Problem 1. Applying the Laplace transform methods, solve the heat conduction problem on a half-line:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \ t > 0, \\
u(0, t) = u_0 \cos \omega t \ (t > 0), \\
u(x, 0) = 0 \ (x > 0),
\]

and the solution should be bounded at \(x = +\infty\). Check if your solution satisfies the boundary condition.

Solution: Applying the Laplace transform, we arrive at

\[
\hat{u}(s, 0) = \frac{u_0 s}{s^2 + \omega^2}, \\
x \to \infty : \ |\hat{u}| < \infty,
\]

which yields

\[
\hat{u}(s, x) = \frac{u_0 s}{s^2 + \omega^2} e^{-\sqrt{\sqrt{s}} x}.
\]
Thus, we just need to find the inverse Laplace transform
\[ u(t, x) = \frac{1}{2\pi i} \int_L \frac{u_0s}{s^2 + \omega^2} e^{-\sqrt{s}x} e^{st} ds. \]

The integrand has two simple poles at \( s = \pm i\omega \) and because of the presence of a square root \( \sqrt{s} \) requires a branch cut, which most conveniently can be done along the negative real axis, as shown in figure 1. To apply the Cauchy theorem, let us choose the contour shown in the same figure 1 and analyze all the portions of the contour integral
\[
\int_L + \int_{\Gamma_1} + \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_0} + \int_{\Gamma_2} = 2\pi i(\text{res}\ s = -i\omega + \text{res}\ s = i\omega).
\]

By Jordan’s lemma, \( \int_{\Gamma_1} = \int_{\Gamma_2} = 0 \), while by direct estimation it can be shown that \( \int_{\gamma_0} \to 0 \) as \( \epsilon \to 0 \), where \( \epsilon \) is the radius of the circle-contour \( \gamma_0 \). Next,
\[
\int_{\gamma_1} = \int_0^{+\infty} u_0 \frac{r e^{-i\sqrt{r}x} e^{-rt}}{\omega^2 + r^2} dr,
\]
\[
\int_{\gamma_2} = \int_0^{+\infty} u_0 \frac{r e^{i\sqrt{r}x} e^{-rt}}{\omega^2 + r^2} dr,
\]
where we introduced \( r = -s \) and took into account that \( \sqrt{s} \) is discontinuous at the branch cut with \( s = re^{i\pi} \) and thus \( \sqrt{s} = i\sqrt{r} \) at \( \gamma_1 \), while \( s = re^{-i\pi} \) and thus \( \sqrt{s} = -i\sqrt{r} \) at \( \gamma_2 \). Since the residues at \( s = \pm i\omega \) are given by
\[
\text{res}\ s = -i\omega \lim_{s \to -i\omega} (s + i\omega)f(s) = \frac{u_0}{2} e^{-\omega t} e^{-i\pi/4\sqrt{\omega}x},
\]
\[
\text{res}\ s = i\omega \lim_{s \to i\omega} (s - i\omega)f(s) = \frac{u_0}{2} e^{i\omega t} e^{-i\pi/4\sqrt{\omega}x},
\]
then the integral over \( L \) is determined by
\[
\int_L = -\int_{\gamma_1} - \int_{\gamma_2} + 2\pi i(\text{res}\ s = -i\omega + \text{res}\ s = i\omega).
\]
As a result, the solution is
\[
u(t, x) = u_0 e^{-\sqrt{\omega^2}x} \cos \left( \omega t - \sqrt{\frac{\omega^2}{2}} x \right) - \frac{u_0}{\pi} \int_0^{+\infty} \frac{re^{-rt}}{\omega^2 + r^2} \sin \sqrt{r}x dr,
\]
which obviously satisfies the boundary condition at \( x = 0 \).

