic Functions

**Theorem 7.1** (Identity Theorem). Let $D \subset \mathbb{C}$ be open and connected, and let $f, g : D \to \mathbb{C}$ be holomorphic. Then the following are equivalent:

(i) $f = g$;

(ii) the set $\{ z \in D : f(z) = g(z) \}$ has a cluster point in $D$;

(iii) there is $z_0 \in D$ such that $f^{(n)}(z_0) = g^{(n)}(z_0)$.

**Proof.** Without loss of generality, let $g \equiv 0$.

(i) $\implies$ (iii) is trivial.

(iii) $\implies$ (ii): Let $z_0 \in D$ be as in (iii), and let $r > 0$ be such that $B_r(z_0) \subset D$. Then we have by Theorem 6.10 that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = 0$$

for all $z \in B_r(z_0)$, so that

$$B_r(z_0) \subset Z(f) := \{ z \in D : f(z) = 0 \}.$$ 

Every point in $B_r(z_0)$ is a cluster point of $Z(f)$.

(ii) $\implies$ (i): Let

$$V := \{ z \in D : z \text{ is a cluster point of } Z(f) \},$$

so that $V \neq \emptyset$ by (ii).

We claim that $V$ is open. Let $z_0 \in V$, and let $r > 0$ be such that $B_r(z_0) \subset D$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

holds for some $a_0, a_1, a_2, \ldots \in \mathbb{C}$ and all $z \in B_r(z_0)$. We claim that $B_r(z_0) \subset Z(f)$, so that $B_r(z_0) \subset V$. Assume that $B_r(z_0) \not\subset Z(f)$. Then there must be $m \in \mathbb{N}_0$ such that $a_m \neq 0$; let $m \in \mathbb{N}_0$ be minimal with this property. As $z_0$ is a cluster point of $Z(f)$, there is a sequence $(z_k)_{k=1}^{\infty}$ in $Z(f) \setminus \{ z_0 \}$ such that $z_0 = \lim_{k \to \infty} z_k$. We obtain:

$$a_m = \lim_{k \to \infty} \sum_{n=m}^{\infty} a_n(z_k - z_0)^{n-m}$$

$$= \lim_{k \to \infty} \frac{1}{(z_k - z_0)^m} \sum_{n=0}^{\infty} a_n(z_k - z_0)^n$$

$$= \lim_{k \to \infty} \frac{f(z_k)}{(z_k - z_0)^m}$$

$$= 0,$$
which is a contradiction. We conclude that \( a_0 = a_1 = a_2 = \cdots = 0 \), so that \( B_r(z_0) \subset \mathbb{Z}(f) \).

We now claim that \( D \setminus V \) is also open. Let \( z_0 \in D \setminus V \), and \( \epsilon > 0 \) be such that 
\[
B_\epsilon(z_0) \subset D \quad \text{and} \quad (B_\epsilon(z_0) \setminus \{z_0\}) \cap \mathbb{Z}(f) = \emptyset.
\]
For any \( z \in B_\epsilon(z_0) \setminus \{z_0\} \) and \( \delta > 0 \) such that 
\[
B_\delta(z) \subset B_\epsilon(z_0) \setminus \{z_0\},
\]
we thus have 
\[
(B_\delta(z) \setminus \{z\}) \cap \mathbb{Z}(f) = \emptyset.
\]
It follows that \( z \notin V \).

Consequently, \( B_\epsilon(z_0) \subset D \setminus V \).

All in all, \( V \) and \( D \setminus V \) are both open and clearly satisfy \( D = V \cup (D \setminus V) \) and 
\[
V \cap (D \setminus V) = \emptyset.
\]
The connectedness of \( D \) yields \( D = V \) and thus \( \mathbb{Z}(f) = D \). \( \square \)

**Examples.**

1. There is no non-zero entire function \( f : \mathbb{C} \rightarrow \mathbb{C} \) such that \( f \left( \frac{1}{n} \right) = 0 \) for 
   \( n \in \mathbb{N} \). For any entire function \( f \) with this property, we have \( f(0) = 0 \) by continuity, 
   so that \( 0 \) is a cluster point of \( \mathbb{Z}(f) \). By the Identity Theorem, this means \( f \equiv 0 \).

2. Since \( \mathbb{R} \) has cluster points in \( \mathbb{C} \), the holomorphic extensions of \( \exp, \cos, \) and \( \sin \) from \( \mathbb{R} \) to \( \mathbb{C} \) are unique. For analogous reasons, \( \log \) is the only holomorphic extension of 
   \( \log \) to \( \mathbb{C}_- \).

3. There is no entire function \( f \) such that 
\[
f \left( \frac{1}{n} \right) = \frac{1}{n} \quad \text{and} \quad f \left( -\frac{1}{n} \right) = \frac{1}{n^2}
\]
for \( n \in \mathbb{N} \). In the view of the identity theorem, the first condition necessitates that 
\( f(z) = z \) whereas the second one implies that \( f(z) = (-z)^2 \) for all \( z \in \mathbb{C} \).

**Lemma 7.2.** Let \( D \subset \mathbb{C} \) be open, let \( f : D \rightarrow \mathbb{C} \) be holomorphic, and let \( z_0 \in D \) and \( r > 0 \) 
be such that \( B_r[z_0] \subset D \). Suppose that 
\[
|f(z_0)| < \inf_{z \in \partial B_r(z_0)} |f(z)|.
\]
Then \( f \) has a zero in \( B_r(z_0) \).

**Proof.** Assume otherwise, i.e., \( f \) has no zero in \( B_r(z_0) \). The hypothesis implies that \( f \) has no zero on \( \partial B_r(z_0) \), so that \( f \) thus has no zero in \( B_r[z_0] \). Assume that, for each \( R > 0 \) such that 
\( B_r[z_0] \subset B_R(z_0) \subset D \), there is a zero of \( f \) in \( B_R(z_0) \). Then we have a sequence \( (R_n)_{n=1}^\infty \) in \((r, \infty)\) with 
\( r = \lim_{n \to \infty} R_n \) such that 
\[
B_r[z_0] \subset B_{R_n}(z_0) \subset D
\]
and \( \mathbb{Z}(f) \cap B_{R_n}(z_0) \neq \emptyset \) for each \( n \in \mathbb{N} \). For each \( n \in \mathbb{N} \), pick \( z_n \in \mathbb{Z}(f) \cap B_{R_n}(z_0) \).

Then \( (z_n)_{n=1}^\infty \) is bounded, and thus has a convergent subsequence \((z_{n_k})_{k=1}^\infty \) with limit \( z' \). 
Clearly, \( z' \in \mathbb{Z}(f) \), and since \( \lim_{k \to \infty} R_{n_k} = r \), we have \( z' \in B_r[z_0] \), which is impossible. 
Consequently, \( f \) has no zero on some \( B_R(z_0) \) with \( B_r[z_0] \subset B_R(z_0) \subset D \). Replacing \( D \) 
with \( B_R(z_0) \), we can thus suppose that \( f \) has no zeros.
From the Cauchy Integral Formula, we obtain
\[
\frac{1}{|f(z_0)|} = \left| \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{1}{f(\zeta)} \frac{1}{\zeta - z_0} d\zeta \right| \\
\leq \frac{1}{2\pi} 2\pi r \sup_{\zeta \in \partial B_r(z_0)} \frac{1}{|f(\zeta)| r} \\
= \frac{\inf_{\zeta \in \partial B_r(z_0)} |f(\zeta)|}{r}
\]
and thus
\[
|f(z_0)| \geq \inf_{\zeta \in \partial B_r(z_0)} |f(\zeta)|,
\]
which is a contradiction. \(\square\)

**Theorem 7.3** (Open Mapping Theorem). Let \(D \subset \mathbb{C}\) be open and connected, and let \(f: D \to \mathbb{C}\) be holomorphic and not constant. Then \(f(D) \subset \mathbb{C}\) is open and connected.

**Proof.** By the continuity of \(f\), it is clear that \(f(D)\) is connected.

Let \(w_0 \in f(D)\), and let \(z_0 \in D\) be such that \(w_0 = f(z_0)\). Choose \(r > 0\) such that \(B_r[z_0] \subset D\) and such that \(\{z \in B_r[z_0] : f(z) = w_0\} = \{w_0\}\). (This can be accomplished with the help of the Identity Theorem.) Choose \(\epsilon > 0\) such that \(|f(z) - w_0| \geq 3\epsilon\) for all \(z \in \partial B_r(z_0)\). We claim that \(B_r(w_0) \subset f(D)\). Let \(w \in B_r(w_0)\). For \(z \in \partial B_r(z_0)\), we have
\[
|f(z) - w| \geq |f(z) - w_0| - |w - w_0| \geq 3\epsilon - \epsilon = 2\epsilon.
\]
It follows that
\[
|f(z_0) - w| < \epsilon < 2\epsilon \leq \inf_{z \in \partial B_r(z_0)} |f(z) - w|.
\]
By Lemma 7.2, this means that
\[
D \to \mathbb{C}, \quad z \mapsto f(z) - w
\]
is constant. It follows that \(w \in f(D)\). \(\square\)

**Theorem 7.4** (Maximum Modulus Principle). Let \(D \subset \mathbb{C}\) be open and connected, and let \(f: D \to \mathbb{C}\) be holomorphic such that the function
\[
|f|: D \to \mathbb{C}, \quad z \mapsto |f(z)|
\]
attains a local maximum on \(D\). Then \(f\) is constant.

**Proof.** Let \(z_0 \in D\) be such that \(|f|\) attains a local maximum at \(z_0\), i.e., there is \(\epsilon > 0\) such that \(B_r(z_0) \subset D\) and \(|f(z_0)| \geq |f(z)|\) for all \(z \in B_r(z_0)\). Then \(f(z_0)\) is not an interior point of \(f(B_r(z_0))\), so that \(|f|_{B_r(z_0)}\) is constant by the Open Mapping Theorem. The Identity Theorem then yields that \(f\) is constant. \(\square\)
Corollary 7.5. Let \( D \subset \mathbb{C} \) be open and connected, and let \( f : D \to \mathbb{C} \) be holomorphic such that \( |f| \) attains a local minimum on \( D \). Then \( f \) is constant or \( f \) has a zero.

Proof. Suppose that \( f \) has no zero. Applying the Maximum Modulus Principle to \( \frac{1}{f} \) then yields that \( f \) is constant. \( \square \)

Corollary 7.6 (Maximum Modulus Principle for Bounded Domains). Let \( D \subset \mathbb{C} \) be open, connected, and bounded, and let \( f : \overline{D} \to \mathbb{C} \) be continuous such that \( f\mid_D \) is holomorphic. Then \( |f| \) attains its maximum on \( \overline{D} \) on \( \partial D \).

Proof. The claim is trivial if \( f \) is constant, so suppose that \( f \) is not constant.

Since \( f \) is continuous and \( \overline{D} \) is compact, there is a point \( z_0 \in \overline{D} \) with \( |f(z_0)| = \max \{|f(z)| : z \in \overline{D}\} \). If \( z_0 \in D \), then \( |f| \) would attain a local maximum at \( z_0 \), which is impossible by the Maximum Modulus Principle. Therefore \( z_0 \in \partial D \) must hold. \( \square \)

From now on, we shall use \( \mathbb{D} \) to denote the open unit disc \( B_1(0) \).

Theorem 7.7 (Schwarz’ “Lemma”). Let \( f : \mathbb{D} \to \mathbb{C} \) be holomorphic such that \( f(\mathbb{D}) \subset \mathbb{D} \) and \( f(0) = 0 \). Then one has

\[
|f(z)| \leq |z| \quad \text{for} \quad z \in \mathbb{D} \quad \text{and} \quad |f'(0)| \leq 1.
\]

Moreover, if there is \( z_0 \in \mathbb{D} \setminus \{0\} \) such that \( |f(z_0)| = |z_0| \) or if \( |f'(0)| = 1 \), then the there is \( c \in \mathbb{C} \) with \( |c| = 1 \) such that \( f(z) = cz \) for \( z \in \mathbb{D} \).

Proof. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be the power series expansion of \( f \). Since \( f(0) = 0 \), we have \( a_0 = 0 \). Define

\[
g : \mathbb{D} \to \mathbb{C}, \quad z \mapsto \sum_{n=1}^{\infty} a_n z^{n-1}.
\]

Then \( g \) is holomorphic with \( g(0) = a_1 = f'(0) \) and \( f(z) = zg(z) \) for \( z \in \mathbb{D} \). Let \( r \in (0, 1) \). Then we have

\[
|g(z)| = \frac{|f(z)|}{r} \leq \frac{1}{r}
\]

for \( z \in \partial B_r(0) \) and thus for all \( z \in B_r[0] \) by the Maximum Modulus Principle. Letting \( r \to 1 \), we obtain that \( |g(z)| \leq 1 \) for all \( z \in \mathbb{D} \) and thus \( |f(z)| \leq |z| \) for all \( z \in \mathbb{D} \) as well as \( |f'(0)| = |g(0)| = 1 \).

Suppose that there is \( z_0 \in \mathbb{D} \setminus \{0\} \) such that \( |f(z_0)| = |z_0| \) or \( |f'(0)| = |g(0)| = 1 \). Then \( |g| \) has a maximum at \( z_0 \) or 0, respectively, so that \( g \) is constant. Hence, there is \( c \in \mathbb{C} \) with \( |c| = 1 \) such that \( f(z) = zg(z) = cz \) for \( z \in \mathbb{D} \). \( \square \)

Definition 7.8. Let \( D_1, D_2 \subset \mathbb{C} \) be open. Then \( f : D_1 \to D_2 \) is called biholomorphic (or conformal) if

(a) \( f \) is bijective and
(b) both \( f \) and \( f^{-1} \) are holomorphic.

**Corollary 7.9.** Let \( f: \mathbb{D} \to \mathbb{D} \) be biholomorphic such that \( f(0) = 0 \). Then there is \( c \in \mathbb{C} \) with \( |c| = 1 \) such that \( f(z) = cz \) for \( z \in \mathbb{D} \).

**Proof.** Let \( z \in \mathbb{D} \). Then \( |f(z)| \leq |z| \) holds by Schwarz’ Lemma as does

\[
|z| = |f^{-1}(f(z))| \leq |f(z)|.
\]

The claim then follows from Schwarz’ Lemma. \( \square \)

**Lemma 7.10.** Let \( w \in \mathbb{D} \), and define

\[
\phi_w: \mathbb{D} \to \mathbb{C}, \quad z \mapsto \frac{z - w}{\bar{w}z - 1}.
\]

Then:

(i) \( \phi_w \) maps \( \mathbb{D} \) bijectively onto \( \mathbb{D} \);

(ii) \( \phi_w(w) = 0 \);

(iii) \( \phi_w(0) = w \);

(iv) \( \phi_w^{-1} = \phi_w \).

**Proof.** Obviously, \( \phi_w \) is holomorphic and extends continuously to \( \overline{\mathbb{D}} \).

Let \( z \in \partial \mathbb{D} \). Then we have

\[
|\phi_w(z)| = \left| \frac{z - w}{\bar{w}z - 1} \right| = \left| \frac{z - w}{-(z - w)} \right| = 1.
\]

By the Maximum Modulus Principle, thus \( \phi_w(\mathbb{D}) \subset \overline{\mathbb{D}} \) holds. Since \( \phi_w \) is not constant, \( \phi_w(\mathbb{D}) \) is open and thus contained in the interior of \( \overline{\mathbb{D}} \), i.e., in \( \mathbb{D} \).

It is obvious that \( \phi_w(w) = 0 \) and \( \phi_w(0) = w \).

Moreover, we have for \( z \in \mathbb{D} \):

\[
(\phi_w \circ \phi_w)(z) = \frac{\frac{z - w}{\bar{w}z - 1} - w}{\frac{z - w}{\bar{w}z - 1} - 1} = \frac{(z - w) - w(\bar{w}z - 1)}{\bar{w}(z - w) - (\bar{w}z - 1)} = \frac{z - w - w\bar{w}z + w}{\bar{w}z - w\bar{w} + 1} = \frac{z(1 - |w|^2)}{1 - |w|^2} = z.
\]

Hence, \( \phi_w \) is bijective with \( \phi_w^{-1} = \phi_w \). \( \square \)
Theorem 7.11. Let $\phi : \mathbb{D} \to \mathbb{D}$ be biholomorphic. Then there are $w \in \mathbb{D}$ and $c \in \partial \mathbb{D}$ with
\[
\phi(z) = \frac{z - w}{wz - 1}
\]
for $z \in \mathbb{D}$.

Proof. Set $w := \phi^{-1}(0)$. Then $\phi \circ \phi_w : \mathbb{D} \to \mathbb{D}$ is biholomorphic with $(\phi \circ \phi_w)(0) = 0$. By Corollary 7.9, there is $c \in \mathbb{C}$ with $|c| = 1$ such that $\phi(\phi_w(z)) = cz$ for $z \in \mathbb{D}$, so that
\[
\phi(z) = \phi(\phi_w(\phi_w(z))) = c \phi_w(z)
\]
for $z \in \mathbb{D}$.

Definition 7.12. Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be holomorphic. We call $z_0 \in \mathbb{C} \setminus D$ an isolated singularity for $f$ if there is $\epsilon > 0$ such that $B_\epsilon(z_0) \setminus \{z_0\} \subset D$. We say that the isolated singularity $z_0$ is removable if there is a holomorphic function $g : D \cup \{z_0\} \to \mathbb{C}$ such that $g|_D = f$.

Theorem 7.13 (Riemann’s Removability Condition). Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be an isolated singularity for $f$. Then the following are equivalent:

(i) $z_0$ is removable;

(ii) there is a continuous function $g : D \cup \{z_0\} \to \mathbb{C}$ such that $g|_D = f$;

(iii) there is $\epsilon > 0$ with $B_\epsilon(z_0) \setminus \{z_0\} \subset D$ such that $f$ is bounded on $B_\epsilon(z_0) \setminus \{z_0\}$.

Proof. (i) $\implies$ (ii) $\implies$ (iii) is clear.

(iii) $\implies$ (i): Let $C \geq 0$ be such that $|f(z)| \leq C$ for $z \in B_\epsilon(z_0) \setminus \{z_0\}$. Define
\[
h : D \cup \{z_0\} \to \mathbb{C}, \quad z \mapsto \begin{cases} (z - z_0)^2 f(z), & z \neq z_0, \\ 0, & z = z_0. \end{cases}
\]
Then we have for $z \in B_\epsilon(z_0) \setminus \{z_0\}$ that
\[
\left| \frac{h(z) - h(z_0)}{z - z_0} \right| = |(z - z_0)f(z)| \leq C|z - z_0|.
\]
Hence, $h$ is holomorphic with $h'(z_0) = h(z_0) = 0$. Let $h(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ be the power series representation of $h$ on $B_\epsilon(z_0)$. Then $h'(z_0) = h(z_0) = 0$ means that $a_0 = a_1 = 0$, so that $h(z) = \sum_{n=2}^{\infty} a_n (z - z_0)^n$ for $z \in B_\epsilon(z_0)$ and thus $f(z) = \sum_{n=0}^{\infty} a_{n+2}(z - z_0)^n$ for $z \in B_\epsilon(z_0) \setminus \{z_0\}$.

Define
\[
g : D \cup \{z_0\} \to \mathbb{C}, \quad z \mapsto \begin{cases} \sum_{n=0}^{\infty} a_{n+2}(z - z_0)^n, & z \in B_\epsilon(z_0), \\ f(z), & z \in D. \end{cases}
\]
Then $g$ is a holomorphic function extending $f$. \qed
8 Analytic Continuation along a Curve

Example. Let
\[ D_1 := \{ z \in \mathbb{C} : \text{Re } z > 0 \}, \]
\[ D_2 := \{ z \in \mathbb{C} : \text{Im } z > \text{Re } z \}, \]
and
\[ D_3 := \{ z \in \mathbb{C} : \text{Im } z < -\text{Re } z \}, \]
so that
\[ D_1 \cup D_2 \cup D_3 = \mathbb{C} \setminus \{ 0 \}. \]

Let
\[ g: \mathbb{C} \setminus \{ 0 \} \to \mathbb{C}, \quad z \mapsto \frac{1}{z}, \]
and let \( f_1 = \text{Log}|_{D_1} \), so that \( f_1 \) is an anti-derivative of \( g \) on \( D_1 \). Since \( D_2 \) is star shaped, \( g \) also has an anti-derivative on \( D_2 \): since \( f_1 - f_2 \) is constant, and by altering \( f_2 \) by an additive constant, we can achieve that \( f_1|_{D_1 \cap D_2} = f_2|_{D_1 \cap D_2} \). In the same fashion, we can find an anti-derivative \( f_3 \) of \( g \) on \( D_3 \) such that \( f_2|_{D_2 \cap D_3} = f_3|_{D_2 \cap D_3} \). However, \( f_1|_{D_1 \cap D_3} \neq f_3|_{D_1 \cap D_3} \) because otherwise, we would have an anti-derivative of \( g \) on all of \( \mathbb{C} \setminus \{ 0 \} \), which we know to be impossible.

