Deterministic and Stochastic Structured Population Models with Application to Green Tree Frogs

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Qihua Huang

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Deterministic and Stochastic Structured Population Models with Application to Green Tree Frogs

Qihua Huang

APPROVED:

Azmy S. Ackleh, Co-chair
Professor of Mathematics

Keng Deng, Co-chair
Professor of Mathematics

Christo I. Christov
Professor of Mathematics

Ping Wong Ng
Assistant Professor of Mathematics

Carolyn Bruder
Dean of the Graduate School
DEDICATION

To my dear family.
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Recently, major declines and even extinctions in some amphibian populations around the world have been reported. There has been much discussion in the literature about causes and general nature of the reported declines or extinctions. There is now recognition of the need for long-term monitoring of amphibian populations. The University of Louisiana at Lafayette and the United States Geological Survey National Wetlands Research Center have initiated a project in partnership to monitor and model frog population at the National Wetlands Research Center/Estuarine Habitat and Coastal Fisheries Center research complex, with an initial focus on *Hyla cinerea* (Schneider) (Green Tree Frog).

In this dissertation, we consider the dynamics of an amphibian population which is divided into two groups: juveniles (tadpoles) and adults (frogs). We assume that juveniles are structured by their age while adults are structured by their size (since often in such population adults become sexually mature when they reach a certain length).

The dynamics of the above population is represented by a system of first order hyperbolic partial differential equations in Chapter 1. An explicit finite differential approximation to this partial differential equation system is developed. Existence and uniqueness of the weak solution to the model are established and convergence of the finite difference approximation to this unique solution is proved.

In Chapter 2, we derive several stochastic models based on the above
deterministic model. Numerical simulation results of the stochastic models are compared with the solution of the deterministic model. These models are then used to understand the effect of demographic stochasticity on the dynamics of an urban green tree frog (*Hyla cinerea*) population.

In Chapter 3, an infinite-dimensional least-squares technique is developed for identifying unknown parameters of the deterministic model and its convergence results are established. We compare the mathematical population model to the statistical population estimates obtained from the field data. Numerical results for the model sensitivity with respect to these parameters are given. Furthermore, the above-mentioned parameter estimates are used to illustrate the long-time behavior of the population under investigation.

Finally, in Chapter 4 we develop a dispersal model for amphibian population which extends the deterministic model developed in Chapter 1 as it models the population in multi-ponds where individual disperse between ponds. An implicit finite difference approximation is developed to establish the existence-uniqueness of the weak solution to the model.
In this chapter\textsuperscript{1}, we consider an amphibian population where individuals are divided into two groups: juveniles (tadpoles) and adults (frogs). We assume that juveniles are structured by age and adults are structured by size. Since juveniles (tadpoles) live in water and adults (frogs) live on land we assume that competition occurs within stage only. This leads to a system of nonlinear and nonlocal hyperbolic equations of first order. An explicit finite difference approximation to this partial differential equation system is developed. Existence and uniqueness of the weak solution to the model are established and convergence of the finite difference approximation to this unique solution is proved.

1.1 Introduction

In this chapter, we consider the dynamics of an amphibian population divided into two groups 1) juveniles (tadpoles) and 2) adults (frogs). We assume that juveniles are structured by their age while adults are structured by their size (since often in such population adults become sexually mature when they reach a certain length, e.g., see [18] for the green tree frogs). Let $J(a, t)$ be the density of juveniles of age $a \in [0, a_{\text{max}}]$ at time $t \in [0, T]$ and $A(x, t)$ be the density of adults having size $x \in [x_{\text{min}}, x_{\text{max}}]$ at

\textsuperscript{1}The results of this chapter have been published by AMS Contemporary Mathematics Series [513 (2010), 1-23].
time $t \in [0, T]$. Here, $a_{\text{max}}$ denotes the age at which a juvenile (tadpole) metamorphoses into a frog ($a_{\text{max}}$ approximately equals five weeks for the green treefrog [8, 12, 13, 16]), and $x_{\text{min}}$ and $x_{\text{max}}$ denote the minimum size and the maximum size of a frog, respectively (green treefrog 15mm to 60mm [14]). Let $P(t) = \int_0^{a_{\text{max}}} J(a, t) da$ be the total number of juveniles in the population at time $t$ and $Q(t) = \int_{x_{\text{min}}}^{x_{\text{max}}} A(x, t) dx$ be the total number of adults in the population at time $t$. The function $\nu(a, t, P)$ denotes the mortality rate of a juvenile of age $a$ at time $t$ which depends on the number of tadpoles $P$ due to competition for resources. The function $\mu$ represents the mortality rate of an adult of size $x$, $g$ represents the growth rate of an adult of size $x$, and $\beta$ represents the reproduction rate of an adult of size $x$. The adult vital rates depend on $t$ due to seasonality of such populations and depend on the total number of adults (frogs) due to competition for resources. However, they do not depend on the total number of juveniles, since juveniles live in water while adults live on land they do not compete for resources. We represent the dynamics of the above population by the following system of first order hyperbolic partial differential equations:

\begin{align}
J_t + J_a + \nu(a, t, P(t))J &= 0, \quad (a, t) \in (0, a_{\text{max}}) \times (0, T), \\
A_t + (g(x, t, Q(t))A)_x + \mu(x, t, Q(t))A &= 0, \quad (x, t) \in (x_{\text{min}}, x_{\text{max}}) \times (0, T), \\
J(0, t) &= \int_{x_{\text{min}}}^{x_{\text{max}}} \beta(x, t, Q(t))A(x, t) dx, \quad t \in (0, T), \\
g(x_{\text{min}}, t, Q(t))A(x_{\text{min}}, t) &= J(a_{\text{max}}, t), \quad t \in (0, T), \\
J(a, 0) &= J^0(a), \quad a \in [0, a_{\text{max}}], \\
A(x, 0) &= A^0(x), \quad x \in [x_{\text{min}}, x_{\text{max}}].
\end{align}

The above model extends the model we developed in [2] as it allows for the mortality rate $\nu$ to depend on $P$ and more importantly allows the growth rate function $g$ to be a function of $Q$. The approach used in [2] to establish the existence of a weak solution is...
in the spirit of those used in [3, 4, 5] and relies on developing a comparison principle and utilizing this principle to construct a monotone sequence of linear partial differential equations. It is then shown that the limit of this sequence is a weak solution for the original problem. This approach does not apply for the above quasilinear system due to the dependency of the growth rate function $g$ on the total population of adults (frogs) $Q$. Thus, here we apply a totally different approach which is in the spirit of the one initially used in [9, 17] for conservation laws and later extended to nonlocal first order hyperbolic initial-boundary value problems arising in population ecology [1, 6, 7]. In [1, 6, 7] the authors used an implicit finite difference scheme to solve a partial differential equation describing the dynamics of a single population. Here we develop an explicit finite difference scheme to solve a system of partial differential equations. In general, explicit schemes are computationally more practical and faster schemes for such problems (e.g., see [15]).

Autonomous continuous structured juvenile-adult models have been developed and studied in the literature. For example, in [10] the authors study a semilinear juvenile-adult model where both juveniles and adults are age-structured. They tackle the question of whether juvenile versus adult intra-specific competition is stabilizing or destabilizing. It is shown that suppressed adult fertility due to juvenile competition is destabilizing in that equilibrium levels are lowered and equilibrium resilience is weakened. However, the effect of increased juvenile mortality due to adult competition is complicated because when equilibrium levels are lowered the resilience can be weakened or strengthened. In [11] the authors consider a nonlinear size-structured
juvenile-adult model. They study the linearized dynamical behavior of stationary solutions using semigroup theory. However, the approaches discussed above do not apply to models with time-dependent parameters as in (1.1.1).

