MATH 209—Calculus, III

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Edmonton, Fall 2011

Definition of limit

Definition

Let f be a function of two variables, and let $(a, b) \in D_f$ or close to D_f. Then we call L the limit of f as (x, y) approaches (a, b), denoted by

$$L = \lim_{(x,y)\to(a,b)} f(x, y),$$

if $f(x, y)$ gets closer and closer to L as (x, y) gets closer and closer to (a, b) along any path in D_f.

Important

In one variable, we have $L = \lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$ because x can approach a only from two directions. In two variables, matters are much more complicated.

Examples, I

Example

Let

$$f(x,y) = x^2 + y^2.$$

Then

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} x^2 + y^2 \\ &= \lim_{(x,y)\to(0,0)} x^2 + \lim_{(x,y)\to(0,0)} y^2 \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Examples, II

Example

Let

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2},$$

so that $D_f = \mathbb{R}^2 \setminus \{(0,0)\}$.

What is $\lim_{(x,y)\to(0,0)} f(x,y)$?

Along the x-axis ($y = 0$): $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^2}{x^2} = 1$.

Along the y-axis ($x = 0$): $\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$.

Therefore, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Examples, III

Example

Let

$$f(x,y) = \frac{xy}{x^2+y^2},$$

so that $D_f = \mathbb{R}^2 \setminus \{(0,0)\}$.

What is $\lim_{(x,y)\to(0,0)} f(x,y)$?

Along the x-axis ($y=0$): $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{0}{x^2} = 0$.

Along the y-axis ($x=0$): $\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0}{y^2} = 0$.

Along the line $y=x$: $\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}$.

Therefore, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Examples, IV

Example

Let

$$f(x,y) = \frac{x^2 y}{x^4 + y^2},$$

so that $D_f = \mathbb{R}^2 \setminus \{(0,0)\}$.

What is $\lim_{(x,y)\to(0,0)} f(x,y)$?

Along the x-axis ($y = 0$): $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{0}{x^4} = 0$.

Along the y-axis ($x = 0$): $\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0}{y^2} = 0$.

Along any line $y = mx$:

$\lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{mx^3}{x^4+m^2x^2} = \lim_{x\to 0} \frac{mx}{x^2+m^2} = 0$.

Along the parabola $y = x^2$:

$\lim_{x\to 0} f(x,x^2) = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \frac{1}{2}$.

Therefore, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Examples, V

Example

What is

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}?$$

Use the estimate:

$$\left|\frac{3x^2y}{x^2+y^2}\right| = \frac{3x^2|y|}{x^2+y^2} \leq \frac{3(x^2+y^2)|y|}{x^2+y^2} = 3|y|.$$

As $\lim_{(x,y)\to(0,0)} 3|y| = 0$, we have $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$.

Continuity in two variables

Definition

A function f of two variables is called continuous at $(a, b) \in D$ if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$. We call f continuous on D if it is continuous at each $(a, b) \in D$.

Remark

Like in one variable: sums, products, differences, quotients, and compositions of continuous functions are continuous again (wherever they are defined).

Examples, VI

Example

A polynomial (in two variables) is a sum of terms of the form $cx^n y^m$, e.g.,

$$p(x, y) = 2011x^2y^7 + x - 42y^3 + 333x^6y^4.$$

A rational function is a quotient of two polynomials, e.g.,

$$r(x, y) = \frac{7x^2y - 9}{x^3 - 4x^5y^7}.$$

Polynomials are continuous on all of $\mathbb{R}^2$, and rational functions are continuous wherever they are defined.

Examples, VII

Example

What is

$$\lim_{(x,y)\to(1,-2)} x^4y + 6x^3 - y^3?$$

The function

$$p(x,y) = x^4y + 6x^3 - y^3$$

is a polynomial and thus continuous.
It follows that

$$\lim_{(x,y)\to(1,-2)} x^4y + 6x^3 - y^3 = p(1,-2) = 12.$$

Examples, VIII

Example

Let

$$r(x,y) = \frac{x^2 - y^2}{x^2 + y^2},$$

so that $D_r = \mathbb{R}^2 \setminus \{(0,0)\}$. Then r is continuous on D_r as a rational function.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by letting:

$$f(x,y) := \begin{cases} r(x,y), & (x,y) \in D_r, \\ 0, & (x,y) = (0,0). \end{cases}$$

Then $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} r(x,y)$ does not exist. Hence, f is not continuous at $(0,0)$.

Examples, IX

Example

Let

$$f(x,y) := \begin{cases} \frac{3x^2y}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Then f is continuous at each $(a,b) \neq (0,0)$. Since $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$: f is also continuous at $(0,0)$. Hence, f is continuous on all of $\mathbb{R}^2$.

Examples, X

Example

Let

$$f(x,y) := \frac{\sin(xy)}{e^x - y^2}.$$

Where is f continuous? We have

$$D_f = \left\{(x,y) \in \mathbb{R}^2 : y \neq \pm e^{\frac{x}{2}}\right\}.$$

The function $g(x,y) = xy$ is a polynomial and thus continuous. The function $h(t) = \sin t$ is continuous. Hence, $h(g(x,y)) = \sin(xy)$, the enumerator of $f(x,y)$, is continuous.

Examples, XI

Example (continued)

The functions $k(x, y) = e^x$ and $j(x, y) = y^2$ are continuous. Thus $e^x - y^2 = k(x, y) - j(x, y)$, the denominator of $f(x, y)$, is continuous. Finally

$$f(x, y) = \frac{h(g(x, y))}{k(x, y) - j(x, y)}$$

is continuous on D_f.
For instance,

$$\lim_{(x,y)\to\left(1,\frac{\pi}{2}\right)} f(x, y) = \frac{\sin\left(\frac{\pi}{2}\right)}{e - \frac{\pi^2}{4}} = \frac{4}{4e - \pi^2}.$$

Limit and continuity in three or more variables

Example

Let

$$f(x,y,z,t) = \frac{(x^2+y^2)(z^2+t^2)}{x^2+y^2+z^2+t^2}.$$

Then f is continuous on $D_f = \mathbb{R}^4 \setminus \{(0,0,0,0)\}$.
Since

$$|f(x,y,z,t)| \leq \frac{(x^2+y^2+z^2+t^2)(x^2+y^2+z^2+t^2)}{x^2+y^2+z^2+t^2} = x^2+y^2+z^2+t^2,$$

we have $\lim_{(x,y,z,t)\to(0,0,0,0)} f(x,y,z,t) = 0$.