Since \( f_1 - f_3 \) is constant, for \( z \in D_1 \cap D_2 \), we claim that \( c = 2\pi i \). To see this, let \( z_1, z_2, z_3 \in \partial \mathbb{D} \) be such that \( z_1 \in D_1 \cap D_3 \), \( z_2 \in D_2 \cap D_1 \), and \( z_3 \in D_3 \cap D_2 \). Let \( \gamma_{z_1,z_2} \), \( \gamma_{z_2,z_3} \), and \( \gamma_{z_3,z_1} \) be the arc segments of \( \partial \mathbb{D} \) from \( z_1 \) to \( z_2 \), from \( z_2 \) to \( z_3 \), and from \( z_3 \) to \( z_1 \), respectively. Since \( f_j \) is an anti-derivative of \( g \) on \( D_j \) for \( j = 1, 2, 3 \), we obtain
\[ \int_{\gamma_{z_1,z_2}} g = f_1(z_2) - f_1(z_1), \quad \int_{\gamma_{z_2,z_3}} g = f_2(z_3) - f_2(z_2), \]
and
\[ \int_{\gamma_{z_3,z_1}} g = f_3(z_1) - f_3(z_3). \]

It follows that
\[ c = f_3(z_1) - f_1(z_1) \]
\[ = f_3(z_1) - f_3(z_3) + f_2(z_3) - f_2(z_2) + f_1(z_2) - f_1(z_1) \]
\[ = \int_{\gamma_{z_3,z_1}} g + \int_{\gamma_{z_2,z_3}} g + \int_{\gamma_{z_1,z_2}} g \]
\[ = \int_{\gamma_{z_1,z_2} \oplus \gamma_{z_2,z_3} \oplus \gamma_{z_3,z_1}} g \]
\[ = \int_{\partial \mathbb{D}} \frac{1}{\zeta} \, d\zeta \]
\[ = 2\pi i. \]
Definition 8.1. A function element is a pair $(D, f)$, where $D \subset \mathbb{C}$ is open and connected, and $f : D \to \mathbb{C}$ is a holomorphic function. For a given function element $(D, f)$ and $z_0 \in D$, the germ of $f$ at $z_0$—denoted by $\langle f \rangle_{z_0}$—is the collection of all function elements $(E, g)$ such that $z_0 \in E$ and there is a an open neighborhood $U \subset D \cap E$ of $z_0$ such that $f(z) = g(z)$ for all $z \in U$.

Definition 8.2. Let $\gamma : [0, 1] \to \mathbb{C}$ be a path, and suppose that, for each $t \in [0, 1]$, there is a function element $(D_t, f_t)$ such that:

(a) $\gamma(t) \in D_t$ for $t \in [0, 1]$;

(b) for each $t \in [0, 1]$, there is $\delta > 0$ such that, whenever $s \in [0, 1]$ is such that $|s - t| < \delta$, then $\gamma(s) \in D_t$ and $\langle f_s \rangle_{\gamma(s)} = \langle f_t \rangle_{\gamma(s)}$.

Then we call $\{(D_t, f_t) : t \in [0, 1]\}$ an analytic continuation along $\gamma$ and say that $(D_1, f_1)$ is obtained by analytic continuation of $(D_0, f_0)$ along $\gamma$.

Remark. Since $\gamma$ is continuous and $D_t$ is open for each $t \in D_t$, it is clear that there is $\delta > 0$ such that $\gamma(s) \in D_t$ for all $s \in [0, 1]$ such that $|s - t| < \delta$. What is important about Definition 8.2(b) is that $\langle f_s \rangle_{\gamma(s)} = \langle f_t \rangle_{\gamma(s)}$, i.e., there is an open neighborhood $U_s \subset D_s \cap D_t$ of $\gamma(s)$ such that $f_s(z) = f_t(z)$ for $z \in U_s$. 
Theorem 8.3. Let \( \gamma : [0, 1] \to \mathbb{C} \) be a path, and let \( \{(D_t, f_t) : t \in [0, 1]\} \) and \( \{(E_t, g_t) : t \in [0, 1]\} \) be analytic continuations along \( \gamma \) such that \( \langle f_0 \rangle_{\gamma(0)} = \langle g_0 \rangle_{\gamma(0)} \). Then we have \( \langle f_1 \rangle_{\gamma(1)} = \langle g_1 \rangle_{\gamma(1)} \).

Proof. Let

\[ I = \{ t \in [0, 1] : \langle f_t \rangle_{\gamma(t)} = \langle g_t \rangle_{\gamma(t)} \} , \]

so that \( 0 \in I \).

We first claim that \( I \) is closed. Let \( t \in I \), and let \( \delta > 0 \) be such that \( \gamma(s) \in D_t \cap E_t \) and

\[ \langle f_s \rangle_{\gamma(s)} = \langle f_t \rangle_{\gamma(s)} \quad \text{and} \quad \langle g_s \rangle_{\gamma(s)} = \langle g_t \rangle_{\gamma(s)} \]

for all \( s \in [0, 1] \) with \( |s - t| < \delta \). Since \( t \in I \), there is \( s \in I \) with \( |s - t| < \delta \). There is thus an open neighborhood \( U \subset D_t \cap D_s \cap E_t \cap E_s \) of \( \gamma(s) \) such that \( f_s(z) = g_s(z) \) for all \( z \in U \) by the definition of \( I \). From the choice of \( \delta \), we also have—after possibly making \( U \) smaller—that \( f_s(z) = f_t(z) \) and \( g_s(z) = g_t(z) \) for \( z \in U \). It follows that \( f_t(z) = g_t(z) \) for \( z \in U \), so that \( t \in I \).

Let \( t_0 := \sup I \). Let \( \delta > 0 \) be such that \( \gamma(s) \in D_{t_0} \cap E_{t_0} \) and

\[ \langle f_s \rangle_{\gamma(s)} = \langle f_{t_0} \rangle_{\gamma(s)} \quad \text{and} \quad \langle g_s \rangle_{\gamma(s)} = \langle g_{t_0} \rangle_{\gamma(s)} \]

for all \( s \in [0, 1] \) with \( |s - t| < \delta \). Since \( I \) is closed, we have \( t_0 \in I \) and thus \( f_{t_0}(z) = g_{t_0}(z) \) for all \( z \) in some neighborhood \( V \) of \( \gamma(t_0) \) contained in \( D_{t_0} \cap E_{t_0} \). It follows that \( \langle f_{t_0} \rangle_{\gamma(s)} = \langle g_{t_0} \rangle_{\gamma(s)} \) for all \( s \in [0, 1] \) such that \( \gamma(s) \in V \). For \( \delta > 0 \) sufficiently small, we thus have \( \langle f_s \rangle_{\gamma(s)} = \langle g_s \rangle_{\gamma(s)} \) for any \( s \in [0, 1] \) with \( |s - t| < \delta \). It follows that \( [0, 1] \cap (t_0 - \delta, t_0 + \delta) \subset I \). Since \( t_0 = \sup I \), this means that \( t_0 = 1 \), so that \( I = [0, 1] \). \( \square \)
9 Harmonic Functions

Definition 9.1. Let $D \subset \mathbb{R}^N$ be open, and let $u : D \to \mathbb{R}$ be twice continuously partially differentiable. Then $f$ is called harmonic if

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_N^2} \equiv 0.$$

In this course, we will only be concerned with harmonic functions on $\mathbb{R}^2$, i.e., on $\mathbb{C}$.

Proposition 9.2. Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be holomorphic. Then $\text{Re } f$ and $\text{Im } f$ are harmonic.

Proof. Clearly, $\text{Re } f$ and $\text{Im } f$ are twice continuously differentiable.

We have

$$\frac{\partial^2 (\text{Re } f)}{\partial x^2} = \frac{\partial}{\partial x} \frac{\partial}{\partial x} \text{Re } f = \frac{\partial}{\partial y} \frac{\partial}{\partial y} \text{Im } f,$$

by Cauchy–Riemann,

$$= \frac{\partial}{\partial y} \frac{\partial}{\partial x} \text{Im } f$$

$$= -\frac{\partial^2 (\text{Re } f)}{\partial y^2},$$

by Cauchy–Riemann again,

so that $\Delta \text{Re } f \equiv 0$, i.e., $\text{Re } f$ is harmonic.

Similarly, one sees that $\text{Im } f$ is harmonic. \qed

Example. Consider

$$u : \mathbb{C} \setminus \{0\} \to \mathbb{R}, \quad z \mapsto \log |z|,$$

so that

$$u(x, y) = \log \sqrt{x^2 + y^2} = \frac{1}{2} \log (x^2 + y^2)$$

for $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Then the first partial derivatives of $u$ are computed as

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} \quad \text{and} \quad \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2},$$

so that

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2} = -\frac{x^2 + y^2}{(x^2 + y^2)^2}$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$
Assume that there is a holomorphic function \( f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \) such that \( \Re f = u \). On \( \mathbb{C} \setminus \{0\} \), we then have that \( \Re f = \Re \Log \). The Cauchy–Riemann Equations thus yield
\[
\frac{\partial (\Im f)}{\partial x}(z) = -\frac{\partial u}{\partial y}(z) = \frac{\partial (\Im \Log)}{\partial x}(z),
\]
so that
\[
f'(z) = \frac{\partial u}{\partial x}(z) + i \frac{\partial (\Im f)}{\partial x}(z) = \frac{\partial u}{\partial x}(z) + i \frac{\partial (\Im \Log)}{\partial x}(z) = \Log' z = \frac{1}{z}
\]
for \( z \in \mathbb{C} \setminus \{0\} \). By continuity, it follows that \( f'(z) = \frac{1}{z} \) for all \( z \in \mathbb{C} \setminus \{0\} \), so that \( f \) is an anti-derivative of \( \mathbb{C} \setminus \{0\} \ni z \mapsto \frac{1}{z} \). This is impossible.

**Definition 9.3.** Let \( D \subset \mathbb{C} \) be open, and let \( u : D \to \mathbb{R} \) be harmonic. We call a harmonic function \( v : D \to \mathbb{C} \) a harmonic conjugate of \( u \) if \( u + iv \) is holomorphic.

**Theorem 9.4.** Let \( D \subset \mathbb{C} \) be open and suppose that there is \( (x_0, y_0) \in D \) with the following property: for each \( (x, y) \in D \), we have
\[
\begin{itemize}
  \item \( (x, t) \in D \) for each \( t \) between \( y \) and \( y_0 \) and
  \item \( (s, y_0) \in D \) for each \( s \) between \( x \) and \( x_0 \).
\end{itemize}

Then every harmonic function on \( D \) has a harmonic conjugate.

**Proof.** Let \( u : D \to \mathbb{R} \) be harmonic. We will find a harmonic \( v : D \to \mathbb{R} \) such that
\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad (*)
\]
For \( (x, y) \in D \), define
\[
v(x, y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x, t) \, dt + \phi(x),
\]
where \( \phi \) will be specified later. First, note that
\[
\frac{\partial v}{\partial x}(x, y) = \int_{y_0}^{y} \frac{\partial^2 u}{\partial x^2}(x, t) \, dt + \phi'(x), \quad \text{by Lemma 5.12},
\]
\[
= -\int_{y_0}^{y} \frac{\partial^2 u}{\partial y^2}(x, t) \, dt + \phi'(x)
\]
\[
= -\frac{\partial u}{\partial y}(x, y) + \frac{\partial u}{\partial y}(x, y_0) + \phi'(x).
\]
Hence, if we want the Cauchy–Riemann Differential Equations to hold for \( u + iv \), we need that \( \phi'(x) = -\frac{\partial u}{\partial y}(x, y_0) \). We thus set
\[
v(x, y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x, t) \, dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(s, y_0) \, ds.
\]
Then \((\ast)\) holds, so that
\[
\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x} \frac{\partial v}{\partial x} = -\frac{\partial}{\partial y} \frac{\partial u}{\partial x} = -\frac{\partial}{\partial y} \frac{\partial u}{\partial x} = -\frac{\partial^2 v}{\partial y^2},
\]
i.e., \(v\) is harmonic.

**Example.** Let
\[
u: \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto xy.
\]
Then \(u\) is harmonic and
\[
v(x, y) = \int_0^y t \, dt - \int_0^x s \, ds = \frac{y^2}{2} - \frac{x^2}{2}
\]
is a harmonic conjugate for \(u\).

**Corollary 9.5.** Let \(D \subset \mathbb{C}\) be open, and let \(u: D \to \mathbb{R}\) be harmonic. Then, for each \(z_0 \in D\), there is an open neighborhood \(U \subset D\) of \(z_0\) such that \(u|_U\) has a harmonic conjugate.

**Corollary 9.6.** Let \(D \subset \mathbb{C}\) be open, and let \(u: D \to \mathbb{R}\) be harmonic. Then \(u\) is infinitely often partially differentiable.

**Corollary 9.7.** Let \(D \subset \mathbb{C}\) be open and connected, and let \(u: D \to \mathbb{R}\) be harmonic. Then the following are equivalent:

(i) \(u \equiv 0\);

(ii) there is \(\emptyset \neq U \subset D\) open with \(u|_U \equiv 0\).

**Proof.** Of course, only \(\text{(ii)} \implies \text{(i)}\) needs proof. Let
\[
V := \{z \in D : u \equiv 0\ \text{on an open neighborhood of} \ z\}.
\]
Clearly, \(V\) is open and not empty.

We will show that \(D \setminus V\) is open, too. As \(D\) is connected, this establishes \(D = V\).

Let \(z_0 \in D \setminus V\) and \(\epsilon > 0\) be such that \(B_\epsilon(z_0) \subset D\). Assume that \(B_\epsilon(z_0) \cap V \neq \emptyset\). Let \(z \in B_\epsilon(z_0) \cap V\), and let \(\delta > 0\) be such that \(B_\delta(z) \subset B_\epsilon(z_0) \cap V\). By Theorem 9.4, \(u\) has a harmonic conjugate \(v\) on \(B_\epsilon(z_0)\). Set \(f := u + iv\), so that \(f\) is holomorphic with
\[
f' = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \equiv 0
\]
on \(B_\delta(z)\). Consequently, \(f\) is constant on \(B_\delta(z)\) and thus—by the Identity Theorem—on all of \(B_\epsilon(z_0)\). Hence, \(u = \text{Re} f\) is constant on \(B_\epsilon(z_0)\), and since \(u \equiv 0\) on \(B_\delta(z)\), this means \(u \equiv 0\) on \(B_\epsilon(z_0)\), which contradicts \(z_0 \notin V\). \qed
Corollary 9.8. Let $D \subset \mathbb{C}$ be open, let $u: D \to \mathbb{R}$ be harmonic, and let $z_0 \in D$ and $r > 0$ be such that $B_r(z_0) \subset D$. Then we have

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it}) \, dt.$$ 

Corollary 9.9. Let $D \subset \mathbb{C}$ be open and connected, and let $u: D \to \mathbb{R}$ be harmonic with a local maximum or minimum on $D$. Then $u$ is constant.

Proof. It is enough to consider the case of a local maximum: otherwise, replace $u$ by $-u$.

Let $z_0 \in D$ be a point where $u$ attains a local maximum. Let $\epsilon > 0$ be such that $B_\epsilon(z_0) \subset D$ and $u(z) \leq u(z_0)$ for all $z \in B_\epsilon(z_0)$. Let $v$ be a harmonic conjugate of $u$ on $B_\epsilon(z_0)$. Hence, $f := u + iv: B_\epsilon(z_0) \to \mathbb{C}$ is holomorphic such that $\text{Re } f$ has a local maximum at $z_0$. By a midterm practice problem, this means that $f$ is constant, i.e., there is $c \in \mathbb{C}$ such that $u - c|_{B_\epsilon(z_0)} \equiv 0$. By Corollary 9.7, this means that $u \equiv c$ on $D$. \qed

Corollary 9.10. Let $D \subset \mathbb{C}$ be open, connected, and bounded, and let $u: \overline{D} \to \mathbb{R}$ be continuous such that $u|_D$ is harmonic. Then $u$ attains its maximum and minimum on $\partial D$.

The Dirichlet Problem. Let $D \subset \mathbb{C}$ be open, connected, and bounded, and let $f: \partial D \to \mathbb{R}$ be continuous. Is there a continuous $g: D \to \mathbb{R}$ such that $g|_{\partial D} = f$ and $f|_D$ is harmonic?

Remark. If the Dirichlet problem has a solution, then it is unique. To see this, let $g_1, g_2: \overline{D} \to \mathbb{R}$ be such that $g_j|_{\partial D} = f$ and $g_j|_D$ is harmonic for $j = 1, 2$. Then $g_1 - g_2$ vanishes on $\partial D$. Since $g_1 - g_2$ attains both its maximum and minimum on $\partial D$, it follows that $g_1 - g_2 \equiv 0$ on $\partial D$.

Definition 9.11. Let $r > 0$. The Poisson kernel for $B_r(0)$ is defined as

$$P_r(\zeta, z) := \frac{1}{2\pi} \frac{r^2 - |z|^2}{|\zeta - z|^2}$$

for $z \in B_r(0)$ and $\zeta \in \partial B_r(0)$.

Lemma 9.12. Let $D \subset \mathbb{C}$ be open, let $r > 0$ be such that $B_r[0] \subset D$, and let $f: D \to \mathbb{C}$ be holomorphic. Then we have

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) \frac{r^2 - |z|^2}{|re^{it} - z|^2} \, dt = \int_0^{2\pi} f(re^{it}) P_r(re^{it}, z) \, dt.$$ 

for $z \in B_r(0)$. 

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Proof. The Cauchy Integral Formula yields
\[
f(z) = \frac{1}{2\pi i} \int_{\partial B_r(0)} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{it})ire^{it}}{re^{it} - z} dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{it})r^2}{r^2 - re^{it}z} dt \quad (**) \]
for \( z \in B_r(0) \).

Fix \( z \in B_r(0) \), and define \( g(w) := \frac{f(w)}{r^2 - w^2} \) for \( w \) in a neighborhood of \( B_r(0) \). Then \( g \) is holomorphic, and from (**)—with \( f \) replaced by \( g \)—we obtain
\[
\frac{f(z)}{r^2 - |z|^2} = g(z)
= \frac{1}{2\pi} \int_0^{2\pi} g(re^{it})r^2 \frac{dt}{r^2 - re^{it}z} \n= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{it})r^2}{(r^2 - re^{it}z)(r^2 - re^{it}z)} dt \n= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{it})r^2}{|r^2 - re^{it}z|^2} dt,
\]
so that
\[
f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) \frac{r^2 - |z|^2}{|r^2 - re^{it}z|^2} dt = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) \frac{r^2 - |z|^2}{|re^{it} - z|^2} dt. \quad \square
\]

**Theorem 9.13** (Poisson’s Integral Formula). Let \( r > 0 \), and let \( u : B_r(0) \to \mathbb{R} \) be continuous such that \( u|_{B_r(0)} \) is harmonic. Then
\[
u(z) = \int_0^{2\pi} u(re^{it})P_r(re^{it}, z) dt
\]
holds for all \( z \in B_r(0) \).

**Proof.** Suppose first that \( u \) extends to \( B_R(0) \) for some \( R > r \) as a harmonic function. Then \( u \) has a harmonic conjugate \( v \) on \( B_R(0) \), so that \( f := u + iv \) is holomorphic. By Lemma 9.12, we have, for \( z \in B_r(0) \), that
\[
u(z) + iv(z) = f(z)
= \int_0^{2\pi} f(re^{it})P_r(re^{it}, z) dt = \int_0^{2\pi} u(re^{it})P_r(re^{it}, z) dt + \int_0^{2\pi} v(re^{it})P_r(re^{it}, z) dt,
\]
so that
\[
u(z) = \int_0^{2\pi} u(re^{it})P_r(re^{it}, z) dt.
\]

Suppose now that \( u \) is arbitrary. For \( \theta \in (0, 1) \), define
\[
u_\theta : B_{\frac{r}{\theta}}(0) \to \mathbb{R}, \quad z \mapsto u(\theta z).\]
Then $u_\theta$ is harmonic, and by the foregoing we have
\[
 u_\theta(z) = \int_0^{2\pi} u_\theta(re^{it}) P_r(re^{it}, z) \, dt
\]
for $z \in B_r(0)$. Letting $\theta \to 1$ (compare a related homework problem on the Cauchy Integral Formula), we obtain for $z \in B_r(0)$ that
\[
 u(z) = \lim_{\theta \to 1} u_\theta(z) = \lim_{\theta \to 1} \int_0^{2\pi} u_\theta(re^{it}) P_r(re^{it}, z) \, dt = \int_0^{2\pi} u(re^{it}) P_r(re^{it}, z) \, dt.
\]

**Theorem 9.14.** Let $r > 0$, and let $f : \partial B_r(0) \to \mathbb{R}$ be continuous. Define
\[
g : B_r[0] \to \mathbb{C}, \quad z \mapsto \begin{cases} f(z), & z \in \partial B_r(0), \\ \int_0^{2\pi} f(re^{it}) P_r(re^{it}, z) \, dt, & z \in B_r(0). \end{cases}
\]

Then $g$ is continuous and harmonic on $B_r(0)$.