This chapter is organized as follows. In Section 1.2, we define a weak solution of (1.1.1) and develop an explicit finite difference approximation to the solution. In Section 1.3 we establish some estimates for this approximation. In Section 1.4, we prove the existence of a weak solution of (1.1.1). Finally, uniqueness of the weak solution of (1.1.1) is the topic of Section 1.5.

1.2 Weak solution and finite difference approximation

Throughout the discussion we let $D_1 = [0, a_{\text{max}}] \times [0, T] \times [0, \infty)$, $D_2 = [x_{\text{min}}, x_{\text{max}}] \times [0, T] \times [0, \infty)$, and $\omega_1$ be a sufficiently large positive constant. We assume that the parameters in (1.1.1) satisfy the following assumptions:

(H1) $\nu(a, t, P)$ is a nonnegative bounded total variation function with respect to $a$ (uniformly in $t$ and $P$) and continuously differentiable with respect to $t$ and $P$.

Furthermore, $\sup_{(a, t, P) \in D_1} \nu(a, t, P) \leq \omega_1$.

(H2) $g(x, t, Q)$ is twice continuously differentiable with respect to $x$ and continuously differentiable with respect to $t$ and $Q$, $g_x$ is continuously differentiable with respect to $Q$ and $g(x, t, Q) > 0$ for $x \in [x_{\text{min}}, x_{\text{max}}]$, $g(x_{\text{max}}, t, Q) = 0$.

Furthermore, $\sup_{(x, t, Q) \in D_2} g(x, t, Q) \leq \omega_1$.

(H3) $\mu(x, t, Q)$ is a nonnegative bounded total variation function with respect to $x$ (uniformly in $t$ and $Q$) and continuously differentiable with respect to $t$ and $Q$. 

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Furthermore, \( \sup_{(x,t,Q) \in \mathcal{D}_2} \mu(x,t,Q) \leq \omega_1 \).

(H4) \( \beta(x,t,Q) \) is a nonnegative bounded total variation function with respect to \( x \) (uniformly in \( t \) and \( Q \)) and continuously differentiable with respect to \( t \) and \( Q \).

Furthermore, \( \sup_{(x,t,Q) \in \mathcal{D}_2} \beta(x,t,Q) \leq \omega_1 \).

(H5) \( J^0 \in BV[0,a_{\max}] \) and \( J^0(a) \geq 0 \).

(H6) \( A^0 \in BV[x_{\min},x_{\max}] \) and \( A^0(x) \geq 0 \).

Multiplying the first and second equations in (1.1.1) by \( \varphi(a,t) \) and \( \psi(x,t) \), respectively, and then formally integrating by parts and utilizing the initial and boundary conditions, we define a weak solution of (1.1.1) as follows:

**Definition 1.1.** A set \( (J,A) \in BV([0,a_{\max}] \times [0,T]) \times BV([x_{\min},x_{\max}] \times [0,T]) \) is called a weak solution to problem (1.1.1) if this set satisfies the following:

\[
\int_0^{a_{\max}} J(a,t)\varphi(a,t)da - \int_0^{a_{\max}} J^0(a)\varphi(a,0)da \\
= \int_0^t \int_0^{a_{\max}} J(\varphi_s + \varphi_a - \nu\varphi)da ds \\
+ \int_0^t \varphi(0,s) \int_{x_{\min}}^{x_{\max}} \beta(x,s,Q(s))A(x,s)dx ds \\
- \int_0^t \varphi(a_{\max},s)J(a_{\max},s)ds,
\]

(1.2.1)

\[
\int_{x_{\min}}^{x_{\max}} A(x,t)\psi(x,t)dx - \int_{x_{\min}}^{x_{\max}} A^0(x)\psi(x,0)dx \\
= \int_0^t \int_{x_{\min}}^{x_{\max}} A(\psi_s + g\psi_x - \mu\psi)dx ds + \int_0^t J(a_{\max},s)\psi(x_{\min},s)ds
\]

for every test function \( \varphi \in C^1([0,a_{\max}] \times (0,T)) \) and \( \psi \in C^1((x_{\min},x_{\max}) \times (0,T)) \) and \( t \in [0,T] \).
We divide the intervals \([0, a_{\text{max}}], [x_{\text{min}}, x_{\text{max}}]\) and \([0, T]\) into \(m\), \(n\) and \(l\) subintervals, respectively. The following notation will be used throughout this:

\[\Delta a = a_{\text{max}}/m, \quad \Delta x = (x_{\text{max}} - x_{\text{min}})/n\] and \(\Delta t = T/l\) denote the age, size, and time mesh lengths, respectively. The mesh points are given by: \(a_i = i\Delta a, i = 0, 1, \ldots, m, \quad x_j = x_{\text{min}} + j\Delta x, j = 0, 1, \ldots, n, \quad t_k = k\Delta t, k = 0, 1, \ldots, l\). We denote by \(J_i^k, A_j^k, P^k\) and \(Q^k\) the finite difference approximation of \(J(a_i, t_k), A(x_j, t_k), P(t_k)\) and \(Q(t_k)\), respectively, and let

\[\nu_i^k = \nu(a_i, t_k, P^k), \quad g_j^k = g(x_j, t_k, Q^k), \quad \mu_j^k = \mu(x_j, t_k, Q^k), \quad \beta_j^k = \beta(x_j, t_k, Q^k).\]

We define the difference operators

\[D_{\Delta a}(J_i^{k+1}) = \frac{J_i^{k+1} - J_i^{k-1}}{\Delta a}, \quad 1 \leq i \leq m, \quad D_{\Delta x}(A_j^{k+1}) = \frac{A_j^{k+1} - A_j^{k-1}}{\Delta x}, \quad 1 \leq j \leq n,\]

and the \(\ell^1\) and \(\ell^\infty\) norms of \(J^k\) and \(A^k\) by

\[\|J^k\|_1 = \sum_{i=1}^{m} |J_i^k|\Delta a, \quad \|A^k\|_1 = \sum_{j=1}^{n} |A_j^k|\Delta x, \]

\[\|J^k\|_\infty = \max_{0 \leq i \leq m} |J_i^k|, \quad \|A^k\|_\infty = \max_{0 \leq j \leq n} |A_j^k|.\]

We then discretize the partial differential equation system (1.1.1) using the following finite difference approximation

\[
\frac{J_i^{k+1} - J_i^k}{\Delta t} + \frac{J_i^k - J_i^{k-1}}{\Delta a} + \nu_i^k J_i^k = 0, \quad 0 \leq k \leq l - 1, \quad 1 \leq i \leq m, \\
\frac{A_j^{k+1} - A_j^k}{\Delta t} + \frac{g_j^k A_j^k - g_j^{k-1}A_j^{k-1}}{\Delta x} + \mu_j^k A_j^k = 0, \quad 0 \leq k \leq l - 1, \quad 1 \leq j \leq n, \\
J_0^{k+1} = \sum_{j=1}^{n} \beta_j^{k+1} A_j^{k+1} \Delta x, \quad g_0^{k+1} A_0^{k+1} = J_m^{k+1}, \quad 0 \leq k \leq l - 1, \\
P_i^{k+1} = \sum_{i=1}^{m} J_i^{k+1} \Delta a, \quad Q_i^{k+1} = \sum_{j=1}^{n} A_j^{k+1} \Delta x, \quad 0 \leq k \leq l - 1.
\]
with the initial conditions

\[ J_0^0 = J^0(0), \quad J_i^0 = \frac{1}{\Delta a} \int_{i-1}^i J^0(a) da, \quad i = 1, 2, \ldots, m, \]

\[ A_0^0 = A^0(0), \quad A_j^0 = \frac{1}{\Delta x} \int_{j-1}^j A^0(x) dx, \quad j = 1, 2, \ldots, n. \]

