Lectures on Amenability
(errata and updates)

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Errata

1. page 5 (discovered by Adam Skalski): In the proof of Proposition 0.1.8, it is tacitly assumed that, if E is paradoxical and has a paradoxical decomposition $E = A_1 \cup \cdots \cup A_n \cup B_1 \cup \cdots \cup B_m$, then the sets $g_1 A_1, \ldots, g_n A_n$ are automatically pairwise disjoint. This does not follow immediately from the definition. However, we can always find a decomposition with this property (this follows easily from Theorem 0.1.13).

2. pages 8–9 (also discovered by Adam Skalski): The definition of a good sequence in the proof of Theorem 0.2.4 is not the right one, and the argument on page 9, lines 6–8, is faulty. Nevertheless, the statement of the last paragraph of the proof is indeed correct: for details, see the reference [Wag].

3. page 10 (discovered by Alex Trichtchenko): Definition 0.2.6(iv) should read: "Let $A, B \subset X^*$ be bounded, and let $k \in \mathbb{N}_0$ be such that

   $$B' := \{(b, n+k) : (b, n) \in B\}$$

   has empty intersection with A. Define $[A] + [B] := [A \cup B']$."

4. page 33, line 1: It should read "Let G be a locally compact group, let E be a Banach space, ..."

5. page 40, proof of Proposition 2.1.5 (discovered by Prakash Rajendran): As it stands, the proof of (ii) $\Longrightarrow$ (i) is flawed. It can be corrected as follows:

   Let E be a Banach $\mathfrak{A}$-bimodule, and let

   $$E_0 := \{a \cdot x \cdot b : a, b \in \mathfrak{A},\ x \in E\} \qquad \text{and} \qquad E_1 := \{x \cdot b : b \in \mathfrak{A},\ x \in E\}.$$

   Then E_0 and E_1 are closed submodules of E, and E_0 is pseudo-unital.

Let $D \in \mathcal{Z}^1(\mathfrak{A}, E_1^*)$. Let $\pi_0 \colon E_1^* \to E_0^*$ be the restriction map. Since E_0 is pseudo-unital, $\pi_0 \circ D \in \mathcal{Z}^1(\mathfrak{A}, E_0^*) = \mathcal{B}^1(\mathfrak{A}, E_0^*)$ holds, so that there is $\phi_0 \in E_0^*$ such that $\pi_0 \circ D = \mathrm{ad}_{\phi_0}$. Use the Hahn–Banach theorem to extend ϕ_0 to E_1, and let that extension be denoted by ϕ_1. It follows that $D - \mathrm{ad}_{\phi_1} \in \mathcal{Z}^1(\mathfrak{A}, E_1^* \cap E_0^\perp)$. Since $E_1^* \cap E_0^\perp \cong (E_1/E_0)^*$ as Banach $\mathfrak{A}$-bimodules, and since $\mathfrak{A} \cdot (E_1/E_0) = \{0\}$, it follows from Proposition 2.1.3 that $\mathcal{H}^1(\mathfrak{A}, E_1^* \cap E_0^\perp) = \{0\}$. Hence, there is $\psi_1 \in E_1^* \cap E_0^\perp$ such that $D - \mathrm{ad}_{\phi_1} = \mathrm{ad}_{\psi_1}$, i.e., $D = \mathrm{ad}_{\psi_1 + \psi_1}$. Since $D \in \mathcal{Z}^1(\mathfrak{A}, E_1^*)$ was arbitrary, this means that $\mathcal{H}^1(\mathfrak{A}, E_1^*) = \{0\}$.

Now, let $D \in \mathcal{Z}^1(\mathfrak{A}, E^*)$. Let $\pi_1 \colon E^* \to E_1^*$ be the restriction map, so that $\pi_1 \circ D_1 \in \mathcal{Z}^1(\mathfrak{A}, E_1^*)$. By the foregoing, there is $\phi_1 \in E_1^*$ such that $\pi_1 \circ D = \mathrm{ad}_{\phi_1}$. Via Hahn–Banach, we obtain an extension $\phi \in E^*$ of ϕ_1, so that $D - \mathrm{ad}_\phi \in \mathcal{Z}^1(\mathfrak{A}, E_1^\perp)$. Since $E_1^\perp \cong (E/E_1)^*$, and since $(E/E_1) \cdot \mathfrak{A} = \{0\}$, (the obvious modification of) Proposition 2.1.3 yields $\psi \in E_1^\perp$ such that $D = \mathrm{ad}_{\phi+\psi}$. This means that $D \in \mathcal{B}^1(\mathfrak{A}, E^*)$, which completes the proof.

