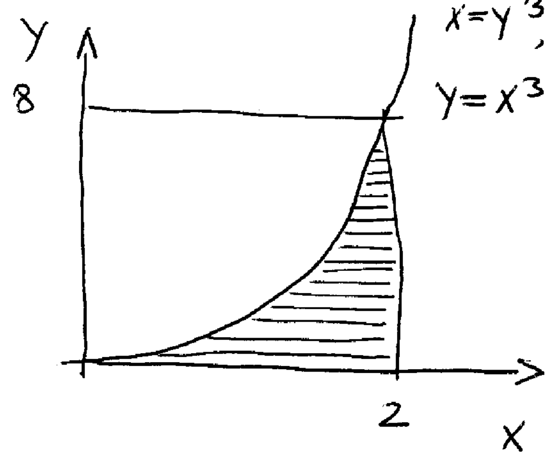
MATH 209 REVIEW – PART II (20 Dec 2011)

Multiple choice questions

MCQ 1

$$\int_0^8 \int_{y^{1/3}}^2 e^{x^4}\,dx\,dy$$



$$= \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx =$$

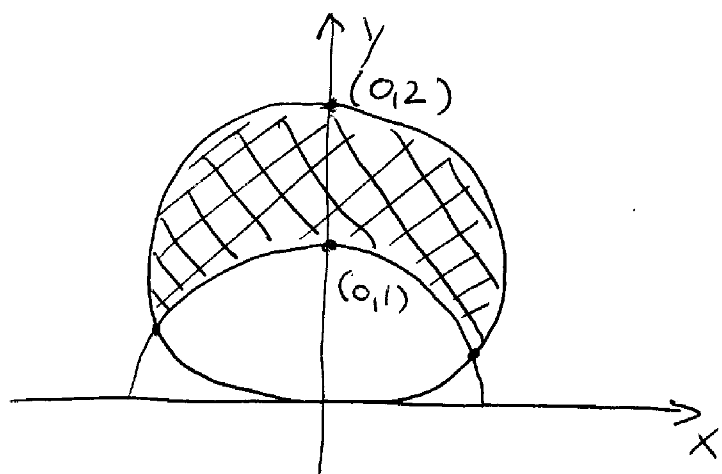
$$= \int_0^2 x^3 e^{x^4}\,dx = \frac{1}{4} e^{x^4}\Big|_{x=0}^{x=2} = \frac{1}{4}\left(e^{16}-1\right) \qquad \boxed{(d)}$$

MCQ 2

$$x^2+y^2=2y$$

$$x^2+(y-1)^2=1$$

$$\rho(x,y)=\frac{K}{\sqrt{x^2+y^2}}$$



$$D=\{(x,y)\mid x^2+y^2\geq 1,\ x^2+(y-1)^2\leq 1\}$$

Polar coordinates: $x = r\cos\vartheta$, $y = r\sin\vartheta$

$$x^2+y^2\geq 1 \iff r\geq 1$$

$$x^2+(y-1)^2\leq 1 \iff r^2-2r\sin\vartheta\leq 0$$

$$\iff r\leq 2\sin\vartheta$$

$$\tilde{D}=\{(r,\vartheta)\mid \tfrac{\pi}{6}\leq\vartheta\leq\tfrac{5\pi}{6},\ 1\leq r\leq 2\sin\vartheta\}$$

$$m = \iint_D \rho\, dA = \iint_{\tilde{D}} \rho\, r\, dr\, d\vartheta = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\vartheta} \frac{K}{r}\, r\, dr\, d\vartheta$$

$$= \int_{\pi/6}^{5\pi/6} K(2\sin\vartheta - 1)\, d\vartheta = K(-2\cos\vartheta - \vartheta)\Big|_{\vartheta=\pi/6}^{\vartheta=\frac{5\pi}{6}}$$

$$= K\left(2\sqrt{3} - \frac{2\pi}{3}\right) = 2K\left(\sqrt{3} - \pi/3\right)$$

$$M_x = \iint_D \rho\, \cancel{x}\, dA = \iint_{\tilde{D}} \rho\, r\sin\vartheta\, r\, dr\, d\vartheta = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\vartheta} \frac{K}{r}\, r\sin\vartheta\, r\, dr\, d\vartheta$$

$$= \int_{\pi/6}^{5\pi/6} K\sin\vartheta\, \frac{r^2}{2}\Big|_{r=1}^{r=2\sin\vartheta} d\vartheta = \frac{K}{2}\int_{\pi/6}^{5\pi/6} \sin\vartheta\,(4\sin^2\vartheta - 1)\, d\vartheta$$