The following condition concerning $\Delta t$, $\Delta a$ and $\Delta x$ is imposed throughout the:

(H7) Assume that $\Delta t$, $\Delta a$ and $\Delta x$ are chosen such that

\[ \Delta t \left( \frac{1}{\Delta a} + \omega_1 \right) \leq 1 \quad \text{and} \quad \omega_1 \Delta t \left( \frac{1}{\Delta x} + 1 \right) \leq 1. \]

We can equivalently write (1.2.2) as the following system of linear equations:

\[
\begin{align*}
J_{i+1}^k &= \frac{\Delta t}{\Delta a} J_{i}^k + \left( 1 - \frac{\Delta t}{\Delta a} - \Delta t v_i^k \right) J_i^k, \quad 0 \leq k \leq l - 1, \quad 1 \leq i \leq m, \\
A_{j+1}^k &= \frac{\Delta t}{\Delta x} g_j^k A_{j-1}^k + \left( 1 - \frac{\Delta t}{\Delta x} g_j^k - \Delta t \mu_j^k \right) A_j^k, \quad 0 \leq k \leq l - 1, \quad 1 \leq j \leq n, \\
J_0^{k+1} &= \sum_{j=1}^n \beta_j^{k+1} A_j^{k+1} \Delta x, \quad g_0^{k+1} A_0^{k+1} = J_m^{k+1}, \quad 0 \leq k \leq l - 1, \\
P^{k+1} &= \sum_{i=1}^m J_{i+1}^k \Delta a, \quad Q^{k+1} = \sum_{j=1}^n A_j^{k+1} \Delta x, \quad 0 \leq k \leq l - 1.
\end{align*}
\]

Since $J_i^0 \geq 0, i = 0, 1, \ldots, m$, $A_j^0 \geq 0, j = 0, 1, \ldots, n$, from the first two equations of (1.2.3), one can easily see that under the assumption (H7),

\[ J_{i+1}^k \geq 0, A_{j+1}^k \geq 0, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, k = 0, 1, \ldots, l - 1. \]  Thus, from (1.2.3), we find $J_0^{k+1}, A_0^{k+1} \geq 0, k = 0, 1, \ldots, l - 1$. That is to say, the system (1.2.3) has a unique solution satisfying $[J_0^{k+1}, J_1^{k+1}, \ldots, J_m^{k+1}, A_0^{k+1}, A_1^{k+1}, \ldots, A_n^{k+1}] \geq \overrightarrow{0}$, $k = 0, 1, \ldots, l - 1$. 

7
1.3 Estimates for the difference approximations

We first show that the difference approximation is bounded in $\ell^1$ norm.

**Lemma 1.3.1.** The following estimate holds:

$$\|J^k\|_1 + \|A^k\|_1 \leq (1 + \omega_1 \Delta t)^k (\|J^0\|_1 + \|A^0\|_1) \leq (1 + \omega_1 \Delta t)^l (\|J^0\|_1 + \|A^0\|_1) \equiv M_1.$$  

**Proof.** Multiplying the first equation of (1.2.3) by $\Delta a$ and summing over $i = 1, 2, \cdots, m$, we have

$$\|J^{k+1}\|_1 = \|J^k\|_1 + \Delta t (J^k_0 - J^k_m) - \Delta t \sum_{i=1}^m \nu^k_i J^k_i \Delta a.$$  

Treating the second equation of (1.2.3) similarly, and noticing that $g^k_n = 0$, we find

$$\|A^{k+1}\|_1 = \|A^k\|_1 + \Delta t g^k_0 A^k_0 - \Delta t \sum_{j=1}^n \mu^k_j A^k_j \Delta x.$$  

Hence, using the boundary condition given in the third and fourth equations of (1.2.3) and (H4), we get

$$\|J^{k+1}\|_1 + \|A^{k+1}\|_1 = \|J^k\|_1 + \|A^k\|_1 + \Delta t J^k_0 - \Delta t \left( \sum_{i=1}^m \nu^k_i J^k_i \Delta a + \sum_{j=1}^n \mu^k_j A^k_j \Delta x \right)$$

\[
\leq \|J^k\|_1 + \|A^k\|_1 + \Delta t \sum_{j=1}^n \beta^k_j A^k_j \Delta x
\leq \|J^k\|_1 + \|A^k\|_1 + \omega_1 \Delta t \|A^k\|_1
\leq (1 + \omega_1 \Delta t)(\|J^k\|_1 + \|A^k\|_1),
\]

which implies the estimate. \hfill \Box

We now define $\mathbb{D}_3 = [x_{\min}, x_{\max}] \times [0, T] \times [0, M_1]$. We then establish $\ell^\infty$ bound on the difference approximation.
Lemma 1.3.2. The following estimates hold:

\[ \| J^k \|_\infty \leq \max\{\| J^0 \|_\infty, \omega_1 M_1 \}, \]

\[ \| A^k \|_\infty \leq \max \left\{ (1 + \omega_2 \Delta t) \| A^0 \|_\infty, \frac{\| J^0 \|_\infty}{\alpha}, \frac{\omega_1 M_1}{\alpha} \right\}, \]

where \( \alpha \leq g(x_{\text{min}}, t, Q) \) for \( t \in [0, T] \) and \( Q \in [0, M_1] \).

Proof. If \( J^k_{0+1} = \max_{0 \leq q \leq m} J^k_q \), then from the third equation of (1.2.3) and (H4) we get

\[ J^k_{0+1} = \sum_{j=1}^{n} \beta^k_{j} A^k_j \Delta x \leq \omega_1 \| A^k \|_1 \leq \omega_1 M_1. \quad (1.3.1) \]

Otherwise, suppose that for some \( 1 \leq i \leq m, J^k_{i+1} = \max_{0 \leq q \leq m} J^k_q \), then from the first equation of (1.2.3) and (H7) we have

\[ J^k_{i+1} \leq \left( 1 - \frac{\Delta t}{\Delta a} \right) \max_{0 \leq q \leq m} J^k_q + \frac{\Delta t}{\Delta a} \max_{0 \leq q \leq m} J^k_q = \| J^k \|_\infty. \quad (1.3.2) \]

A combination of (1.3.1) and (1.3.2) then yields

\[ \| J^k \|_\infty \leq \max\{\| J^0 \|_\infty, \omega_1 M_1 \}. \]

Similarly, if \( A^k_{0+1} = \max_{0 \leq r \leq n} A^k_r \), then from the fourth equation of (1.2.3) and (H4) we find

\[ A^k_{0+1} \leq \frac{\| J^k \|_\infty}{\alpha} \leq \frac{\max\{\| J^0 \|_\infty, \omega_1 M_1 \}}{\alpha}. \quad (1.3.3) \]

Now, suppose that for some \( 1 \leq j \leq n, A^k_{j+1} = \max_{0 \leq r \leq n} A^k_r \), then from the second equation of (1.2.3) and (H2) we get

\[ A^k_{j+1} \leq \left( 1 - \frac{\Delta t}{\Delta x} g^k_j \right) \max_{0 \leq r \leq n} A^k_r + \frac{\Delta t}{\Delta x} g^k_{j-1} \max_{0 \leq r \leq n} A^k_r \]
\[ = \left[ 1 + \frac{\Delta t}{\Delta x} (g(x_{j-1}, t_k, Q^k) - g(x_j, t_k, Q^k)) \right] \| A^k \|_\infty \quad (1.3.4) \]
\[ \leq \left[ 1 + \Delta t|g(x_j, t_k, Q^k)| \right] \| A^k \|_\infty \]
\[ \leq (1 + \omega_2 \Delta t) \| A^k \|_\infty, \]
where $x_j \in [x_{j-1}, x_j]$ and $\omega_2 = \sup_{(x,t,Q) \in \mathcal{D}_3} |g_t(x, t, Q)|$. A combination of (1.3.3) and (1.3.4) leads to the desired result.