Alternatively, the solution can be constructed in terms of convolutions. Since
\[
\mathcal{L}\{u_0 \cos \omega t\} = \frac{u_0s}{s^2 + \omega^2} \quad \text{and} \quad \mathcal{L}\left\{ \frac{x}{2\sqrt{\pi t^3}} \exp \left( -\frac{x^2}{4t} \right) \right\} = e^{-\sqrt{\pi}x},
\]
the solution reads
\[ u(x, t) = (u_0 \cos \omega t) \ast \left[ \frac{x}{2\sqrt{\pi t^3}} \exp \left( -\frac{x^2}{4t} \right) \right] = u_0 \frac{x}{2\sqrt{\pi}} \int_0^t \cos(\omega(t - \tau)\tau^{-3/2}e^{-\frac{x^2}{4\tau}} d\tau, \]
or, with a new variable of integration
\[ u(x, t) = \frac{2u_0}{\sqrt{\pi}} \int_{\frac{1}{2}x^2}^\infty \cos \left[ \omega \left( t - \frac{x^2}{4\lambda^2} \right) \right] e^{-\lambda^2} d\lambda. \]
Check the boundary condition at \( x = 0 \):
\[ u(0, t) = \frac{2u_0}{\sqrt{\pi}} \int_0^\infty \cos \omega t e^{-\lambda^2} d\lambda = u_0 \cos \omega t. \]

**Problem 2.** Find a solution of the Laplace equation
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad x > 0, \quad y > 0, \]
\[ u(x, 0) = 0(x > 0), \]
\[ \frac{\partial u}{\partial x}(0, y) = \begin{cases} -q & (0 < y < b), \\ 0 & (y > b), \end{cases} \]
where \( q = \text{const} \). Moreover, find the magnitude of the flow \( \frac{\partial u}{\partial y}(x, 0) \) through the horizontal boundary of the domain, \( y = 0 \).

**Solution:** The particular solution satisfying \( u(x, 0) = 0 \) is \( e^{-\lambda x} \sin \lambda y \), where \( \lambda > 0 \), if the solution is sought in the class of bounded functions. Therefore, it is possible to find the solution in the form
\[ u(x, y) = \int_0^\infty A(\lambda) e^{-\lambda x} \sin \lambda y \, d\lambda, \]
which can be seen as the Laplace transform of \( A(\lambda) \sin \lambda y \), where \( A(\lambda) \) is the function to be determined. From the boundary condition at \( x = 0 \) by the inversion we find
\[ A(\lambda) \lambda = \frac{2}{\pi} \int_0^b q \sin \lambda y \, dy = \frac{2q(1 - \cos b\lambda)}{\pi \lambda}, \]
and hence
\[ u(x, y) = \frac{2q}{\pi} \int_0^\infty (1 - \cos b\lambda) \frac{1}{\lambda^2} e^{-\lambda x} \sin \lambda y \, d\lambda, \]
and
\[ \frac{\partial u}{\partial y}(x, 0) = \frac{2q}{\pi} \int_0^\infty \frac{(1 - \cos b\lambda)}{\lambda^2} e^{-\lambda x} \, d\lambda = \frac{q}{\pi} \ln \left( 1 + \frac{b^2}{x^2} \right). \]
Problem 3. Consider a homogeneous horizontal beam of mass $M$ and length $L$, which is clamped at both ends, i.e. the boundary conditions are $y(0) = y'(0) = y(L) = y'(L) = 0$, and subjected to a transverse point load $Q$ at the distance $2L/3$ from the left end. If the beam deflection $y(x)$ is governed by

$$EIy^{(iv)} = f(x),$$

where the constants $E$ and $I$ are Young’s modulus and the moment of inertia, respectively, $f(x)$ is the transverse loading, solve the problem using Laplace transform.

Solution. Extending the problem from the domain $[0, L]$ to $[0, +\infty)$ we can use the Laplace transform to solve the problem. Then the transverse load can be extended as

$$f(x) = \begin{cases} \frac{Mg}{L} + Q\delta\left(x - \frac{2L}{3}\right), & 0 \leq x \leq L, \\ 0, & x > L. \end{cases}$$

The Laplace transform then gives

$$EIs^4Y - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = F(s), \quad F(s) = \frac{Mg}{sL} + Qe^{-\frac{2s}{3L}} - \frac{e^{-sL}Mg}{sL},$$

where $y''(0) \equiv A$ and $y'''(0) \equiv B$ are not known. Using the boundary conditions at the left end of the beam, we find

$$EIs^4Y = As + B + \frac{Mg}{L} \frac{1}{s} + Qe^{-\frac{2s}{3L}} - \frac{Mg e^{-sL}}{sL}.$$}

