MATH 217 (Fall 2017)
Honors Advanced Calculus, I

Assignment #2

1. Let \(x, y \in \mathbb{R}\) with \(x < y\). Show that there is \(z \in \mathbb{R} \setminus \mathbb{Q}\) such that \(x < z < y\).

2. Show that \(\mathbb{Z}\) is closed in \(\mathbb{R}\), but not open, and that \(\mathbb{Q} \subset \mathbb{R}\) is neither open nor closed.

3. Let \(a_1, b_1, \ldots, a_N, b_N \in \mathbb{R}\) such that \(a_j < b_j\) for \(j = 1, \ldots, n\). Show that \((a_1, b_1) \times \cdots \times (a_N, b_N)\) is open and that \([a_1, b_1] \times \cdots \times [a_N, b_N]\) is closed in \(\mathbb{R}^N\).

4. For \(x = (x_1, \ldots, x_N) \in \mathbb{R}^N\), set

\[
\|x\|_1 := |x_1| + \cdots + |x_N| \quad \text{and} \quad \|x\|_\infty := \max\{|x_1|, \ldots, |x_N|\}.
\]

(a) Show that the following are true for \(j = 1, \infty, x, y \in \mathbb{R}^N\) and \(\lambda \in \mathbb{R}:

(i) \(\|x\|_j \geq 0\) and \(\|x\|_j = 0\) if and only if \(x = 0\);

(ii) \(\|\lambda x\|_j = |\lambda|\|x\|_j\);

(iii) \(\|x + y\|_j \leq \|x\|_j + \|y\|_j\).

(b) For \(N = 2\), sketch the sets of those \(x\) for which \(\|x\|_1 \leq 1\), \(\|x\| \leq 1\), and \(\|x\|_\infty \leq 1\).

(c) Show that

\[
\|x\|_1 \leq \sqrt{N}\|x\| \leq N\|x\|_\infty
\]

for all \(x \in \mathbb{R}^N\).

5. Let \(x, y \in \mathbb{R}^N\). Show that \(|x \cdot y| = \|x\|\|y\|\) holds if and only if \(x\) and \(y\) are linearly dependent.

6*. For any set \(S\), its power set \(\mathcal{P}(S)\) is defined to be the set consisting of all subsets of \(S\). Show that there is no surjective map from \(S\) to \(\mathcal{P}(S)\). (Hint: Assume that there is a surjective map \(f : S \to \mathcal{P}(S)\) and consider the set \(\{x \in S : x \notin f(x)\}\).)

Due Monday, September 25, 2017, at 10:00 a.m.; no late assignments.