The next lemma is necessary to show that the approximations $J_i^k$ and $A_j^k$ have bounded total variation.

**Lemma 1.3.3.** There exists a positive constant $M_2$ such that

$$\left| \frac{J_{0}^{k+1} - J_0^k}{\Delta t} \right| \leq M_2, \ k = 1, \ldots, l - 1.$$

**Proof.** We have from the second and third equations of (1.2.3) that

$$J_0^{k+1} - J_0^k = \sum_{j=1}^{n} (\beta_j^{k+1} A_j^{k+1} - \beta_j^k A_j^k) \Delta x$$

$$= \sum_{j=1}^{n} \beta_j^{k+1} (A_j^{k+1} - A_j^k) \Delta x + \sum_{j=1}^{n} (\beta_j^{k+1} - \beta_j^k) A_j^k \Delta x$$

$$= \sum_{j=1}^{n} \beta_j^{k+1} [(g_j^k A_j^{k+1} - g_j^k A_j^k) - \mu_j^k A_j^k \Delta x] \Delta t$$

$$+ \sum_{j=1}^{n} [\beta_i(x_j, t_k, Q^{k+1}) \Delta t + \beta_Q(x_j, t_k, Q^k)(Q^{k+1} - Q^k)] A_j^k \Delta x,$$

where $t_k \in [t_{k-1}, t_k]$.

Since $g_0^k = 0$, simple calculations yield

$$\sum_{j=1}^{n} \beta_j^{k+1} (g_j^{k+1} A_{j-1}^k - g_j^k A_j^k)$$

$$= \beta_1^{k+1} g_0^k A_0^k - \beta_1^k g_1^k A_1^k + \sum_{j=2}^{n} \beta_j^{k+1} g_j^k A_{j-1}^k - \sum_{j=2}^{n-1} \beta_j^{k+1} g_j^k A_j^k$$

$$= \beta_1^{k+1} g_0^k A_0^k + \sum_{j=1}^{n-1} (\beta_j^{k+1} A_j^k - \beta_j^{k+1} A_j^k).$$
Hence,
\[
\frac{J^{k+1}_0 - J^k_0}{\Delta t} = \left| \beta^{k+1} g^k A^k_0 + \sum_{j=1}^{n-1} (\beta^{j+1}_k - \beta^k_j) g^k_j A^k_j - \sum_{j=1}^{n} \beta^{k+1}_j \mu^k_j A^k_j \Delta x \right| \\
+ \sum_{j=1}^{n} \left[ \beta(x_j, t_k, Q^{k+1}) + \beta Q(x_j, t_k, Q^k) \left( \frac{Q^{k+1} - Q^k}{\Delta t} \right) \right] A^k_j \Delta x \\
\leq \sup_{(x,t,Q) \in D_2} (\beta g) \|A^k\|_\infty + \sup_{(x,t,Q) \in D_2} g \|A^k\|_\infty \sum_{j=1}^{n-1} |\beta^{j+1}_k - \beta^k_j| \\
+ \sup_{(x,t,Q) \in D_2} (\beta \mu) \|A^k\|_1 + \sup_{(x,t,Q) \in D_3} |\beta Q| \|A^k\|_1 \left| \frac{Q^{k+1} - Q^k}{\Delta t} \right|.
\]
(1.3.6)

Note that from (H4) it follows that there exists a \( c_1 > 0 \) such that
\[
\sum_{j=1}^{n-1} |\beta^{j+1}_k - \beta^k_j| \leq c_1.
\]
(1.3.7)

Furthermore,
\[
\left| \frac{Q^{k+1} - Q^k}{\Delta t} \right| = \left| \sum_{j=1}^{n} (A^k_{j+1} - A^k_j) \frac{\Delta x}{\Delta t} \right| \\
= \left| \sum_{j=1}^{n} (g^j_{j+1} A^k_{j+1} - A^k_j - g^k_j \mu^k_j A^k_j \Delta x) \right| \\
\leq \sup_{(x,t,Q) \in D_2} g \|A^k\|_\infty + \sup_{(x,t,Q) \in D_2} \mu \|A^k\|_1.
\]

Thus, by Lemmas 1.3.1-1.3.2 and (H2)-(H3), there exists a constant \( c_2 > 0 \) such that
\[
\frac{|Q^{k+1} - Q^k|}{\Delta t} \leq c_2.
\]
(1.3.8)

Applying the bounds (1.3.7) and (1.3.8) to (1.3.6), we conclude that there exists a positive constant \( M_2 \) such that \(|(J^{k+1}_0 - J^k_0)/\Delta t| \leq M_2 \) for each \( k \).
With the help of the above lemmas, we will show that approximations $J^k_i$ and $A^k_j$ have bounded total variation. The total variation bound plays an important role in establishing the sequential convergence of the difference approximation (1.2.2) to a weak solution of (1.1.1).

**Lemma 1.3.4.** There exists a positive constant $M_3$ such that

$$
\|D_{\Delta a}(J^k)\|_1 + \|D_{\Delta x}(A^k)\|_1 \leq M_3.
$$

**Proof.** Set $\xi^k_i = D_{\Delta a}(J^k_i)$ and apply the operator $D_{\Delta a}$ to the first equation of (1.2.3) to get

$$
\xi^{k+1}_i = \frac{\Delta t}{\Delta a} \xi^k_{i-1} + \left(1 - \frac{\Delta t}{\Delta a}\right) \xi^k_i - \Delta t D_{\Delta a}(\nu^k_i J^k_i), \quad 2 \leq i \leq m.
$$

By (H7), we have

$$
|\xi^{k+1}_i| \leq \frac{\Delta t}{\Delta a} |\xi^k_{i-1}| + \left(1 - \frac{\Delta t}{\Delta a}\right) |\xi^k_i| + \Delta t |D_{\Delta a}(\nu^k_i J^k_i)|, \quad 2 \leq i \leq m.
$$

Multiplying the above equation by $\Delta a$, and summing over the indices $i = 2, 3, \cdots, m$, we find

$$
\sum_{i=2}^m |\xi^{k+1}_i| \Delta a \leq \sum_{i=2}^m |\xi^k_i| \Delta a + \Delta t (|\xi^k_1| - |\xi^k_m|) + \Delta t \sum_{i=2}^m |D_{\Delta a}(\nu^k_i J^k_i)| \Delta a. \quad (1.3.9)
$$

For $i = 1$, again using the first equation of (1.2.3) and (H7), we obtain

$$
|\xi^{k+1}_1| \Delta a = |J^{k+1}_1 - J^{k+1}_0| = \left|\frac{\Delta t}{\Delta a} J^k_0 + \left(1 - \frac{\Delta t}{\Delta a} - \Delta t \nu^k_1\right) J^k_1 - J^{k+1}_0\right|
$$

$$
= \left|\left(1 - \frac{\Delta t}{\Delta a}\right) \left(J^k_1 - J^k_0\right) - \Delta t \nu^k_1 J^k_1 - (J^{k+1}_1 - J^k_0)\right| \leq \left(1 - \frac{\Delta t}{\Delta a}\right) |\xi^k_1| \Delta a + \Delta t \nu^k_1 J^k_1 + |J^{k+1}_0 - J^k_0|.
$$