Inverting, we get

$$EIy(x) = A\frac{x^2}{2} + B\frac{x^3}{3!} + \frac{Mg}{4!} \frac{x^4}{L} + QH\left(x - \frac{2L}{3}\right) \frac{(x - 2L/3)^3}{3!} - \frac{Mg}{L} H(x - L) \frac{(x - L)^4}{4!},$$

where $H$ is the Heaviside step-function. Applying the boundary conditions at $x = L$ we have

$$EIy(L) = \frac{AL^2}{2} + \frac{BL^3}{6} + \frac{MgL^3}{24} + \frac{QL^3}{162} = 0,$$

$$EIy'(L) = AL + \frac{BL^2}{2} + \frac{MgL^2}{6} + \frac{QL^2}{18} = 0,$$

which yields

$$A = \frac{MgL}{12} + \frac{2QL}{27}, \quad B = -\frac{Mg}{2} - \frac{7Q}{27}. $$
Problem 4. Using the Fourier transform solve
\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \pm k^2 \psi = H(x, y), \quad (x, y) \in \mathbb{R}^2.
\]

Solution. Applying the Fourier transform in both \(x\) and \(y\), we get
\[
-(\xi^2 + \eta^2) \hat{\psi} \pm k^2 \hat{\psi} = \hat{H}, \quad \hat{\psi}(\xi, \eta) = \iint \psi(x, y) e^{-i\xi x - i\eta y} \, dx \, dy.
\]
Taking inverse FT and using the FT formula for \(\hat{H}(\xi, \eta)\):
\[
\hat{H}(\xi, \eta) = \iint H(x', y') e^{-i\xi x' - i\eta y'} \, dx' \, dy',
\]
we get the solution of the original problem:
\[
\psi(x, y) = \frac{1}{4\pi^2} \iint H(x', y') \, dx' \, dy' \iint \frac{e^{i\xi(x-x') + i\eta(y-y')}}{\pm k^2 - (\xi^2 + \eta^2)} \, d\xi \, d\eta.
\]
Denoting the last two integrals as a Green’s function
\[
G(x, y; x', y') = \frac{1}{4\pi^2} \iint e^{i\rho R \cos \phi} \, d\rho \, d\phi,
\]
it can be written in polar coordinates as
\[
G(x, y; x', y') = \frac{1}{4\pi^2} \int_0^\infty \int_0^{2\pi} e^{i\rho R \cos \phi} \frac{1}{\pm k^2 - \rho^2} \, d\rho \, d\phi,
\]
where
\[
\rho = i\xi + j\eta, \quad \rho = |\rho|,
\]
\[
R = i(x - x') + j(y - y'), \quad R = |R|.
\]
Since \(\frac{1}{2\pi} \int_0^{2\pi} e^{i\rho R \cos \phi} \, d\phi = J_0(\rho R)\) with \(J_0\) being Bessel’s function of zero order, we get
\[
G(x, y; x', y') = \frac{1}{4\pi^2} \int_0^\infty J_0(\rho R) \rho \, d\rho \begin{cases} -\frac{1}{2\pi} K_0(-ikR) = -\frac{i}{4} H_0^{(1)}(kR) & \text{for } +k^2, \\ -\frac{1}{2\pi} K_0(kR) & \text{for } -k^2, \end{cases}
\]
where \(K_0\) is the modified Bessel function and \(H_0\) is the Henkel function.
Problem 5. Using the Fourier transform method solve

\[ \phi_{xx} + \phi_{zz} = 0, \]

\[ z = 0 : \quad \phi_t + g\phi_z = 0, \]

\[ t = 0 : \quad \phi = -\frac{\varepsilon}{\rho} \delta(x), \phi_t = 0, \]

\[ z = -\infty : \phi_z = 0, \]

where \( \delta(x) \) is the Dirac delta-function and one can assume coefficients \( g, p, \rho \) to be constant. Determine the quantity

\[ \int_0^t \phi_z|_{z=0} \, dt, \]

and find its expression in terms of the Fresnel integrals.

