MATH 209—Calculus, III

Volker Runde

University of Alberta

Edmonton, Fall 2011

Examples, I

Example

Let T be the temparature at a point on Earth with latitude x and longitude y. Then T depends on both x and y. We can think of T as a function of x and y: $T = f(x, y)$.

Example

The volume V of a circular cone with radius r and height h depends on both r and h: $V = \frac{1}{3}\pi r^2 h$. We can view V as a function of r and h: $V(r, h) = \frac{1}{3}\pi r^2 h$.

Definitions and notation

Definition

Let $D \subset \mathbb{R}^2$. A function f of two variables is a rule that assigns to each pair $(x, y) \in D$ a unique real number $z = f(x, y)$.

Notation and concepts

We write $f : D \to \mathbb{R}$ and call:

- D the domain of f. (Sometimes, we write D_f if we want to emphasize f);
- x and y the independent variables;
- z the dependent variable.

The range of f is defined as

$$R_f = \{z \in \mathbb{R} : z = f(x, y) \text{ with } (x, y) \in D\}.$$

Examples, II

Example

Find the domain of

$$f(x, y) = x \ln(y^2 - x)$$

and evaluate $f(8, 3)$.
We know: $\ln(y^2 - x)$ is defined if and only if $y^2 - x > 0$ if and only if $x < y^2$.
Hence,

$$D_f = \{(x, y) : x < y^2\}.$$

Also,

$$f(8, 3) = 8 \cdot \ln(9 - 8) = 0.$$

Examples, III

Example

Sometimes, one wants to consider on a smaller domain. Take

$$V(r,h) = \frac{1}{3}\pi r^2 h.$$

The largest possible domain is $\mathbb{R}^2$, but its physical domain is

$$\{(r,h) \in \mathbb{R}^2 : r,h \geq 0\}.$$

Examples, IV

Example

Find D_g and R_g for

$$g(x, y) = \sqrt{9 - x^2 - y^2}.$$

Then $(x, y) \in D_g$ if and only if $9 - x^2 - y^2 \geq 0$ if and only if $x^2 + y^2 \leq 9$. Hence,

$$D_g = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}.$$

As $x^2 + y^2 \leq 9$ if and only if $\sqrt{9 - x^2 - y^2} \leq 3$:

$$R_g = \{g(x, y) : (x, y) \in D_g\} = [0, 3].$$

Definition of a graph

Definition

Let f be a function of two variables with domain D. Then the graph of f is the set of all points $(x, y, z) \in \mathbb{R}^3$ with $(x, y) \in D$ and $z = f(x, y)$.

Example

Let D be the set of the geographical coordinates (x, y) of all points on the map of Alberta. Let $f(x, y)$ the elevation above sea level of the point with coordinates (x, y). Then the graph of f is the relief map of Alberta.

Visualization

If $f \geq 0$, we can visualize the graph of f as the surface S given by $z = f(x, y)$ lying above D.

Examples, V

Example

Let

$$g(x, y) = \sqrt{9 - x^2 - y^2}.$$

Then the graph of g is the top half of the sphere around $(0, 0, 0)$ with radius 3.

Example

Let

$$f(x, y) = 6 - 3x - 2y.$$

Then the graph of f has the equation $z = 6 - 3x - 2y$, i.e.,

$$3x + 2y + z = 6.$$

It is a plane with intercepts $x = 2$, $y = 3$, and $z = 6$.

Examples, VI

Example

Let

$$h(x, y) = 4x^2 + y^2.$$

It is easy to see that $D_h = \mathbb{R}^2$ and $R_h = [0, \infty)$.
The graph of h has the equation $z = 4x^2 + y^2$, an elliptic paraboloid.

Definition and examples

Definition

Let f be a function of two variables. The level curves of f are the curves with the equation $f(x, y) = c$ where c is a constant in the range of f.

Example

Let

$$g(x, y) = \sqrt{9 - x^2 - y^2}$$

For $c \in [0, 3]$, the level curves have the equation $\sqrt{9 - x^2 - y^2} = c$, i.e.,

$$x^2 + y^2 = 9 - c^2.$$

Hence, the level curves of g are circles centered at $(0, 0)$ with radius $\sqrt{9 - c^2}$.

Examples, VII

Example

Let

$$f(x,y) = 8 - x^2 - y^2 + 2y.$$

As

$$f(x,y) = 9 - x^2 - (y^2 - 2y + 1) = 9 - x^2 - (y-1)^2,$$

The level curves have the equation

$$x^2 + (y-1)^2 = 9 - c.$$

These are circles centered at $(0,1)$ with radius $\sqrt{9-c}$.

Examples of functions of three or more variables

Example

Let T be the temparature at a point of the Earth with latitude x and longitude y at the time t. Then $T = f(x, y, t)$ is a function of three variables.

Example

Pause let T be the temparature at a point of the Earth with latitude x and longitude y, at the time t, and at height h above ground level. Then $T = f(x, y, t, h)$ is a function of four variables.

Examples, VIII

Example

Let

$$f(x, y, z) = \frac{\ln(y - x)}{z}.$$

Then

$$D_f = \{(x, y, z) \in \mathbb{R}^3 : y > x,\ z \neq 0\}$$

and $R_f = \mathbb{R}$.

Important difference

The graph of a function of three variables is an object in $\mathbb{R}^4$, i.e., in four dimensional space, and therefore difficult to visualize.

Level surfaces

Definition

Let f be a function of three variables. Then a level surface of f is a suface with the equation $f(x, y, z) = c$, with c in the range of f.

Example

Let

$$f(x, y, z) = x^2 + y^2 + z^2 - 4x$$
$$= x^2 - 4x + 4 + y^2 + z^2 - 4 = (x - 2)^2 + y^2 + z^2 - 4.$$

Then the level surfaces of f have the equation

$$(x - 2)^2 + y^2 + z^2 = 4 + c,$$

i.e., they are spheres centered at $(2, 0, 0)$ with radius $\sqrt{4 + c}$.