Homework 3
(due at 11:00 pm on March 27, 2018)

Problem 1. Solve

\[ x^2 + \epsilon x - 1 = 0, \ \epsilon \ll 1, \]

up to \(O(\epsilon^2)\) and compare with exact solutions (provide plots).

Problem 2. Solve

\[ \epsilon x^2 + x - 1 = 0, \ \epsilon \ll 1, \]

up to \(O(\epsilon^2)\) and compare with exact solutions (provide plots).

Problem 3. Solve

\[ \cos x = \epsilon \sin (x + \epsilon), \ \epsilon \ll 1, \]

up to \(O(\epsilon)\).

Problem 4. Solve

(a) \( y'' + \epsilon y^2 = 0, \) with \( y(0) = 1, \ y'(0) = 0, \)

(b) \( y'' + \epsilon y^2 = 0, \) with \( y(0) = 1, \ y'(0) = \epsilon, \)

for \( \epsilon \ll 1 \) up to \(O(\epsilon^2)\) and compare with exact solutions (provide plots).

Problem 5. Find the amplitude of the limit cycle oscillations\(^1\) of the van der Pol equation

\[ \ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0, \] \( x(0) = A, \ \dot{x}(0) = 0, \ \epsilon \ll 1, \)

using the method of strained coordinates. Here \( A \) is the amplitude and is considered to be an adjustable parameter in this problem.

Problem 6. Using the method of multiple scales solve

\[ \ddot{y} + (1 + \delta + \epsilon \cos 2t)y = 0, \ \epsilon \ll 1, \ \delta = \epsilon \delta_1, \]

where \( \delta_1 = O(1). \)

\(^1\)A limit cycle is a closed trajectory in phase space \((x, \dot{x})\) having the property that at least one other trajectory spirals into it either as time approaches infinity or as time approaches negative infinity.
Problem 7. Using the method of matched asymptotic expansion, solve

$$\epsilon \ddot{y} - \dot{y} + y = 0, \text{ with } y(0) = 0, y(1) = 1, \epsilon \ll 1.$$ 

Compare with the exact solution.

Problem 8. Using the method of matched asymptotic expansion, solve

$$\epsilon \ddot{y} + \dot{y} = 2x, \text{ with } y(0) = 1, y(1) = 1, \epsilon \ll 1.$$ 

Problem 9. Using WKBJ method, solve

$$\epsilon^2 \ddot{y} + xy = 0, \text{ for } x > 0, \epsilon \ll 1.$$ 

Problem 10. Using WKBJ method, solve

$$x^3 \ddot{y} = y,$$

for small positive $x$.

Problem 11. Deduce the averaged version of Newton’s second law, when a particle $\mathbf{x} \in \mathbb{R}^n$ is subject to a time-dependent force of the following form

$$\ddot{\mathbf{x}} = a(t, \epsilon) \mathbf{f}(\mathbf{x}), \epsilon \ll 1,$$

where the scalar ‘acceleration’ $a$ is rapidly oscillating

$$a(t, \epsilon) = e^\alpha A(t/\epsilon), \alpha > -2,$$

i.e. $A(\tau + 1) = A(\tau) = O(1)$ is a periodic function.