Solution. Applying FT in \( x \) to the Laplace equation we get \( \hat{\phi}_{zz} - |k|^2 \hat{\phi} = 0 \), the relevant solution of which is

\[ \hat{\phi}(t; k, z) = C(t; k)e^{\sqrt{|k|z}}, \]

substitution of which into the boundary condition at \( z = 0 \) yields

\[ C_{tt} + g|k|C = 0, \]

\[ t = 0 : \quad C(0; k) = -\frac{p}{\rho}, \quad C_t(0; k) = 0. \]

The latter problem has the solution

\[ C(t; k) = -\frac{p}{\rho} \cos \sqrt{g|k|t}, \]

which defines the solution for \( \hat{\phi} \). Taking into account the symmetry \( k \to -k \) of \( \hat{\phi} \):

\[ \phi(t; x, z) = \frac{1}{\pi} \int_0^{+\infty} \hat{\phi}(t; k, z) \cos kx \, dk. \]

Finally, evaluation of the required quantity produces

\[ \int_0^t \phi_z|_{z=0} \, dt = -\frac{p}{\pi \rho g} \frac{\partial^2}{\partial t^2} J, \]

where \( J \), after introduction of an auxiliary variable \( \eta = \sqrt{gk} \), is

\[ J = \int_0^{+\infty} \left\{ \sin \left( \frac{\eta^2 x}{g} + \eta t \right) - \sin \left( \frac{\eta^2 x}{g} - \eta t \right) \right\} \, d\eta \]

\[ = \sqrt{\frac{2\pi g}{x}} \left[ C(u) \sin \frac{\pi u^2}{2} - S(u) \cos \frac{\pi u^2}{2} \right], \]
where \( u = \frac{t}{2} \sqrt{\frac{2}{\pi}} \) and

\[
C(u) = \int_0^u \cos \frac{\pi}{2} \alpha^2 \, d\alpha, \quad S(u) = \int_0^u \sin \frac{\pi}{2} \alpha^2 \, d\alpha,
\]

are Fresnel integrals.

**Problem 6.** Find asymptotic approximation of the probability integral

\[
\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt
\]

for \( x \to +\infty \).

**Solution.** First note that

\[
\Phi(x) = 1 - \frac{2}{\sqrt{\pi}} F(x), \quad F(x) = \int_x^\infty e^{-t^2} \, dt.
\]

Letting \( t = x \tau \) and then \( \tau^2 = 1 + u \), we obtain the integral of the Laplace type:

\[
F(x) = \frac{1}{2} x e^{-x^2} \int_0^\infty e^{-x^2 u} (1 + u)^{-1/2} \, du,
\]

the asymptotics of which can be studied with the theorem proved in the class. As a result,

\[
\Phi(x) = 1 - \frac{1}{x \sqrt{\pi}} e^{-x^2} + \frac{1}{2 x^3 \sqrt{\pi}} e^{-x^2} + \text{h.o.t.}
\]

**Problem 7.** Find asymptotic approximation of

\[
\Phi(x) = \int_x^\infty e^{it^2} \, dt
\]

for \( x \to +\infty \).

**Solution.** Integrating by parts,

\[
\Phi(x) = e^{ix^2} \left( \frac{i}{2x} + \frac{1}{4x^3} \right) + O(x^{-5}).
\]

**Problem 8.** Find asymptotic approximation of the Bessel function

\[
J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(x \sin \phi - n\phi)} \, d\phi
\]

for \( x \to +\infty \).
Solution. The phase \( \sin \phi \) has two stationary points \( \phi_1 = \frac{\pi}{2} \) and \( \phi_2 = \frac{3\pi}{2} \), which gives

\[
J_n(x) = \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{n\pi}{2} - \frac{\pi}{4} \right) + O(x^{-1}), \quad x \to +\infty.
\]

Problem 9. Find asymptotic approximation of

\[
I(n) = \int_0^{\pi/2} \sin^n x \, dx
\]

for \( n \to \infty \).