$$= \frac{K}{2}\int_{\pi/6}^{5\pi/6} \sin\vartheta\,(3 - 4\cos^2\vartheta)\, d\vartheta =$$

$$= \frac{K}{2}\left(-3\cos\vartheta + \frac{4}{3}\cos^3\vartheta\right)\Big|_{\vartheta=\pi/6}^{\vartheta=5\pi/6}$$

$$= \frac{K}{2}\left(-3\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) + \frac{4}{3}\left(-\frac{3\sqrt{3}}{8} - \frac{3\sqrt{3}}{8}\right)\right)$$

$$= \frac{K}{2}\left(3\sqrt{3} - \sqrt{3}\right) = K\sqrt{3}$$

$$\bar{y} = \frac{M_x}{m} = \frac{K\sqrt{3}}{2K(\sqrt{3} - \pi/3)} = \frac{\sqrt{3}}{2(\sqrt{3} - \pi/3)} \qquad \boxed{(b)}$$

MCQ3 $\quad F = (P, Q, R) \qquad fF = (fP, fQ, fR)$

$$\text{Curl}(fF) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (fP, fQ, fR)$$

$$= \left((fR)_y - (fQ)_z, (fP)_z - (fR)_x, (fQ)_x - (fP)_y \right)$$

$$= \left(fR_y - fQ_z + f_y R - f_z Q, fP_z - fR_x + f_z P - f_x R, \right.$$
$$\left. fQ_x - fP_y + f_x Q - f_y P \right)$$

$$= f\left(R_y - Q_z, P_z - R_x, Q_x - P_y \right) + (f_x, f_y, f_z) \times (P, Q, R)$$

$$= f \operatorname{curl} F + \nabla f \times F \qquad \boxed{(a)}$$

MCQ4 $\quad F(x,y,z) = (\underbrace{y e^{xy} + z\cos(xz) + 2x}_{P}, \underbrace{y + x e^{xy}}_{Q}, \underbrace{x\cos(xz)}_{R})$

$$\frac{\partial R}{\partial y} = 0 = \frac{\partial Q}{\partial z}$$
$$\frac{\partial P}{\partial z} = \cos(xz) - xz\sin(xz) = \frac{\partial R}{\partial x}$$
$$\frac{\partial Q}{\partial x} = e^{xy} + xy e^{xy} = \frac{\partial P}{\partial y}$$

domain of $F = \mathbb{R}^3$ simply connected $\Rightarrow$ F is conservative

$$\frac{\partial f}{\partial x} = y e^{xy} + z\cos(xz) + 2x$$
$$\Rightarrow \quad f = e^{xy} + \sin(xz) + x^2 + C(y,z)$$
$$\frac{\partial f}{\partial y} = x e^{xy} + \frac{\partial C}{\partial y} = y + x e^{xy} \quad \Rightarrow C(y,z) = \tfrac{1}{2} y^2 + \tilde{C}(z)$$
$$\frac{\partial f}{\partial z} = x\cos(xz) + \tilde{C}'(z) = x\cos(xz) \quad \Rightarrow \tilde{C}(z) \equiv C_0$$

$$f(x,y,z) = e^{xy} + \sin(xz) + x^2 + \frac{1}{2}y^2 + C_0$$

$$\left(\text{check:} \quad \frac{\partial f}{\partial x} = y e^{xy} + z \cos(xz) + 2x = P \checkmark \right.$$

$$\frac{\partial f}{\partial y} = x e^{xy} + y = Q \checkmark$$

$$\left. \frac{\partial f}{\partial z} = x \cos(xz) = R \checkmark \right)$$

$$\int_{\mathcal{C}} F \cdot dr = f(\text{endpoint of } \mathcal{C}) - f(\text{starting point of } \mathcal{C})$$

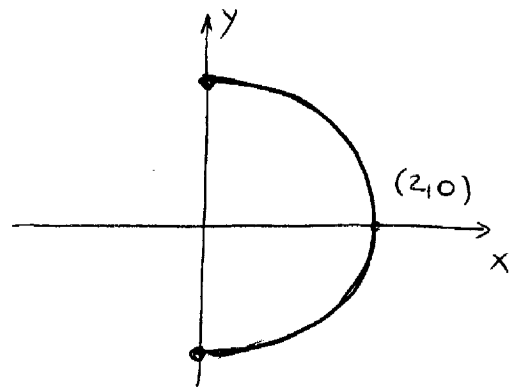
$$= f(0,0,1) - f(1,0,0)$$

$$= 1 + C_0 - (1 + 1 + C_0) = -1 \qquad \boxed{(c)}$$

MC Q5

$r(t) = (2\cos t, 2\sin t)$

$-\pi/2 \le t \le \pi/2$



$$\frac{dr}{dt} = 2(-\sin t, \cos t)$$

$$\left|\frac{dr}{dt}\right| \equiv 2$$

$$m = \int_{\mathcal{C}} \rho \, ds = \int_{-\pi/2}^{\pi/2} k \, 2 \, dt = 2k\pi$$

