

MATH 209—
Calculus,
III

Volker Runde

The
divergence
theorem

MATH 209—Calculus, III

Volker Runde

University of Alberta

Edmonton, Fall 2011

The divergence theorem, I

MATH 209—
Calculus,
III

Volker Runde

The
divergence
theorem

Physical motivation

Let S be an oriented surface in \mathbb{R}^3 in the path of a fluid with density $\rho(x, y, z)$ and velocity $\mathbf{v}(x, y, z)$. The **rate of flow** is the vector field $\mathbf{F}(x, y, z) := \rho(x, y, z)\mathbf{v}(x, y, z)$. The mass of the fluid crossing S in the direction of \mathbf{n} is the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Suppose that S is the boundary of a solid region E .

Suppose that E is split into small rectangular boxes $E_{j,k,l}$. Let $\langle x_0, y_0, z_0 \rangle$ be the lower left corner of $E_{j,k,l}$, and let Δx , Δy , and Δz the lengths of its edges.

The divergence theorem, II

MATH 209—
Calculus,
III

Volker Runde

The
divergence
theorem

Physical motivation (continued)

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Then:

$$\begin{aligned} & \text{mass flowing through } E_{j,k,l} \text{ in the direction of } \mathbf{i} \\ & \approx (P(x_0 + \Delta x, y_0, z_0) - P(x_0, y_0, z_0))\Delta y\Delta z \\ & = \frac{P(x_0 + \Delta x, y_0, z_0) - P(x_0, y_0, z_0)}{\Delta x}\Delta x\Delta y\Delta z \\ & \approx \frac{\partial P}{\partial x}(x_0, y_0, z_0)\Delta x\Delta y\Delta z. \end{aligned}$$

Similarly for \mathbf{j} and \mathbf{k} .

The divergence theorem, III

MATH 209—
Calculus,
III

Volker Runde

The
divergence
theorem

Physical motivation (continued)

Therefore:

$$\begin{aligned} & \text{mass flowing through } E_{j,k,l} \\ & \approx \left(\frac{\partial P}{\partial x}(x_0, y_0, z_0) + \frac{\partial Q}{\partial y}(x_0, y_0, z_0) + \frac{\partial R}{\partial z}(x_0, y_0, z_0) \right) \\ & \quad \Delta x \Delta y \Delta z \\ & = \operatorname{div} \mathbf{F}(x_0, y_0, z_0) \Delta x \Delta y \Delta z. \end{aligned}$$

The divergence theorem, IV

MATH 209—
Calculus,
III

Volker Runde

The
divergence
theorem

Physical motivation (continued)

Consequently:

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \text{mass flowing through } E \\ &= \lim_{n,m,\nu \rightarrow \infty} \sum_{j=1}^n \sum_{k=1}^m \sum_{l=1}^{\nu} \operatorname{div} \mathbf{F}(x_0, y_0, z_0) \Delta x \Delta y \Delta z \\ &= \iiint_E \operatorname{div} \mathbf{F} \, dV.\end{aligned}$$

The divergence theorem, ∇

MATH 209—
Calculus,
III

Volker Runde

The
divergence
theorem

Definition

We call a solid region $E \subset \mathbb{R}^3$ **simple** if its of type I, II, and III simultaneously.

Theorem (divergence theorem; Gauss' theorem)

Let $E \subset \mathbb{R}^3$ be a simple region with boundary S with positive orientation, and let \mathbf{F} be a vector field on an open region containing E with continuous partial derivatives. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} dV.$$

Examples, I

MATH 209—
Calculus,
III

Volker Runde

The
divergence
theorem

Example

Let S be the sphere $x^2 + y^2 + z^2 = 1$, and let

$$\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}.$$

Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Let E be the unit ball $x^2 + y^2 + z^2 \leq 1$. Then by the divergence theorem:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E 1 \, dV = \frac{4\pi}{3}.$$

Examples, II

MATH 209—
Calculus,
III

Volker Runde

The
divergence
theorem

Example

Let

$$\mathbf{F}(x, y, z) = xy\mathbf{i} + (y^2 + e^{xz^2})\mathbf{j} + \sin(xy)\mathbf{k},$$

and let S be the boundary of the solid region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$.

Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

View E as a type III region:

$$E = \{(x, y, z) : -1 \leq x \leq 1, 0 \leq z \leq 1 - x^2, 0 \leq y \leq 2 - z\}.$$

Note also:

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 + e^{xz^2}) + \frac{\partial}{\partial z} \sin(xy) = y + 2y = 3y.$$

Examples, III

MATH 209—
Calculus,
III

Volker Runde

The
divergence
theorem

Example (continued)

The divergence theorem yields:

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= 3 \iiint_E y \, dV = 3 \int_{-1}^1 \int_0^{1-x^2} \int_0^{2-z} y \, dy \, dz \, dx \\ &= 2 \int_{-1}^1 \int_0^{1-x^2} \left. \frac{y^2}{2} \right|_{y=0}^{y=2-z} dz \, dx = 3 \int_{-1}^1 \int_0^{1-x^2} \frac{(2-z)^2}{2} dz \, dx \\ &= \frac{3}{2} \int_{-1}^1 \left. -\frac{(2-z)^3}{3} \right|_{z=0}^{z=1-x^2} dx = -\frac{1}{2} \int_{-1}^1 (x^2+1)^3 - 8 \, dx \\ &= -\int_0^1 x^6 + 3x^4 + 3x^2 - 7 \, dx = -\frac{x^7}{7} - \frac{3x^5}{5} - x^3 + 7x \Big|_{x=0}^{x=1} \\ &= -\frac{1}{7} - \frac{3}{5} - 1 + 7 = \frac{184}{35}.\end{aligned}$$

Examples, IV

MATH 209—
Calculus,
III

Volker Runde

The
divergence
theorem

Example

Let S be the boundary of the solid E bounded by the hemispheres

$$z = \sqrt{4 - x^2 - y^2} \quad \text{and} \quad z = \sqrt{1 - x^2 - y^2}.$$

Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F} = (x^3 + y \sin z)\mathbf{i} + (y^3 + z \sin x)\mathbf{j} + 3z\mathbf{k}.$$

Even though E is **not** simple, the divergence theorem applies.
We have

$$\operatorname{div} \mathbf{F} = 3x^2 + 3y^2 + 3 = 3(x^2 + y^2 + 1).$$

Examples, V

MATH 209—
Calculus,
III

Volker Runde

The
divergence
theorem

Example (continued)

In rectangular coordinates:

$$E = \left\{ (x, y, z) : 1 \leq x^2 + y^2 \leq 4, \right. \\ \left. \sqrt{1 - x^2 - y^2} \leq z \leq \sqrt{4 - x^2 - y^2} \right\}.$$

In spherical coordinates:

$$E = \left\{ (\rho, \theta, \phi) : 1 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{2} \right\}.$$

Thus:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV = 3 \iiint_E x^2 + y^2 + 1 \, dV = \dots$$

Examples, VI

MATH 209—
Calculus,
III

Volker Runde

The
divergence
theorem

Example (continued)

$$\begin{aligned}\dots &= 3 \iiint_E x^2 + y^2 + 1 \, dV \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^2 (\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + 1) \\ &\quad \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= 3 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^2 (\rho^2 \sin^2 \phi + 1) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= 6\pi \int_0^{\frac{\pi}{2}} \int_1^2 (\rho^2 \sin^2 \phi + 1) \rho^2 \sin \phi \, d\rho \, d\phi = \dots\end{aligned}$$

Examples, VII

MATH 209—
Calculus,
III

Volker Runde

The
divergence
theorem

Example (continued)

$$\begin{aligned}\dots &= 6\pi \int_0^{\frac{\pi}{2}} \int_1^2 (\rho^2 \sin^2 \phi + 1) \rho^2 \sin \phi \, d\rho \, d\phi \\ &= 6\pi \int_0^{\frac{\pi}{2}} \int_1^2 \rho^4 \sin^3 \phi + \rho^2 \sin \phi \, d\rho \, d\phi \\ &= 6\pi \int_0^{\frac{\pi}{2}} \left. \frac{\rho^5}{5} \sin^3 \phi + \frac{\rho^3}{3} \sin \phi \right|_{\rho=1}^{\rho=2} d\phi \\ &= 6\pi \int_0^{\frac{\pi}{2}} \frac{31}{5} \sin^3 \phi + \frac{7}{3} \sin \phi \, d\phi \\ &= \frac{186\pi}{5} \int_0^{\frac{\pi}{2}} (1 - \cos^2 \phi) \sin \phi \, d\phi + 14\pi \int_0^{\frac{\pi}{2}} \sin \phi \, d\phi = \dots\end{aligned}$$

Examples, VIII

MATH 209—
Calculus,
III

Volker Runde

The
divergence
theorem

Example (continued)

$$\begin{aligned}\dots &= \frac{186\pi}{5} \int_0^{\frac{\pi}{2}} (1 - \cos^2 \phi) \sin \phi \, d\phi + 14\pi \int_0^{\frac{\pi}{2}} \sin \phi \, d\phi \\ &= \frac{186\pi}{5} \int_0^1 1 - u^2 \, du + 14\pi \left(-\cos \phi \Big|_{\phi=0}^{\phi=\frac{\pi}{2}} \right) \\ &= \frac{186\pi}{5} \left(u - \frac{u^3}{3} \Big|_{u=0}^{u=1} \right) + 14\pi = \frac{124\pi}{5} + \frac{70\pi}{5} = \frac{194\pi}{5}.\end{aligned}$$