Solution. Rewriting the integral as

\[
I(n) = \int_0^{\pi/2} \exp (n \ln \sin x) \, dx
\]

we find stationary points

\[
f'(x) = \frac{n}{\sin x} \cos x = n \cot x = 0 \Rightarrow x_n = \frac{\pi}{2} + \pi n,
\]

from which only \( x_0 \) belongs to the interval of integration. Since \( f''(x_0) = -n \sin^{-2} x_0 = -n \), then

\[
f(x) \simeq -\frac{1}{2} n \left( x - \frac{\pi}{2} \right)^2,
\]

but because the stationary point is exactly at the end of the integral of integration and thus only half of the Gaussian bell belongs to the interval of integration, we must divide the result by 2:

\[
I(n) \simeq \frac{1}{2} \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} n \left( x - \frac{\pi}{2} \right)^2 \right) \, dx = \sqrt{\frac{\pi}{2n}}.
\]

Problem 10. Find asymptotic behavior of

\[
I(a, x) = \int_0^x \exp (a \cdot \sin t) \, dt
\]

for \( a \gg 1 \) and \( x \leq a \).

Solution. There are infinite number of stationary points defined from

\[
f'(t) = a \cos t = 0 \Rightarrow t_n = \frac{\pi}{2} + \pi n,
\]
half of which are local maximums \( f''(t_n) = -a \sin t_n = (-1)^n a \) corresponding to 
\( t_{2n} = \frac{\pi}{2} + 2\pi n \). Contribution of each stationary point is

\[
I_{2n} \simeq \int_{-\infty}^{+\infty} \exp \left( a - \frac{a}{2} (t - t_{2n}^2) \right) dt \simeq \sqrt{\frac{2\pi}{a}} e^a.
\]

As the limit of integration \( x \) in the original integral increases, more stationary points contribute to the value of the integral with the contribution of each stationary point being the same, i.e. upon reaching \( t_{2n} \) the value of the integral abruptly increases by 
\( \sim \sqrt{\frac{2\pi}{a}} e^a \). Thus, the plot of \( I(a, x) \) behaves as a staircase. Note that for large \( x \) the errors of approximation \( \sqrt{\frac{2\pi}{a}} e^a \cdot O(\frac{1}{a}) \) accumulate and cannot be neglected.

**Problem 11.** Find asymptotic behavior of

\[
F(x) = \int_{-\infty}^{+\infty} \cos xt \frac{dt}{\cosh t \sqrt{t^2 + \pi^2}}
\]

for \( x \to \infty \).

![Figure 2: Contour of integration.](image_url)

**Solution.** Let us write

\[
F(x) = \int_{-\infty}^{+\infty} \frac{e^{ixt}}{\cosh t} \frac{dt}{\sqrt{t^2 + \pi^2}}
\]

and notice that the integrand has a pole at \( t = \pi i/2 \) and branch point at \( t = \pi i \).

The quantity

\[
I(x) = \int_{-\infty}^{+\infty} \frac{e^{ixt}}{\cosh t} \frac{dt}{\sqrt{t^2 + \pi^2}} - \int_{\pi i - \infty}^{\pi i + \infty} \frac{e^{ixt}}{\cosh t} \frac{dt}{\sqrt{t^2 + \pi^2}}
\]

can be considered as an integral over a closed contour shown in figure 2. Inside the strip \( 0 < \text{Im} t < \pi \) bounded by this contour, the integrand has a simple pole at
\[ t = \pi i/2 \] with the residue equal to \( e^{-\pi x/\pi i \sqrt{3}} \), so that from the residue theorem
\[ I(x) = \frac{4}{\sqrt{3}} e^{-\frac{\pi}{2} x}. \]

Since
\[ \int_{\pi i - \infty}^{\pi i + \infty} \frac{e^{ixt}}{\cosh t} \frac{dt}{\sqrt{t^2 + \pi^2}} = -e^{-\pi x} \int_{-\infty}^{+\infty} \frac{e^{ixu}}{\cosh u} \frac{du}{\sqrt{u(u + 2\pi i)}} = O(e^{-\pi x}), \quad x \to +\infty, \]
where \( u = t - \pi i \), so that
\[ F(x) = \frac{4}{\sqrt{3}} e^{-\frac{\pi}{2} x} + O(e^{-\pi x}), \quad x \to +\infty. \]

**Problem 12.** Determine asymptotic behavior of the integral
\[ \zeta(r, t) = \text{Re} \int_0^{+\infty} J_0(kr) e^{-i\sqrt{g} t} \, k \, dk \]
for sufficiently large \( r \) and \( t \). \( J_0(kr) \) is the Bessel function of zeroth order.

