1. Let \( y \) solve \( e^t(y - t) dt + (1 + e^t) dy = 0 \) with \( y(0) = -\frac{1}{2} \). Then \( y \) at \( t = 1 \) equals 0.

Solution: The equation is exact. Integrating \( 1 + e^t \) with respect to \( y \) yields
\[
F(t, y) = y(1 + e^t) + g(t).
\]
Differentiating with respect to \( t \) yields
\[
e^t(y - t) = \frac{\partial F}{\partial t} = ye^t + g'(t)
\]
and thus
\[
g'(t) = -te^t.
\]
Pick \( g(t) = -te^t + e^t \), so that
\[
F(t, y) = y(1 + e^t) - te^t + e^t.
\]
As \( F(0, -\frac{1}{2}) = -\frac{1}{2}2 + 1 = 0 \), we obtain
\[
0 = F(1, y(1)) = y(1)(1 + e) - e + e
\]
and thus \( y(1) = 0 \).

2. If \( \frac{dy}{dx} = (x + y + 1)^2 - (x + y - 1)^2 \) and \( y \left(-\frac{1}{4}\right) = 0 \), then \( y \) equals \( -\frac{1}{2} \) at \( x = \frac{1}{4} \).

Solution: Substitute \( z = x + y \), so that \( \frac{dy}{dx} = \frac{dz}{dx} - 1 \) and thus
\[
\frac{dz}{dx} - 1 = (z + 1)^2 - (z - 1)^2 = 4z
\]
This equation is separable. Integrating, we obtain
\[
\frac{1}{4} \log |4z + 1| = x + C
\]
and thus
\[
4z + 1 = C e^{4x}
\]
For \( x = -\frac{1}{4} \) and \( y = 0 \), i.e., \( z = -\frac{1}{4} \), the left hand side vanishes, so that \( C = 0 \). We conclude that
\[
y = -x - \frac{1}{4},
\]
and therefore \( y \left(\frac{1}{4}\right) = -\frac{1}{2} \).
3. Let \( y \) solve \( xy' = y(\ln y - \ln x + 1) \) with \( y(1) = 2 \). Then \( y \) at \( x = 4 \) equals 64.

**Solution:** The equation is homomogeneous

\[
y' = \frac{y}{x} \left( \ln \frac{y}{x} + 1 \right).
\]

Substitute \( v := \frac{y}{x} \), so that

\[
v + x \frac{dv}{dx} = v(\ln v + 1)
\]

and thus

\[
\frac{dx}{x} = \frac{dv}{v \ln v}.
\]

Integrating yields

\[
\ln |\ln v| = \ln |x| + C
\]

and taking the exponential, we obtain

\[
\ln v = Cx
\]

Using the intial condition, we obtain that \( C = \ln 2 \) and conclude that

\[
y = xe^{\ln 2} = 2x.
\]

It follows that \( y(4) = 64 \).

4. If the Laplace transform of \( f \) is \(-\frac{1}{s^2(s^2+1)}\) then \( f \) at \( t = \pi \) equals \(-\pi\).

**Solution:** Find \( A, B \) and \( C \) such that

\[
-\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 1},
\]

i.e.,

\[
-1 = As(s^2 + 1) + B(s^2 + 1) + (Cs + D)s^2 = (A + C)s^3 + (B + D)s^2 + As + B.
\]

Comparing coefficients yields \( B = -1, A = 0, D = 1, \) and \( C = 0 \), so that

\[
-\frac{1}{s^2(s^2+1)} = -\frac{1}{s^2} + \frac{1}{s^2 + 1}.
\]

Its inverse Laplace transform is

\[
\mathcal{L}^{-1} \left\{-\frac{1}{s^2(s^2+1)}\right\} (t) = -t + \sin t.
\]

Its value \( t = \pi \) is \(-\pi\).
5. If $y$ solves $x^2y'' + 5xy' + 5y = 0$ and $y(1) = 1$ and $y'(1) = -1$, then $y$ at $x = e^\pi$ equals $e^{-2\pi}$.

**Solution:** The differential equation is Cauchy–Euler with characteristic equation

$$0 = r^2 + 4r + 5 = (r + 2)^2 + 1$$

which has the complex conjugate roots $r = -2 \pm i$. Two linearly independent solutions are therefore

$$y_1(t) = t^{-2} \cos(\ln t) \quad \text{and} \quad y_2(t) = t^{-2} \sin(\ln t)$$

Let $c_1$ and $c_2$ be such that $y = c_1y_1 + c_2y_2$, letting $t = 1$, we see immediately that $c_1 = 1$. Differentiating $y$, we get

$$y'(t) = t^{-2} (-\frac{\sin(\ln t)}{t} + c_2 \frac{\cos(\ln t)}{t}) - 2t^{-3} (\cos(\ln t) + c_2 \sin(\ln t))$$

Letting $t = 1$, yields

$$-1 = c_2 - 2,$$

i.e., $c_2 = 1$, and thus

$$y(e^\pi) = -e^{-2\pi}$$

6. Let $x$ solve $x'' - 2x' + x = 24t^2 e^t$ with $x(0) = x'(0) = 0$. Then $x$ equals $\frac{1}{6} e^{\frac{3}{2}}$ at $t = \frac{1}{2}$.