6. page 46, line -1: Replace “necessary” by “sufficient”.

7. page 52, first line of Exercise 2.3.8: It should read “Let $\mathfrak{A}$ be a Banach algebra with a bounded approximate identity.”

8. page 53, line -12: This displayed formula should read:

$$\begin{aligned}
|\langle f, \mu \rangle|^2 &= |\langle f - f_n, \mu \rangle|^2 \\
&= \left| \int_\Omega (f - f_n)\, d\mu \right|^2 \\
&\leq \left(\int_\Omega |f - f_n|\, d|\mu| \right)^2 \\
&\leq |\mu|(\Omega) \int_\Omega |f - f_n|^2\, d|\mu| \\
&= |\mu|(\Omega) \|f - f_n\|_2^2 \\
&\to 0.
\end{aligned}$$

9. page 60, second paragraph of 2.5: The last sentence should read: “If G is is an amenable, locally compact group, then $L^1(G)$ is already symmetrically amenable.”

10. pages 73–74 (discovered by Matthew Mazowita): In the proof of Lemma 3.2.6, the natural number occurring in line 17 on page 73 is different from the one in line 20 on the same page even though both are denoted by n; one of them, should therefore be relabled. Also, the basis $a_1, \ldots, a_n$ of $\mathfrak{A}$ should consist of unit vectors.

11. page 92, line −11: The displayed formula should read

$$|\langle x, T_{f_{F,n}}\psi - \mathrm{tr}\rangle| < \frac{1}{n} \qquad (x \in F).$$

12. page 98: The last line of the proof of Theorem 4.3.5 should read: "For (vi) $\Longrightarrow$ (i)..."

13. page 112, line −10: It should read "...that $\mathcal{Q} := \mathrm{id}_{\mathfrak{B}} - \mathcal{P}$ is..."

14. page 117, line 2: It should read: "...for which $VN(G)$ is nevertheless Connes-amenable..."

15. pages 117 and 225: It is claimed that in [Run 5] the equivalence of the following statements about a locally compact group G was proved:

    (i) G is amenable;

    (ii) $M(G)$ is Connes-amenable;

    (iii) $M(G)$ has a normal, virtual diagonal.

    As S. Tabaldyev pointed out, the proof of (ii) $\Longrightarrow$ (iii) in [Run 5] contains a gap. Nevertheless, we could repair [Run 5], so that this paper contains at least the equivalence of (i) and (ii). The equivalence of (i), (ii), and (iii) is proved in

    V. Runde, Connes-amenability and normal, virtual diagonals for measure algebras, II. *Bull. Austral. Math. Soc.* **68** (2003), 325–328.

16. page 120, line 6: Replace ${}_{\mathfrak{A}}\mathcal{L}(E, F)$ by ${}_{\mathfrak{A}}\mathcal{L}(\mathfrak{A}^{\#}\hat{\otimes} E, E)$

17. page 149, line −3: It should read $\cdots \geq \frac{1}{2}\|pyp + np + py^*y + np\| = \cdots$.

18. page 150, line 3: It should read $\frac{1}{2i}(pyp - py^*p)$.