**Solution.** Since for small \( k \) the integrand tends to zero, the interval of integration should be away from the origin. Taking into account that
\[ J_0(kr) \simeq \sqrt{\frac{2}{\pi kr}} \sin \left( kr + \frac{\pi}{4} \right), \]
the integral can be represented as
\[ \zeta(r, t) = \text{Re} \frac{1}{2i} \sqrt{\frac{2}{\pi r}} \int_0^{+\infty} \left[ e^{i(kr - t\sqrt{g} k + \frac{\pi}{4})} - e^{-i(kr + t\sqrt{g} k + \frac{\pi}{4})} \right] \sqrt{k} \, dk. \]
The integral from the second term in the square brackets does not contribute since the expression \( kr + t\sqrt{g} k \) does not have a maximum or minimum for \( r > 0, t > 0 \). On the other hand, from the first term in square brackets
\[ \frac{\partial}{\partial k} \left( kr - t\sqrt{g} k \right) = 0 \Rightarrow k_1 = \frac{g t^2}{4 r^2}, \quad \frac{\partial^2}{\partial k^2} \left( kr - t\sqrt{g} k \right) \bigg|_{k=k_1} = \frac{1}{4} \frac{g^{1/2} t}{k_1^{3/2}}, \]
so that
\[ \zeta(r, t) \simeq \text{Re} \frac{1}{2i} \sqrt{\frac{2}{\pi r}} \sqrt{k_1} \sqrt{\frac{8\pi k_1^{3/2}}{g^{1/2} t}} e^{i(k_1 r - t\sqrt{2\pi k_1} + \frac{\pi}{4})} = \frac{g t^2}{2^{3/2} r^3} \cos \frac{g t^2}{4 r}. \]
Problem 13. Under which limit, $\omega \to 0$ or $\omega \to \infty$, does the function

$$\delta_\omega(x) = \frac{1}{\pi} \frac{\sin^2(\omega x)}{\omega x^2}$$

approximate the Dirac delta-function.

Solution. The region where the function $\delta_\omega(x)$ substantially differs from zero has the width $\sim \omega^{-1}$ and therefore the integral

$$I = \int \delta_\omega(x)f(x)dx$$

is determined in the neighborhood of the origin as $\omega \to \infty$:

$$I \simeq \int_{-\infty}^{+\infty} \delta_\omega(x)f(0)dx = f(0) \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin^2(\omega x)}{\omega x^2}dx,$$

where

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin^2(\omega x)}{\omega x^2}dx = 1$$

is the sign-integral calculated from the observation that it can be reduced to the Dirichlet integral

$$S(a) = \int_{-\infty}^{+\infty} \frac{\sin^2(ax)}{x^2}dx \Rightarrow \frac{dS(a)}{da} = \int_{-\infty}^{+\infty} \frac{\sin 2ax}{x}dx = \pi,$$

and then integrating with respect to the parameter $a$: $S(a) = \pi a + \text{const}$ with $\text{const} = 0$ due to $S(0) = 0$. Altogether, since

$$\lim_{\omega \to \infty} \int_{-\infty}^{+\infty} \delta_\omega(x)f(x)dx = f(0)$$

than $\lim_{\omega \to \infty} \delta_\omega(x) = \delta(x)$.

Note. The Dirichlet integral can be calculated with the help of Laplace transform

$$\int_0^{+\infty} \frac{\sin t}{t}dt = \int_0^{+\infty} \mathcal{L}\{\sin t\}(s)ds = \int_0^{+\infty} \frac{1}{1+s^2}ds = \arctan s\big|_0^{+\infty} = \frac{\pi}{2}.$$

Problem 14. Find asymptotic approximation of the cylindrical function

$$I_n(\lambda) = \frac{1}{2\pi i} \int_{|z|=1} e^{\frac{\lambda}{2}(z-\frac{1}{z})} \frac{dz}{z^{n+1}}, \ n \in \mathbb{N}$$
for $\lambda \to +\infty$.