Adding (1.3.9) and (1.3.10), we have

$$
\|\xi^{k+1}\|_1 \leq \|\xi^k\|_1 - \Delta t |\xi^k_m| + \Delta t \left|\sum_{i=2}^m |D_{\Delta a}(\nu^k_i J^k_i)| \Delta a + \nu^k_i J^k_1 + \left|\frac{J^{k+1}_0 - J^k_0}{\Delta t}\right|\right|.
$$
Note that
\[
\sum_{i=2}^{m} |D_{\Delta x}(v_i J_i)| \Delta a + v_i J_i = \sum_{i=2}^{m} |(v_i - v_{i-1})J_i + v_{i-1}(J_i - J_{i-1})| + v_1 J_1
\leq \sum_{i=2}^{m} |v_i - v_{i-1}| ||J_i||_{\infty} + \max_i (v_i) ||\xi||_1 + \max_i (v_i) ||J||_{\infty}.
\]

Therefore, by Lemmas 1.3.2-1.3.3 and (H1), there exist positive constants \(c_3\) and \(c_4\) such that
\[
\sum_{i=2}^{m} |D_{\Delta x}(v_i J_i)| \Delta a + v_i J_i = \sum_{i=2}^{m} |(v_i - v_{i-1})J_i + v_{i-1}(J_i - J_{i-1})| + v_1 J_1
\leq \sum_{i=2}^{m} |v_i - v_{i-1}| ||J_i||_{\infty} + \max_i (v_i) ||\xi||_1 + \max_i (v_i) ||J||_{\infty}.
\]

Thus,
\[
||\xi^{k+1}||_1 \leq (1 + \Delta t c_3) ||\xi||_1 + \Delta t c_4 - \Delta t ||\xi_m||.
\] (1.3.11)

Set \(\eta^k_j = D_{\Delta x}(A^k_j)\) and apply the operator \(D_{\Delta x}\) to the second equation of (1.2.3) to get
\[
\eta^{k+1}_j = \eta^k_j - \Delta t \frac{\partial}{\partial x} \left[D_{\Delta x}(g^k J^k_j) - D_{\Delta x}(g^k J^k_{j-1})\right] - \Delta t D_{\Delta x}(\mu^k A^k_j)
\leq \eta^k_j - \Delta t \frac{\partial}{\partial x} \left[D_{\Delta x}(g^k J^k_j) - D_{\Delta x}(g^k J^k_{j-1})\right] - \Delta t D_{\Delta x}(\mu^k A^k_j)
\leq \left(1 - \frac{\Delta t}{\Delta x} g^k_{j-1}\right) \eta^k_j + \frac{\Delta t}{\Delta x} g^k_{j-2} \eta^k_{j-1} - \frac{\Delta t}{\Delta x} \left[D_{\Delta x}(g^k J^k_j) - D_{\Delta x}(g^k J^k_{j-1})\right]
\leq \left(1 - \frac{\Delta t}{\Delta x} g^k_{j-1}\right) \eta^k_j + \frac{\Delta t}{\Delta x} g^k_{j-2} \eta^k_{j-1} - \frac{\Delta t}{\Delta x} \left[D_{\Delta x}(g^k J^k_j) - D_{\Delta x}(g^k J^k_{j-1})\right]
\]

By (H7), we have
\[
|\eta^{k+1}_j| \leq \left(1 - \frac{\Delta t}{\Delta x} g^k_{j-1}\right) |\eta^k_j| + \frac{\Delta t}{\Delta x} g^k_{j-2} |\eta^k_{j-1}| + \Delta t |D_{\Delta x}(D_{\Delta x}(g^k J^k_j))|\\n+ \Delta t |D_{\Delta x}(\mu^k A^k_j)|, \quad 2 \leq j \leq n.
\]

Multiplying the above equation by \(\Delta x\), and summing over the indices \(j = 2, 3, \cdots, n\),
we get
\[
\sum_{j=2}^{n} |\eta_{j}^{k+1}|\Delta x \leq \sum_{j=2}^{n} |\eta_{j}^{k}|\Delta x + \Delta t(g_{0}^{k}|\eta_{1}| - g_{n-1}^{k}|\eta_{n}|)
+ \Delta t\sum_{j=2}^{n} |D_{\Delta x}^{-}(D_{\Delta x}^{-}(g_{j}^{k})A_{j}^{k})|\Delta x + \Delta t\sum_{j=2}^{n} |D_{\Delta x}^{-}(\mu_{j}^{k}A_{j}^{k})|\Delta x.
\]
(1.3.12)

For \(j = 1\), using the second equation of (1.2.3) and (H7) we have
\[
|\eta_{1}^{k+1}|\Delta x = |A_{1}^{k+1} - A_{0}^{k+1}| = \left| \frac{\Delta t}{\Delta x} g_{0}^{k}A_{0}^{k} + \left(1 - \frac{\Delta t}{\Delta x} g_{1}^{k} - \Delta t\mu_{1}^{k}\right) A_{1}^{k} - A_{0}^{k+1}\right|
\leq \left(1 - \frac{\Delta t}{\Delta x} g_{0}^{k}\right) |A_{1}^{k} - A_{0}^{k}| + \Delta t \left[ |D_{\Delta x}^{-}(g_{1}^{k})|A_{1}^{k} + \mu_{1}^{k}A_{1}^{k} + \left|\frac{A_{0}^{k+1} - A_{0}^{k}}{\Delta t}\right|\right].
\]
(1.3.13)

Adding (1.3.12) and (1.3.13) we get
\[
\|\eta_{1}^{k+1}\|_{1} \leq \|\eta_{1}^{k}\|_{1} - \Delta t g_{n-1}^{k}|\eta_{n}| + \Delta t \left[ \sum_{j=2}^{n} |D_{\Delta x}^{-}(D_{\Delta x}^{-}(g_{j}^{k})A_{j}^{k})|\Delta x
+ \sum_{j=2}^{n} |D_{\Delta x}^{-}(\mu_{j}^{k}A_{j}^{k})|\Delta x + |D_{\Delta x}^{-}(g_{1}^{k})|A_{1}^{k} + \mu_{1}^{k}A_{1}^{k} + \left|\frac{A_{0}^{k+1} - A_{0}^{k}}{\Delta t}\right|\right].
\]
(1.3.14)

Furthermore, we find
\[
\sum_{j=2}^{n} |D_{\Delta x}^{-}(D_{\Delta x}^{-}(g_{j}^{k})A_{j}^{k})|\Delta x + \sum_{j=2}^{n} |D_{\Delta x}^{-}(\mu_{j}^{k}A_{j}^{k})|\Delta x + |D_{\Delta x}^{-}(g_{1}^{k})|A_{1}^{k} + \mu_{1}^{k}A_{1}^{k}
= \sum_{j=2}^{n} |D_{\Delta x}^{-}(D_{\Delta x}^{-}(g_{j}^{k})A_{j}^{k})|\Delta x + D_{\Delta x}^{-}(g_{j-1}^{k})D_{\Delta x}^{-}(A_{j}^{k})|\Delta x
+ \sum_{j=2}^{n} |D_{\Delta x}^{-}(\mu_{j}^{k}A_{j}^{k}) + \mu_{j-1}^{k}D_{\Delta x}^{-}(A_{j}^{k})|\Delta x + |D_{\Delta x}^{-}(g_{1}^{k})|A_{1}^{k} + \mu_{1}^{k}A_{1}^{k}
\leq \max_{j} |D_{\Delta x}^{-}(D_{\Delta x}^{-}(g_{j}^{k}))||A_{1}^{k}| + \max_{j} |D_{\Delta x}^{-}(g_{j-1}^{k})||A_{1}^{k}|
+ \|A_{1}^{k}\|_{\infty} \sum_{j=2}^{n} |\mu_{j}^{k} - \mu_{j-1}^{k}|
+ \max_{j}(\mu_{j-1}^{k})\|\eta_{1}^{k}\|_{1} + \max_{j} |D_{\Delta x}^{-}(g_{j}^{k})||A_{1}^{k}||\infty + \max_{j}(\mu_{j}^{k})\|A_{1}^{k}\|_{\infty}.
\]