19. page 150, line 6: It should read

$$\|y + npy(e_{\mathfrak{A}} - p)\| = \cdots .$$

20. page 161, Corollary 6.3.16: The displayed formula should read

$$\mathfrak{M}\tilde{\otimes}_{W^*}\mathfrak{N} \cong (\pi \otimes \rho)(\mathfrak{M} \otimes \mathfrak{N})''.$$

21. page 164: The displayed formula in Definition 6.4.6(i) should read

$$S_{\mathrm{nor}}(\mathfrak{M} \otimes \mathfrak{B}) := \{\phi \in S(\mathfrak{M}\tilde{\otimes}_{\max}\mathfrak{B}) : T_\phi\mathfrak{B} \subset \mathfrak{M}_*\}.$$

22. page 169, line 1 of Theorem 6.4.17 should read: “Let $\mathfrak{M}$ be a von Neumann algebra. . . ”

23. page 172, line 1 of Corollary 6.4.21 should read’: “. . . Let $N \in \mathcal{N}(\mathfrak{H})$ be positive, . . . ”

24. page 182, line −2: z_j instead of x_j.

25. page 183, line −3: Replace $\mathcal{E} \in {}_{\mathfrak{M}'}\mathcal{L}_{\mathfrak{M}}(\mathcal{L}(\mathfrak{H}))$ by $\mathcal{E} \in {}_{\mathfrak{M}'}\mathcal{L}_{\mathfrak{M}'}(\mathcal{L}(\mathfrak{H}))$.

26. pages 183 and 184 (discovered by Matt Daws): In Propositions 6.5.8 as well as in the two paragraphs immediately preceding and following it, respectively, it should read $\mathcal{L}^2_{h,w^*}(\mathfrak{M}, \mathbb{C})^*$ throughout instead of $\mathcal{L}^2_{h,w^*}(\mathfrak{M}, \mathbb{C})$.

27. page 184: In Lemma 6.5.9, replace $\mathcal{L}^2_{h,w^*}(\mathfrak{M}, \mathbb{C})$ and $\mathcal{L}^2_{w^*}(\mathfrak{M}, \mathbb{C})$ by $\mathcal{L}^2_{h,w^*}(\mathfrak{M}, \mathbb{C})^*$ and $\mathcal{L}^2_{w^*}(\mathfrak{M}, \mathbb{C})^*$, respectively.

28. page 186, Lemma 6.5.10: as for Lemma 6.5.9.

29. page 186, line 1 of Lemma 6.5.10 should start with: “Let $\mathfrak{M}$ be a properly infinite W^*-algebra.”

30. pages 193 and 194 (discovered by Ross Stokke): There is a gap in the proof of (i) $\Longrightarrow$ (ii) of Leptin’s Theorem (Theorem 7.1.3). The problem arises in the last paragraph of the proof of Lemma 7.1.2: There, it is claimed that if ϕ as in (i) is a state, then ϕ as in (ii) can also be chosen as a state. It is indeed true that, if ϕ as in (i) is positive, then so is ϕ as in (ii); unless G is compact, however, it is not clear why ϕ in (ii) should be non-zero.

    An alternative proof of (i) $\Longrightarrow$ (ii) of Theorem 7.1.3, which avoids the use of Følner type conditions, is as follows: Let $(f_\alpha)_\alpha$ be a net in $P(G)$ such that $\|\delta_g * f_\alpha - f_\alpha\|_1 \to 0$ uniformly on compact subsets of G. Let $\xi_\alpha := f_\alpha^{\frac{1}{2}}$, and $e_\alpha := \xi_\alpha * \check{\xi}_\alpha$. Then $(e_\alpha)_\alpha$ is a net in the unit sphere of $A(G)$ such that $e_\alpha \to 1$ uniformly on compact subsets of G. This, however, is already sufficient for $(e_\alpha)_\alpha$ to be a bounded approximate identity for $A(G)$ by Theorem B_2 of

    E. E. Granirer and M. Leinert, On some topologies which coincide on the unit sphere of the Fourier–Stieltjes algebra $B(G)$ and of the measure algebra $M(G)$. *Rocky Mountain J. Math.* **11** (1981), 459–472.

31. page 223, line 18: Replace “Problem 6” by “Problem 8”.

32. page 229: The displayed formula should read

$$\mathcal{K}(\sigma)\theta(a)\mathcal{K}(\sigma)^{-1} = \sigma(a) \qquad (\sigma \in U,\ a \in \mathfrak{A}).$$

33. page 252, line −8: Replace $f = \sum_{j=1}^{\infty} f_j x_j$ by $f = \sum_{j=1}^{n} f_j x_j$.