**Proof.** There is no loss of generality to suppose that $r = 1$.

For $z \in \mathbb{D}$ and $\zeta \in \partial \mathbb{D}$, note that
\[
\text{Re} \left( \frac{\zeta + z}{\zeta - z} \right) = \frac{1}{|\zeta - z|^2} \text{Re}(|\zeta|^2 - |z|^2 + z\bar{\zeta} - \zeta\bar{z}) = \frac{1 - |z|^2}{|\zeta - z|^2}.
\]

As the real part of a holomorphic function,
\[
\mathbb{D} \to \mathbb{R}, \quad z \mapsto P_1(\zeta, z)
\]
is therefore harmonic for each $\zeta \in \partial \mathbb{D}$. We thus obtain for $z \in \mathbb{D}$:
\[
(\Delta g)(z) = \frac{\partial^2 g}{\partial x^2}(z) + \frac{\partial^2 g}{\partial y^2}(z) = \int_0^{2\pi} f(e^{it}) \left( \frac{\partial^2}{\partial x^2} P_1(e^{it}, z) + \frac{\partial^2}{\partial y^2} P_1(e^{it}, z) \right) \, dt = 0.
\]

Consequently, $g$ is harmonic on $B_r(0)$.

What remains to be shown is that $g$ is continuous at any point $z_0 \in \partial \mathbb{D}$.

Let $z_0 = e^{i\theta_0}$, and suppose without loss of generality that $\theta_0 \in (0, 2\pi)$. Let $\epsilon > 0$. We need to find $\delta > 0$ such that $|g(z_0) - g(z)| < \epsilon$ for all $z \in \mathbb{D}$ with $|z_0 - z| < \delta$.

For $\delta_0 > 0$, let $J := [\theta_0 - 2\delta_0, \theta_0 + 2\delta_0]$. Making $\delta_0 > 0$ sufficiently small, we can achieve that $J \subset [0, 2\pi]$ and $|f(z_0) - f(e^{i\theta})| < \frac{\epsilon}{2}$ for $\theta \in J$. Set
\[
S := \{ se^{i\theta} : s \in [0, 1), \, \theta \in [\theta_0 - \delta_0, \theta_0 + \delta_0] \},
\]
and note that $C := \inf\{|e^{i\theta} - z| : \theta \in [0, 2\pi] \setminus J, \, z \in S\} > 0$.  

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Since \( \int_0^{2\pi} P_1(e^{it}, z) \, dt = 1 \) for all \( z \in \mathbb{D} \), we have

\[
g(z) - g(z_0) = \int_0^{2\pi} (f(e^{it}) - f(z_0)) P_1(e^{it}, z) \, dt
= \int_J (f(e^{it}) - f(z_0)) P_1(e^{it}, z) \, dt + \int_{[0,2\pi] \setminus J} (f(e^{it}) - f(z_0)) P_1(e^{it}, z) \, dt.
\]

Note that

\[
|I_1| \leq \int_J |f(e^{it}) - f(z_0)| \, dt \leq \frac{\epsilon}{2} \int_0^{2\pi} P_1(e^{it}, z) \, dt = \frac{\epsilon}{2}.
\]

Set \( K := \sup_{\zeta \in \partial \mathbb{D}} |f(\zeta)| \). For \( z \in S \), we then have

\[
|I_2| \leq \int_{[0,2\pi] \setminus J} \left( |f(e^{it})| + |f(z_0)| \right) P_1(e^{it}, z) \, dt
\]

\[
= \frac{1}{2\pi} \int_{[0,2\pi] \setminus J} \left( |f(e^{it})| + |f(z_0)| \right) \frac{1 - |z|^2}{|e^{it} - z|^2} \, dt
\]

\[
\leq \frac{K}{\pi} \int_{[0,2\pi] \setminus J} \frac{1 - |z|^2}{|e^{it} - z|^2} \, dt
\]

\[
\leq \frac{K}{\pi C^2} \int_{[0,2\pi] \setminus J} 1 - |z|^2 \, dt, \quad \text{because} \ z \in S,
\]

\[
\leq \frac{2K}{C^2} (1 - |z|^2)
\]

Choose \( \delta \in (0, \delta_0) \) so small that \( |z_0 - z| < \delta \) for \( z \in \mathbb{D} \) implies \( z \in S \) and

\[
1 - |z|^2 < \frac{C^2 \epsilon}{2K^2}.
\]
For $z \in \mathbb{D}$ with $|z_0 - z| < \delta$, we then have $|J_2| < \frac{\varepsilon}{2}$ and therefore, all in all, $|g(z_0) - g(z)| < \varepsilon$.

**Definition 9.15.** Let $D \subseteq \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be continuous. We say that $f$ has the **mean value property** if, for every $z_0 \in D$, there is $R > 0$ with $B_R[z_0] \subset D$ such that $$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \, dt =: \mu_r(f, z_0)$$ for all $r \in [0, R]$.

**Theorem 9.16.** Let $D \subseteq \mathbb{C}$ be open, and let $f : D \to \mathbb{R}$ have the mean value property such that $|f|$ attains a local maximum at $z_0 \in D$. Then $f$ is constant on a neighborhood of $z_0$.

**Proof.** Without loss of generality, suppose that $f(z_0) > 0$.

Choose $R > 0$ with $B_R[z_0] \subset D$ such that $f(z_0) \geq |f(z)|$ for all $z \in B_R[z_0]$ and $f(z_0) = \mu_r(f, z_0)$ for all $r \in [0, R]$. Set $g := \text{Re} f - f(z_0)$. Then $g$ has the mean value property and satisfies $$g(z) \leq |f(z)| - f(z_0) \leq 0$$ for $z \in B_R[z_0]$. It follows that $$0 = g(z_0) = \mu_r(g, z_0) = \int_0^{2\pi} g(z_0 + re^{i\theta}) \, dt \leq 0$$ for all $r \in [0, R]$. As the integrand is continuous, we conclude that $g(z_0 + re^{i\theta}) = 0$ for all $r \in [0, R]$ and $\theta \in [0, 2\pi]$, i.e., $g \equiv 0$ on $B_R[z_0]$. This means that, for $z \in B_R[z_0]$, we have $f(z_0) = \text{Re} f(z)$ as well as $$|f(z)| \leq f(z_0) = \text{Re} f(z) \leq |f(z)|,$$ so that $|f(z)| = \text{Re} f(z)$ for $z \in B_R[z_0]$ and thus $f(z) = f(z_0)$.

**Corollary 9.17.** Let $D \subseteq \mathbb{C}$ be open, let $f : D \to \mathbb{R}$ have the mean value property, and suppose that $f$ has a local maximum or minimum at $z_0 \in D$. Then $f$ is constant in a neighborhood of $z_0$.

**Proof.** We only consider the case of a local maximum (for a local minimum, replace $f$ by $-f$).

Let $R > 0$ be such that $B_R[z_0] \subset D$ and $f(z) \leq f(z_0)$ for all $z \in B_R[z_0]$. Choose $C$ such that $f(z) + C \geq 0$ for all $z \in B_R[z_0]$. It follows that $|f + C|$ has a local maximum at $z_0$. Hence, $f + C$ is constant in a neighborhood of $z_0$, as is $f$. 

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Corollary 9.18. Let $D \subset \mathbb{C}$ be open, connected, and bounded, and let $f : \overline{D} \to \mathbb{R}$ be continuous such that $f|_D$ has the mean value property. Then $f$ attains its maximum and minimum on $\partial D$.

Proof. Without loss of generality, suppose that $f$ is not constant. Let $z_0 \in \overline{D}$ be such that $f(z_0)$ is maximal. Set

$$V := \{z \in D : f(z) < f(z_0)\}.$$ 

Then $V$ is open and not empty. Let $z \in D \setminus V$, i.e., $f(z) = f(z_0)$. Then $f$ has a local maximum at $z$, so that, by Corollary 9.17, $f(w) = f(z) = f(z_0)$ for $w$ in an open neighborhood, say $W \subset D$, of $z$. Consequently, $W \subset D \setminus V$ holds, so that $z$ is an interior point of $D \setminus V$. Since $z \in D \setminus V$ is arbitrary, this shows that $D \setminus V$ is open. Since $D$ is connected, and $V \neq \emptyset$, we must have $D \setminus V = \emptyset$, i.e., $V = D$.

The case of the minimum is treated analogously. 

Corollary 9.19. Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{R}$ be continuous. Then the following are equivalent:

(i) $f$ is harmonic;

(ii) $f$ has the mean value property.

Proof. Only (ii) $\implies$ (i) needs proof.

Let $z_0 \in D$, and let $R > 0$ be such that $B_R[z_0] \subset D$. By Theorem 9.14, there is a continuous function $g : B_R[z_0] \to \mathbb{R}$ such that $g|_{\partial B_R[z_0]} = f|_{\partial B_R[z_0]}$ and $g|_{B_R(z_0)}$ is harmonic. Consequently, $f - g|_{B_R(z_0)}$ has the mean value property. By Corollary 9.18, this means that $f - g$ attains its maximum on $B_R[z_0]$ on $\partial B_R[z_0]$, so that $g = f|_{B_R[z_0]}$. Hence, $f|_{B_R(z_0)}$ is harmonic, i.e., $\Delta f \equiv 0$ on $B_R[z_0]$. Since $z_0 \in D$ was arbitrary, this means that $\Delta f \equiv 0$. 

\[\square\]
10 The Singularities of a Holomorphic Function

**Definition 10.1.** Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be an isolated singularity for $f$. Then $z_0$ is called a pole for $f$ if $\lim_{z \to z_0} |f(z)| = \infty$.

**Example.** For $n \in \mathbb{N}$, the function $\mathbb{C} \setminus \{0\} \to \mathbb{C}, \ z \mapsto \frac{1}{z^n}$ has a pole at 0.

**Theorem 10.2.** Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be an isolated singularity for $f$. Then the following are equivalent:

(i) $z_0$ is a pole for $f$;

(ii) there are unique $k \in \mathbb{N}$ and holomorphic $g : D \cup \{z_0\} \to \mathbb{C}$ such that $g(z_0) \neq 0$ and

$$f(z) = \frac{g(z)}{(z - z_0)^k}$$

for $z \in D$.

**Proof.** (ii) $\implies$ (i) is obvious.

(i) $\implies$ (ii): We prove the uniqueness first.

Suppose that there are $k_1, k_2 \in \mathbb{N}$ and holomorphic $g_1, g_2 : D \cup \{z_0\} \to \mathbb{C}$ such that $g_j(z_0) \neq 0$ and

$$f(z) = \frac{g_j(z)}{(z - z_0)^{k_j}}$$

for $z \in D$ and $j = 1, 2$. Assume that $k_1 \neq k_2$ and without loss of generality that $k_1 < k_2$.

For all $z \in D$, we then have

$$g_1(z) = \frac{g_2(z)}{(z - z_0)^{k_2 - k_1}}$$

It follows that

$$|g_1(z_0)| = \lim_{z \to z_0} \frac{|g_2(z)|}{|z - z_0|^{k_2 - k_1}} = \infty,$$

which is impossible. Hence, $k_1 = k_2$ must hold and thus $g_1 = g_2$ on $D$ and—by continuity—on all of $D \cup \{z_0\}$.

To establish the existence of $k$ and $g$, choose $r > 0$ such that $B_r(z_0) \setminus \{z_0\} \subset D$ and $|f(z)| \geq 1$ for all $z \in B_r(z_0) \setminus \{z_0\}$. Then

$$B_r(z_0) \setminus \{z_0\} \to \mathbb{C}, \ \ z \mapsto \frac{1}{f(z)}$$

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is holomorphic and bounded and thus—by Riemann’s Removability Criterion—, has a holomorphic extension \( h: B_r(z_0) \to \mathbb{C} \) with \( h(z_0) = \lim_{z \to z_0} \frac{1}{f(z)} = 0. \) (Note that \( z_0 \) is the only zero of \( h \).) Let
\[
h(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n
\]
for \( z \in B_r(z_0) \) be the power series representation of \( h \). Set \( k := \min\{n \in \mathbb{N}_0 : a_n \neq 0 \} \).

Since \( a_0 = h(z_0) = 0 \), we have \( k \geq 1 \). Define
\[
\tilde{h}: B_r(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=k}^{\infty} a_n(z - z_0)^{n-k}.
\]

Then \( \tilde{h} \) is holomorphic, has no zeros, and satisfies \( h(z) = (z - z_0)^k \tilde{h}(z) \) for \( z \in B_r(z_0) \).

For \( z \in B_r(z_0) \setminus \{z_0\} \), we thus have
\[
f(z) = \frac{1}{\tilde{h}(z)(z - z_0)^k}.
\]

Define
\[
g: D \cup \{z_0\} \to \mathbb{C}, \quad z \mapsto \begin{cases} (z - z_0)^k f(z), & z \neq z_0, \\ \frac{1}{\tilde{h}(z)}, & z \in B_r(z_0). \end{cases}
\]

**Definition 10.3.** Let \( D \subset \mathbb{C} \) be open, let \( f: D \to \mathbb{C} \) be holomorphic, and let \( z_0 \in \mathbb{C} \setminus D \) be a pole for \( f \). Then the positive integer \( k \) in Theorem 10.2(ii) is called the **order** of \( z_0 \) and denoted by \( \text{ord}(f, z_0) \). If \( \text{ord}(f, z_0) = 1 \), we call \( z_0 \) a **simple pole** of \( f \).

**Example.** For \( m \in \mathbb{N} \), consider
\[
f_m: \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad z \mapsto \frac{\sin z}{z^m}.
\]

We claim that \( f_1 \) has a removable singularity at 0 whereas \( f_n \) has a pole of order \( m - 1 \) at 0 for \( m \geq 2 \).

Recall that
\[
\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}
\]
for \( z \in \mathbb{C} \). For \( z \neq 0 \), we thus have
\[
\frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}.
\]

Define
\[
g: \mathbb{C} \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}.
\]

Then \( g \) is holomorphic with \( g(0) \neq 0 \) and clearly extends \( f_1 \). Hence, \( f_1 \) has a removable singularity at 0.

For \( m \geq 2 \) and \( z \neq 0 \), note that
\[
f_m(z) = g(z) \frac{1}{z^{m-1}}.
\]
Hence, \( f_m \) has a pole of order \( m - 1 \) at 0.
Example. Consider
\[ f : \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad z \mapsto e^{\frac{1}{z}}. \]
Then 0 is not removable because \( \lim_{n \to \infty} f \left( \frac{1}{n} \right) = \lim_{n \to \infty} e^n = \infty. \) But 0 is not a pole for \( f \) either: for \( n \in \mathbb{N}, \) we have
\[ \left| f \left( \frac{i}{n} \right) \right| = |e^{-in}| = 1. \]

**Definition 10.4.** Let \( D \subset \mathbb{C} \) be open, let \( f : D \to \mathbb{C} \) be holomorphic, and let \( z_0 \in \mathbb{C} \setminus D \) be an isolated singularity for \( f. \) Then \( z_0 \) is called *essential* if it is neither removable nor a pole.

**Theorem 10.5** (Casorati–Weierstraß Theorem). Let \( D \subset \mathbb{C} \) be open, let \( f : D \to \mathbb{C} \) be holomorphic, and let \( z_0 \in \mathbb{C} \setminus D \) be an isolated singularity for \( f. \) Then the following are equivalent:

(i) \( z_0 \) is essential;

(ii) \( \overline{f(B_\epsilon(z_0) \cap D)} = \mathbb{C} \) for each \( \epsilon > 0. \)

**Proof.** (ii) \( \implies \) (i): For each \( n \in \mathbb{N} \) choose \( z_n \in B_{\frac{1}{n}}(z_0) \cap D \) such that \( |f(z_n) - n| < \frac{1}{n}. \) It follows that \( \lim_{n \to \infty} |f(z_n)| = \infty. \) Hence, \( z_0 \) cannot be removable.

For each \( n \in \mathbb{N}, \) choose \( z'_n \in B_{\frac{1}{n}}(z_0) \cap D \) such that \( |f(z'_n)| < \frac{1}{n}. \) This means that \( \lim_{n \to \infty} f(z'_n) = 0, \) so that \( z_0 \) is not a pole either.

(i) \( \implies \) (ii): Assume otherwise, i.e., there is \( \epsilon_0 > 0 \) such that \( \overline{f(B_{\epsilon_0}(z_0) \cap D)} \neq \mathbb{C}. \)

Without loss of generality, suppose that \( B_{\epsilon_0}(z_0) \setminus \{z_0\} \subset D. \) Let \( w_0 \in \mathbb{C} \) and \( \delta > 0 \) be such that \( B_\delta(w_0) \subset \mathbb{C} \setminus f(B_{\epsilon_0}(z_0) \setminus \{z_0\}). \) Consider
\[ g : B_{\epsilon_0}(z_0) \setminus \{z_0\} \to \mathbb{C}, \quad z \mapsto \frac{1}{f(z) - w_0}. \]
Then \( g \) is holomorphic with
\[ |g(z)| = \frac{1}{|f(z) - w_0|} \leq \frac{1}{\delta} \]
for \( z \in B_{\epsilon_0}(z_0) \setminus \{z_0\}. \) Hence, \( z_0 \) is a removable singularity for \( g. \) Let \( \tilde{g} : B_{\epsilon_0}(z_0) \to \mathbb{C} \) be a holomorphic extension of \( g. \)

**Case 1:** \( g(z_0) \neq 0. \) Since \( f(z) = \frac{1}{g(z)} + w_0 \) for \( z \in B_{\epsilon_0}(z_0) \setminus \{z_0\}, \) this means that \( z_0 \) is a removable singularity for \( f, \) which contradicts (i).

**Case 2:** \( g(z_0) = 0. \) For \( z \neq z_0, \) we have
\[ |f(z)| \geq \frac{1}{|g(z)|} - |w_0| \xrightarrow{z \to z_0} \infty. \]
Hence, \( z_0 \) is a pole for \( f, \) again contradicting (i). \( \square \)
11 Holomorphic Functions on Annuli

**Definition 11.1.** Let \( z_0 \in \mathbb{C} \), and let \( r, R \in [0, \infty] \) be such that \( r < R \). Then the **annulus** centered at \( z_0 \) with inner radius \( r \) and outer radius \( R \) is defined as

\[
A_{r,R}(z_0) := \{ z \in \mathbb{C} : r < |z - z_0| < R \}.
\]

**Theorem 11.2 (Cauchy’s Integral Theorem for Annuli).** Let \( z_0 \in \mathbb{C} \), let \( r, \rho, P, R \in [0, \infty] \) be such that \( r < \rho < P < R \), and let \( f : A_{r,R}(z_0) \to \mathbb{C} \) be holomorphic. Then we have

\[
\int_{\partial B_\rho(z_0)} f(\zeta) \, d\zeta = \int_{\partial B_P(z_0)} f(\zeta) \, d\zeta.
\]

**Proof.** The claim is equivalent to

\[
\int_{\partial B_P(z_0)} f(\zeta) \, d\zeta + \int_{\partial B_\rho(z_0)} f(\zeta) \, d\zeta = 0.
\]

Consider

Split \( \partial B_P(z_0) \) and \( \partial B_\rho(z_0) \) into finitely many arc segments—say \( \alpha_1, \ldots, \alpha_n \) and \( \beta_1, \ldots, \beta_n \)—, and connect them with line segments \( \gamma_1, \ldots, \gamma_n \) as shown (for \( n = 3 \)): 
We thus obtain

\[
\int_{\partial B_{r}(z_0)} f(\zeta) \, d\zeta + \int_{\partial B_{R}(z_0)} f(\zeta) \, d\zeta = \sum_{j=1}^{n} \int_{\alpha_j} f(\zeta) \, d\zeta + \sum_{j=1}^{n} \int_{\beta_j} f(\zeta) \, d\zeta \\
= \sum_{j=1}^{n-1} \int_{\alpha_j \oplus \gamma_j \oplus \beta_j \oplus \gamma_{j+1}} f(\zeta) \, d\zeta + \int_{\alpha_n \oplus \gamma_n \oplus \beta_n \oplus \gamma_1} f(\zeta) \, d\zeta
\]

Making the arc segments $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ sufficiently, we can achieve that each of the closed curves $\alpha_1 \oplus \gamma_1 \oplus \beta_1 \oplus \gamma_2, \ldots, \alpha_{n-1} \oplus \gamma_{n-1} \oplus \beta_{n-1} \oplus \gamma_n, \alpha_n \oplus \gamma_n \oplus \beta_n \oplus \gamma_1$ lies inside a star shaped open subset of $A_{r,R}(z_0)$:
It follows that
\[
\int_{\partial B_r(z_0)} f(\zeta) \, d\zeta + \int_{\partial B_R(z_0)^-} f(\zeta) \, d\zeta = \sum_{j=1}^{n-1} \int_{\alpha_j \oplus \gamma_j \oplus \beta_j \oplus \gamma_{j+1}} f(\zeta) \, d\zeta + \int_{\alpha_n \oplus \gamma_n \oplus \beta \oplus \gamma_1} f(\zeta) \, d\zeta = 0
\]

as claimed.

\[\square\]

**Theorem 11.3** (Laurent Decomposition). Let \( z_0 \in \mathbb{C} \), let \( r, R \in [0, \infty) \) be such that \( r < R \), and let \( f: A_{r,R}(z_0) \to \mathbb{C} \) be holomorphic. Then there are holomorphic \( g: B_R(z_0) \to \mathbb{C} \) and \( h: \mathbb{C} \setminus B_r[z_0] \to \mathbb{C} \) with \( f = g + h \) on \( A_{r,R}(z_0) \). Moreover, \( h \) can be chosen such that \( \lim_{|z| \to \infty} |h(z)| = 0 \), in which case \( g \) and \( h \) are uniquely determined.

