Final Exam
(9:00 am – 12:00 pm on April 15, 2016)

Problem 1. [15] Consider the following initial-value problem on $x \in \mathbb{R}$:

\[ \begin{align*}
    u_t &= u_{xxx}, \\
    u(x,0) &= f(x).
\end{align*} \tag{1} \]

(a) [5] Construct the solution of (1) with the help of Fourier transform and express it in terms of Fourier harmonics.

(b) [10] Determine the asymptotic behavior of this solution as $t \to \infty$ with $x/t = v > 0$, i.e. in the reference frame moving with velocity $v$ in the positive $x$-direction.

Problem 2. [20] Consider the problem of free oscillations with small damping written in the dimensional form

\[ \begin{align*}
    \ddot{x} + 2\beta \dot{x} + \omega_0^2 x &= 0, \\
    x(0) &= A, \quad \dot{x}(0) = 0.
\end{align*} \tag{2} \]

(a) [5] Identify what are the two time scales in the problem, introduce two corresponding non-dimensional time variables, and define a dimensionless small parameter as a ratio of the two dimensional time scales.

(b) [15] Construct the leading order solution of (2) using multiple-scale approach. \textit{Hint}: you may need to consider the first three approximations.

Problem 3. [15] Construct the leading-order solution of the following boundary-value problem for the second-order nonlinear equation with the help of matched asymptotic analysis:

\[ \begin{align*}
    \dddot{y} + \frac{20}{\cos x} \dot{y} + 10 y^2 &= 0, \\
    y(0) &= 1, \quad y(1) = 1.
\end{align*} \tag{3} \]

\footnote{The maximum score for the final exam is 50.}
Bonus problems²

Problem 4. [20] Consider the Cauchy-type integral

\[ \Phi(z) = \frac{1}{2\pi i} \int_L \frac{\phi(\tau)}{\tau - z} \, d\tau, \quad (4) \]

where \( \phi(\tau) \) satisfies the Hölder condition, i.e. \( |\phi(t_2) - \phi(t_1)| < A |t_2 - t_1|^{\lambda}, \) with \( t \in L, \ A > 0 \) and \( 0 < \lambda \leq 1, \) and the contour \( L \) is smooth and closed. Determine the jump \( \Phi^+(t) - \Phi^-(t) \) in the value of \( \Phi(z) \) when the point \( z \) tends to the point on the contour \( t \in L \) from inside yielding \( \Phi^+(t) \) and from outside yielding \( \Phi^-(t), \) respectively.

Problem 5. [15] Consider the vertical (along the Earth radius) motion of an object with initial conditions \( x(0) = 0 \) and \( \dot{x}(0) = v_0, \) where \( x(t) \) is the object position relative to the Earth surface.

(a) [1] Formulate Newton’s second law in the dimensional form in terms of \( x(t). \) As a reminder, Newton’s law of gravity is \( F = -\gamma m M/r^2, \) where \( \gamma \) is the gravitational constant and \( r \) is the distance between the centers of mass of two bodies with masses \( m \) and \( M. \)

(b) [4] Non-dimensionalize the problem and identify the non-dimensional small parameter \( \epsilon \) in the problem.

(c) [10] Determine the maximum height achieved by the object and the time it takes to get to this height using perturbation theory including the correction of \( O(\epsilon). \)

²Bonus Problems 4 and 5 are optional – only one of them will be counted towards your final exam score, which cannot exceed 50.