34. page 278, line 5: Replace $L_+^\infty(X,\mu)$ by $L_+^1(X,\mu)$.

Updates

1. A recent popular science book on the Banach–Tarski paradox is

   L. M. Wapner, *The Pea and The Sun.* A K Peters, 2005.

2. Problems 4 and 5 — even though Problem 5 appears to be a rather special case of Problem 4 — are in fact equivalent: Every reflexive, amenable Banach algebra is finite-dimensional (and then automatically semisimple) if and only if every weakly compact homomorphism from an amenable Banach algebra into another Banach algebra has finite rank. This is proven in:

   A. Blanco, S. Kaijser, and T. J. Ransford, Real interpolation of Banach algebras and factorization of weakly compact homomorphisms. *J. Funct. Anal.* **217** (2004), 126–141.

3. In

   S. A. Argyros and R. G. Haydon, A hereditarily indecomposable L_∞-space that solves the scalar-plus-compact problem. Preprint (2009), `arXiv:0903.3921 [math.FA]`,

   a solution to the "scalar plus compact" problem was presented. More precisely, the authors constructed a Banach space E with $E^* = \ell^1$ such that $\mathcal{L}(E) = \mathcal{K}(E) + \mathbb{C}\,\mathrm{id}$ (and thus $\mathcal{L}(E) = \mathcal{A}(E) + \mathbb{C}\,\mathrm{id}$). By Examples 2.1.12(b) and Theorem 3.1.10, E has property ($\mathbb{A}$), so that $\mathcal{A}(E)$ is amenable as is its unitization $\mathcal{L}(E)$. This—surprisingly!—gives a positive answer to Problem 8.

4. On the other hand, $\mathcal{L}(\ell^p)$ is not amenable for any $p \in (1,\infty) \setminus \{2\}$: this was shown in

   V. Runde, $\mathcal{B}(\ell^p)$ can never be amenable. *J. Amer. Math. Soc.* **23** (2010), 1175–1185.

   The proof combines methods from the paper below by N. Ozawa with reductions steps obtained in

   M. Daws and V. Runde, Can $\mathcal{B}(\ell^p)$ ever be amenable? *Studia Math.* **188** (2008), 151–174.

5. An alternative proof for the non-amenability of $\mathcal{L}(\ell^2)$ — one that simultaneously establishes the non-amenability of $\mathcal{L}(\ell^p)$ for $p = 1, 2, \infty$ — is given in

   N. Ozawa, A note on non-amenability of $\mathcal{B}(\ell_p)$ for $p = 1, 2$. *Internat. J. Math.* **15** (2004), 557–565.

   The author uses ideas by G. Pisier to modify and simplify Read's approach in [Rea 2]. (A closer inspection of his proof yields the non-amenability of $\mathcal{L}(c_0)$ as well.) The paper also contains a partial answer to Problem 26 (see below).

6. A self-contained exposition of the equivalence of injectivity (and thus Connes-amenability) and being approximately finite-dimensional for a von Neumann algebra (acting on a separable Hilbert space) is given in Chapter XVI of

   M. Takesaki, *Theory of Operator Algebras*, III. Encyclopedia of Mathematical Sciences **127**, Springer Verlag, 2003.

7. A recent thorough — yet very accessible — introduction to Banach space tensor products is

   R. A. Ryan, *Introduction to Tensor Products of Banach Spaces*. Springer Monographs in Mathematics, Springer Verlag, 2002.

8. Problem 14 was solved in

   B. E. Forrest and V. Runde, Amenability and weak amenability of the Fourier algebra. *Math. Z.* **250** (2005), 731–744.

   The conjectured answer is indeed true: A locally compact group has an amenable Fourier algebra if and only if it has an abelian subgroup of finite index.