**Proof.** We prove the uniqueness assertion first.

Let \( g_1, g_2 : B_R(z_0) \to \mathbb{C} \) and \( h_1, h_2 : \mathbb{C} \setminus B_r[z_0] \to \mathbb{C} \) be holomorphic such that \( \lim_{|z| \to \infty} |h_j(z)| = 0 \) for \( j = 1, 2 \) and

\[ f = g_1 + h_1 = g_2 + h_2. \]

It follows that \( g_1 - g_2 = h_2 - h_1 \) on \( A_{r,R}(z_0) \). Define

\[ F: \mathbb{C} \to \mathbb{C}, \quad z \mapsto \begin{cases} g_1(z) - g_2(z), & z \in B_R(z_0), \\ h_2(z) - h_1(z), & z \in \mathbb{C} \setminus B_r[z_0]. \end{cases} \]
Then $F$ is entire with $\lim_{|z| \to \infty} |F(z)| = \lim_{|z| \to \infty} |h_2(z) - h_1(z)| = 0$. Hence, $F$ is bounded and entire and thus bounded by Liouville’s theorem. Since $\lim_{|z| \to \infty} |F(z)| = 0$, this means that $F \equiv 0$, so that $g_1 = g_2$ and $h_1 = h_2$.

To show that $g$ and $h$ exists, let $\rho \in (r, R)$, and define

$$g_\rho: B_\rho(z_0) \to \mathbb{C}, \quad z \mapsto \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$  

Then $g_\rho$ is holomorphic. Let $\rho_1, \rho_2 \in (r, R)$ be such that $\rho_1 < \rho_2$. Let $z \in B_{\rho_1}(z_0)$, and choose $r_0 \in (|z - z_0|, \rho_1)$. Since

$$A_{r_0, R}(z_0) \to \mathbb{C}, \quad w \mapsto \frac{1}{2\pi i} \frac{f(w)}{w - z}$$

is holomorphic, Cauchy’s Integral Theorem for Annuli yields that

$$g_{\rho_1}(z) = \frac{1}{2\pi i} \int_{\partial B_{\rho_1}(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_{\partial B_{\rho_2}(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta = g_{\rho_2}(z).$$

Hence, we can define $g: B_R(z_0) \to \mathbb{C}$ as follows: for $z \in B_R(z_0)$, choose $\rho \in (r, R)$ with $z \in B_\rho(z_0)$, and set

$$g(z) := \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$  

Similarly, define $h: \mathbb{C} \setminus B_r[z_0] \to \mathbb{C}$ as follows: for $z \in \mathbb{C} \setminus B_r[z_0]$, choose $\sigma \in (r, R)$ such that $z \in \mathbb{C} \setminus B_\sigma[z_0]$ and set

$$h(z) = \frac{1}{2\pi i} \int_{\partial B_\sigma(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$  

Then $h$ is holomorphic and satisfies

$$|h(z)| \leq \sigma \sup_{\zeta \in \partial B_\sigma(z_0)} \left| \frac{f(\zeta)}{\zeta - z} \right| \leq \sigma \frac{\sup_{\zeta \in \partial B_\sigma(z_0)} |f(\zeta)|}{\operatorname{dist}(z, \partial B_\sigma(z_0))} \to 0 \text{ as } |z| \to \infty.$$

Let $z \in A_{r, R}(z_0)$ and choose $\sigma$ and $\rho$ such that $r < \sigma < |z - z_0| < \rho < R$. Define

$$G: A_{r, R}(z_0) \to \mathbb{C}, \quad w \mapsto \begin{cases} \frac{f(w) - f(z)}{w - z}, & w \neq z, \\ f'(z), & z = w. \end{cases}$$

Then $G$ is holomorphic on $A_{r, R}(z_0) \setminus \{z\}$ and continuous on $A_{r, R}(z_0)$. By Riemann’s Removability Criterion, this means that $G$ is holomorphic on all of $A_{r, R}(z_0)$. It follows that

$$\int_{\partial B_\sigma(z_0)} G(\zeta) \, d\zeta = \int_{\partial B_\rho(z_0)} G(\zeta) \, d\zeta,$$

i.e.,

$$\int_{\partial B_\sigma(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta - f(z) = \int_{\partial B_\rho(z_0)} \frac{1}{\zeta - z} \, d\zeta = \int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta + f(z) = \int_{\partial B_\rho(z_0)} \frac{1}{\zeta - z} \, d\zeta.$$  

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so that
\[ -2\pi i h(z) = 2\pi i g(z) - 2\pi i f(z) \]
and thus
\[ f(z) = g(z) + h(z). \]

**Terminology.** The function \( h \) in Theorem 11.3 is called the *principal part* and \( g \) is called the *secondary part* of the Laurent decomposition \( f = g + h \).

**Theorem 11.4.** Let \( z_0 \in \mathbb{C} \), let \( r, R \in [0, \infty] \) be such that \( r < R \), and let \( f : A_{r, R}(z_0) \to \mathbb{C} \) be holomorphic. Then \( f \) has a representation
\[
f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n
\]
for \( z \in A_{r, R}(z_0) \) as a Laurent series, which converges uniformly and absolutely on compact subsets of \( A_{r, R}(z_0) \). Moreover, for any \( n \in \mathbb{Z} \) and \( \rho \in (r, R) \), we have
\[
a_n = \frac{1}{2\pi i} \int_{\partial B_{\rho}(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta.
\]

**Proof.** Let \( g \) and \( h \) be as in Theorem 11.3 (in particular, with \( \lim_{|z| \to \infty} |h(z)| = 0 \)).

For \( z \in B_R(z_0) \), we have the Taylor series
\[
g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n
\]
with uniform and absolute convergence on the compact subsets of \( B_R(z_0) \).

Define
\[
\tilde{h} : A_{0, \frac{1}{r}}(0) \to \mathbb{C}, \quad z \mapsto h \left( z_0 + \frac{1}{z} \right),
\]
so that \( \tilde{h} \) is holomorphic with \( \lim_{z \to 0} \tilde{h}(z) = 0 \). Hence, \( \tilde{h} \) has a removable singularity at \( 0 \) and thus extends to \( B_1(0) \) as a holomorphic function (which we denote likewise by \( \tilde{h} \)).

For \( z \in B_1(0) \), we have the Taylor series
\[
\tilde{h}(z) = \sum_{n=0}^{\infty} b_n z^n = \sum_{n=1}^{\infty} b_n z^n,
\]
so that
\[
h(z) = \sum_{n=1}^{\infty} b_n(z - z_0)^{-n}
\]
for \( z \in \mathbb{C} \setminus B_r[z_0] \) with uniform and absolute convergence on compact subsets.

Set \( a_n := b_{-n} \) for \( n < 0 \). For \( z \in A_{r, R}(z_0) \), we obtain
\[
f(z) = g(z) + h(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n} = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n.
\]
Finally, pick \( m \in \mathbb{Z} \) and \( \rho \in (r, R) \). Note that

\[
\frac{f(z)}{(z - z_0)^{m+1}} = \sum_{n=-\infty}^{-1} a_{n+m+1}(z - z_0)^n + \sum_{n=0}^{\infty} a_{n+m+1}(z - z_0)^n
\]

converges uniformly on \( \partial B_\rho(z_0) \). Hence, we obtain:

\[
\int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{m+1}} d\zeta = \sum_{n=-\infty}^{-1} a_{n+m+1} \int_{\partial B_\rho(z_0)} (\zeta - z_0)^n d\zeta + \sum_{n=0}^{\infty} a_{n+m+1} \int_{\partial B_\rho(z_0)} (\zeta - z_0)^n d\zeta
\]

\[
= a_m \int_{\partial B_\rho(z_0)} \frac{1}{\zeta - z_0} d\zeta
\]

\[
= 2\pi i a_m,
\]

i.e.,

\[
a_m = \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{1}{\zeta - z_0} d\zeta.
\]

**Corollary 11.5.** Let \( z_0 \in \mathbb{C} \), let \( r > 0 \), and let \( f : B_r(z_0) \setminus \{z_0\} \to \mathbb{C} \) be holomorphic with Laurent representation \( f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \). Then the singularity \( z_0 \) of \( f \) is

(i) removable if and only if \( a_n = 0 \) for \( n < 0 \);

(ii) a pole of order \( k \in \mathbb{N} \) if and only if \( a_{-k} \neq 0 \) and \( a_n = 0 \) for all \( n < -k \);

(iii) essential if and only if \( a_n \neq 0 \) for infinitely many \( n < 0 \).

**Proof.** (i) The “if” part is clear.

Conversely, suppose that \( z_0 \) is a removable singularity, and let \( \tilde{f} : B_r(z_0) \to \mathbb{C} \) be a holomorphic extension of \( f \) with Taylor expansion \( \tilde{f}(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n \) for \( z \in B_r(z_0) \). Uniqueness of the Laurent representation yields \( a_n = b_n \) for \( n \in \mathbb{N}_0 \) and \( a_n = 0 \) for \( n < 0 \).

(ii) For the “if” part, set

\[
g(z) := (z - z_0)^k f(z) = \sum_{n=-k}^{\infty} a_n(z - z_0)^{n+k}
\]

for \( z \in B_r(z_0) \setminus \{z_0\} \). Then \( g \) extends holomorphically to \( B_r(z_0) \) with \( g(z_0) = a_{-k} \neq 0 \). By definition, we have \( f(z) = \frac{g(z)}{(z - z_0)^k} \) for \( z \in B_r(z_0) \setminus \{z_0\} \). Hence, \( f \) has a pole of order \( k \) at \( z_0 \).
For the converse, let \( g : B_r(z_0) \to \mathbb{C} \) be holomorphic such that \( g(z_0) \neq 0 \) and \( f(z) = \frac{g(z)}{(z-z_0)^k} \) for \( z \in B_r(z_0) \setminus \{z_0\} \). Let \( g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n \) for \( z \in B_r(z_0) \) be the Taylor series of \( g \), so that

\[
f(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^{n-k}
\]

for \( z \in B_r(z_0) \setminus \{z_0\} \). Uniqueness of the Laurent representation yields that \( a_n = b_{n+k} \) for \( n \geq -k \) and \( a_n = 0 \) for \( n < -k \).

(iii) This follows by default from (i) and (ii).

\[\square\]

**Examples.**

1. Let

\[ f : \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad z \mapsto e^{-\frac{1}{z^2}}. \]

Then \( f \) has the Laurent representation

\[
f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n! z^{2n}}
\]

for \( z \in \mathbb{C} \setminus \{0\} \) and thus an essential singularity at 0.

2. Let

\[ f : \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad z \mapsto e^z - \frac{1}{z^3} \]

so that

\[
f(z) = \frac{1}{z^3} \sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-3}}{n!}
\]

for \( z \in \mathbb{C} \setminus \{0\} \). Hence, \( f \) has a pole of order two at 0.

The Laurent representation of a holomorphic function on an annulus \( A_r,R(z_0) \) does not only depend on \( z_0 \), but also on \( r \) and \( R \).

**Example.** Consider the function

\[ f : \mathbb{C} \setminus \{1, 3\} \to \mathbb{C}, \quad z \mapsto \frac{2}{z^2 - 4z + 3}, \]

and note that

\[
f(z) = \frac{1}{1-z} - \frac{1}{3-z}.
\]

Then \( f \) has the following Laurent representations:

(a) On \( A_{0,1}(0) \): For \( |z| < 1 \), we have

\[
\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n
\]
and for $|z| < 3$ that
\[
\frac{1}{3 - z} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n.
\]

We thus have for $z \in A_{0,1}(0)$:
\[
f(z) = \sum_{n=0}^{\infty} \left(1 - \frac{1}{3^{n+1}}\right) z^n.
\]

(b) On $A_{1,3}(0)$: For $|z| > 1$, we have
\[
-\frac{1}{1-z} = \frac{1}{z-1} = \frac{1}{z} \left( \frac{1}{1 - \frac{1}{z}} \right) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}},
\]
so that, for $z \in A_{1,3}(0)$:
\[
f(z) = -\left( \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}} \right).
\]

(c) On $A_{3,\infty}(0)$: For $|z| > 3$, we have
\[
-\frac{1}{3 - z} = \frac{1}{z - 3} = \frac{1}{z} \left( \frac{1}{1 - \frac{3}{z}} \right) = \sum_{n=0}^{\infty} \frac{3^n}{z^{n+1}}
\]
and thus, for $z \in A_{3,\infty}(0)$:
\[
f(z) = \sum_{n=1}^{\infty} \left(3^{n-1} - 1\right) \frac{1}{z^n}.
\]
12 The Winding Number of a Curve

Definition 12.1. Let $\gamma$ be a closed curve in $\mathbb{C}$, and let $z \in \mathbb{C} \setminus \{\gamma\}$. Then the winding number of $\gamma$ with respect to $z$ is defined as

$$\nu(\gamma, z) := \frac{1}{2\pi i} \int_{\gamma} \frac{1}{\zeta - z} d\zeta.$$ 

Remark. Geometrically, $\nu(\gamma, z)$ is the number of times $\gamma$ winds around $z$ in counterclockwise direction.

Lemma 12.2. Let $\gamma: [0, 1] \to \mathbb{C}$ be curve, and let $z \in \mathbb{C} \setminus \{\gamma\}$. Then there are a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and open discs $D_1, \ldots, D_n \subset \mathbb{C} \setminus \{z\}$ such that $\gamma([t_{j-1}, t_j]) \subset D_j$ for $j = 1, \ldots, n$.

Proof. Let $\epsilon := \text{dist}(z, \{\gamma\}) > 0$. Since $\gamma$ is uniformly continuous, there is $\delta > 0$ such that $|\gamma(t) - \gamma(t')| < \epsilon$ for all $t, t' \in [0, 1]$ such that $|t - t'| < \delta$. Choose $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $|t_{j-1} - t_j| < \delta$ for $j = 1, \ldots, n$, and set $D_j := B_\epsilon(\gamma(t_j))$ for $j = 1, \ldots, n$. By the choice of $\epsilon$, it is clear that $D_1, \ldots, D_n \subset \mathbb{C} \setminus \{z\}$. For $j = 1, \ldots, n$, let $t \in [t_{j-1}, t_j]$, and note that $|t - t_j| \leq |t_{j-1} - t_j| < \delta$, so that $|\gamma(t) - \gamma(t_j)| < \epsilon$, i.e., $\gamma(t) \in D_j$; consequently, $\gamma([t_{j-1}, t_j]) \subset D_j$ holds.

Proposition 12.3. Let $\gamma$ be a closed curve in $\mathbb{C}$, and let $z \in \mathbb{C} \setminus \{\gamma\}$. Then $\nu(\gamma, z) \in \mathbb{Z}$.

Proof. Choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and open discs $D_1, \ldots, D_n$ as Lemma 12.2.

Let $j \in \{1, \ldots, n\}$. Since $z \notin D_j$, there is a holomorphic function $L_j: D_j \to \mathbb{C}$ such that 

$$e^{L_j(w)} = w - z$$

for $w \in D_j$. Differentiation yields $L_j'(w)e^{L_j(w)} = 1$ and thus 

$$L_j'(w) = \frac{1}{w - z}$$

for $w \in D_j$. It follows that 

$$\int_{\gamma} \frac{1}{\zeta - z} d\zeta = \sum_{j=1}^{n} \int_{\gamma([t_{j-1}, t_j])} \frac{1}{\zeta - z} d\zeta = \sum_{j=1}^{n} (L_j(\gamma(t_j)) - L_j(\gamma(t_{j-1}))) = L_n(\gamma(t_n)) - L_1(\gamma(t_0)) + \sum_{j=1}^{n-1} (L_j(\gamma(t_j)) - L_{j+1}(\gamma(t_j))).$$

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Since 
\[ \exp(L_j(\gamma(t_j))) = \exp(L_{j+1}(\gamma(t_j))) = \gamma(t_j) - z, \]
we have 
\[ \exp(L_j(\gamma(t_j))) - L_{j+1}(\gamma(t_j))) = 1 \]
and thus 
\[ L_j(\gamma(t_j)) - L_{j+1}(\gamma(t_j))) \in 2\pi i \mathbb{Z} \]
for \( j = 1, \ldots, n - 1 \). With a similar argument—using that \( \gamma(t_n) = \gamma(t_0) \)—, we see that 
\[ L_n(\gamma(t_n)) - L_1(\gamma(t_0)) \in 2\pi i \mathbb{Z}, \]
too. 

All in all, \( \int_\gamma \frac{d\zeta}{\zeta - z} \in 2\pi i \mathbb{Z} \) holds. \( \square \)

**Definition 12.4.** Let \( \gamma \) be a closed curve in \( \mathbb{C} \). We define the *interior* and *exterior* of \( \gamma \) to be 
\[ \text{int } \gamma := \{ z \in \mathbb{C} \setminus \{ \gamma \} : \nu(\gamma, z) \neq 0 \} \]
and 
\[ \text{ext } \gamma := \{ z \in \mathbb{C} \setminus \{ \gamma \} : \nu(\gamma, z) = 0 \}. \]

**Proposition 12.5.** Let \( \gamma \) be a closed curve in \( \mathbb{C} \). Then:

(i) the map 
\[ \mathbb{C} \setminus \{ \gamma \} \to \mathbb{C}, \quad z \mapsto \nu(\gamma, z) \]
is locally constant;

(ii) there is \( R > 0 \) such that \( \mathbb{C} \setminus B_R[z_0] \subset \text{ext } \gamma \).

**Proof.** (i): Let \( z_0 \in \mathbb{C} \setminus \{ \gamma \} \). Let \( R > r > 0 \) be such that \( B_R(z_0) \subset \mathbb{C} \setminus \{ \gamma \} \).

Consider the function 
\[ F: \{ \gamma \} \times B_r[z_0] \to \mathbb{C}, \quad (\zeta, z) \mapsto \frac{1}{\zeta - z}. \]

Then \( F \) is continuous and thus uniformly continuous. Choose \( \delta \in (0, r) \) such that 
\[ |F(\zeta', z') - F(\zeta, z)| < \frac{\pi}{\ell(\gamma) + 1} \]
for all \( (\zeta', z'), (\zeta, z) \in \{ \gamma \} \times B_r[z_0] \) such that \( ||(\zeta', z') - (\zeta, z)|| < \delta \).

Let \( z \in B_\delta(z_0) \) and note that 
\[ |F(\zeta, z) - F(\zeta, z_0)| < \frac{\pi}{\ell(\gamma) + 1} \]
holds for all $\zeta \in \{\gamma\}$. Consequently, we have

$$\left|\nu(\gamma, z) - \nu(\gamma, z_0)\right| = \left|\frac{1}{2\pi i} \int_{\gamma} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0}\right) d\zeta\right|$$

$$\leq \frac{\ell(\gamma)}{2\pi} \sup_{\zeta \in \{\gamma\}} |F(\zeta, z) - F(\zeta, z_0)|$$

$$\leq \frac{\ell(\gamma)}{2\pi} \frac{\pi}{\ell(\gamma) + 1}$$

$$< \frac{1}{2}.$$ 

Since $\nu(\gamma, z) - \nu(\gamma, z_0) \in \mathbb{Z}$, this means that $\nu(\gamma, z) = \nu(\gamma, z_0)$.

(ii): For any $z \in \mathbb{C} \setminus \{\gamma\}$, we have

$$\left|\nu(\gamma, z)\right| = \left|\frac{1}{2\pi i} \int_{\gamma} \frac{1}{\zeta - z} d\zeta\right| \leq \frac{\ell(\gamma)}{2\pi} \frac{1}{\operatorname{dist}(z, \{\gamma\})}.$$ 

Since $\lim_{|z| \to \infty} \operatorname{dist}(z, \{\gamma\}) = \infty$, there is $R > 0$ such that $|\nu(\gamma, z)| \leq \frac{\ell(\gamma)}{2\pi} \frac{1}{\operatorname{dist}(z, \{\gamma\})} < 1$ for all $z \in \mathbb{C}$ such that $|z| > R$. Since $\nu(\gamma, z) \in \mathbb{Z}$ for all $z \in \mathbb{C} \setminus \{\gamma\}$, this implies that $\nu(\gamma, z) = 0$ for all $z \in \mathbb{C}$ with $|z| > R$. 

$\square$
13 The Residue Theorem and Applications

Definition 13.1. Let $z_0 \in \mathbb{C}$, let $r > 0$, and let $f : B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic with Laurent series representation

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

for $z \in B_r(z_0) \setminus \{z_0\}$. Then $a_{-1}$ is called the residue of $f$ at $z_0$ and denoted by $\text{res}(f, z_0)$.