By Lemmas 1.3.1-1.3.2 and (H2)-(H3), there exist positive constants \(c_{5}\) and \(c_{6}\) such
that
\[
\sum_{j=2}^{n} |D_{\Delta x}(D_{\Delta x}(g_j^k)A_j^k)| \Delta x + \sum_{j=2}^{n} |D_{\Delta x}(\mu_j^k A_j^k)| \Delta x + |D_{\Delta x}(g_1^k)|A_1^k + \mu_1^k A_1^k \leq c_5 \|\eta^k\|_1 + c_6. \tag{1.3.15}
\]

By virtue of the first and fourth equations of (1.2.2), we obtain
\[
\frac{A_{k+1}^0 - A_0^k}{\Delta t} = \frac{J_{m+1}^k/g_{0}^k - J_m^k/g_0^k}{\Delta t} = \left| \frac{(g_{0}^k - g_{0}^{k+1})J_{m+1}^k + g_{0}^{k+1}(J_{m}^k - J_{m}^{k+1})}{\Delta t g_{0}^{k+1} g_{0}^k} \right|
\]
\[
= \left| \frac{J_{m+1}^k}{g_{0}^{k+1} g_{0}^k} \left[ -g(x_{\text{min}}, t_k, Q^k) + g(x_{\text{min}}, t_{k+1}, Q^{k+1}) \right] Q_{k} \frac{Q^k - Q^{k+1}}{\Delta t} - \frac{1}{g_{0}^k} \left( \xi_{m}^{k} + \nu_{m}^{k} J_{m}^k \right) \right|
\]
\[
\leq \left\| \frac{J_{k+1}^1}{g_{0}^{k+1} g_{0}^k} \right\|_{\infty} \left[ \sup_{(x, t, Q) \in \mathbb{D}_3} |g_t| + \sup_{(x, t, Q) \in \mathbb{D}_3} |g_Q| \right] \left( \frac{Q^k - Q^{k+1}}{\Delta t} \right) \right]
\]
\[
+ \frac{1}{g_{0}^k} (|\xi_{m}^{k}| + |\nu_{i}^{k}|) \|J_{k}^1\|_{\infty},
\]

where $t_k \in [t_k, t_{k+1}]$, and $Q^k$ is between $Q^k$ and $Q^{k+1}$.

Then, by (H1)-(H2) and Lemmas 1.3.2-1.3.3, there exists a positive constant $c_7$ such that
\[
\frac{A_{0}^{k+1} - A_0^k}{\Delta t} \leq c_7 + \frac{|\xi_{m}^{k}|}{g_{0}^k}. \tag{1.3.16}
\]

Applying (1.3.15) and (1.3.16) to (1.3.14) we get
\[
\|\eta^{k+1}\|_1 \leq (1 + \Delta tc_5)\|\eta^k\|_1 + \Delta t(c_6 + c_7) + \frac{\Delta t|\xi_{m}^{k}|}{g_{0}^k}. \tag{1.3.17}
\]
Now dividing (1.3.11) by $g_k^0$, we have

\[
\frac{\|\xi^{k+1}\|_1}{g_k^0} \leq (1 + \Delta tc_3) \frac{\|\xi^k\|_1}{g_k^0} + \frac{\Delta tc_4}{g_k^0} - \frac{\Delta t|\xi^k_m|}{g_k^0}
\]

\[
= (1 + \Delta tc_3) \frac{\|\xi^k\|_1}{g_k^0}
\]

\[
+ \left( \frac{1}{g_k^0} - \frac{1}{g_k^{-1}} \right) \|\xi^k\|_1 + \Delta t \left[ c_3 \left( \frac{1}{g_k^0} - \frac{1}{g_k^{-1}} \right) \|\xi^k\|_1 + \frac{c_4}{g_k^0} - \frac{|\xi^k_m|}{g_k^0} \right]
\]

\[
= (1 + \Delta tc_3) \frac{\|\xi^k\|_1}{g_k^{-1}}
\]

\[
- g_0(x_{\text{min}}, t_k, Q_k^{-1}) \Delta t + g_Q(x_{\text{min}}, t_k, Q_k^{-1}, Q_k^{-1}, Q_k^{-1}) \|\xi^k\|_1
g_0^k g_0^k
\]

\[
+ \Delta t \left[ c_3 \left( \frac{1}{g_k^0} - \frac{1}{g_k^{-1}} \right) \|\xi^k\|_1 + \frac{c_4}{g_k^0} - \frac{|\xi^k_m|}{g_k^0} \right]
\]

\[
\leq (1 + \Delta tc_3) \frac{\|\xi^k\|_1}{g_k^0} + \Delta t \left[ \sup_{(x, t,\bar{Q}) \in \mathbb{D}_3} |g_l| + \sup_{(x, t,\bar{Q}) \in \mathbb{D}_3} |g_Q| (Q_k^{-1} - Q_k^{-1}) \right. \Delta t
g_0^k g_0^k
\]

\[
+ c_3 \left( \frac{1}{g_k^0} - \frac{1}{g_k^{-1}} \right) \|\xi^k\|_1 + \frac{c_4}{g_k^0} - \frac{\Delta t|\xi^k_m|}{g_k^0}.
\]

By (H2), Lemma 1.3.3, and (1.3.11) which implies that $\|\xi^k\|_1$ is bounded, there exists a positive constant $c_8$ such that

\[
\frac{\|\xi^{k+1}\|_1}{g_k^0} \leq (1 + \Delta tc_3) \frac{\|\xi^k\|_1}{g_k^0} + \Delta tc_8 - \frac{\Delta t|\xi^k_m|}{g_k^0}.
\]  

(1.3.18)

Adding (1.3.17) and (1.3.18) we obtain

\[
\frac{\|\xi^{k+1}\|_1}{g_k^0} + \frac{\|\eta^{k+1}\|_1}{g_k^0} \leq [1 + \Delta t(c_3 + c_5)] \left( \frac{\|\xi^k\|_1}{g_k^{-1}} + \frac{\|\eta^k\|_1}{g_k^0} \right) + \Delta t(c_6 + c_7 + c_8).
\]

The result now easily follows from the above inequality.

The next result shows that the difference approximations satisfy a Lipschitz-type condition in $t$. 

\[16\]
Lemma 1.3.5. There exist positive constants $M_4$ and $M_5$ such that for any $q > p$, we have

$$\sum_{i=1}^{m} \left| \frac{J_i^q - J_i^p}{\Delta t} \right| \Delta a \leq M_4(q - p), \quad \sum_{j=1}^{n} \left| \frac{A_j^q - A_j^p}{\Delta t} \right| \Delta x \leq M_5(q - p).$$

Proof. Summing the first equation in (1.2.2) over $i$ and multiplying by $\Delta a$, we obtain

$$\sum_{i=1}^{m} \left| \frac{J_i^{k+1} - J_i^k}{\Delta t} \right| \Delta a = \sum_{i=1}^{m} \left| \frac{J_i^k - J_i^{k-1}}{\Delta a} + \nu_i \left( J_i^k - J_i^{k-1} \right) \right| \Delta a \leq \|D^-\Delta a(J_i^k)\|_1 + \max_i |\nu_i|^1\|J_i^k\|_1.$$

By Lemmas 1.3.1 and 1.3.4, there exists a positive constant $M_4$ such that

$$\sum_{i=1}^{m} \left| \frac{J_i^{k+1} - J_i^k}{\Delta t} \right| \Delta a \leq M_4.$$