9. Problem 18 (due to B. E. Johnson) has a positive solution as was shown by V. Losert: For every locally compact group G and for every derivation $D : L^1(G) \to L^1(G)$, there is $\mu \in M(G)$ such that $Df = f * \mu - \mu * f$ for all $f \in L^1(G)$. This solution is presented in:

   V. Losert, The derivation problem for group algebras. *Annals of Math.* **168** (2008), 221–246.

   Losert's theorem follows from a more general result on crossed homomorphisms that can also be used to answer Problem 17 affirmatively, as was done in

   Y. Choi, F. Ghahramani, and Y. Zhang, Approximate and pseudo-amenability of various classes of Banach algebras. *J. Funct. Anal.* **256** (2009), 3158–3191 and — independently — in

V. Losert, On derivations and crossed homomorphisms. In: R. J. Loy, V. Runde, and A. Sołtysiak (ed.s), *Banach Algebras 2009*, pp. 199–217. Polish Academy of Sciences, Banach Center Publications **91**, 2010.

10. A very recent — and stunningly simple — proof of Losert's theorem that every derivation on a group algebra is an inner derivation implemented by a measure was given in

    U. Bader, T. Gelander, and N. Monod, A fixed point theorem for L^1 spaces. Preprint (2010), `arXiv:1012.1488 [math.FA]`,

    as an application of a fixed point theorem in so-called *L-embedded* Banach spaces. That very same fixed point theorem also immediately yields a proof for the weak amenability of C^*-algebras that does not require the non-commutative Grothendieck inequality, thus answering Problem 19.

11. In

    M. Daws, Dual Banach algebras: representations and injectivity. *Studia Math.* **178** (2007), 231–275,

    a dual Banach algebra $\mathfrak{A}$ is defined to be injective if, for every reflexive Banach space E and every w^*-continuous homomorphism $\theta\colon \mathfrak{A} \to \mathcal{L}(E)$, there is a quasi-expectation $\mathcal{Q}\colon \mathcal{L}(E) \to Z_{\mathcal{L}(E)}(\theta(\mathfrak{A}))$. Daws shows that $\mathfrak{A}$ is injective if and only if it is Connes-amenable; in particular, this provides an affirmative answer to Problem 22.

12. Problem 23 was answered in the negative in

    V. Runde, A Connes-amenable, dual Banach algebra need not have a normal, virtual diagonal. *Trans. Amer. Math. Soc.* **358** (2006), 391–402.

    It is shown that, for a $[SIN]$-group G, the dual Banach algebra $\mathcal{WAP}(G)^*$ does not have a normal, virtual diagonal unless G is compact. In particular, $\mathcal{WAP}(G)^*$ is Connes-amenable without a normal, virtual diagonal for any amenable, but non-compact $[SIN]$-group.

13. Problem 24 (due to A. Ya. Helemskiĭ) was solved for infinite, discrete groups in the negative by S. Tabaldyev in

    S. Tabaldyev, Non-injectivity of the predual of the measure algebra for infinite discrete groups (in Russian). *Mat. Zametki* **73** (2003), 735–742. English tranlation: *Math. Notes* **73** (2003), 690–696.

More generally, the $M(G)$-bimodule $\mathcal{C}_0(G)$ is injective for a locally compact group G if and only if G is finite: this is shown in

> V. Runde, Dual Banach algebras: Connes-amenability, normal, virtual diagonals, and injectivity of the predual bimodule. *Math. Scand.* **95** (2004), 124–144.

The proof relies on recent work by H. G. Dales and M. Polyakov.

14. Using ideas of G. Pisier and from [Rea 2], N. Ozawa in his aforementioned paper gives a proof that simultaneously shows the non-amenability of $\prod_{n=1}^{\infty} \mathcal{L}(\ell_n^p)$ for *all* $p \in [1,\infty]$. Even more generally, if $(p_n)_{n=1}^{\infty}$ is *any* sequence in $[1,\infty]$, Ozawa's methods show that $\prod_{n=1}^{\infty} \mathcal{L}(\ell_n^{p_n})$ is not amenable (even though this is not explictly stated in his paper). This partially answers Problem 26.

15. For subalgebras of $\mathcal{L}(\mathfrak{H})$ generated by a single operator, an affirmative answer to Problem 30 was claimed to be given in

> D. R. Farenick, B. E. Forrest, and L. W. Marcoux, Amenable operators on Hilbert spaces. *J. reine angew. Math.* **582** (2005), 201–228.

