MATH 209—Calculus, III

Volker Runde

University of Alberta

Edmonton, Fall 2011

The curl of a vector field, I

Definition

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field on $\mathbb{R}^3$ such that P, Q, and R have partial derivatives. The curl of $\mathbf{F}$ is defined as

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}.$$

If $\operatorname{curl} \mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is called irrotational.

The curl of a vector field, II

How to remember it...

Let

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Then:

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.\end{aligned}$$

Thus:

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F}$$

Examples, I

Example

Let

$$\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}.$$

Then:

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} \\ &= (-2y - xy)\mathbf{i} - (0 - x)\mathbf{j} + (yz - 0)\mathbf{k} \\ &= -y(2 + x)\mathbf{i} + x\mathbf{j} + yz\mathbf{k}. \end{aligned}$$

Examples, II

Example

Let $\mathbf{F}$ be conservative, i.e., $\mathbf{F} = \nabla f$, and suppose that f has continuous second order partial derivatives. Then:

$$\begin{aligned}
\operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial x} \end{vmatrix} \\
&= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \mathbf{i} \\
&\quad - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right) \mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \mathbf{k} \\
&= 0.
\end{aligned}$$

Examples, III

Theorem

Let $\mathbf{F}$ *be a vector field defined on all of* $\mathbb{R}^3$ *such that* $\operatorname{curl}\mathbf{F} = \mathbf{0}$. *Then* $\mathbf{F}$ *is conservative.*

Example

Let

$$\mathbf{F}(x, y, z) = y^2 z^3 \mathbf{i} + 2xyz^3 \mathbf{j} + 3xy^2 z^2 \mathbf{k}.$$

Then:

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{vmatrix}$$

$$= (6xyz^3 - 6xyz^3)\mathbf{i} - (3y^2z^2 - 3y^2z^2)\mathbf{j} + (2yz^3 - 2yz^3)\mathbf{k} = 0.$$

Examples, IV

Example (continued)

Hence, $\mathbf{F}$ is conservative, i.e., there is a function f with $\mathbf{F} = \nabla f$. This means:

$$f_x = y^2 z^3,$$
$$f_y = 2xyz^3,$$
$$f_z = 3xyz^2.$$

Thus:

$$f(x,y,z) = \int y^2 z^3 \, dx = xy^2 z^3 + g(y,z).$$

Examples, V

Example (continued)

Differentiate with respect to y:

$$2xyz^3 = f_y(x,y,z) = 2xyz^3 + g_y(y,z).$$

Hence, $g_y(y,z) = 0$ and thus $g(y,z) = h(z)$. We obtain:

$$f(x,y,z) = xy^2z^3 + h(z).$$

Differentiate with respect to z:

$$3xy^2z^3 = f_z(x,y,z) = 3xy^2z^2 + h'(z).$$

Hence, $h'(z) = 0$, i.e., $h(z) = C$, so that

$$f(x,y,z) = xy^2z^3 + C.$$

The divergence of a vector field, I

Definition

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field on $\mathbb{R}^3$ such that P, Q, and R have partial derivatives. The divergence of $\mathbf{F}$ is defined as

$$\operatorname{div} \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}.$$

If $\operatorname{div} \mathbf{F} = 0$, $\mathbf{F}$ is called incompressible.

The divergence of a vector field, II

Shorthand

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F}$$

Example

Let

$$\mathbf{F}(x,y,z) = e^x\mathbf{i} + e^{-xy}\mathbf{j} - xyz\mathbf{k}.$$

Then:

$$\operatorname{div}\mathbf{F} = e^x - xe^{-xy} - xy.$$

The divergence of a curl, I

Computing the divergence of a curl

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be such that P, Q, and R have continuous second order partial derivatives. Then:

$$\begin{aligned}
\operatorname{div}\operatorname{curl}\mathbf{F} &= \nabla \cdot (\nabla \times \mathbf{F}) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \\
&\qquad + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} \\
&= 0
\end{aligned}$$

The divergence of a curl, II

Theorem

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be such that P, Q, and R have continuous second order partial derivatives. Then $\operatorname{div} \operatorname{curl} \mathbf{F} = 0$.

Example

Let

$$\mathbf{F} = xy\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}.$$

Is there a vector field $\mathbf{G}$ with $\mathbf{F} = \operatorname{curl} \mathbf{G}$?

No, because

$$\operatorname{div} \mathbf{F} = y + xz \neq 0.$$

Curl and div and Green's theorem, I

The vector form of Green's theorem

Let C and D be as in Green's theorem, and consider the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, so that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy.$$

View $\mathbf{F}$ as a vector field on $\mathbb{R}^3$ with the third component 0. Then:

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}.$$

Curl and div and Green's theorem, II

The vector forms of Green's theorem (continued)

As

$$(\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y},$$

Green's theorem becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA.$$

Curl and div and Green's theorem, III

The vector forms of Green's theorem (continued)

Let C be given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \qquad t \in [a, b].$$

Then the unit tangent vector is

$$\mathbf{T}(t) = \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{j}.$$

The outward unit normal vector to C is given by

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{j}.$$

Curl and div and Green's theorem, IV

The vector forms of Green's theorem (continued)

Then

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{n}\,ds &= \int_a^b (\mathbf{F}\cdot\mathbf{n})(t)|\mathbf{r}'(t)|\,dt \\ &= \int_a^b \left(\frac{P(x(t),y(t))y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t),y(t))x'(t)}{|\mathbf{r}'(t)|}\right)|\mathbf{r}'(t)|\,dt \\ &= \int_a^b P(x(t),y(t))y'(t) - Q(x(t),y(t))x'(t)\,dt \\ &= \int_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA,\end{aligned}$$

and thus

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}\,dA.$$