Remarks. 1. By Theorem 11.4, we have

$$\text{res}(f, z_0) = \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} f(\zeta) \, d\zeta$$

for any $\rho \in (0, r)$.

2. If $f$ has a removable singularity at $z_0$, then $\text{res}(f, z_0) = 0$.

3. Suppose that $f$ has a simple pole at $z_0$, i.e.,

$$f(z) = \sum_{n=-1}^{\infty} a_n(z - z_0)^n$$

with $a_{-1} \neq 0$, then

$$\text{res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

4. Suppose that $f$ has a pole of order $k$ at $z_0$, and let $g$ be as specified in Theorem 10.2(ii). Then we have

$$\text{res}(f, z_0) = \frac{g^{(k-1)}(z_0)}{(k-1)!}.$$

Examples. 1. Let

$$f(z) = \frac{e^{iz}}{z^2 + 1},$$

so that $f$ has a simple pole at $z_0 = i$. It follows that

$$\text{res}(f, i) = \lim_{z \to i} (z - i) f(z) = \lim_{z \to i} \frac{e^{iz}}{z + i} = -\frac{i}{2e}.$$  

2. Let

$$f(z) = \frac{\cos(\pi z)}{\sin(\pi z)},$$

so that $f$ has a simple pole at each $n \in \mathbb{Z}$. For $n \in \mathbb{Z}$, we thus have:

$$\text{res}(f, n) = \lim_{z \to n} (z - n) \frac{\cos(\pi z)}{\sin(\pi z)}$$

$$= \lim_{z \to n} \frac{\cos(\pi z)}{\sin(\pi z) - \sin(\pi n)}$$

$$= \frac{1}{\pi} \lim_{z \to n} \frac{\pi z - \pi n}{\sin(\pi z) - \sin(\pi n)} \cos(\pi z)$$

$$= \frac{1}{\pi}.$$
3. Let

\[ f(z) = \frac{1}{(z^2 + 1)^3}; \]

then \( f \) has a pole of order 3 at \( z_0 = i \). With

\[ g(z) = (z - i)^3 f(z) = \frac{1}{(z + i)^3}, \]

we have

\[ g'(z) = -\frac{3}{(z + i)^4} \quad \text{and} \quad g''(z) = \frac{12}{(z + i)^5}, \]

so that

\[ \text{res}(f, i) = \frac{1}{2} \frac{12}{(2i)^5} = -\frac{3i}{16}. \]

**Theorem 13.2** (Residue Theorem). Let \( D \subset \mathbb{C} \) be open and star shaped, let \( z_1, \ldots, z_n \in D \) be such that \( z_j \neq z_k \) for \( j \neq k \), let \( f: D \setminus \{z_1, \ldots, z_n\} \to \mathbb{C} \) be holomorphic, and let \( \gamma \) be a closed curve in \( D \setminus \{z_1, \ldots, z_n\} \). Then we have

\[ \int_{\gamma} f(\zeta) \, d\zeta = 2\pi i \sum_{j=1}^{n} \nu(\gamma, z_j) \text{res}(f, z_j). \]

**Proof.** Let \( \epsilon > 0 \) be such that \( B_{\epsilon}(z_j) \subset D \) for \( j = 1, \ldots, n \). For \( j = 1, \ldots, n \), we have Laurent representations

\[ f(z) = \sum_{k=-\infty}^{\infty} a_k^{(j)} (z - z_j)^k \]

for \( z \in B_{\epsilon}(z_j) \setminus \{z_j\} \), so that \( \text{res}(f, z_j) = a_{-1}^{(j)} \). For \( j = 1, \ldots, n \), define

\[ h_j: \mathbb{C} \setminus \{z_j\} \to \mathbb{C}, \quad z \mapsto \sum_{k=\infty}^{-1} a_k^{(j)} (z - z_j)^k, \]

so that \( h_j \) is holomorphic on \( \mathbb{C} \setminus \{z_j\} \). Define

\[ g: D \setminus \{z_1, \ldots, z_n\} \to \mathbb{C}, \quad z \mapsto f(z) - \sum_{j=1}^{n} h_j(z), \]

and note that \( z_1, \ldots, z_n \) are removable singularities for \( g \).
Since $D$ is star shaped, Cauchy’s Integral Theorem yields:

$$0 = \int_\gamma g(\gamma) \, d\zeta$$

$$= \int_\gamma f(\zeta) \, d\zeta \quad - \sum_{j=1}^n \int_\gamma h_j(\zeta) \, d\zeta$$

$$= \int_\gamma f(\zeta) \, d\zeta \quad - \sum_{j=1}^n \int_\gamma \sum_{k=-\infty}^{-1} a_k^{(j)} (\zeta - z_j)^k \, d\zeta$$

$$= \int_\gamma f(\zeta) \, d\zeta \quad - \sum_{j=1}^n \int_\gamma \frac{1}{\zeta - z_j} \, d\zeta$$

$$= \int_\gamma f(\zeta) \, d\zeta \quad - \sum_{j=1}^n a_j^{(j)} 2\pi i \nu(\gamma, z_j). \quad \Box$$

**Corollary 13.3.** Let $D \subset \mathbb{C}$ be open, and star shaped, let $f : D \to \mathbb{C}$ be holomorphic, and let $\gamma$ be a closed curve in $D$. Then we have

$$\nu(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

for $z \in D \setminus \{\gamma\}$.

**Proof.** Fix $z \in D \setminus \{\gamma\}$, and define

$$g : D \setminus \{z\} \to \mathbb{C}, \quad w \mapsto \frac{f(w)}{w - z}.$$  

Then $g$ is holomorphic with an isolated singularity at $z$. Let

$$f(w) = \sum_{n=0}^{\infty} a_n (w - z)^n$$

be the Taylor series expansion of $f$ near $z$, so that

$$g(w) = \sum_{n=-1}^{\infty} a_{n+1} (w - z)^n,$$

and thus $\text{res}(g, z) = a_0 = f(z)$. The Residue Theorem then yields:

$$2\pi i \nu(\gamma, z) f(z) = 2\pi i \nu(\gamma, z) \text{res}(g, z) = \int_\gamma g(\zeta) \, d\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta. \quad \Box$$
Applications of the Residue Theorem to Real Integrals

Proposition 13.4. Let $p$ and $q$ be polynomials of two real variables such that $q(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$ with $x^2 + y^2 = 1$. Then we have

$$\int_0^{2\pi} \frac{p(\cos t, \sin t)}{q(\cos t, \sin t)} \, dt = 2\pi i \sum_{z \in \mathbb{D}} \text{res}(f, z),$$

where

$$f(z) = \frac{1}{iz} \frac{p\left(\frac{1}{2} (z + \frac{1}{z}), \frac{1}{2i} (z - \frac{1}{z})\right)}{q\left(\frac{1}{2} (z + \frac{1}{z}), \frac{1}{2i} (z - \frac{1}{z})\right)}.$$

Proof. Just note that, by the Residue Theorem,

$$2\pi i \sum_{z \in \mathbb{D}} \text{res}(f, z) = \int_{\partial \mathbb{D}} f(\zeta) \, d\zeta$$

$$= \int_0^{2\pi} f(e^{it}) e^{it} \, dt$$

$$= \int_0^{2\pi} \frac{p(\cos t, \sin t)}{q(\cos t, \sin t)} \, dt. \quad \Box$$

Examples. 1. Let $a > 1$. What is $\int_0^\pi \frac{dt}{a + \cos t}$?

First, note that

$$\int_0^\pi \frac{dt}{a + \cos t} = \frac{1}{2} \int_{-\pi}^\pi \frac{dt}{a + \cos t} = \frac{1}{2} \int_0^{2\pi} \frac{dt}{a + \cos t}.$$

Let

$$p(x, y) = 1 \quad \text{and} \quad q(x, y) = a + x,$$

so that

$$f(z) = \frac{1}{iz} \frac{1}{a + \frac{1}{2} (z + \frac{1}{z})}$$

$$= \frac{1}{iz} \frac{1}{a + \frac{1}{2} \left(\frac{z^2+1}{z}\right)}$$

$$= \frac{1}{iz} \frac{1}{a z + \frac{z^2}{2} + \frac{1}{2}}$$

$$= \frac{2}{iz^2 + 2az + 1}$$

$$= \frac{2}{iz \left(z - z_1\right)\left(z - z_2\right)},$$

where

$$z_1 = -a + \sqrt{a^2 - 1} \in \mathbb{D} \quad \text{and} \quad z_2 = -a - \sqrt{a^2 - 1} \notin \mathbb{D}.$$
By Proposition 13.4, we thus obtain:
\[
\int_0^{\pi} \frac{dt}{a + \cos t} = \frac{1}{2} \int_0^{2\pi} \frac{dt}{a + \cos t}
= \pi i \text{res}(f, z_1)
= \pi i \frac{1}{i (z_1 - z_2)}
= 2\pi \frac{1}{2\sqrt{a^2 - 1}}
= \frac{\pi}{\sqrt{a^2 - 1}}.
\]

2. Let \(a > b > 0\). What is \(\int_0^{2\pi} \frac{dt}{(a + b \cos t)^2}\)?

Let \(p(x, y) = 1\) and \(q(x, y) = (a + bx)^2\), so that
\[
f(z) = \frac{1}{iz} \left( \frac{1}{a + \frac{b}{2} \left( z + \frac{1}{z} \right)} \right)^2
= \frac{1}{iz} \left( \frac{1}{a + \frac{bz^2 + b}{2z}} \right)^2
= \frac{1}{iz} \left( \frac{z}{az + \frac{b}{2} + \frac{b}{2} z} \right)^2
= \frac{z}{i} \frac{b^2}{4} (z - z_1)^2 (z - z_2)^2,
\]
where
\[
z_1 = -\frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1} \in \mathbb{D} \quad \text{and} \quad z_2 = -\frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1} \notin \mathbb{D}.
\]
At \(z_1\), the function \(f\) has a pole of order two. In order to calculate \(\text{res}(f, z_1)\), set
\[
g(z) := (z - z_1)^2 f(z) = \frac{4z}{b^2 i} \frac{1}{(z - z_2)^2},
\]
so that
\[
g'(z) = \frac{4}{b^2 i} \left( \frac{1}{(z - z_2)^2} - \frac{2z}{(z - z_2)^3} \right)
= \frac{4}{b^2 i} \frac{1}{(z - z_2)^2} \left( 1 - \frac{2z}{z - z_2} \right)
= \frac{4}{b^2 i} \frac{1}{(z - z_2)^3} ((z - z_2) - 2z)
= -\frac{4}{b^2 i} \frac{z + z_2}{(z - z_2)^3};
\]
it follows that
\[
\text{res}(f, z_1) = g'(z_1) = -\frac{4}{b^3 i} \frac{-2a}{8 \left(\sqrt{\frac{a^2}{b^2} - 1}\right)^3} = \frac{a}{b^3 i} \frac{1}{\left(\sqrt{\frac{a^2}{b^2} - 1}\right)^3} = \frac{1}{i} \frac{a}{\left(\sqrt{a^2 - b^2}\right)^3}.
\]

From Proposition 13.4, we conclude that
\[
\int_0^{2\pi} \frac{dt}{(a + b \cos t)^2} = 2\pi i \text{res}(f, z_1) = 2\pi a \left(\sqrt{a^2 - b^2}\right)^3.
\]

**Proposition 13.5.** Let \(p\) and \(q\) be polynomials of one real variable with \(\deg q \geq \deg p + 2\) and such that \(q(x) \neq 0\) for \(x \in \mathbb{R}\). Then we have
\[
\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \, dx = 2\pi i \sum_{z \in \mathbb{H}} \text{res} \left( \frac{p}{q} , z \right),
\]
where
\[
\mathbb{H} := \{ z \in \mathbb{C} : \text{Im} \, z > 0 \}.
\]

**Proof.** Since \(\deg q \geq \deg p + 2\), the comparison test yields that the indefinite integral exists.

Let \(r > 0\), and define
\[
\gamma_r : [0, \pi] \to \mathbb{C}, \quad t \mapsto re^{it}.
\]

Let \(\epsilon > 0\) be such that, for \(D := \{ z \in \mathbb{C} : \text{Im} \, z > -\epsilon \}\), we have
\[
\{ z \in \mathbb{H} : q(z) = 0 \} = \{ z \in D : q(z) = 0 \}.
\]

Then \(D\) is star shaped, and \(\frac{p}{q}\) is holomorphic on \(D\) except at the zeros of \(q\) in \(\mathbb{H}\). For large enough \(r\)—such that all zeros of \(q\) in \(D\) lie in the interior of \([-r, r] \oplus \gamma_r\)—, we have by the Residue Theorem:
\[
\int_{[-r,r] \oplus \gamma_r} \frac{p(\zeta)}{q(\zeta)} \, d\zeta = 2\pi i \sum_{z \in D} \text{res} \left( \frac{p}{q} , z \right) = 2\pi i \sum_{z \in \mathbb{H}} \text{res} \left( \frac{p}{q} , z \right).
\]

There are \(R > 0\) and \(C \geq 0\) such that
\[
\left| \frac{p(z)}{q(z)} \right| \leq \frac{C}{|z|^2}
\]

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for all \( z \in \mathbb{C} \) with \( |z| \geq R \). It follows that

\[
\left| \int_{\gamma_r} \frac{p(\zeta)}{q(\zeta)} \, d\zeta \right| \leq \pi r \sup_{\zeta \in \{\gamma_r\}} \frac{C}{|\zeta|^2} \leq \frac{\pi C}{r}
\]

for \( r \geq R \) and thus

\[
\lim_{r \to \infty} \int_{\gamma_r} \frac{p(\zeta)}{q(\zeta)} \, d\zeta = 0.
\]

All in all, we obtain:

\[
\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \, dx = \lim_{r \to \infty} \int_{[-r,r]} \frac{p(\zeta)}{q(\zeta)} \, d\zeta = \lim_{r \to \infty} \int_{[-r,r]} \frac{p(\zeta)}{q(\zeta)} \, d\zeta + \lim_{r \to \infty} \int_{\gamma_r} \frac{p(\zeta)}{q(\zeta)} \, d\zeta = 2\pi i \sum_{z \in \mathbb{H}} \text{res} \left( \frac{p}{q}, z \right).
\]

**Examples.**

1. What is \( \int_{0}^{\infty} \frac{1}{1 + x^6} \, dx \)?

The zeros of \( q(z) = 1 + z^6 \) are of the form \( e^{i\theta} \) where \( \theta \in [0, 2\pi) \) is such that \( e^{i6\theta} = -1 = e^{i\pi} \), i.e., \( 6\theta - \pi \in 2\pi \mathbb{Z} \), so that \( \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6} \). For \( k = 1, \ldots, 6 \), let

\[ z_k = e^{i(2k-1)\frac{\pi}{6}}. \]

Then \( \frac{1}{q} \) has a simple pole at \( z_k \) for \( k = 1, \ldots, 6 \).

By a homework problem, we have

\[ \text{res} \left( \frac{1}{q}, z_k \right) = \frac{1}{6z_k^5} = -\frac{z_k}{6}. \]

so that by, Proposition 13.5,

\[
\int_{0}^{\infty} \frac{1}{1 + x^6} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1 + x^6} \, dx = \frac{\pi i}{6} \sum_{k=1}^{3} \text{res} \left( \frac{1}{q}, z_k \right) = \frac{-\pi i}{6} \left( e^{i\frac{\pi}{6}} + e^{i\frac{5\pi}{6}} + e^{i\frac{5\pi}{6}} \right) = \frac{-\pi i}{6} \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} + i + \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) = \frac{\pi}{6} \left( 2 \sin \frac{\pi}{6} + 1 \right) = \frac{\pi}{3}.
\]
2. What is \( \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n} \, dx \), where \( n \in \mathbb{N} \)?

The polynomial \( q(z) := (z^2 + 1)^n \) has zeros of order \( n \) at \( \pm i \). Define

\[
g(z) = (z - i)^n \frac{1}{q(z)} = (z + i)^{-n},
\]

so that

\[
g^{(n-1)}(z) = (-n) \cdots (-2n + 2)(z + i)^{-2n+1}
\]

and thus

\[
\text{res}\left( \frac{1}{q}, i \right) = \frac{g^{(n-1)}(i)}{(n-1)!}
\]

\[
= \frac{1}{(n-1)!} \frac{1}{2^{2n-1}} \frac{1}{i^n} \frac{1}{(2n - 2)!}
\]

\[
= \frac{1}{i} \frac{1}{2^{2n-1}} \frac{(2n - 2)!}{((n-1)!)^2}.
\]

It follows that

\[
\int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)^n} \, dx = 2\pi i \text{res}\left( \frac{1}{q}, i \right) = \frac{\pi}{2^{2n-2} ((n-1)!)^2};
\]

in particular, we have

\[
\int_{-\infty}^{\infty} \frac{1}{x^2 + 1} \, dx = \pi,
\]

\[
\int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)^2} \, dx = \frac{\pi}{2}, \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)^3} \, dx = \frac{3\pi}{8}.
\]
14 Function Theoretic Consequences of the Residue Theorem

Definition 14.1. Let $D \subset \mathbb{C}$ be open. We call $S \subset D$ discrete in $D$ if it has no cluster points in $D$.

Example. If $D$ is connected, and $f : D \to \mathbb{C}$ is holomorphic, then $\mathcal{Z}(f)$ is discrete if $f \not\equiv 0$.

Remark. Let $S \subset D$ be discrete, and let $K \subset D$ be compact. Assume that $K \cap S$ is infinite. Then $K \cap S$ has cluster points, which must lie in $K$ in thus in $D$. Hence, $K \cap S$ must be finite. In the homework, it was shown that there are compact $K_1 \subset K_2 \subset K_3 \subset \cdots \subset D$ with $D = \bigcup_{n=1}^{\infty} K_n$. It follows that

$$S = \bigcup_{n=1}^{\infty} K_n \cap S$$

is countable.

Proposition 14.2. Let $D \subset \mathbb{C}$ be open, let $\gamma$ be a closed curve in $D$, and let $S \subset D$ be discrete. Then $S \cap \text{int} \gamma$ is finite.

Proof. By Proposition 12.5(ii), there is $R > 0$ such that $\text{int} \gamma \subset B_R[0]$.

Definition 14.3. Let $D \subset \mathbb{C}$ be open. A meromorphic function on $D$ is a holomorphic function $f : D \setminus \mathcal{P}(f) \to \mathbb{C}$, where $\mathcal{P}(f)$ is a discrete subset of $D$ such that each point in $\mathcal{P}(f)$ is a pole of $f$.

Remark. If $D$ is connected, and $f, g : D \to \mathbb{C}$ are holomorphic, then $f \cdot g$ is meromorphic if $g \not\equiv 0$.

Proposition 14.4. Let $D \subset \mathbb{C}$ be open, and let $f$ be meromorphic on $D$. Then, for each $z_0 \in D$, there are $\epsilon > 0$ with $B_{\epsilon}(z_0) \subset D$ as well as holomorphic $g, h : B_{\epsilon}(z_0) \to \mathbb{C}$ such that $f(z) = \frac{g(z)}{h(z)}$ for $z \in B_{\epsilon}(z_0) \setminus \{z_0\}$.

Proof. If $z_0$ is not a pole of $f$, the claim is clear.

Otherwise, choose $\epsilon > 0$ so small that $B_{\epsilon}(z_0) \subset D$ and $\mathcal{P}(f) \cap B_{\epsilon}(z_0) = \{z_0\}$. We can then find holomorphic $g : B_{\epsilon}(z_0) \to \mathbb{C}$ with $g(z_0) \neq 0$ and $k \in \mathbb{N}$ such that

$$f(z) = \frac{g(z)}{(z - z_0)^k}$$

for $z \in B_{\epsilon}(z_0) \setminus \{z_0\}$. Setting $h(z) := (z - z_0)^n$ yields the claim. \qed

Lemma 14.5. Let $D \subset \mathbb{C}$ be open and connected, and let $S \subset D$ be discrete. Then $D \setminus S$ is open and connected.
Lemma 14.5. From the Identity Theorem, we conclude that poles and zeros, respectively, of which contradicts the connectedness of $D$. Then $	ilde{\gamma}$ and $(B_D)$—with pointwise addition and multiplication—form a field.

Proof. It is routinely checked that the meromorphic functions do indeed form a commutative ring with identity.

Let $f$ be a non-zero meromorphic function on $D$, and let $P(f)$ and $Z(f)$ denote the poles and zeros, respectively, of $f$ in $D$. As $P(f)$ is discrete, $D \setminus P(f)$ is connected by Lemma 14.5. From the Identity Theorem, we conclude that $Z(f)$ is discrete, too.

Define

$$\tilde{f}: D \setminus Z(f) \to C, \quad z \mapsto \begin{cases} \frac{1}{f(z)}, & z \notin P(f), \\ 0, & z \in P(f). \end{cases}$$

Then $\tilde{f}$ is holomorphic, and we have $f(z)\tilde{f}(z) = 1$ for $z \in D \setminus (P(f) \cup Z(f))$. \hfill $\square$

Theorem 14.6. Let $D \subset C$ be open and connected. Then the meromorphic functions on $D$—with pointwise addition and multiplication—form a field.

