Homework 3

(due at 11:00 am on March 11, 2014)

Problem 1. Let $\langle x, y\rangle$ be an inner product and $\alpha \in \mathbb{C}$. Prove that

$$\langle \alpha x, y\rangle = \overline{\alpha}\,\langle x, y\rangle\,,$$

where $\overline{\alpha}$ is a complex conjugate of α.

Problem 2. Consider $\mathrm{d}/\mathrm{d}x$ as a linear operator acting on all functions f in $C[0,1]$ such that $f'(t)$ exists at each $t \in (0,1)$.

(a) How to define the domain D of this operator such that the operator maps a subset of $C[0,1]$ into $C[0,1]$? *Hint*: appeal to uniform continuity.

(b) Is this operator bounded or unbounded? Prove your assertion.

Problem 3. Using affine transforms, find the solution for a submerged jet (liquid ejected from a pipe in the space filled with the same liquid) in a half-plane $x > 0$, $-\infty < y < +\infty$:

$$\begin{aligned} u\,u_x + v\,u_y &= u_{yy},\\ u_x + v_y &= 0,\\ |u| \to 0,&\ \ y \to \pm\infty. \end{aligned}$$

Here (u, v) is the velocity field with (x, y)-components, respectively.

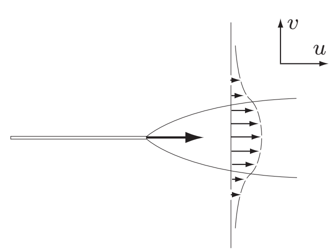

Figure 1: Submerged jet.

Hint: introduce a stream-function ψ: $u = \psi_y$, $v = -\psi_x$ and assume symmetry w.r.t. $y = 0$.

Problem 4. Using affine transforms, find the solution for an axisymmetric drop spreading on a flat surface, cf. figure 2, described by the following equation

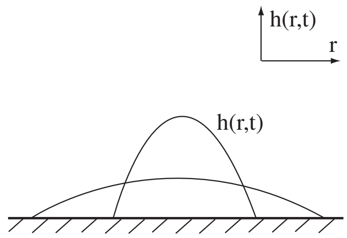

Figure 2: Spreading drop.

$$\frac{\partial h}{\partial t} = \frac{2}{3r}\frac{\partial}{\partial r}\left(rh^3\frac{\partial h}{\partial r}\right),$$

with the boundary condition $h = 0$ at $r = \infty$ and a mass conservation condition, i.e. mass of the drop should be constant.

Problem 5. Find the Fourier transformation of the following functions

1. $-2xe^{-x^2}$
2. $e^{-a\,|x|}$
3. $-x^2e^{-ax^2}$
4. $f(x) = \begin{cases} 1 - |x|, & |x| < 1, \\ 0, & |x| \geq 1 \end{cases}$

Problem 6. Prove that

$$\xi(x, a) * \xi(x, b) = 2\xi(x, a + b),$$

where $\xi(x, \alpha) = (\pi\alpha)^{-1/2}e^{-x^2/(4\alpha)}$ is a heat kernel and $*$ is the convolution sign.

Problem 7. Using the Fourier transform solve

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} \pm k^2\psi = H(x, y),$$
$$x^2 + y^2 \to \infty,$$

by constructing the appropriate Green's function.

Hint. An explicit representation of Green's function should be given in terms of the modified Bessel functions.

Problem 8. Using the Fourier transform method solve

$$
\begin{aligned}
&\psi_{xx} + \psi_{zz} = 0, \\
&z = 0 : \left|\begin{array}{r} \phi_{tt} + g\phi_z = 0, \\ t = 0 :\ \phi = -\frac{p}{\rho}\delta(x), \phi_t = 0, \end{array}\right. \\
&z = -\infty : \phi_z = 0,
\end{aligned}
$$

where $\delta(x)$ is the Dirac delta-function and one can assume coefficients g, p, ρ to be constant. Determine the quantity

$$
\int_0^t \phi_z\big|_{z=0}\, \mathrm{d}t,
$$

and find its expression in terms of the Fresnel integrals.