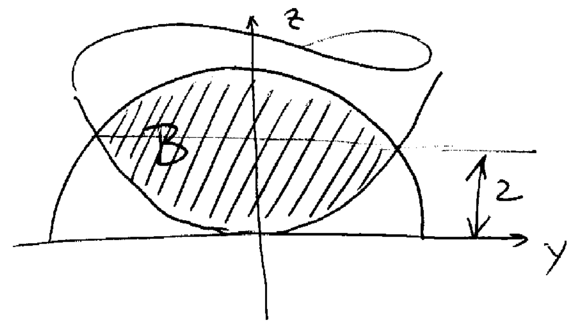
$$M_y = \int_{\mathcal{C}} \rho x \, ds = \int_{-\pi/2}^{\pi/2} k \, 2\cos t \, 2 \, dt = 4k \sin t \Big|_{t=-\pi/2}^{t=\pi/2} = 8k$$

$$\bar{x} = \frac{M_y}{m} = \frac{8k}{2k\pi} = \frac{4}{\pi} \qquad \boxed{(e)}$$

MCQ6 $B = \{(x,y,z) \mid x^2+y^2+z^2 \le 6,\ z \ge x^2+y^2\}$

intersection?

$$x^2+y^2+z^2 = 6$$
$$x^2+y^2 = z$$



$$z^2+z-6=0$$

$$(z-2)(z+3)=0 \rightarrow z=2$$
$$\rightarrow z=-3 \text{ (not relevant!)}$$

Cylindrical coordinates:

$$\tilde{B} = \{(r,\vartheta,z) \mid 0 \le r \le \sqrt{2},\ 0 \le \vartheta \le 2\pi,\ r^2 \le z \le \sqrt{6-r^2}\}$$

$$\text{Vol}(B) = \iiint_B dV = \iiint_{\tilde{B}} r\,dr\,d\vartheta\,dz = \int_0^{\sqrt{2}}\int_0^{2\pi}\int_{r^2}^{\sqrt{6-r^2}} r\,dz\,d\vartheta\,dr$$

$$= \int_0^{\sqrt{2}}\int_0^{2\pi} r\left(\sqrt{6-r^2}-r^2\right) d\vartheta\,dr = 2\pi\int_0^{\sqrt{2}}\left(r\sqrt{6-r^2}-r^3\right)dr$$

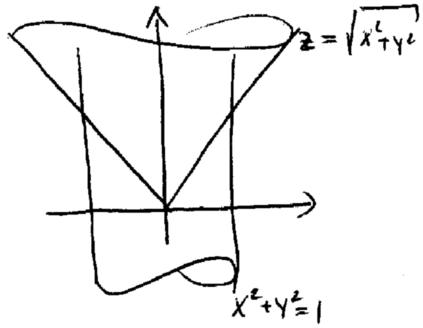
$$= 2\pi\left(-\frac{1}{3}(6-r^2)^{3/2} - \frac{1}{4}r^4\right)\Bigg|_{r=0}^{r=\sqrt{2}} = 2\pi\left(-\frac{8}{3} - 1 - \left(-\frac{1}{3}6\sqrt{6}\right)\right)$$

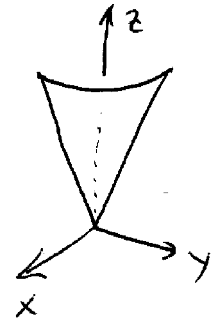
$$= 2\pi\left(2\sqrt{6} - \frac{11}{3}\right) = \frac{2\pi}{3}\left(6\sqrt{6}-11\right)$$