Solution. There are two saddle points $z_{1,2} = \pm i$ (of the same level with $\text{Re} f(z) = 0$) at which the function $f(z) = \frac{1}{2} \left( z - \frac{1}{z} \right)$ assumes $f(\pm i) = \pm i$ and $|f''(\pm i)| = 1$. Level sets $u = \text{Re} f(z) = \frac{z}{2} \left( 1 - \frac{1}{x^2+y^2} \right) = 0$ consist of the circle $|z| = 1$ and straight line $x = 0$. This geometry of $f(z)$ dictates the contour deformation as in figure 3. Given

$\phi(z) = 1/z^{n+1}$ and its values at the saddle points $\phi(\pm i) = \pm i \exp(\pm in\pi/2)$, the resulting value of the integral is

$$I_n(\lambda) \sim \frac{1}{\sqrt{\pi \lambda}} \cos \left( \lambda - n \frac{\pi}{2} - \frac{\pi}{4} \right).$$

**Problem 15.** Find asymptotic approximation of

$$I(\lambda) = \int_0^1 \log t e^{i\lambda t} dt,$$

for $\lambda \to +\infty$ using the method of steepest descent.

Solution. Let

$$I(\lambda) = \int_C e^{\lambda h(z)} f(z) \, dz, \quad f(z) = \log z, \quad h(z) = iz = i(x-y).$$

There are no saddle points. The stepeast descent paths are given by $\text{Im} h(z) = x = \text{const}$. For $f(z) = \log z$ we take a branch cut along the negative real axis. Using
Cauchy’s theorem for the contour in figure 4 we can write
\[ \int_0^1 \log t \, e^{i\lambda t} \, dt = - \int_{C_1 + C_2 + C_3} \log z \, e^{i\lambda z} \, dz. \]

With \( z = it \):

\[
\lim_{R \to \infty} \int_{C_3} e^{\lambda h(z)} f(z) \, dz = - \lim_{R \to \infty} \int_0^R e^{-\lambda t} \log (it) \, i \, dt = - \lim_{R \to \infty} \int_0^R e^{-\lambda t} \left( \log t + \frac{i\pi}{2} \right) \, i \, dt
\]
\[= \frac{i \log \lambda}{\lambda} + \frac{i \gamma}{\lambda} + \frac{\pi}{2\lambda}, \]

where \( \gamma = - \int_0^\infty \log x \, e^{-x} \, dx \) is the Euler-Mascheroni constant. Next, with \( z = t + iR \), we have

\[
\int_{C_2} e^{\lambda h(z)} f(z) \, dz = - \int_0^1 \log (t + iR) e^{i\lambda(t + iR)} \, dt = -e^{-\lambda R} \int_0^1 e^{i\lambda t} \log (t + iR) \, dt \to 0 \text{ as } R \to \infty.
\]

Similarly evaluating the integral over the contour \( C_1 \) with \( z = 1 + it \) and \( R \to +\infty \), we find

\[
\int_{C_1} e^{\lambda h(z)} f(z) \, dz = \int_0^\infty \log (1 + it) e^{i\lambda(1 + it)} \, i \, dt = i e^{i\lambda} \int_0^\infty e^{-\lambda t} \log (1 + it) \, dt =
\]
\[= i e^{i\lambda} \int_0^\infty e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(-it)^n}{n} \, dt = e^{i\lambda} \sum_{n=1}^{\infty} \frac{(-1)^n i^{n+1} \Gamma(n)}{\lambda^{n+1}}, \]

Figure 4: Contour of integration.
where we took into account that
\[ \log(1 + it) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(it)^n}{n}, \quad \Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt, \quad \Gamma(n+1) = n\Gamma(n). \]

As a result,
\[
I(\lambda) = -\frac{i \log \lambda}{\lambda} - \frac{i \gamma}{\lambda} - \frac{\pi}{2\lambda} + e^{i\lambda} \sum_{n=1}^{\infty} \frac{(-1)^{n} i^{n+1} \Gamma(n)}{\lambda^{n+1}}.
\]