MATH 209—Calculus, III

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Integrals over intervals

Question

What is $\int_a^b f(x)\,dx$?

Recall...

Divide $[a,b]$ into n subintervals $[x_{j-1},x_j]$ of length $\Delta x = \frac{b-a}{n}$. In each subinterval $[x_{j-1},x_j]$ find a support point x_j^*. Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^n f(x_j^*)\Delta x.$$

Intuition

For $f \geq 0$: $\int_a^b f(x)\,dx$ is the area under the graph of f.

Double integrals over rectangles, I

Question

Let $R = [a, b] \times [c, d]$. What is $\iint_R f(x, y)\, dA$?

As in one variable...

Divide $[a, b]$ into n subintervals $[x_{j-1}, x_j]$ of length $\Delta x = \frac{b-a}{n}$.
Divide $[c, d]$ into m subintervals $[y_{k-1}, y_k]$ of length $\Delta x = \frac{d-c}{m}$.
For $(j, k) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$, chose a support point $(x_j^*, y_k^*) \in [x_{j-1}, x_j] \times [y_{k-1}, y_k]$. Then

$$\iint_R f(x, y)\, dA = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=1}^{n} \sum_{k=1}^{m} f(x_j^*, y_k^*) \Delta x \Delta y.$$

Double integrals over rectangles, II

Intuition

For $f \geq 0$: $\iint_R f(x,y)\,dA$ is the volume of the solid over R and below $z = f(x,y)$.

Theorem (Midpoint rule)

If $\bar{x}_j$ and $\bar{y}_k$ are the midpoints of $[x_{j-1}, x_j]$ and $[y_{k-1}, y_k]$, respectively, then

$$\iint_R f(x,y)\,dA \approx \sum_{j=1}^{n} \sum_{k=1}^{m} f(\bar{x}_j, \bar{y}_k) \underbrace{\Delta x \Delta y}_{=:\Delta A}.$$

Examples, I

Example

Let $R = [0,2] \times [1,2]$. What is $\iint_R (x - 3y^2)\, dA$?
Let $n = m = 2$, so that $\Delta x = 1$ and $\Delta y = \frac{1}{2}$, and thus $\Delta A = \frac{1}{2}$.
Also:

$$\bar{x}_1 = \frac{1}{2}, \qquad \bar{x}_2 = \frac{3}{2}, \qquad \bar{y}_1 = \frac{5}{4}, \qquad \text{and} \qquad \bar{y}_2 = \frac{7}{4}.$$

Examples, II

Example (continued)

Hence:

$$
\begin{aligned}
\iint_R (x - 3y^2)\, dA &\approx \sum_{j,k=1}^{2} f(\bar{x}_j, \bar{y}_k)\Delta A \\
&= f\left(\frac{1}{2}, \frac{5}{4}\right)\Delta A + f\left(\frac{1}{2}, \frac{7}{4}\right)\Delta A \\
&\qquad + f\left(\frac{3}{2}, \frac{5}{4}\right)\Delta A + f\left(\frac{3}{2}, \frac{7}{4}\right)\Delta A \\
&= \left(-\frac{67}{16} - \frac{139}{16} - \frac{51}{16} - \frac{123}{16}\right)\frac{1}{2} \\
&= -\frac{95}{8} = -11.875.
\end{aligned}
$$

Properties of the integral

Proposition

1. $\iint_R [f(x,y) + g(x,y)]\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA;$
2. *for c constant:*
$$\iint_R c\, f(x,y)\, dA = c \iint_R f(x,y)\, dA;$$
3. *for $f(x,y) \geq g(x,y)$ on R:*
$$\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA.$$

Fubini's theorem

Theorem (Fubini's theorem)

Let $R = [a, b] \times [c, d]$, and let $f : R \to \mathbb{R}$ be continuous. Then

$$\iint_R f(x, y)\, dA = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy.$$