$\boxed{(b)}$

Long answer questions

LAQ 1 $\iint_S xy\,dS = ?$ S = part of $z = \sqrt{x^2+y^2}$ inside $x^2+y^2=1$, first octant





$$S: \ r(u,v) = (u\cos v, u\sin v, u)$$

$$0 \le u \le 1, \quad 0 \le v \le \pi/2$$

$$r_u = (\cos v, \sin v, 1)$$
$$r_v = (-u\sin v, u\cos v, 0)$$

$$r_u \times r_v = (-u\cos v, -u\sin v, u)$$

$$|r_u \times r_v|^2 = u^2\cos^2 v + u^2\sin^2 v + u^2 = 2u^2$$

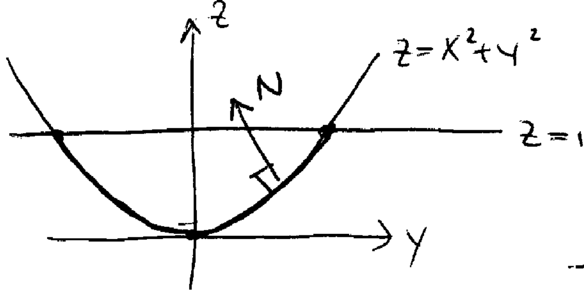
$$|r_u \times r_v| = \sqrt{2}\,u$$

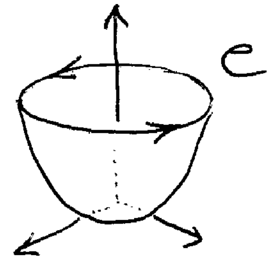
$$\iint_S xy\, dS = \int_0^1 \int_0^{\pi/2} u\cos v \, u\sin v \, \sqrt{2}\, u \, dv\, du$$

$$= \int_0^1 \int_0^{\pi/2} \sqrt{2}\, u^3 \sin v\cos v \, dv\, du = \sqrt{2} \int_0^1 u^3 du \int_0^{\pi/2} \sin v\cos v\, dv$$

$$= \sqrt{2} \cdot \left. \frac{u^4}{4} \right|_{u=0}^{u=1} \cdot \left. \frac{\sin^2 v}{2} \right|_{v=0}^{v=\pi/2} = \sqrt{2} \cdot \frac{1}{4} \cdot \frac{1}{2} = \underline{\underline{\frac{\sqrt{2}}{8}}}$$

LAQ2 $S =$ part of $z = x^2 + y^2$, oriented UPWARD





Stokes' Thm.: $\int_{C} F \cdot dr = \iint_{S} \text{Curl } F \cdot dS$

C: $r(t) = (\cos t, \sin t, 1)$ $\quad 0 \leq t \leq 2\pi$

$\frac{dr}{du} = (-\sin t, \cos t, 0)$

$$\int_{C} F \cdot dr = \int_0^{2\pi} (\sin^2 t, \cos t, 1) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= \int_0^{2\pi} (\cos^2 t - \sin^3 t)\, dt = \int_0^{2\pi} \left(\frac{1+\cos 2t}{2} - \sin t\,(1-\cos^2 t)\right) dt$$

$$= \left(\frac{t}{2} + \frac{\sin 2t}{4} + \cos t - \frac{1}{3}\cos^3 t \right) \Bigg|_{t=0}^{t=2\pi}$$

$$= \underline{\underline{\pi}}$$

S: $r(u,v) = (u\cos v, u \sin v, u^2)$ $\quad 0 \leq u \leq 1,\ 0 \leq v \leq 2\pi$

$r_u = (\cos v, \sin v, 2u)$

$r_v = (-u\sin v, u\cos v, 0)$

$r_u \times r_v = (-2u^2\cos v, -2u^2 \sin v, u)$

Points UP

$F = (y^2, x, z^2)$

$\text{Curl } F = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (y^2, x, z^2)$

$= (0, 0, 1-2y)$

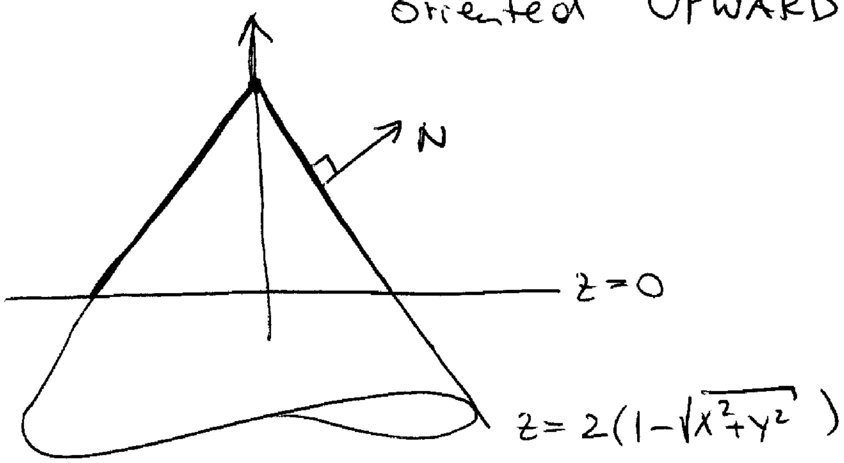
$$\iint_S \operatorname{curl} F \cdot dS = \int_0^1 \int_0^{2\pi} (0, 0, 1 - 2u\sin v) \cdot (-2u^2\cos v, -2u^2\sin v, u)\, dv\, du$$

