EXAM1 - MC (white)

1. $$\lim_{(x,y)\to(0,0)} \frac{x^2\cos(y)}{x^2+y^2}$$

If $x=0,\ y\to 0^+$ then $f(x,y)=0 \to 0$

If $y=0,\ x\to 0^+$ then $f(x,y)=\dfrac{x^2\cos(0)}{\cancel{2}x^2} = 1 \to 1$

$\Big\}$ No limit (e)

2. $$3x^2+3z^2\frac{\partial z}{\partial x} + 6yz^2 + 6xy\cdot 2z\frac{\partial z}{\partial x} = 0$$

$\therefore$ at $(1,0,1)$: $\quad 3 + 3\dfrac{\partial z}{\partial x} = 0 \Rightarrow \dfrac{\partial z}{\partial x} = -1$ $\quad$ (c)

3. $$f(1.96,1.04) - f(2,1) = \Delta f \simeq df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$= \frac{1}{\cancel{2}}\frac{1}{(21-x^2-y^2)^{1/2}}(-\cancel{2}x)\,dx + \frac{1}{\cancel{2}}\frac{1}{(21-x^2-y^2)^{1/2}}(-\cancel{2}y)\,dy$$

$$= \frac{1}{4}(-2)(-.04) + \frac{1}{4}(-1)(.04)$$

$$= .02 - .01 = .01$$

$f(1.96,1.04) \simeq 4.01$ $\quad$ (d)

4. $$\vec{\nabla} f = \left(1+e^y\cos(xz)z\right)\vec{i} + \left(1+e^y\sin(xz)\right)\vec{j} + \left(1+e^y\cos(xz)x\right)\vec{k}\,\Big|_{(1,2,0)}$$

$$= \vec{i} + \vec{j} + (1+e^2)\vec{k}$$

directional derivative $= \left(\vec{i}+\vec{j}+(1+e^2)\vec{k}\right)\cdot\dfrac{(3,-3,c)}{\sqrt{9+9+c^2}} = 0$

$\therefore c(1+e^2) = 0 \Rightarrow c = 0$ $\quad \boxed{(c)}$

Exam1 LA-1 (White)

$\frac{\partial f}{\partial x} = 5y - 2xy - y^2$ $\qquad$ $\frac{\partial f}{\partial y} = -4 - x^2 - 2xy + 2y + 5x$

$f_{xx} = -2y$ $\qquad$ $f_{yy} = -2x+2$ $\qquad$ $f_{xy} = -2x - 2y + 5$

So critical points:

$5y - 2xy - y^2 = 0 \implies y = 0$ or $5 - 2x - y = 0$

$y=0$: $-4 - x^2 - 2xy + 2y + 5x = 0 \rightarrow 5x - 4 - x^2 = 0$

$-(x-4)(x-1) = 0$

$\therefore x = 4, x = 1$

$y = -2x+5$: $\rightarrow 5x - 4 - x^2 + 2y(1-x) = 0$

$5x - 4 - x^2 + (1-x)(5-2x)2 = 0$

$5x - 4 - x^2 + [5 - 5x - 2x + 2x^2]2 = 0$

$3x^2 - 9x + 6 = 0$

$x^2 - 3x + 2 = 0 \quad (x-2)(x-1) = 0$

$x = 1, x = 2$

$\therefore$

So critical pts: $(1,0), (4,0)$
$(1,3), (2,1)$

Thus $(1,0)$ $\quad f_{xx}f_{yy} - f_{xy}^2 < 0$ $\quad$ Saddle

$(4,0)$ $\quad < 0$ $\quad$ Saddle

$(1,3)$ $\quad = (-6)(0) - (5-2-6)^2 < 0$ $\quad$ Saddle

$(2,1)$ $\quad = \underbrace{(-2)}_{<0}(-2) - (-1)^2 > 0$ $\quad$ Max.

LA2 WHITE

Let $F(x,y,z) = \sqrt{4x} + \sqrt{y} + \sqrt{z}$.

The normal vector to S at $P(x_0, y_0, z_0)$ is:

$\vec{N} = \nabla F(P) = \langle \frac{1}{\sqrt{x_0}}, \frac{1}{2\sqrt{y_0}}, \frac{1}{2\sqrt{z_0}} \rangle$

The EQ. of tangent plane to S at P:

(a) $\frac{1}{\sqrt{x_0}}(x - x_0) + \frac{1}{2\sqrt{y_0}}(y - y_0) + \frac{1}{2\sqrt{z_0}}(z - z_0) = 0$

OR $\frac{x}{\sqrt{x_0}} + \frac{y}{2\sqrt{y_0}} + \frac{z}{2\sqrt{z_0}} = \sqrt{x_0} + \frac{\sqrt{y_0}}{2} + \frac{\sqrt{z_0}}{2}$

NOTE: $\sqrt{x_0} + \frac{\sqrt{y_0}}{2} + \frac{\sqrt{z_0}}{2} = \frac{\sqrt{C}}{2}$

(b) $y = z = 0 \Rightarrow$ $x_1 = \sqrt{x_0}\frac{\sqrt{C}}{2} \Rightarrow \boxed{4x_1 = 2\sqrt{x_0}\sqrt{C}}$

$x = z = 0 \Rightarrow$ $\boxed{y_1 = \sqrt{y_0} \cdot \sqrt{C}}$

$x = y = 0 \Rightarrow$ $z_1 = \sqrt{z_0}\sqrt{C}$

Hence, $4x_1 + y_1 + z_1 = \sqrt{4x_0}\sqrt{C} + \sqrt{y_0}\sqrt{C} + \sqrt{z_0}\sqrt{C}$

$= \sqrt{C}\sqrt{C} = C$, does not depend on P

LA3 **Problem** (white). Find the maximum and minimum values of $f(x, y, z) = xy + z^2$ on the sphere $x^2 + y^2 + z^2 = 2$ and the points at which they are attained.

Solution: Set $g(x, y, z) = x^2 + y^2 + z^2$, so that

$$\nabla f = \langle y, x, 2z \rangle \quad \text{and} \quad \nabla g = \langle 2x, 2y, 2z \rangle.$$

We need to find x, y, z, and λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, i.e.,

$$y = \lambda 2x, \tag{1}$$
$$x = \lambda 2y, \tag{2}$$
$$2z = \lambda 2z, \tag{3}$$

and

$$x^2 + y^2 + z^2 = 2. \tag{4}$$

If $x = 0$, then $y = 0$ by (1), so that $z = \pm\sqrt{2}$ by (4). Evaluating f at $(0, 0, \pm\sqrt{2})$, we obtain the value 2 at both points.

Suppose that $x \neq 0$. Plugging (1) into (2), we obtain $x = 4\lambda^2 x$ and thus $\lambda = \pm\frac{1}{2}$. From (3), we conclude that $2z = \pm z$ and thus $z = 0$. By (1), $y = \pm x$. Plugging into (4) yields $2x^2 = 2$, i.e., $x = \pm 1$. As $y = \pm x$, we thus have to test f at the points $(1, 1, 0)$, $(-1, -1, 0)$, $(1, -1, 0)$, and $(-1, 1, 0)$; the value of f at the first two of those points in 1, and it is -1 and the two last ones.

Hence, the maximum value is 2, attained at $(0, 0, \pm\sqrt{2})$, and the minimum value is -1, attained at $(\pm 1, \mp 1, 0)$.