Proof. It is routinely checked that the meromorphic functions do indeed form a commutative ring with identity.

Let $f$ be a non-zero meromorphic function on $D$, and let $P(f)$ and $Z(f)$ denote the poles and zeros, respectively, of $f$ in $D$. As $P(f)$ is discrete, $D \setminus P(f)$ is connected by Lemma 14.5. From the Identity Theorem, we conclude that $Z(f)$ is discrete, too.

Define

$$\tilde{f}: D \setminus Z(f) \to C, \quad z \mapsto \begin{cases} \frac{1}{f(z)}, & z \notin P(f), \\ 0, & z \in P(f). \end{cases}$$

Then $\tilde{f}$ is holomorphic, and we have $f(z)\tilde{f}(z) = 1$ for $z \in D \setminus (P(f) \cup Z(f))$. \hfill $\square$

Theorem 14.7 (Argument Principle). Let $D \subset C$ be open and star shaped, let $f$ be meromorphic on $D$, and let $\gamma$ be a closed curve in $D \setminus (P(f) \cup Z(f))$. Then we have

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum_{z \in Z(f)} \nu(\gamma, z) \mathrm{ord}(f, z) - \sum_{z \in P(f)} \nu(\gamma, z) \mathrm{ord}(f, z).$$

(The order of a zero of a holomorphic function was defined on the homework.)

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Proof. By the Residue Theorem, we have

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta = \sum_{\zeta \in \mathcal{Z}(f)} \nu(\gamma, z) \text{res} \left( \frac{f'}{f}, z \right) + \sum_{\zeta \in \mathcal{P}(f)} \nu(\gamma, z) \text{res} \left( \frac{f'}{f}, z \right).
\]

Let \( z_0 \in \mathcal{Z}(f) \), and let \( k := \text{ord}(f, z_0) \). Then there is a holomorphic function \( g \) with \( g(z_0) \neq 0 \) such that \( f(z) = (z - z_0)^k g(z_0) \) and thus

\[
f'(z) = k (z - z_0)^{k-1} g(z) + (z - z_0)^k g'(z).
\]

It follows that

\[
\frac{f'(z)}{f(z)} = \frac{k}{z - z_0} + \frac{g'(z)}{g(z)}
\]

for \( z \) near \( z_0 \), so that

\[
\text{res} \left( \frac{f'}{f}, z \right) = k.
\]

Let \( z_0 \in \mathcal{P}(f) \), and let \( k := \text{ord}(f, z_0) \). Then \( f(z) = \frac{g(z)}{(z - z_0)^k} \) holds with \( g \) holomorphic such that \( g(z_0) \) and, consequently,

\[
f'(z) = -k(z - z_0)^{-(k+1)} g(z) + (z - z_0)^{-k} g'(z).
\]

It follows that

\[
\frac{f'(z)}{f(z)} = \frac{-k}{z - z_0} + \frac{g'(z)}{g(z)}
\]

for \( z \neq z_0 \) near \( z_0 \), so that

\[
\text{res} \left( \frac{f'}{f}, z \right) = -k. \quad \square
\]

**Definition 14.8.** Let \( D \subset \mathbb{C} \) be open, and let \( f : D \to \mathbb{C} \) be holomorphic. We say that \( f \) attains \( w_0 \in \mathbb{C} \) with multiplicity \( k \in \mathbb{N} \) at \( z_0 \in D \) if the function

\[
D \to \mathbb{C}, \quad z \mapsto f(z) - w_0
\]

has a zero of order \( k \) at \( z_0 \).

**Theorem 14.9.** Let \( D \subset \mathbb{C} \) be open, let \( f : D \to \mathbb{C} \) be holomorphic, and suppose that, at \( z_0 \in D \), the function \( f \) attains \( w_0 \) with multiplicity \( k \in \mathbb{N} \). Then there are open neighborhoods \( V \subset D \) of \( z_0 \) and \( W \subset \mathcal{f}(V) \) of \( w_0 \) such that, for each \( w \in W \setminus \{w_0\} \), there are pairwise distinct \( z_1, \ldots, z_k \in V \) with \( f(z_1) = \cdots = f(z_k) = w \), where \( f \) attains \( w \) at each \( z_j \) with multiplicity one.

Proof. Without loss of generality, suppose that \( D \) is star shaped.

Choose \( \epsilon > 0 \) with \( B_\epsilon[z_0] \subset D \) such that \( f(z) \neq w_0 \) and \( f'(z) \neq 0 \) for all \( z \in B_\epsilon[z_0] \setminus \{z_0\} \).
Set $V := B_{\epsilon}(z_0)$ and $\gamma := \partial B_{\epsilon}(z_0)$. Choose $\delta > 0$ such that $B_{\delta}(w_0) \subset \mathbb{C} \setminus \{f \circ \gamma\}$, and set $W := B_{\delta}(w_0)$. Let $w \in W$. By the Argument Principle, the number of times $w$ is attained in $V$ (with multiplicity) is
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(\zeta)}{f(\zeta) - w} d\zeta = \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{d\zeta}{\zeta - w} = \nu(f \circ \gamma, w).
\]
As $\nu(f \circ \gamma, \cdot)$ is constant on $W$, the number of times $w$ is attained in $V$ is the same as the number of times $w_0$ is attained in $V$, i.e., $k$. Suppose that $w \in W \setminus \{w_0\}$. Since $f'(z) \neq 0$ for all $z \in V \setminus \{z_0\}$, there are pairwise distinct $z_1, \ldots, z_k \in V \setminus \{z_0\}$ such that $f(z_1) = \cdots = f(z_k) = w$; necessarily, $f$ attains $w$ at each $z_j$ with multiplicity one.

**Theorem 14.10** (Hurwitz' Theorem). Let $D \subset \mathbb{C}$ be open and connected, let $f, f_1, f_2, \ldots : D \to \mathbb{C}$ be holomorphic such that $(f_n)_{n=1}^{\infty}$ converges to $f$ compactly on $D$, and suppose that $\mathbf{Z}(f_n) = \emptyset$ for $n \in \mathbb{N}$. Then $f \equiv 0$ or $\mathbf{Z}(f) = \emptyset$.

**Proof.** Suppose that $f \neq 0$, but that there is $z_0 \in \mathbf{Z}(f)$. Choose $\epsilon > 0$ such that $B_{\epsilon}(z_0) \subset D$ and $f(z) \neq 0$ for all $z \in B_{\epsilon}(z_0) \setminus \{z_0\}$, and note that
\[
0 = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\partial B_{\epsilon}(z_0)} \frac{f_n'(\zeta)}{f_n(\zeta)} d\zeta = \frac{1}{2\pi i} \int_{\partial B_{\epsilon}(z_0)} \frac{f'(\zeta)}{f(\zeta)} d\zeta = \text{ord}(f, z_0),
\]
which is a contradiction. \(\square\)

**Corollary 14.11.** Let $D \subset \mathbb{C}$ be open and connected, let $f, f_1, f_2, \ldots : D \to \mathbb{C}$ be holomorphic such that $(f_n)_{n=1}^{\infty}$ converges to $f$ compactly on $D$, and suppose that $f_n$ is injective for $n \in \mathbb{N}$. Then $f$ is constant or injective.

**Proof.** Suppose that $f$ is not constant. Let $z_0 \in D$ be arbitrary, and define
\[
g_n : D \setminus \{z_0\} \to \mathbb{C}, \quad z \mapsto f_n(z) - f_n(z_0)
\]
for $n \in \mathbb{N}$. Then $g_1, g_2, \ldots$ have no zeros. Since $f$ is not constant, the function
\[
D \setminus \{z_0\} \to \mathbb{C}, \quad z \mapsto f(z) - f(z_0)
\]
is not zero, so that that is has no zeros by Hurwitz's theorem, i.e., $f(z) \neq f(z_0)$ for all $z \in D$, $z \neq z_0$. \(\square\)

**Theorem 14.12** (Rouché's Theorem). Let $D \subset \mathbb{C}$ be open and star shaped, and let $f, g : D \to \mathbb{C}$ be holomorphic. Suppose that $\gamma$ is a closed curve in $D$ such that $\text{int} \gamma = \{z \in D \setminus \gamma : \nu(\gamma, z) = 1\}$ and that
\[
|f(\zeta) - g(\zeta)| < |f(\zeta)|
\]
for $\zeta \in \gamma$. Then $f$ and $g$ have the same number of zeros in $\text{int} \gamma$ (counted with multiplicity).
Proof. For \( t \in [0, 1] \), define \( h_t := f + t(g - f) \), so that \( h_0 = f \) and \( h_1 = g \). Also, since

\[
|t(g - f)| \leq |g - f| < |f|
\]

for any \( t \in [0, 1] \) on \( \{ \gamma \} \), the functions \( h_t \) have no zeros on \( \{ \gamma \} \). For \( t \in [0, 1] \), let \( n_t \in \mathbb{N}_0 \) denote the number of zeros of \( h_t \) in \( \text{int} \gamma \). From the Argument Principle, we obtain that

\[
n_t = \frac{1}{2\pi i} \int_{\gamma} h_t'(\zeta) h_t(\zeta) d\zeta = \frac{1}{2\pi i} \int_{\gamma} f'(\zeta) + t(g'(\zeta) - f'(\zeta)) - t(g(\zeta) - f(\zeta)) d\zeta.
\]

Since the integral on the right hand side depends continuously on \( t \), we conclude that \( n_0 = n_1 \).

Example. How many zeros does \( z^4 - 4z + 2 \) have in \( \mathbb{D} \)?

Set \( g(z) := z^4 - 4z + 2 \) and \( f(z) = -4z + 2 \).

For \( \zeta \in \partial \mathbb{D} \), we have \( |f(\zeta)| \geq 4 - 2 = 2 \), so that

\[
|f(\zeta) - g(\zeta)| = |\zeta^4| = 1 < 2 \leq |f(\zeta)|.
\]

Since \( f \) has precisely one zero in \( \mathbb{D} \), so has \( g \).

**Corollary 14.13.** Let \( p \) be a polynomial with \( n := \deg p \geq 1 \). Then \( p \) has \( n \) zeros (counted with multiplicity).

**Proof.** Let

\[
p(z) = a_n z^n + \cdots + a_1 z + a_0
\]

with \( a_n \neq 0 \), and let \( g(z) := a_n z^n \), so that \( \lim_{|z| \to \infty} \left| \frac{p(z) - g(z)}{g(z)} \right| = 0 \). Choose \( R > 0 \) such that

\[
\left| \frac{p(z) - g(z)}{g(z)} \right| < 1
\]

for \( z \in \mathbb{C} \) with \( |z| \geq R \). Consequently, if \( \zeta \in \partial B_R(0) \), we have \( |p(\zeta) - g(\zeta)| < |g(\zeta)| \). By Rouche’s Theorem, \( p \) thus has as many zeros in \( B_R(0) \) as \( g \), namely \( n \). Since \( p \) has at most \( n \) zeros, these are all the zeros of \( p \).
15 The General Cauchy Integral Theorem

Definition 15.1. Let $D \subset \mathbb{C}$ be open. We call a closed curve $\gamma$ in $D$ homologous to zero if $\nu(\gamma, z) = 0$ for each $z \in \mathbb{C} \setminus D$.

Theorem 15.2 (Cauchy’s Integral Theorem). Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $\gamma$ be a closed curve in $D$ that is homologous to zero. Then $\int_{\gamma} f(\zeta) \, d\zeta = 0$ holds.

Theorem 15.3 (Cauchy’s Integral Formula). Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $\gamma$ be a closed curve in $D$ that is homologous to zero. Then, for $n \in \mathbb{N}_0$ and $z \in D \setminus \{\gamma\}$, we have

$$\nu(\gamma, z)f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta.$$

Proof. It is enough to prove the claim for $n = 0$: for general $n \in \mathbb{N}_0$, use induction and differentiation under the integral.

Define

$$g : D \times D \to \mathbb{C}, \quad (w, z) \mapsto \begin{cases} \frac{f(w) - f(z)}{w - z}, & w \neq z, \\ f'(z), & w = z. \end{cases}$$

We claim that $g$ is continuous. To see this, let $(w_0, z_0) \in D \times D$. As $g$ is clearly continuous at $(w_0, z_0)$ if $w_0 \neq z_0$, we suppose without loss of generality that $w_0 = z_0$. Let $\delta > 0$ be such that $B_\delta[z_0] \subset D$. For $(w, z) \in B_\delta(z_0) \times B_\delta(z_0)$, we then have:

- if $w = z$:
  $$g(w, z) - g(z_0, z_0) = f'(z) - f'(z_0);$$
- if $w \neq z$:
  $$g(w, z) - g(z_0, z_0) = \frac{f(w) - f(z)}{w - z} - f'(z_0) = \frac{1}{w - z} \int_{[z, w]} (f'(\zeta) - f'(z_0)) \, d\zeta.$$

Let $\epsilon > 0$, and choose $\delta > 0$ so small that $|f'(z) - f'(z_0)| < \epsilon$ for all $z \in B_\delta[z_0]$. Let $z, w \in B_\delta(z_0)$. If $w = z$, we have

$$|g(w, z) - g(z_0, z_0)| = |f'(z) - f'(z_0)| < \epsilon$$

and, if $w \neq z$,

$$|g(w, z) - g(z_0, z_0)| \leq \frac{1}{|w - z|} |z - w| \sup_{\zeta \in \{z, w\}} |f'(\zeta) - f'(z_0)| \leq \epsilon.$$

All in all, $g$ is continuous at $(z_0, w_0)$. 

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Next, define
\[ h_0 : D \to \mathbb{C}, \quad z \mapsto \int_\gamma g(\zeta, z) \, d\zeta. \]

We claim that \( h_0 \) is holomorphic. It is easy to see that \( h_0 \) is continuous. To see that it is indeed holomorphic, we shall show that it satisfies the Morera condition. Let \( \Delta \subset D \) be a triangle. For fixed \( \zeta \in \{\gamma\} \), the function
\[ D \to \mathbb{C}, \quad z \mapsto g(\zeta, z) \]
is holomorphic as a consequence of Riemann’s Removability Condition. Goursat’s Lemma thus yields that
\[ \int_{\partial\Delta} g(\zeta, z) \, dz = 0 \]
for each \( \zeta \in \{\gamma\} \). As a consequence, we obtain that
\[
0 = \int_\gamma \left( \int_{\partial\Delta} g(\zeta, z) \, dz \right) d\zeta \\
= \int_{\partial\Delta} \left( \int_\gamma g(\zeta, z) \, d\zeta \right) dz \\
= \int_{\partial D} h_0(z) \, dz,
\]
so that \( h_0 \) is holomorphic as claimed.

Define
\[ h_1 : \text{ext} \gamma \to \mathbb{C}, \quad z \mapsto \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta. \]

Then \( h_1 \) is holomorphic. For \( z \in D \cap \text{ext} \gamma \), we have
\[
h_0(z) = \int_\gamma g(\zeta, z) \, d\zeta \\
= \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta \\
= \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta + f(z) \int_\gamma \frac{1}{\zeta - z} \, d\zeta \\
= \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta.
\]

Define
\[ h : D \cup \text{ext} \gamma, \quad z \mapsto \begin{cases} h_0(z), & z \in D, \\ h_1(z), & z \in \text{ext} \gamma. \end{cases} \]

Then \( h \) is holomorphic. Since \( \gamma \) is homologous to zero, we have \( \mathbb{C} \setminus D \subset \text{ext} \gamma \). Hence, \( h \) is entire.
For any \( z \in \text{ext} \gamma \), we have the estimate
\[
|h(z)| = |h_1(z)| \leq \frac{\ell(\gamma)}{\text{dist}(z, \{\gamma\})} \sup_{\zeta \in \{\gamma\}} |f(\zeta)|.
\]

Let \( R > 0 \) be such that \( \mathbb{C} \setminus B_R(0) \subset \text{ext} \gamma \). Then (\ast) yields that \( h \) is bounded on \( \mathbb{C} \setminus B_R(0) \); by continuity, \( h \) is trivially bounded on \( B_R(0) \). All in all, \( h \) is bounded and thus constant by Liouville’s Theorem. From (\ast) again, we see that \( \lim_{|z| \to \infty} |h(z)| = 0 \). Hence, \( h \equiv 0 \) holds.

All in all, we have for \( z \in D \setminus \{\gamma\} \) that
\[
0 = h(z) = h_0(z) = \int_{\gamma} \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta.
\]

\textit{Proof of Cauchy’s Integral Theorem.} Let \( z_0 \in D \setminus \{\gamma\} \) be arbitrary, and define
\[
g: D \to \mathbb{C}, \quad z \mapsto (z - z_0)f(z),
\]
so that
\[
0 = \nu(\gamma, z_0)g(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_{\gamma} f(\zeta) \, d\zeta
\]

\textbf{Definition 15.4.} Let \( D \subset \mathbb{C} \) be open. We call \( D \) \textit{simply connected} if \( D \) is connected, and every closed curve in \( D \) is homologous to zero.

\textbf{Corollary 15.5.} The following are equivalent for an open, connected set \( D \subset \mathbb{C} \):

(i) \( D \) is simply connected;

(ii) every holomorphic function on \( D \) has an anti-derivative.

\textbf{Corollary 15.6 (Existence of Holomorphic Logarithms).} Let \( D \subset \mathbb{C} \) be simply connected, and let \( f: D \to \mathbb{C} \) be holomorphic such that \( \mathcal{Z}(f) = \emptyset \). Then there is a holomorphic \( g: D \to \mathbb{C} \) with \( f = \exp \circ g \).

\textit{Proof.} This was proven in the homework for star shaped \( D \); the argument carries over verbatim.

\textbf{Corollary 15.7 (Existence of Holomorphic Roots).} Let \( D \subset \mathbb{C} \) be simply connected, and let \( f: D \to \mathbb{C} \) be holomorphic such that \( \mathcal{Z}(f) = \emptyset \). Then, for each \( n \in \mathbb{N} \), there is a holomorphic \( h_n : D \to \mathbb{C} \) such that \( h_n(z)^n = f(z) \) for \( z \in D \).

\textit{Proof.} Let \( g \) be as in the previous corollary, and set \( h_n := \exp \circ \left( \frac{g}{n} \right) \) for \( n \in \mathbb{N} \).
16 Montel’s Theorem

Definition 16.1. Let $S \subseteq \mathbb{R}^N$. A family $\mathcal{F}$ of functions on $S$ into $\mathbb{R}^M$ is called equicontinuous if, for each $\epsilon > 0$, there is $\delta > 0$ such that $\|f(x) - f(y)\| < \epsilon$ for all $f \in \mathcal{F}$ and for all $x, y \in S$ such that $\|x - y\| < \delta$.

Lemma 16.2. Let $S \subseteq \mathbb{R}^N$. Then $S$ contains a countable dense subset.

Proof. Let \{x_1, x_2, x_3, \ldots\} be a dense, countable subset of $\mathbb{R}^N$, e.g., $\mathbb{Q}^N$. For $n, m \in \mathbb{N}$ with $S \cap B_{\frac{1}{m}}(x_n) \neq \emptyset$, choose $y_{n,m} \in S \cap B_{\frac{1}{m}}(x_n)$. Then

$$S_0 := \left\{ y_{n,m} : S \cap B_{\frac{1}{m}}(x_n) \neq \emptyset \right\} \subset S$$

is countable.

Let $\epsilon > 0$ and $x \in S$. Choose $m \in \mathbb{N}$ so large that $\frac{1}{m} < \frac{\epsilon}{2}$. Since \{x_1, x_2, x_3, \ldots\} is dense in $\mathbb{R}^N$, there is $n \in \mathbb{N}$ such that $\|x_n - x\| < \frac{1}{m}$ and thus $x \in S \cap B_{\frac{1}{m}}(x_n) \neq \emptyset$. It follows that

$$\|y_{n,m} - x\| \leq \|y_{n,m} - x_n\| + \|x_n - x\| < \frac{2}{m} < \epsilon \quad \square$$

Theorem 16.3 (Arzelà–Ascoli Theorem). Let $K \subseteq \mathbb{R}^N$ be compact, and let $\mathcal{F}$ be a bounded, equicontinuous family of functions from $K$ to $\mathbb{R}^M$. Then every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on $K$.

Proof. Let $(f_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{F}$, and let \{x_1, x_2, x_3, \ldots\} be a countable dense subset of $K$.

Since $(f_n(x_1))_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}^M$, there is a subsequence $(f_{n,1})_{n=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ such that $(f_{n,1}(x_1))_{n=1}^{\infty}$ converges.

Since $(f_{n,1}(x_2))_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}^M$, there is a subsequence $(f_{n,2})_{n=1}^{\infty}$ of $(f_{n,1})_{n=1}^{\infty}$ such that $(f_{n,2}(x_2))_{n=1}^{\infty}$ converges.