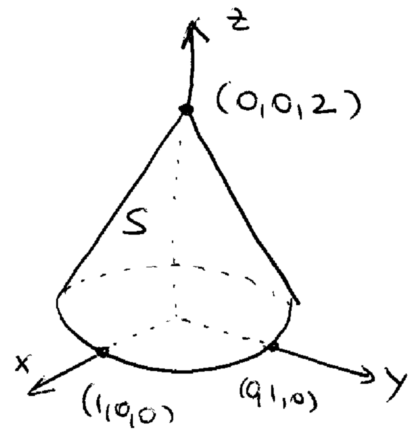
$$= \int_0^1 \int_0^{2\pi} (u - 2u^2 \sin v)\, dv\, du = \int_0^1 2\pi u\, du = \pi u^2 \Big|_{u=0}^{u=1}$$

$$= \underline{\underline{\pi}}$$

LAQ 3

$S =$ part of $z = 2(1 - \sqrt{x^2+y^2})$ above $z = 0$, oriented UPWARD





Note: S is NOT closed!

to apply Divergence Thm: add bottom disc $S_1 = \{(x,y,z) \mid x^2+y^2 \le 1, z = 0\}$.

(oriented DOWNWARD)

$\Rightarrow$ Divergence Thm: $\displaystyle\iint_{S \cup S_1} F \cdot dS = \iiint_B \operatorname{div} F\, dV$

$\Rightarrow \displaystyle\iint_S F \cdot dS = \iiint_B \operatorname{div} F\, dV - \iint_{S_1} F \cdot dS$

S_1: $r(u,v) = (u\cos v, u\sin v, 0)$ $\quad 0 \le u \le 1, 0 \le v \le 2\pi$

$r_u = (\cos v, \sin v, 0)$

$r_v = (-u\sin v, u\cos v, 0)$

$r_u \times r_v = (0,0,u)$ points UP

$$\Rightarrow \iint_{S_1} F \cdot dS = -\int_0^1 \int_0^{2\pi} (*,*,1)\cdot(0,0,u)\, dv\, du$$

$$= -\int_0^1 \int_0^{2\pi} u\, dv\, du = -\int_0^1 2\pi u\, du = -\pi$$

$$\operatorname{div} F = 1 + 3x^2 + 3y^2 + 1 = 2 + 3(x^2+y^2)$$

$$B = \{(x,y,z) \mid x^2+y^2 \le 1,\ 0 \le z \le 2(1-\sqrt{x^2+y^2})\}$$

in cylindrical coordinates:

$$\tilde{B} = \{(r,\vartheta,z) \mid 0 \le r \le 1,\ 0 \le \vartheta \le 2\pi,\ 0 \le z \le 2(1-r)\}$$

$$\iiint_B \operatorname{div} F\, dV = \iiint_{\tilde{B}} (2+3r^2)\, r\, dr\, d\vartheta\, dz$$

$$= \int_0^1 \int_0^{2\pi} \int_0^{2(1-r)} (2r+3r^3)\, dz\, d\vartheta\, dr =$$

$$= \int_0^1 \int_0^{2\pi} 2(1-r)(2r+3r^3)\, d\vartheta\, dr = 4\pi \int_0^1 (1-r)(2r+3r^3)\, dr$$

$$= 4\pi \int_0^1 (2r - 2r^2 + 3r^3 - 3r^4)\, dr$$

$$= 4\pi\left(\frac{2}{2} - \frac{2}{3} + \frac{3}{4} - \frac{3}{5}\right) = 4\pi\left(\frac{1}{3} + \frac{15-12}{20}\right)$$

$$= 4\pi \frac{29}{60} = \pi \frac{29}{15}$$

$$\Rightarrow \iint_S F \cdot dS = \pi\frac{29}{15} - (-\pi) = \pi \frac{44}{15}$$