Hence,

$$\sum_{i=1}^{m} \left| \frac{J_i^q - J_i^p}{\Delta t} \right| \Delta a \leq \sum_{k=p}^{q-1} \sum_{i=1}^{m} \left| \frac{J_i^{k+1} - J_i^k}{\Delta t} \right| \Delta a \leq M_4(q - p).$$

Similarly, using the second equation of (1.2.2), Lemma 1.3.1 and Lemma 1.3.4 we get

$$\sum_{j=1}^{n} \left| \frac{A_j^{k+1} - A_j^k}{\Delta t} \right| \Delta x = \sum_{j=1}^{n} \left| \frac{g_j^k A_j^k - g_j^{k-1} A_j^{k-1}}{\Delta x} + \mu_j A_j^k \right| \Delta x$$

$$= \sum_{j=1}^{n} \left| \left( \frac{g_j^k - g_j^{k-1}}{\Delta x} + \mu_j^k \right) A_j^k + g_j^{k-1} \frac{A_j^k - A_j^{k-1}}{\Delta x} \right| \Delta x$$

$$\leq \left( \max_j |D^-\Delta x(g_j^k)| + \max_j |\mu_j^k|^1\|A^k\|_1 + \max_j |g_j^{k-1}|\|D^-\Delta x(A^k)\|_1 \right) \leq M_5.$$

Thus,

$$\sum_{j=1}^{n} \left| \frac{A_j^q - A_j^p}{\Delta t} \right| \Delta x \leq \sum_{k=p}^{q-1} \sum_{j=1}^{n} \left| \frac{A_j^{k+1} - A_j^k}{\Delta t} \right| \Delta x \leq M_5(q - p).$$

1.4 Convergence of difference approximation and existence of a weak solution

Following [17] we define a family of functions $\{U_{\Delta a, \Delta t}\}$ and $\{V_{\Delta x, \Delta t}\}$ by

$$U_{\Delta a, \Delta t}(a, t) = J_i^k, \quad V_{\Delta x, \Delta t}(x, t) = A_j^k$$

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for $a \in [a_{i-1}, a_i), x \in [x_{j-1}, x_j), t \in [t_{k-1}, t_k), i = 1, \cdots, m, j = 1, \cdots, n, k = 1, \cdots, l$.

Then by Lemmas 1.3.1-1.3.5 the set of functions $\{U_{\Delta a, \Delta t}\}, \{V_{\Delta x, \Delta t}\}$ is compact in the topology of $L^1((0, a_{\text{max}}) \times (0, T)) \times L^1((x_{\text{min}}, x_{\text{max}}) \times (0, T))$, respectively, and hence as in the proof of Lemma 16.7 on page 276 in [17], we have the following lemma.

**Lemma 1.4.1.** There exists a sequence of functions $\{U_{\Delta a, \Delta t}\}, \{V_{\Delta x, \Delta t}\} \subset \{U_{\Delta a, \Delta t}\}, \{V_{\Delta x, \Delta t}\}$ which converge to a function $(J, A) \in BV([0, a_{\text{max}}] \times [0, T]) \times BV([x_{\text{min}}, x_{\text{max}}] \times [0, T])$, in the sense that for all $t > 0$

$$
\int_0^{a_{\text{max}}} |U_{\Delta a, \Delta t}(a, t) - J(a, t)| \, da \to 0,
$$

$$
\int_{x_{\text{min}}}^{x_{\text{max}}} |V_{\Delta x, \Delta t}(x, t) - A(x, t)| \, dx \to 0,
$$

$$
\int_0^T \int_0^{a_{\text{max}}} |U_{\Delta a, \Delta t}(a, t) - J(a, t)| \, dadt \to 0,
$$

$$
\int_0^T \int_{x_{\text{min}}}^{x_{\text{max}}} |V_{\Delta x, \Delta t}(x, t) - A(x, t)| \, dxdt \to 0,
$$

as $\gamma \to \infty$ (i.e., $\Delta a_\gamma, \Delta x_\gamma, \Delta t_\gamma \to 0$). Furthermore, there exist constants $M_6$ and $M_7$ (dependent on $\|J^0\|_{BV[0,a_{\text{max}}]}$ and $\|A^0\|_{BV[x_{\text{min}},x_{\text{max}}]}$) such that the limit functions satisfy

$$
\|J\|_{BV([0,a_{\text{max}}] \times [0,T])} \leq M_6, \quad \|A\|_{BV([x_{\text{min}},x_{\text{max}}] \times [0,T])} \leq M_7.
$$

The next theorem shows that the set of limit functions $J(a, t), A(x, t)$ constructed via our difference scheme is actually a weak solution of problem (1.1.1).

**Theorem 1.4.1.** The set of limit functions $J(a, t)$ and $A(x, t)$ defined in Lemma 1.4.1 is a weak solution of (1.1.1) and satisfies

$$
P(t), Q(t) \leq e^{\omega T}(\|J^0\|_1 + \|A^0\|_1).
$$
\[ \| J \|_{L^\infty((0,a_{\text{max}}) \times (0,T))} \leq \max \left\{ \| J^0 \|_{\infty}, \omega_1 e^{\omega_1 T} (\| J^0 \|_1 + \| A^0 \|_1) \right\}, \]

and

\[ \| A \|_{L^\infty((x_{\text{min}}, x_{\text{max}}) \times (0,T))} \leq \max \left\{ e^{\omega_2 T} \| A^0 \|_{\infty}, \frac{\| J^0 \|_{\infty}}{\alpha}, \frac{\omega_1 e^{\omega_1 T} (\| J^0 \|_1 + \| A^0 \|_1)}{\alpha} \right\}. \]

**Proof.** Let \( \varphi \in C^1((0,a_{\text{max}}) \times (0,T)) \) and denote the finite difference approximations \( \varphi(a_i, t_k) \) by \( \varphi_i^k \). Multiplying the first equation of the difference scheme (1.2.3) by \( \varphi_i^{k+1} \), we have

\[ J_i^{k+1} \varphi_i^{k+1} = J_i^k \varphi_i^k + \frac{\Delta t}{\Delta a} \left( J_i^k - J_i^{k-1} \right) \varphi_i^k - \Delta t \nu_i^k J_i^k \varphi_i^{k+1}. \]

Thus,

\[ J_i^{k+1} \varphi_i^{k+1} - J_i^k \varphi_i^k = J_i^k (\varphi_i^{k+1} - \varphi_i^k) + \frac{\Delta t}{\Delta a} \left[ J_i^{k-1} (\varphi_i^{k+1} - \varphi_i^{k-1}) + (J_i^{k-1} \varphi_i^{k+1} - J_i^{k-1} \varphi_i^{k-1}) \right] - \Delta t \nu_i^k J_i^k \varphi_i^{k+1}. \]

