MATH 217 (Fall 2017)  
Honors Advanced Calculus, I

Assignment #6

1. Let  
\[ f : \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto \begin{cases} 
  xy\frac{y^2-x^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\
  0, & \text{otherwise}.
\end{cases} \]

Show that \( f \) is twice partially differentiable everywhere, but that  
\[ \frac{\partial^2 f}{\partial y \partial x}(0,0) \neq \frac{\partial^2 f}{\partial x \partial y}(0,0). \]

Is \( f \) continuous at \((0,0)\)?

2. Let \( f : \mathbb{R} \to \mathbb{R} \) be twice continuously differentiable, let \( c > 0 \) and \( v \in \mathbb{R}^N \) be arbitrary, and let \( \omega := c \|v\| \). Show that  
\[ F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}, \quad (x,t) \mapsto f(x \cdot v - \omega t) \]
solves the wave equation  
\[ \Delta F - \frac{1}{c^2} \frac{\partial^2 F}{\partial t^2} = 0. \]

3. Determine the Jacobians of  
\[ \mathbb{R}^3 \to \mathbb{R}^3, \quad (r,\theta,\phi) \mapsto (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta) \]

and  
\[ \mathbb{R}^3 \to \mathbb{R}^3, \quad (r,\theta,z) \mapsto (r \cos \theta, r \sin \theta, z). \]

4. An \( N \times N \) matrix \( X \) is invertible if there is \( X^{-1} \in M_N(\mathbb{R}) \) such that \( XX^{-1} = X^{-1}X = I_N \) where \( I_N \) denotes the unit matrix.

(a) Show that \( U := \{ X \in M_N(\mathbb{R}) : X \text{ is invertible} \} \) is open. (Hint: \( X \in M_N(\mathbb{R}) \) is invertible if and only if \( \det X \neq 0 \).)

(b) Show that the map  
\[ f : U \to M_N(\mathbb{R}), \quad X \mapsto X^{-1} \]
is totally differentiable on \( U \), and calculate \( Df(X_0) \) for each \( X_0 \in U \). (Hint: You may use that, by Cramer’s Rule, \( f \) is continuous.)
5. Let
\[ p: (\mathbb{R} \setminus \{0\}) \times \mathbb{R} \to \mathbb{R}^2, \quad (r, \theta) \mapsto (r \cos \theta, r \sin \theta) \]
let, \( \emptyset \neq U \subset \mathbb{R}^2 \) be open, and let \( f: U \to \mathbb{R} \) be twice continuously partially differentiable. Show that
\[ (\Delta f) \circ p = \frac{\partial^2 (f \circ p)}{\partial r^2} + \frac{1}{r} \frac{\partial (f \circ p)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 (f \circ p)}{\partial \theta^2} \]
on \( p^{-1}(U) \). (Hint: Apply the chain rule twice.)

6. Let \( \emptyset \neq C \subset \mathbb{R}^N \) be open and connected, and let \( f: C \to \mathbb{R} \) be differentiable such that \( \nabla f \equiv 0 \). Show that \( f \) is constant. (Hint: First, treat the case where \( C \) is convex using the chain rule; then, for general \( C \), assume that \( f \) is not constant, let \( x, y \in C \) such that \( f(x) \neq f(y) \), and show that \( \{U, V\} \) with \( U := \{z \in C : f(z) = f(x)\} \) and \( V := \{z \in C : f(z) \neq f(x)\} \) is a disconnection for \( C \).

Due Monday, October 30, 2017, at 10:00 a.m.; no late assignments.