**Solution:** The auxiliary equation is $(r - 1)^2 = 0$, i.e., it has 1 as double root. Use undetermined coefficients to find a particular solution. As $r = 1$ is a double root, we need to find $A$, $B$, and $C$ such that

$$x_p(t) = (At^4 + Bt^3 + Ct^2)e^t$$

Differentiating twice, we obtain

$$x_p'(t) = (At^4 + Bt^3 + Ct^2)e^t + (4At^3 + 3Bt^2 + 2C)t e^t$$

and

$$x_p''(t) = (At^4 + (4A + B)t^3 + (3B + C)t^2 + 2C)t e^t$$

$$+ (4At^3 + (12A + 3B)t^2 + (6B + 2C)t + 2C)e^t$$

$$= (At^4 + (8A + B)t^3 + (12A + 6B + C)t^2 + (6B + 4C)t + 2C)e^t.$$}

By inspection, we get $C = B = 0$ and $A = 2$, so that $x_p(t) = 2t^4 e^t$. It is obvious that $x_p$ satisfies the initial condition, and evaluating it at $t = \frac{1}{2}$ yields $\frac{1}{6} e^{\frac{3}{2}}$. 

3
Long Answer Problems

1. Use Laplace transforms to solve the initial value problem

\[ y'' + y = h(t); \quad y(0) = 1, \quad y'(0) = 4 \]

where

\[ h(t) = \begin{cases} 
2t, & 0 \leq t < 6, \\
12, & t \geq 6.
\end{cases} \]

**Solution:** Set \( Y(s) := \mathcal{L}\{y\}(s) \), so that \( \mathcal{L}\{y''\}(s) = s^2Y(s) - s - 4 \). Set \( H(s) := \mathcal{L}\{h\}(s) \), and note that

\[ h(t) = 2t + (12 - 2t)u(t - 6). \]

It follows that

\[ H(s) = \frac{2}{s^2} - e^{-6s}\mathcal{L}\{2t\}(s) = \frac{2}{s^2}(1 - e^{-6s}). \]

The initial value problem thus becomes

\[ s^2Y(s) - s - 4 + Y(s) = \frac{2}{s^2}(1 - e^{-6s}) \]

Solving for \( Y(s) \), we obtain

\[
Y(s) = \frac{1}{s^2+1} \left( \frac{2}{s^2}(1 - e^{-6s}) + s + 4 \right) \\
= 2 \left( \frac{1}{s^2} - \frac{1}{s^2+1} \right) (1 - e^{-6s}) + \frac{s}{s^2+1} + \frac{4}{s^2+1} \\
= \frac{2}{s^2} - e^{-6s} \frac{2}{s^2} + \frac{2}{s^2+1} + e^{-6s} \frac{2}{s^2+1} + \frac{s}{s^2+1}.
\]

Applying the inverse Laplace transform yields

\[ y(t) = 2t - 2(t - 6)u(t - 6) + 2 \sin t + 2 \sin(t - 6)u(t - 6) + \cos t. \]

2. Find the general solution to \( y'' - 6y' + 9y = t^{-3}e^{3t} \).

**Solution:** The auxiliary equation is \((r - 3)^2 = 0\), i.e., has a double real root. Two linearly independent solutions of the associated homogeneous equation are

\[ y_1(t) = e^{3t} \quad \text{and} \quad y_2(t) = te^{3t}. \]

To find a particular solution, use variation of parameters: solve

\[ v_1'(t)e^{3t} + v_2'(t)te^{3t} = 0 \]

and

\[ 3v_1'(t)e^{3t} + v_2'(t)(1 + 3t)e^{3t} = t^{-3}e^{3t} \]


Multiplying the first equation by 3 and subtracting it from the second yields.

\[ v_2'(t)e^{3t} = t^{-3}e^{3t} \]

and thus \( v_2'(t) = t^{-3} \). Plugging into the first equation (and dividing by \( e^{3t} \)) we obtain \( v_1'(t) = -t^{-2} \). Choose

\[ v_1(t) = t^{-1} \quad \text{and} \quad v_2(t) = -\frac{t^{-2}}{2}. \]

Then

\[ y_p(t) = \frac{e^{3t}}{t} - \frac{e^{3t}}{2t} = \frac{e^{3t}}{2t}. \]

A general solution is thus of the form

\[ y(t) = \frac{e^{3t}}{2t} + c_1e^{3t} + c_2te^{3t}. \]