Continuing inductively in this fashion, we obtain, for each $k \in \mathbb{N}$, a subsequence $(f_{n,k})_{n=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ such that, for each $k \in \mathbb{N}$,

- $(f_{n,k+1})_{n=1}^{\infty}$ is a subsequence of $(f_{n,k})_{n=1}^{\infty}$, and
- $(f_{n,k}(x_k))_{n=1}^{\infty}$ converges.

For $n \in \mathbb{N}$, set $g_n := f_{n,n}$. Then $(g_n)_{n=1}^{\infty}$ is a subsequence of $(f_n)_{n=1}^{\infty}$, and $(g_n(x_k))_{n=1}^{\infty}$ converges for each $k \in \mathbb{N}$.

We claim that $(g_n)_{n=1}^{\infty}$ is a uniform Cauchy sequence on $K$ (and thus convergent).

Let $\epsilon > 0$. Choose $\delta > 0$ such that $\|f(x) - f(y)\| < \frac{\epsilon}{3}$ for all $f \in \mathcal{F}$ and for all $x, y \in K$ with $\|x - y\| < \delta$. Since $K$ is compact, there are $y_1, \ldots, y_\nu \in K$ such that $K \subseteq \bigcup_{j=1}^{\nu} B_{\frac{3}{2}}(y_j)$. Since \{x_1, x_2, x_3, \ldots\} is dense in $K$, there are $k_1, \ldots, k_\nu \in \mathbb{N}$ such that $x_{k_j} \in B_{\frac{3}{2}}(y_j)$. It follows that $K \subseteq \bigcup_{j=1}^{\nu} B_{\frac{3}{2}}(x_{k_j})$.  

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By definition, \((g_n(x_k))_{n=1}^{\infty}\) is a Cauchy sequence for each \(k \in \mathbb{N}\). Choose \(n_\varepsilon \in \mathbb{N}\) such that
\[
\|g_n(x_{k_j}) - g_m(x_{k_j})\| < \frac{\varepsilon}{3}
\]
for \(n, m \geq n_\varepsilon\) and \(j = 1, \ldots, \nu\). Let \(x \in K\) be arbitrary, and let \(n, m \geq n_\varepsilon\). Choose \(j \in \{1, \ldots, \nu\}\) such that \(x \in B_{\delta}(x_{k_j})\), and note that
\[
\|g_n(x) - g_m(x)\| \leq \|g_n(x) - g_n(x_{k_j})\| + \|g_n(x_{k_j}) - g_m(x_{k_j})\| + \|g_m(x_{k_j}) - g_m(x)\| < \varepsilon.
\]
Hence, \((g_n)_{n=1}^{\infty}\) is a uniform Cauchy sequence as claimed.

**(Definition 16.4)** Let \(D \subset \mathbb{R}^N\) be open. We call a family \(\mathcal{F}\) of functions from \(D\) into \(\mathbb{R}^M\)

(a) **locally bounded** if, for each \(x_0 \in D\), there is \(\varepsilon > 0\) with \(B_\varepsilon(x_0) \subset D\) such that
\[
\{f|_{B_\varepsilon(x_0)} : f \in \mathcal{F}\}
\]
is bounded and

(b) **locally equicontinuous** if, for each \(x_0 \in D\), there is \(\varepsilon > 0\) with \(B_\varepsilon(x_0) \subset D\) such that
\[
\{f|_{B_\varepsilon(x_0)} : f \in \mathcal{F}\}
\]
is equicontinuous.

For any non-empty set \(S \subset \mathbb{R}\), we define its **diameter** as
\[
diam(S) := \sup\{\|x - y\| : x, y \in S\}.
\]

**(Proposition 16.5)** (Lebesgue’s Covering “Lemma”). Let \(K \subset \mathbb{R}^N\) be compact, and let \(\mathcal{U}\) be an open cover for \(K\). Then there is a number \(\ell(\mathcal{U}) > 0\)—the Lebesgue number of \(\mathcal{U}\)—such that, for each \(\emptyset \neq S \subset K\) with \(diam(S) < \ell(\mathcal{U})\), there is \(U \in \mathcal{U}\) such that \(S \subset U\).

**Proof.** Assume towards a contradiction that the assertion is false. Then, for each \(n \in \mathbb{N}\), there is \(\emptyset \neq S_n \subset K\) such that \(diam(S_n) < \frac{1}{n}\) and \(S_n \not\subset U\) for each \(U \in \mathcal{U}\). For \(n \in \mathbb{N}\), choose \(x_n \in S_n\). Since \(K\) is compact, \((x_n)_{n=1}^{\infty}\) has a convergent subsequence, say \((x_n)_{k=1}^{\infty}\), with limit \(x_0 \in K\). Let \(U_0 \in \mathcal{U}\) be such that \(x_0 \in U_0\). Let \(\varepsilon > 0\) be such that
\[
B_\varepsilon(x_0) \subset U_0
\]
(this is possible because \(U_0\) is open). Let \(k_0 \in \mathbb{N}\) be so large that \(\frac{1}{n_{k_0}} < \frac{\varepsilon}{2}\) and \(\|x_n - x_0\| < \frac{\varepsilon}{2}\). Let \(x \in S_{n_{k_0}}\) be arbitrary, and note that
\[
\|x - x_0\| \leq \|x - x_{n_{k_0}}\| + \|x_{n_{k_0}} - x_0\| < \varepsilon.
\]
It follows that \(S_{n_{k_0}} \subset B_\varepsilon(x_0) \subset U_0\), which contradicts the choices of \(S_1, S_2, \ldots\) \(\square\)

**(Lemma 16.6)** Let \(D \subset \mathbb{R}^N\) be open, and let \(\mathcal{F}\) be a family of functions from \(D\) into \(\mathbb{R}^M\).

For \(K \subset D\) compact, set
\[
\mathcal{F}|_K := \{f|_K : f \in \mathcal{F}\}.
\]

Then we have:
(i) if $\mathcal{F}$ is locally bounded, then $\mathcal{F}|_K$ is bounded;

(ii) if $\mathcal{F}$ is locally equicontinuous, then $\mathcal{F}|_K$ is equicontinuous.

Proof. (i): For each $x \in K$, there is $\epsilon_x > 0$ such that $B_{\epsilon_x}(x) \subset D$ and $\{ f|_{B_{\epsilon_x}(x)} : f \in \mathcal{F} \}$ is bounded, i.e., there is $C_x > 0$ such that

$$\sup \{ \| f(y) \| : f \in \mathcal{F}, y \in B_{\epsilon_x}(x) \} \leq C_x.$$ 

Since $\{ B_{\epsilon_x}(x) : x \in K \}$ is an open cover for $K$, and since $K$ is compact, there are $x_1, \ldots, x_n \in K$ such that

$$K \subset \bigcup_{j=1}^n B_{\epsilon_{x_j}}(x_j).$$

Set $C := \max \{ C_{x_1}, \ldots, C_{x_n} \}$. Then

$$\sup \{ \| f(y) \| : f \in \mathcal{F}, y \in K \} \leq C$$

holds, i.e., $\mathcal{F}|_K$ is bounded.

(ii): For each $x \in K$, there is $\epsilon_x > 0$ with $B_{\epsilon_x}(x) \subset D$ such that $\{ f|_{B_{\epsilon_x}(x)} : f \in \mathcal{F} \}$ is equicontinuous.

Let $\epsilon > 0$ be arbitrary. Then, for each $x \in K$, there is $\delta_x > 0$ such that $\| f(y) - f(z) \| < \epsilon$ for all $f \in \mathcal{F}$ and all $y, z \in B_{\epsilon_x}(x)$ with $\| y - z \| < \delta_x$. Since $K$ is compact, there are $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup_{j=1}^n B_{\epsilon_{x_j}}(x_j)$. Let $\ell > 0$ be the Lebesgue number of the open cover $\{ B_{\epsilon_{x_j}}(x_j) : j = 1, \ldots, n \}$ of $K$, and choose $\delta := \min \{ \delta_{x_1}, \ldots, \delta_{x_n}, \ell \}$. Let $f \in \mathcal{F}$, and let $x, y \in K$ be such that $\| x - y \| < \delta$, so that $\text{diam}\{ \{ x, y \} \} < \delta \leq \ell$. By the definition of $\ell$, there is $j \in \{ 1, \ldots, n \}$ such that $\{ x, y \} \subset B_{\epsilon_{x_j}}(x_j)$. Since $\| x - y \| < \delta_{x_j} \leq \delta$, this means that $\| f(x) - f(y) \| < \epsilon$. Hence, $\mathcal{F}|_K$ is indeed equicontinuous. \hfill \Box

**Proposition 16.7.** Let $D \subset \mathbb{R}^N$ be open, and let $\mathcal{F}$ be a locally bounded and locally equicontinuous family of functions from $D$ to $\mathbb{R}^M$. Then every sequence in $\mathcal{F}$ has a compactly convergent subsequence.

Proof. Let $(f_n)_{n=1}^\infty$ be a sequence in $\mathcal{F}$.

Define a sequence $K_1, K_2, \ldots$ of compact sets such that

- $\bigcup_{k=1}^\infty K_k = D$ and
- $K_k \subset \text{int} K_{k+1}$ for $n \in \mathbb{N}$.

(As in the $N = 2$ case, this can be accomplished, for instance, by letting $K_k := B_k[0]$ if $D = \mathbb{C}$ and $K_k := \{ x \in D : \text{dist}(x, \partial D) \geq \frac{1}{k}, \| x \| \leq k \}$ if $D \neq \mathbb{C}$.)

By Lemma 16.6, $\mathcal{F}|_{K_1}$ is bounded and equicontinuous. By the Arzelà–Ascoli Theorem, there is thus a subsequence $(f_{n,1})_{n,1=1}^\infty$ of $(f_n)_{n=1}^\infty$ and a function $g_1 : K_1 \to \mathbb{R}^M$ such that $f_{n,1}|_{K_1} \to g_1$ uniformly on $K_1$. 

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Invoking Lemma 16.6 and the Arzelà–Ascoli Theorem again, we obtain a subsequence \((f_{n,2})^\infty_{n=1}\) of \((f_{n,1})^\infty_{n=1}\) and a function \(g_2 : K_2 \to \mathbb{R}^M\) such that \(f_{n,2}|_{K_2} \to g_2\) uniformly on \(K_2\).

Inductively, we thus obtain, for each \(k \in \mathbb{N}\), a subsequence \((f_{n,k})^\infty_{n=1}\) of \((f_{n})^\infty_{n=1}\) and a function \(g_k : K_k \to \mathbb{R}^M\) such that, for each \(k \in \mathbb{N}\),

- \((f_{n,k+1})^\infty_{n=1}\) is a subsequence of \((f_{n,k})^\infty_{n=1}\), and
- \(f_{n,k}|_{K_k} \to g_k\) uniformly on \(K_k\).

Define \(g : D \to \mathbb{R}^M\) as follows: for \(x \in D\), let \(k \in \mathbb{N}\) be such that \(x \in K_k\), set \(g(x) := g_k(x)\). It is easy to see that \(g\) well defined. From the definition of \(g\), it is obvious that \(f_{n,n}|_{K_k} \to g|_{K_k}\) uniformly on \(K_k\).

Let \(K \subset D\) be compact. By the choices of \(K_1, K_2, \ldots\), we have \(K \subset D \subset \bigcup_{k=1}^\infty \text{int } K_k\), so that \(\{\text{int } K_k : k \in \mathbb{N}\}\) is an open cover for \(K\). Since \(K\) is compact, and since \(\text{int } K_k \subset \text{int } K_{k+1}\) for \(k \in \mathbb{N}\), there is \(k_0 \in \mathbb{N}\) such that \(K \subset \text{int } K_{k_0} \subset K_{k_0}\). Since \(f_{n,n}|_{K_{k_0}} \to g|_{K_{k_0}}\) uniformly on \(K_{k_0}\), it follows that \(f_{n,n}|_{K} \to g|_{K}\) uniformly on \(K\).

**Theorem 16.8** (Montel’s Theorem). Let \(D \subset \mathbb{C}\) be open, and let \(F\) be a locally bounded family of holomorphic functions on \(D\). Then every sequence in \(F\) has a subsequence that converges compactly to a holomorphic function on \(D\).

**Proof.** In view of Proposition 16.7, we only need to show that \(F\) is locally equicontinuous.

Let \(z_0 \in D\), and let \(r > 0\) be such that \(B_{2r}[z_0] \subset D\). By Lemma 16.6(ii), there is \(C > 0\) such that \(|f(\zeta)| \leq C\) for all \(f \in F\) and for all \(\zeta \in \partial B_{2r}(z_0)\).

Let \(f \in F\), and let \(z, w \in B_r(z_0)\). Then we have:

\[
|f(z) - f(w)| = \left| \frac{1}{2\pi i} \int_{\partial B_{2r}(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\partial B_{2r}(z_0)} \frac{f(\zeta)}{\zeta - w} d\zeta \right|
\]

\[
= \frac{1}{2\pi} \left| \int_{\partial B_{2r}(z_0)} \left( \frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w} \right) d\zeta \right|
\]

\[
= \frac{1}{2\pi} \left| \int_{\partial B_{2r}(z_0)} \frac{f(\zeta)(w - z)}{(\zeta - z)(\zeta - w)} d\zeta \right|
\]

\[
= \frac{|z - w|}{2\pi} \left| \int_{\partial B_{2r}(z_0)} \frac{f(\zeta)}{(\zeta - z)(\zeta - w)} d\zeta \right|
\]

\[
\leq \frac{|z - w|}{2\pi} \frac{C}{r^2}
\]

\[
= \frac{2C}{r} |z - w|.
\]

For \(\epsilon > 0\), thus choose \(\delta := \frac{r \epsilon}{2C}\), so that \(|f(z) - f(w)| < \epsilon\) for all \(z, w \in B_r(z_0)\) with \(|z - w| < \delta\). \(\square\)
17 The Riemann Mapping Theorem

Definition 17.1. Let \( D_1, D_2 \subset \mathbb{C} \) be open and connected. We say that \( D_1 \) and \( D_2 \) are biholomorphically equivalent if there is a biholomorphic map from \( D_1 \) onto \( D_2 \).

Examples. 1. Let \( z_1, z_2 \in \mathbb{C} \), and let \( r_1, r_2 > 0 \). Then \( B_{r_1}(z_1) \) and \( B_{r_2}(z_2) \) are biholomorphically equivalent because

\[ B_{r_1}(z_1) \to B_{r_2}(z_2), \quad z \mapsto \frac{r_2}{r_1}(z - z_1) + z_2 \]

is biholomorphic.

2. Consider the Cayley transform \( f: \mathbb{H} \to \mathbb{C}, \quad z \mapsto \frac{z - i}{z + i} \).

Let \( x, y \in \mathbb{R} \) with \( y > 0 \), and let \( z = x + iy \). Then

\[
|z - i|^2 = |x + i(y - 1)|^2 \\
= x^2 + y^2 - 2y + 1 \\
< x^2 + y^2 + 2y + 1 \\
= |x + i(y + 1)|^2 \\
= |z + i|^2
\]

holds, so that \( |f(z)| < 1 \). Consequently, we have \( f(\mathbb{H}) \subset \mathbb{D} \). Consider

\[ g: \mathbb{D} \to \mathbb{C}, \quad z \mapsto i\frac{1 + z}{1 - z}, \]

and note that

\[
g(f(z)) = i\frac{1 + \frac{z - i}{z + i}}{1 - \frac{z - i}{z + i}} \\
= i\frac{z + i + z - i}{z + i - z + i} \\
= i\frac{2z}{2i} \\
= z
\]

for \( z \in \mathbb{H} \). Hence, \( f \) is injective. Let \( x^2 + y^2 < 1 \), and note that

\[
g(x + iy) = i\frac{(1 + x) + iy}{(1 - x) - iy} \\
= i\frac{(1 + x) + iy)((1 - x) + iy)}{(1 - x)^2 + y^2} \\
= -\frac{2y}{(1 - x)^2 + y^2} + i\frac{1 - (x^2 + y^2)}{(1 - x)^2 + y^2} \in \mathbb{H},
\]

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For \( z \in \mathbb{D} \), we can thus evaluate
\[
  f(g(z)) = \frac{i^{\frac{1+z}{1-z}} - i}{i^{\frac{1+z}{1-z}} + i} = \frac{1 + z - 1 + z}{1 + z + 1 - z} = \frac{2z}{2} = z.
\]
Hence, \( f \) is also surjective and thus bijective with inverse \( g \). Since \( f \) and \( g \) are obviously holomorphic, this means that \( \mathbb{D} \) and \( \mathbb{H} \) are biholomorphically equivalent.

3. There is no biholomorphic map \( f : \mathbb{C} \to \mathbb{D} \) because any holomorphic map from \( \mathbb{C} \) to \( \mathbb{D} \) is bounded and thus constant by Liouville's theorem. Hence, \( \mathbb{C} \) and \( \mathbb{D} \) are not biholomorphically equivalent.

**Lemma 17.2.** Let \( D_1, D_2 \subset \mathbb{C} \) be open, let \( f : D_1 \to D_2 \) be holomorphic, let \( g : D_2 \to \mathbb{C} \) be continuous, and let \( \gamma \) be a curve in \( D_1 \). Then we have
\[
  \int_{f \circ \gamma} g(\zeta) \, d\zeta = \int_{\gamma} g(f(\zeta)) f'(\zeta) \, d\zeta.
\]
**Proof.** Let \( a = t_0 < t_1 < \cdots < t_n = b \) be such that \( \gamma|_{[t_{j-1}, t_j]} \) is continuously differentiable for \( j = 1, \ldots, n \), and note that
\[
  \int_{f \circ \gamma} g(\zeta) \, d\zeta = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} g(f(\gamma(t))) f'(\gamma(t)) \gamma'(t) \, dt = \int_{\gamma} g(f(\zeta)) f'(\zeta) \, d\zeta. \quad \square
\]

**Proposition 17.3.** Let \( D_1, D_2 \subset \mathbb{C} \) be open and connected such that \( D_1 \) is simply connected, and suppose that \( D_1 \) and \( D_2 \) are biholomorphically equivalent. Then \( D_2 \) is simply connected.

**Proof.** Let \( f : D_1 \to D_2 \) be biholomorphic. Let \( g : D_2 \to \mathbb{C} \) be holomorphic, and let \( \gamma \) be a closed curve in \( D_2 \). We shall see that \( \int_\gamma g(\zeta) \, d\zeta = 0 \). Note that \( f^{-1} \circ \gamma \) is a closed curve in \( D_1 \), so that by Lemma 17.2 and the simple connectedness of \( D_1 \),
\[
  \int_\gamma g(\zeta) \, d\zeta = \int_{f \circ f^{-1} \circ \gamma} g(\zeta) \, d\zeta = \int_{f^{-1} \circ \gamma} g(f(\zeta)) f'(\zeta) \, d\zeta = 0 \quad \square
\]

**Example.** Let \( r, R \in [0, \infty) \) be such that \( r < R \). Then \( A_{r,R}(0) \) is not biholomorphically equivalent to \( \mathbb{D} \) or \( \mathbb{C} \).

Biholomorphic maps have a very interesting geometric property.

Given an open set \( D \subset \mathbb{R}^N \) and curves \( \gamma_1, \gamma_2 : [0, 1] \to D \), suppose there are \( t_1, t_2 \in (0, 1) \) such that \( \gamma_1(t_1) = \gamma_2(t_2) = x_0 \). In order to define the angle between \( \gamma_1 \) and \( \gamma_2 \) at \( x_0 \),
we further suppose that there is \( \epsilon > 0 \) such that \( \gamma_j \) is differentiable on \((t_j - \epsilon, t_j + \epsilon)\) for \( j = 1, 2 \) with \( \gamma_j'(t_j) \neq 0 \). The angle is then defined to be the unique \( \theta \in [0, \pi] \) such that

\[
\cos \theta = \frac{\gamma_1'(t_1) \cdot \gamma_2'(t_2)}{\|\gamma_1'(t_1)\| \|\gamma_2'(t_2)\|}.
\]

Given two open sets \( D_1, D_2 \subset \mathbb{R}^N \), a differentiable map \( f : D_1 \rightarrow D_2 \) is called angle preserving at \( x_0 \in D_1 \) if, for any two curves \( \gamma_1 \) and \( \gamma_2 \) in \( D_1 \), the angle between \( f \circ \gamma_1 \) and \( f \circ \gamma_2 \) at \( f(x_0) \) is the same between \( \gamma_1 \) and \( \gamma_2 \) at \( x_0 \).

Recall that a real \( N \times N \) matrix \( A \) is called orthogonal if it is invertible with \( A^{-1} = A^t \).

**Lemma 17.4.** Let \( D_1, D_2 \subset \mathbb{R}^N \) be open, let \( x_0 \in D_1 \), and let \( f : D_1 \rightarrow D_2 \) be differentiable such that \( J_f(x_0) \) is orthogonal. Then \( f \) is angle preserving at \( x_0 \).