Examples, III

Example

Let $R = [0,2] \times [1,2]$.
By Fubini's theorem:

$$\iint_R x - 3y^2 \, dA = \int_0^2 \left(\int_1^2 x - 3y^2 \, dy \right) dx.$$

Fix x and compute

$$\begin{aligned} \int_1^2 x - 3y^2 \, dy &= xy - y^3 \Big|_{y=1}^{y=2} \\ &= 2x - 8 - x + 1 \\ &= x - 7. \end{aligned}$$

Examples, IV

Example

It follows that

$$\begin{aligned}\iint_R x - 3y^2 \, dA &= \int_0^2 x - 7 \, dx \\ &= \frac{x^2}{2} - 7x \Big|_{x=0}^{x=2} \\ &= 2 - 14 \\ &= -12.\end{aligned}$$

Examples, V

Example

Let $R = [1,2] \times [0,\pi]$. What is $\iint_R y\sin(xy)\,dA$?
By Fubini's theorem:

$$\iint_R y\sin(x,y)\,dA = \int_1^2 \left(\int_0^\pi y\sin(xy)\,dy \right) dx.$$

Fix x and compute $\int_0^\pi y\sin(xy)\,dy$.
Use parts:

$$u = y \qquad \text{and} \qquad dv = \sin(xy)\,dy.$$

Then

$$du = dy \qquad \text{and} \qquad v = -\frac{\cos(xy)}{x}.$$

Examples, VI

Example (continued)

Therefore:

$$\begin{aligned}\int_0^\pi y\sin(xy)\,dy &= -\frac{y\cos(xy)}{x}\bigg|_{y=0}^{y=\pi} + \frac{1}{x}\int_0^\pi \cos(xy)\,dy\\ &= -\frac{\pi\cos(\pi x)}{x} + \frac{1}{x^2}\sin(xy)\bigg|_{y=0}^{y=\pi}\\ &= -\frac{\pi\cos(\pi x)}{x} + \frac{\sin(\pi x)}{x^2}.\end{aligned}$$

Examples, VII

Example (continued)

Now:

$$\int \left(-\frac{\pi \cos(\pi x)}{x}\right) dx = -\frac{\sin(\pi x)}{x} - \int \frac{\sin(\pi x)}{x^2}\, dx,$$

so that

$$\int \left(-\frac{\pi \cos(\pi x)}{x} + \frac{\sin(\pi x)}{x^2}\right) dx = -\frac{\sin(\pi x)}{x}.$$

Examples, VIII

Example (continued)

Hence:

$$\begin{aligned}\iint_R y\sin(xy)\,dA &= \int_1^2 \left(\int_0^\pi y\sin(xy)\,dy\right)dx\\ &= \int_1^2 \left(-\frac{\pi\cos(\pi x)}{x} + \frac{\sin(\pi x)}{x^2}\right)dx\\ &= -\frac{\sin(\pi x)}{x}\Bigg|_{x=1}^{x=2}\\ &= 0.\end{aligned}$$

Examples, IX

Example (continued)

Another approach:

$$\begin{aligned}
\iint_R y\sin(xy)\,dA &= \int_0^\pi \left(\int_1^2 y\sin(xy)\,dx\right) dy \\
&= \int_0^\pi \left(-\cos(xy)\Big|_{x=1}^{x=2}\right) dx \\
&= \int_0^\pi -\cos(2y) + \cos y\,dy \\
&= -\frac{1}{2}\sin(2y) + \sin y\Big|_{y=0}^{y=\pi} \\
&= 0.
\end{aligned}$$

Examples, X

Example

Find the volume V of the the solid bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

We have:

$$V = \iint_{[0,2]\times[0,2]} 16 - x^2 - 2y^2 \, dA$$

$$= \int_0^2 \int_0^2 (16 - x^2 - 2y^2) \, dx \, dy = \int_0^2 16x - \frac{x^3}{3} - 2y^2 x \Big|_{x=0}^{x=2} dy$$

$$= \int_0^2 \left(\frac{88}{3} - 4y^2 \right) dy = \frac{88}{3} y - \frac{4}{3} y^3 \Big|_{y=0}^{y=2} = 48.$$