However, as G. A. Willis points out in his review (MR2139716) of this paper, there is an error in Proposition 3.8. As a consequence, the proof of the main result (Theorem 1.3) collapses. Nevertheless, the following remains true:

> *Let $T \in \mathcal{L}(\mathfrak{H})$ be a* triangular *operator such that the Banach subalgebra of $\mathcal{L}(\mathfrak{H})$ generated by T is amenable. Then T is similar to a normal operator whose spectrum is Lavrentieff, i.e. has connected complement and no interior.*

In particular, Problem 30 still has an affirmative answer for subalgebras of $\mathcal{L}(\mathfrak{H})$ generated by a single *triangular* operator.

16. Problem 31 has been answered affirmatively in

> V. Runde, The operator amenability of uniform algebras. *Canadian Math. Bull.* **46** (2003), 632–634.

17. A partial answer to Problem 32 is given in

> V. Runde and N. Spronk, Operator amenability of Fourier–Stieltjes algebras. *Math. Proc. Cambridge Phil. Soc.* **136** (2004), 675–686.

In this paper, it is shown that a locally compact group G is compact whenever $B(G)$ is operator C-amenable with $C < 5$. On the other hand, the natural conjecture that $B(G)$ is operator amenable if and only if G is compact is wrong. In

> V. Runde and N. Spronk, Operator amenability of Fourier–Stieltjes algebras, II. *Bull. London Math. Soc.* **39** (2007), 194–202.

it is shown that there are non-compact (necessarily amenable) groups G for which $B(G)$ is operator 5-amenable.

18. In his doctoral thesis

> *Operatorfolgenräume. Eine Kategorie auf dem Weg von den Banach-Räumen zu den Operatorräumen.* Universität des Saarlandes, 2002.

under G. Wittstock's supervision, A. Lambert defines column and row operator spaces over arbitrary Banach spaces (over Hilbert spaces, his definitions yield the usual column and row Hilbertian operator spaces). This can be used to define an operator space structure over $A_p(G)$ for a locally compact group G and *arbitrary* $p \in (1, \infty)$ such that multiplication becomes completely bounded (even though not necessarily completely contractive). For this operator space structure, the following extension of Theorem 7.4.3 holds:

Theorem. *The following are equivalent for a locally compact group G:*

(i) *G is amenable.*

(ii) *$A(G)$ is operator amenable.*

(iii) *$A_p(G)$ is operator amenable for each $p \in (1, \infty)$.*

(iv) *There is $p \in (1, \infty)$ such that $A_p(G)$ is operator amenable.*

This result, which can be viewed as a solution to Problem 34, is contained in

> A. Lambert, M. Neufang, and V. Runde, Operator space structure and amenability for Figà-Talamanca–Herz algebras. *J. Funct. Anal.* **211** (2004), 245–269.

19. Reference [Ari 2] has appeared in: *J. Math. Sci. (New York)* **111** (2002), 3339–3386.

20. Reference [D–Gh–H] has appeared in: *J. London Math. Soc.* (2) **66** (2002), 213–226.

21. Reference [Gif] has appeared in: *J. Austral. Math. Soc.* **80** (2006), 297–315.

22. Reference [Rea 2] has appeared in: *J. Austral. Math. Soc.* **80** (2006), 317–333.

23. Reference [Run 3] has appeared in: *Arch. Math. (Basel)* **77** (2001), 265–272.

24. Reference [Run 4] has appeared in: *Studia Math.* **148** (2001), 47–66.

25. Reference [Run 5] has appeared as: Connes-amenability and normal, virtual diagonals, I. *J. London Math. Soc.* **67** (2003), 643–656.

26. Reference [Run 6] has appeared in: *Studia Math.* **155** (2003), 153–170.

27. Reference [Spr 1] has appeared in: *Proc. Amer. Math. Soc.* **130** (2002), 3609–3617.

28. An English language version of reference [Wit *et al.*] is available under

    `http://www.math.uni-sb.de/~ag-wittstock/projekt2001.html.`

29. Reference [Woo 2] has appeared in : *Canadian J. Math.* **54** (2002), 1100–1120.