**Proof.** Let \( \gamma_1 \) and \( \gamma_2 \) be two curves in \( D_1 \) satisfying the necessary requirements, and note that

\[
\text{cosine of the angle between } f \circ \gamma_1 \text{ and } f \circ \gamma_2 \text{ at } f(x_0) = \frac{\langle f \circ \gamma_1'(t_1), f \circ \gamma_2'(t_2) \rangle}{\| f \circ \gamma_1'(t_1) \| \| f \circ \gamma_2'(t_2) \|}.
\]

By the chain rule,

\[
\frac{\| J_f(x_0) \gamma_1'(t_1) \| \| J_f(x_0) \gamma_2'(t_2) \|}{\| J_f(x_0) \gamma_1'(t_1) \| \| J_f(x_0) \gamma_2'(t_2) \|} = \frac{\| J_f(x_0) \gamma_1'(t_1) \| \| J_f(x_0) \gamma_2'(t_2) \|}{\| J_f(x_0) \gamma_1'(t_1) \| \| J_f(x_0) \gamma_2'(t_2) \|}.
\]

Hence, \( f \) is angle preserving at \( x_0 \).

**Example.** Let \( z \) be a complex number. Then multiplication by \( z \) is a \( \mathbb{R} \)-linear map from \( \mathbb{C} = \mathbb{R}^2 \) into itself and thus uniquely represented by a real \( 2 \times 2 \) matrix \( A \) of the form

\[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix},
\]

where \( a = \text{Re } z \) and \( b = \text{Im } z \). It follows that \( A^t \) is the matrix representing \( \bar{z} \). Hence, \( A \) is orthogonal if and only if \( |z| = 1 \).

**Proposition 17.5.** Let \( D_1, D_2 \subset \mathbb{C} \) be open, and let \( f : D_1 \rightarrow D_2 \) be holomorphic. Then \( f \) is angle preserving at \( z_0 \in D_1 \) whenever \( f'(z_0) \neq 0 \).

**Proof.** Let \( z_0 \in D_1 \) be such that \( f'(z_0) \neq 0 \). In view of Lemma 17.4 and the example following it, the claim is clear if \( |f'(z_0)| < 1 \).

For the general case, let

\[
\frac{1}{|f'(z_0)|} D_2 := \left\{ \frac{z}{|f'(z_0)|} : z \in D_2 \right\},
\]

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and define
\[ g : D_1 \to \frac{1}{|f'(z_0)|}D_2, \quad z \mapsto \frac{f(z)}{|f'(z_0)|} \]
and
\[ h : \frac{1}{|f'(z_0)|}D_2 \to D_2, \quad z \mapsto |f'(z_0)|z. \]
Then \( g \) is angle preserving at \( z_0 \) because \(|g'(z_0)| = 1\), and it is easily seen that \( h \) is angle preserving at \( g(z_0) \). Consequently, \( f = h \circ g \) is angle preserving at \( z_0 \). \( \square \)

**Corollary 17.6.** Let \( D_1, D_2 \subset \mathbb{C} \) be open and connected, and let \( f : D_1 \to D_2 \) be biholomorphic. Then \( f \) is angle preserving at every point of \( D_1 \).

**Proposition 17.7.** Let \( D_1, D_2 \subset \mathbb{C} \) be open and connected, and let \( f : D_1 \to D_2 \) be holomorphic and bijective. Then \( f \) is biholomorphic.

**Proof.** We first show that \( f^{-1} \) is continuous.

Let \( w_0 \in D_2 \), and let \( \epsilon > 0 \). Without loss of generality suppose that \( B_\epsilon(f^{-1}(w_0)) \subset D_1 \). By the Open Mapping Theorem, \( f(B_\epsilon(f^{-1}(w_0))) \) is open. Hence, there is \( \delta > 0 \) such that \( B_{\delta}(w_0) \subset f(B_\epsilon(f^{-1}(w_0))) \). Hence, if \( w \in D_2 \) is such that \(|w - w_0| < \delta\), i.e., \( w \in B_{\delta}(w_0) \), we have \( f^{-1}(w) \in B_\epsilon(f^{-1}(w_0)) \), i.e., \(|f^{-1}(w) - f^{-1}(w_0)| < \epsilon\). Hence, \( f^{-1} \) is continuous at \( w_0 \).

Since \( f \) is not constant, we have \( f' \neq 0 \), so that \( Z(f') \) is discrete. We claim that \( f(Z(f')) \) is also discrete. Assume that \( f(Z(f')) \) is not discrete. Then there are \( w_n \in D_2 \) and a sequence \( (z_n)_{n=1}^\infty \) in \( Z(f') \) such that \( w_0 \neq f(z_n) \) for \( n \in \mathbb{N} \), but \( w_0 = \lim_{n \to \infty} f(z_n) \). By the bijectivity and continuity of \( f^{-1} \), we have \( f^{-1}(w_0) \neq z_n \) for \( n \in \mathbb{N} \) and \( f^{-1}(w_0) = \lim_{n \to \infty} z_n \). Hence, \( f^{-1}(w_0) \) is a cluster point of \( Z(f') \), which is impossible.

Let \( z_0 \in D_1 \) be such that \( f'(z_0) \neq 0 \), and let \( w_0 := f(z_0) \). For \( w \in D_2 \setminus \{w_0\} \), we then have
\[
\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{f^{-1}(w) - f^{-1}(w_0)}{f^{-1}(w) - f^{-1}(w_0)} \frac{1}{f'(z_0)}.
\]
Hence, \( f^{-1} \) is holomorphic on \( D_2 \setminus f(Z(f')) \). Since \( f^{-1} \) is continuous and \( f(Z(f')) \) is discrete, Riemann’s Removability Criterion yields the holomorphy of \( f^{-1} \) on all of \( D_2 \). \( \square \)

**Corollary 17.8.** Let \( D \subset \mathbb{C} \) be open and connected, and let \( f : D \to \mathbb{C} \) be holomorphic and injective. Then \( f'(z) \neq 0 \) for all \( z \in D \).

**Proof.** If \( f \) is injective, it is not constant. By the Open Mapping Theorem, \( f(D) \) is therefore open and connected, and by the previous proposition, \( f : D \to f(D) \) is biholomorphic, so that
\[
1 = (f^{-1} \circ f)'(z) = (f^{-1})'(f(z))f'(z)
\]
for all \( z \in D \), which is possible only if \( f'(z) \neq 0 \) for all \( z \in D \). \( \square \)
**Definition 17.9.** Let $D \subset \mathbb{C}$ be open and connected. We say that $D$ admits

(a) holomorphic logarithms if, for every holomorphic $f : D \to \mathbb{C}$ with $\text{Z}(f) = \emptyset$, there is a holomorphic $g : D \to \mathbb{C}$ with $f = \exp \circ g$;

(b) holomorphic roots if, for every holomorphic $f : D \to \mathbb{C}$ with $\text{Z}(f) = \emptyset$ and each $n \in \mathbb{N}$, there is a holomorphic $h_n : D \to \mathbb{C}$ with $f(z) = h_n(z)^n$ for $z \in D$;

(c) holomorphic square roots if, for every holomorphic $f : D \to \mathbb{C}$ with $\text{Z}(f) = \emptyset$, there is a holomorphic $h : D \to \mathbb{C}$ with $f(z) = h(z)^2$ for $z \in D$.

**Remark.** By Corollary 15.6, every simply connected $D$ admits holomorphic logarithms, and the proof of Corollary 15.7 shows that every $D$ that admits holomorphic logarithms also admits holomorphic roots.

**Theorem 17.10** (Riemann’s Mapping Theorem). Let $D \subset \mathbb{C}$ be open and connected and admit holomorphic square roots, and let $z_0 \in D$. Then there is a unique biholomorphic $f : D \to \mathbb{D}$ with $f(z_0) = 0$ and $f'(z_0) > 0$.

**Proof.** *Uniqueness:* Let $g : D \to \mathbb{D}$ be another such function. Then $f \circ g^{-1} : \mathbb{D} \to \mathbb{D}$ is biholomorphic with $(f \circ g^{-1})(0) = f(z_0) = 0$. By Corollary 7.9, there is thus $c \in \mathbb{C}$ with $|c| = 1$ such that

$$f(g^{-1}(z)) = cz$$

for $z \in \mathbb{D}$ and thus

$$f(z) = f(g^{-1}(g(z))) = cg(z)$$

for $z \in D$. Differentiation yields $f'(z) = cg'(z)$ for $z \in D$. Since $|c| = 1$ and $f'(z_0), g'(z_0) > 0$, we conclude that $c = 1$.

*Existence:* Let

$$\mathcal{F} := \{f : D \to \mathbb{D} : f \text{ is injective and holomorphic with } f(z_0) = 0 \text{ and } f'(z_0) > 0\}.$$

**Claim 1** $\mathcal{F} \neq \emptyset$.

Since $D \neq \mathbb{C}$, there is $w \in \mathbb{C} \setminus D$. Since $D$ admits holomorphic square roots, there is a holomorphic $g : D \to \mathbb{C}$ such that $g(z)^2 = z - w$ for $z \in D$. Let $z_1, z_2 \in D$ be such that $g(z_1) = \pm g(z_2)$. Then $z_1 = z_2$ holds, and, in particular, $g$ is injective and thus not constant.

By the Open Mapping Theorem, there is $r > 0$ with $B_r(g(z_0)) \subset g(D)$.

Assume that there is $z \in D$ with $g(z) \in B_r(-g(z_0))$. This means that

$$r > |g(z) + g(z_0)| = |g(z) - g(z_0)|,$$
so that \(-g(z) \in B_r(g(z_0)) \subset g(D)\). Hence, there is \(\tilde{z} \in D\) with \(g(\tilde{z}) = -g(z)\) and thus \(\tilde{z} = z\), which, in turn, yields that \(g(z) = 0\), so that

\[0 = g(z)^2 = z - w.\]

This contradicts \(w \notin D\). Hence, \(g(D) \cap B_r(-g(z_0)) = \emptyset\) must hold.

Define

\[T: \mathbb{C} \setminus B_r[-g(z_0)] \to \mathbb{C}, \quad z \mapsto \frac{r}{z + g(z_0)},\]

and set \(\tilde{g} := T \circ g\). Then \(\tilde{g}: D \to \mathbb{D}\) is holomorphic and injective. Let \(\tilde{w} := \tilde{g}(z_0)\). Then \(\phi_{\tilde{w}} \circ \tilde{g}: D \to \mathbb{D}\) is holomorphic and injective with \((\phi_{\tilde{w}} \circ \tilde{g})(z_0) = 0\) and \((\phi_{\tilde{w}} \circ \tilde{g})'(z_0) \neq 0\) by Corollary 17.8. Let \(c \in \mathbb{C}\) with \(|c| = 1\) be such that \(c(\phi_{\tilde{w}} \circ \tilde{g})'(z_0) > 0\). Then \(c(\phi_{\tilde{w}} \circ \tilde{g}) \in \mathcal{F}\) holds, so that indeed \(\mathcal{F} \neq \emptyset\).

**Claim 2** Let \((f_n)_{n=1}^\infty\) be a sequence in \(\mathcal{F}\) converging compactly to \(f: D \to \mathbb{C}\). Then either \(f \equiv 0\) or \(f \in \mathcal{F}\).

It is straightforward that \(f(z_0) = 0\), \(f'(z_0) \geq 0\), and \(f(D) \subset \overline{\mathbb{D}}\). By Corollary 14.11, \(f \equiv 0\) or \(f\) is injective. If \(f\) is injective, then \(f'(z_0) \neq 0\) must hold by Corollary 17.8, i.e., \(f'(z_0) > 0\). Also, since \(f(D) \subset \overline{\mathbb{D}}\) is open, we have \(f(D) \subset \mathbb{D}\), so that \(f \in \mathcal{F}\).

**Claim 3** There is \(f \in \mathcal{F}\) such that \(f(D) = \mathbb{D}\).

Choose a sequence \((f_n)_{n=1}^\infty\) such that

\[
\lim_{n \to \infty} f'_n(z_0) = \sup\{(\tilde{f})'(z_0) : \tilde{f} \in \mathcal{F}\} \in (0, \infty].
\]

By Montel’s Theorem, we can suppose, by passing to a subsequence, that \((f_n)_{n=1}^\infty\) converges compactly to some \(f: D \to \mathbb{C}\). In particular,

\[
f'(z_0) = \sup\{(\tilde{f})'(z_0) : \tilde{f} \in \mathcal{F}\} > 0 \quad (\ast)
\]

holds, so that \(f \in \mathcal{F}\) by Claim 2.

Assume that there is \(w \in \mathbb{D} \setminus f(D)\). Since \(D\) admits holomorphic square roots, there is a holomorphic \(h: D \to \mathbb{C}\) such that

\[
h(z)^2 = \frac{f(z) - w}{1 - \overline{w}f(z)} = (\phi_w \circ f)(z) \quad (\ast\ast)
\]

for \(z \in D\). In particular, \(h(D) \subset \mathbb{D}\), and \(h\) is injective. Define

\[f_0: D \to \mathbb{C}, \quad z \mapsto \frac{|h'(z_0)|}{h'(z_0)} \frac{h(z) - h(z_0)}{1 - \overline{h(z_0)}h(z)} = \frac{|h'(z_0)|}{h'(z_0)} (\phi_{h(z_0)} \circ h)(z).\]

Then \(f_0\) is injective with \(f_0(D) \subset \mathbb{D}\) and \(f_0(z_0) = 0\). Differentiation yields

\[f'_0(z) = \frac{|h'(z_0)|}{h'(z_0)} \phi_{h(z_0)}(h(z))h'(z) = \frac{|h'(z_0)|}{h'(z_0)} \frac{1 - |h(z_0)|^2}{(1 - \overline{h(z_0)}h(z))^2} h'(z)\]

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for \( z \in D \) and thus, in particular,
\[
f'_0(z_0) = \frac{|h'(z_0)|}{1 - |h(z_0)|^2} > 0,
\]
so that \( f_0 \in \mathcal{F} \).

Differentiating (**) we obtain
\[
2h(z)h'(z) = \frac{1 - |w|^2}{(1 - \bar{w}f(z))^2}f'(z)
\]
for \( z \in D \) and thus, letting \( z = z_0 \),
\[
2h(z_0)h'(z_0) = (1 - |w|^2)f'(z_0).
\]
Since \(|h(z_0)|^2 = |-w| = |w|\), we obtain
\[
f'_0(z_0) = \frac{|h'(z_0)|}{1 - |h(z_0)|^2}
\]
\[
= \frac{2|h_0(z_0)|}{2|h(z_0)|^2(1 - |h(z_0)|^2)}
\]
\[
= \frac{(1 - |w|^2)f'(z_0)}{2\sqrt{|w|(1 - |w|)}}
\]
\[
= f'(z_0) \frac{1 + |w|}{2\sqrt{|w|}}
\]
\[
> f'(z_0),
\]
which contradicts (*). \(\square\)

**Characterizations of Simply Connected Domains.** The following are equivalent for an open and connected set \( D \subset \mathbb{C} \):

(i) \( D \) is simply connected;

(ii) \( D \) admits holomorphic logarithms;

(iii) \( D \) admits holomorphic roots;

(iv) \( D \) admits holomorphic square roots;

(v) \( D \) is all of \( \mathbb{C} \) or biholomorphically equivalent to \( \mathbb{D} \);

(vi) every holomorphic function \( f: D \to \mathbb{C} \) has an anti-derivative;

(vii) \( \int_\gamma f(\zeta) d\zeta = 0 \) for each holomorphic \( f: D \to \mathbb{C} \) and each closed curve \( \gamma \) in \( D \);
(viii) for every holomorphic function \( f : D \to \mathbb{C} \), we have
\[
\nu(f, z) f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta
\]
for each closed curve \( \gamma \) in \( D \) and all \( z \in D \setminus \{ \gamma \} \);

(ix) every harmonic function \( u : D \to \mathbb{R} \) has a harmonic conjugate.

Proof. (i) \( \implies \) (ii) is Corollary 15.6, (ii) \( \implies \) (iii) is shown in the proof of Corollary 15.7, (iii) \( \implies \) (iv) is trivial, (iv) \( \implies \) (v) follows from Theorem 17.10, and (v) \( \implies \) (i) is implied by Proposition 17.3.

(i) \( \iff \) (vi) is Corollary 15.5, and (vi) \( \iff \) (vii) follows from Theorem 4.10.

(i) \( \implies \) (viii) follows from Theorem 15.3, and (viii) \( \implies \) (vii) is seen as in the proof of Theorem 15.2.

(v) \( \implies \) (ix): Let \( u : D \to \mathbb{R} \) be harmonic. If \( D = \mathbb{C} \), the existence of a harmonic conjugate is immediate by Theorem 9.4. So suppose that \( D \neq \mathbb{C} \). Hence, there is a biholomorphic map \( f : D \to \mathbb{D} \). It is easily seen that \( \tilde{u} := u \circ f^{-1} : \mathbb{D} \to \mathbb{R} \) is harmonic and thus has a harmonic conjugate \( \tilde{v} : \mathbb{D} \to \mathbb{R} \) by Theorem 9.4. Then \( v := \tilde{v} \circ f : D \to \mathbb{R} \) is a harmonic conjugate of \( u \).

(ix) \( \implies \) (ii): Let \( f : D \to \mathbb{R} \) be holomorphic such that \( \mathbb{Z}(f) = \emptyset \). Then \( u := \log |f| \) is harmonic and thus has a harmonic conjugate \( v : D \to \mathbb{R} \), i.e., \( v \) is harmonic such that \( g := u + i v \) is holomorphic. For \( z \in D \), we have
\[
|\exp(g(z))| = |\exp(u(z) + i v(z))| = |\exp(u(z))||\exp(i v(z))| = \exp(u(z)) = |f(z)|.
\]
Thus,
\[
D \to \mathbb{C}, \quad z \mapsto \frac{f(z)}{\exp(g(z))}
\]
is a holomorphic function whose range lies in \( \partial \mathbb{D} \) and therefore isn’t open. By the Open Mapping Theorem, this means that there is \( c \in \mathbb{C} \)—necessarily of modulus one—such that \( f(z) = c \exp(g(z)) \) for \( z \in D \). Choose \( \theta \in \mathbb{R} \) with \( \exp(i \theta) = c \), and note that \( f(z) = \exp(g(z) + i \theta) \) for \( z \in D \).

Further Characterizations of Simply Connected Domains. There are more conditions that characterize the simply connected domains. We will only mention them without giving proofs.

(x) for every holomorphic function \( f : D \to \mathbb{C} \), there is a sequence of polynomials converging to \( f \) compactly on \( D \);

It was shown in the homework that, if every holomorphic function on \( D \) can be approximated by polynomials, then \( D \) is indeed simply connected. The proof of the converse relies on Runge’s Approximation Theorem, which we didn’t prove in this course.
Two (not necessarily piecewise smooth) curves $\gamma_1, \gamma_2 : [0, 1] \to D$ are with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$ are called path homotopic if there is a continuous function $\Gamma : [0, 1] \times [0, 1] \to D$ such that,

$$\Gamma(0,t) = \gamma_1(t) \quad \text{and} \quad \Gamma(1,t) = \gamma_2(t)$$

for $t \in [0, 1]$ and

$$\Gamma(s,0) = \gamma_1(0) \quad \text{and} \quad \Gamma(s,1) = \gamma_1(1)$$

for $s \in [0, 1]$. A closed curve $\gamma$ with starting point/endpoint $z_0$ is called homotopic to zero if $\gamma$ and the constant curve $z_0$ are path homotopic.

Simple connectedness is equivalent to:

(xi) every (not necessarily smooth) curve in $D$ is homotopic to zero;

This conditions makes no reference to holomorphic functions and is entirely topological in nature, like:

(xii) $D$ is homeomorphic to $\mathbb{D}$;

This means that there is a bijective, continuous map $f : D \to \mathbb{C}$ with continuous inverse. Since (xi) is preserved under homeomorphisms, it is clear that $D$ satisfies (xi) if it is homeomorphic to $\mathbb{D}$. For the converse, it is sufficient to show that $\mathbb{C}$ is homeomorphic to $\mathbb{D}$ (for $D \neq \mathbb{C}$, this is clear by Theorem 17.10). Since

$$\mathbb{C} \to \mathbb{D}, \quad z \mapsto \frac{z}{1 + |z|}$$

and

$$\mathbb{D} \to \mathbb{C}, \quad z \mapsto \frac{z}{1 - |z|}$$

are continuous and inverse to each other, this is indeed the case.

Let $\mathbb{C}_\infty$ denote the unit sphere in $\mathbb{R}^3$. We can connect every point $z$ in $xy$-plane, which we identify with $\mathbb{C}$, with $(0,0,1)$ through a straight line that intersects $\mathbb{C}_\infty$ in a point $\iota(z)$. Thus, we obtain an injective map $\iota : \mathbb{C} \to \mathbb{C}_\infty$. 

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One can show that \( \iota \) is continuous, and that the inverse \( \iota^{-1} : \iota(\mathbb{C}) \rightarrow \mathbb{C} \) is also continuous. Hence, we may identify \( \mathbb{C} \) with \( \iota(\mathbb{C}) \).

With regards to simple connectedness, we have another equivalent conditions involving \( \mathbb{C}_\infty \):

(xiii) \( \mathbb{C}_\infty \setminus D \) is connected.