Homework 2

(due at 2:00 pm on May 4, 2009)

Problem 1. *Double-diffusive convection.*

- Derive the governing equations for an infinite fluid layer of thickness d (analogous to the considered in the class Rayleigh-Benard convection), which is heated and salted below. Assume the Boussinesq approximation for the density dependence on temperature T and salt concentration S, i.e. $\rho = \rho_0(1 - \alpha T + \beta S)$.
- Determine a base state, write equations for the disturbance field, and non-dimensionalize the resulting equations.
- Demonstrate that the linear operator is non-self-adjoint. Discuss implications of this fact for the stability of the base state.

Solution. Consider a two-dimensional problem with z being a vertical coordinate and x along the layer (with v and u velocity components, respectively). Since the base state is just linear profiles for temperature T and concentration S, the evolution equations for a perturbation become

$$Pr^{-1}\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = -\nabla p + (R_T\theta - R_C c)\mathbf{k} + \nabla^2\mathbf{u}, \tag{1a}$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta - v = \nabla^2\theta, \tag{1b}$$

$$\frac{\partial c}{\partial t} + (\mathbf{u}\cdot\nabla)c - v = Le\nabla^2 c, \tag{1c}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1d}$$

with the boundary conditions at the top and bottom rigid boundaries

$$z = 0, 1: \ \ \theta = c = 0, \ \mathbf{u} = \mathbf{0}. \tag{2}$$

The linear part of the above system can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & R_T & 0 \\ 0 & 0 & 0 & 0 & R_C \end{pmatrix}\begin{pmatrix} p \\ u \\ v \\ \theta \\ c \end{pmatrix} = L\begin{pmatrix} p \\ u \\ v \\ \theta \\ c \end{pmatrix}, \tag{3}$$

with

$$L = \begin{pmatrix} 0 & -\partial_x & -\partial_y & 0 & 0 \\ -\partial_x & \nabla^2 & 0 & 0 & 0 \\ -\partial_y & 0 & \nabla^2 & R_T & 0 \\ 0 & 0 & R_T & R_T\nabla^2 & 0 \\ 0 & 0 & 0 & 0 & LeR_C\nabla^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -R_C \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_C & 0 & 0 \end{pmatrix}, \qquad (4)$$

where symmetric and skew-symmetric components of the linear operator are distinguished. Thus, for $R_C > 0$, the linear operator L is not self-adjoint and thus the principle of exchange of stabilities does not apply.

Problem 2. Apply the Rayleigh criterion to the base state of the viscous Couette flow between rotating cylinders.

Solution. The base state is given by

$$\Omega(r) = A(\mu, \eta) + B(\mu, \eta)/r^2,$$

where

$$A = \Omega_1 \frac{\mu - \eta^2}{1 - \eta^2},\ B = \Omega_1 R_1^2 \frac{1 - \mu}{1 - \eta^2},\ \mu = \Omega_2/\Omega_1,\ \eta = R_1/R_2,$$

so that the Rayleigh discriminant is $\Phi = 4A\Omega$. Then, if the cylinders rotate in the same direction, the Rayleigh discriminant changes its sign at $\mu = \eta^2$, which is the stability boundary (Rayleigh line).

Problem 3. *Oscillations of a rotating liquid column.* Consider a cylindrical column of liquid of radius R_0 rotating about its axis with a constant angular velocity Ω_0. Using Lagrangian approach (cf. §15.1 of D&R) find the frequencies of oscillation of the rotating liquid column.

Solution. The frequencies of oscillations, $\omega = \mathrm{Im}(\lambda)$, are given by

$$\frac{\omega}{\Omega_0} = -n \pm \frac{2}{\sqrt{1 + \alpha^2/a^2}},$$

where $a = kR_0$ and α is any root of the equation $\alpha J_n'(\alpha) \pm n\sqrt{1 + \alpha^2/a^2}\, J_n(\alpha) = 0$.

Problem 4. Why is the Fjortoft theorem a stronger condition than the Rayleigh theorem? Give an example.

Solution. The Fjortoft theorem a stronger condition than the Rayleigh theorem because there can be $U'' = 0$ at some z_s (which is a necessary condition for instability according to the Rayleigh theorem), but $U''(U - U_s) \geq 0$, i.e. the flow must be stable according to the Fjortoft theorem.