Multiplying the above equation by \( \Delta a \), summing over \( k = 0, 1, \cdots, l-1 \) and \( i = 1, 2, \cdots, m \) and using the third equation of (1.2.3), we obtain

\[ \sum_{i=1}^{m} (J_i^l \varphi_i^l - J_i^0 \varphi_i^0) \Delta a \]

\[ = \sum_{k=0}^{l-1} \sum_{i=1}^{m} \left[ J_i^k (\varphi_i^{k+1} - \varphi_i^k) \Delta a + J_i^{k-1} (\varphi_i^{k+1} - \varphi_i^{k-1}) \Delta t - \nu_i^k J_i^k \varphi_i^{k+1} \Delta a \Delta t \right] \]

\[ + \sum_{k=0}^{l-1} \left( J_0^k \varphi_0^{k+1} - J_m^k \varphi_m^{k+1} \right) \Delta t \]

\[ = \sum_{k=0}^{l-1} \sum_{i=1}^{m} \left( J_i^k \frac{\varphi_i^{k+1} - \varphi_i^k}{\Delta t} + J_i^{k-1} \frac{\varphi_i^{k+1} - \varphi_i^{k-1}}{\Delta a} - \nu_i^k J_i^k \varphi_i^{k+1} \right) \Delta a \Delta t \]

\[ + \sum_{k=0}^{l-1} \varphi_0^{k+1} \left( \sum_{j=1}^{n} \beta_j^k A_j^k \Delta x \right) \Delta t - \sum_{k=0}^{l-1} \sum_{i=1}^{m} J_m^k \varphi_m^{k+1} \Delta t. \]
On the other hand, let $\psi \in C^1((x_{\min}, x_{\max}) \times (0, T))$ and denote the finite difference approximations $\psi(x_j, t_k)$ by $\psi_j^k$. Multiply the second equation of (1.2.3) by $\psi_j^{k+1}$ to find

$$A_j^{k+1}\psi_j^{k+1} = A_j^k\psi_j^k + \frac{\Delta t}{\Delta x} (g_j^{k+1}A_{j-1}^k - g_j^kA_j^k)\psi_j^{k+1} - \Delta t\mu_j^kA_j^k\psi_j^{k+1}. $$

Hence,

$$A_j^{k+1}\psi_j^{k+1} - A_j^k\psi_j^k = A_j^k(\psi_j^{k+1} - \psi_j^k) + \frac{\Delta t}{\Delta x} (g_j^{k+1}A_{j-1}^k(\psi_j^{k+1} - \psi_j^{k-1}))$$

$$+ (g_j^{k+1}A_{j-1}^k\psi_j^{k+1} - g_j^kA_j^k\psi_j^{k+1}) - \Delta t\mu_j^kA_j^k\psi_j^{k+1}. $$

Multiplying the above equation by $\Delta x$, summing over $k = 0, 1, \cdots, l - 1$ and $j = 1, 2, \cdots, n$, and using $g_n^k = 0$ and $g_0^kA_0^k = J_m^k$, we have

$$\sum_{j=1}^n (A_j^l\psi_j^l - A_j^0\psi_j^0)\Delta x$$

$$= \sum_{k=0}^{l-1} \sum_{j=1}^n [A_j^k(\psi_j^{k+1} - \psi_j^k)\Delta x + g_j^{k+1}A_{j-1}^k(\psi_j^{k+1} - \psi_j^{k-1})\Delta t]$$

$$- \mu_j^kA_j^k\psi_j^{k+1}\Delta x\Delta t] + \sum_{k=0}^{l-1} (g_0^kA_0^k\psi_0^{k+1} - g_n^kA_n^k\psi_n^{k+1})\Delta t$$

$$= \sum_{k=0}^{l-1} \sum_{j=1}^n \frac{(A_j^k\psi_j^{k+1} - \psi_j^k)}{\Delta t} + g_j^{k+1}A_{j-1}^k \frac{\psi_j^{k+1} - \psi_j^{k-1}}{\Delta x}$$

$$- \mu_j^kA_j^k\psi_j^{k+1}\Delta x\Delta t] + \sum_{k=0}^{l-1} J_m^k\psi_0^{k+1}\Delta t.$$ 

Using (1.4.1) and (1.4.2) and following an argument similar to that used in the proof of Lemma 16.9 on page 280 of [17] we obtain, by letting $m, n, l \to \infty$, that the limit of the difference approximations defined in Lemma 1.4.1 is a weak solution of (1.1.1). Taking
the limit in the bounds obtained in Lemmas 1.3.1-1.3.2, we get the bounds on \( P(t), Q(t), \| J \|_{L^\infty((0,t_{\text{max}}) \times (0,T))} \) and \( \| A \|_{L^\infty((x_{\text{min}},x_{\text{max}}) \times (0,T))} \).

1.5 Uniqueness of the weak solution

The following theorem guarantees the continuous dependence of the solution \( J_i^k \) and \( A_j^k \) of (1.2.3) with respect to the initial condition \( J_i^0 \) and \( A_j^0 \).

**Theorem 1.5.1.** Let \( \{ J_i^k, A_j^k \} \) and \( \{ \hat{J}_i^k, \hat{A}_j^k \} \) be the solutions of (1.2.3) corresponding to the initial conditions \( \{ J_i^0, A_j^0 \} \) and \( \{ \hat{J}_i^0, \hat{A}_j^0 \} \), respectively. Then there exists a positive constant \( \sigma \) such that

\[
\| J_i^{k+1} - \hat{J}_i^{k+1} \|_1 + \| A_j^{k+1} - \hat{A}_j^{k+1} \|_1 \leq (1 + \sigma t)(\| J_i^k - \hat{J}_i^k \|_1 + \| A_j^k - \hat{A}_j^k \|_1)
\]

for all \( k \geq 0 \).

**Proof.** Let \( u_i^k = J_i^k - \hat{J}_i^k, v_j^k = A_j^k - \hat{A}_j^k \) for \( i = 0, 1, \ldots, m, j = 0, 1, \ldots, n \) and \( k = 0, 1, \ldots, l \). Then \( u_i^k, v_j^k \) satisfy the following:

\[
\begin{align*}
 u_i^{k+1} &= \frac{\Delta t}{\Delta a} u_i^{k} + \left( 1 - \frac{\Delta t}{\Delta a} \right) u_i^{k} - \Delta t (\nu_i^k J_i^k - \hat{\nu}_i^k \hat{J}_i^k), \\
 v_j^{k+1} &= \frac{\Delta t}{\Delta x} (g_j^{k} A_j^{k-1} - \hat{g}_j^{k} \hat{A}_j^{k-1}) + v_j^{k} - \frac{\Delta t}{\Delta x} (g_j^{k} A_j^{k} - \hat{g}_j^{k} \hat{A}_j^{k}) - \Delta t (\mu_j^k A_j^{k} - \hat{\mu}_j^k \hat{A}_j^{k}), \\
 u_0^k &= \sum_{j=1}^{n} \beta_j^k A_j^k \Delta x - \sum_{j=1}^{n} \beta_j^k \hat{A}_j^k \Delta x, \\
 g_0^k A_0^k - \hat{g}_0^k \hat{A}_0^k &= u_m^k,
\end{align*}
\]

(1.5.1)

where \( \hat{\nu}_i^k = \nu(a_i, t_k, \hat{P}_k) \) and similar notations are used for the rest of the parameters.

Using the first equation of (1.5.1) and (H7), we have

\[
|u_i^{k+1}| \leq \frac{\Delta t}{\Delta a} |u_i^{k}| + \left( 1 - \frac{\Delta t}{\Delta a} \right) |u_i^{k}| + \Delta t |\nu_i^k J_i^k - \hat{\nu}_i^k \hat{J}_i^k|, \quad i = 1, 2, \ldots, m.
\]
Multiplying the above equation by $\Delta a$ and summing over the indices $i = 1, 2, \ldots, m$, we find

$$\| u^{k+1} \|_1 \leq \| u^k \|_1 + \Delta t \left( |u^k_0| - |u^k_m| + \sum_{i=1}^m |\nu_i^k J_i^k - \hat{\nu}_i^k \hat{J}_i^k| \Delta a \right),$$

(1.5.2)

Furthermore, we have

$$|u^k_0| + \sum_{i=1}^m |\nu_i^k J_i^k - \hat{\nu}_i^k \hat{J}_i^k| \Delta a$$

$$= \left| \sum_{j=1}^n (\beta_j^k A_j^k - \hat{\beta}_j^k \hat{A}_j^k) \Delta x \right| + \sum_{i=1}^m |\nu_i^k J_i^k - \hat{\nu}_i^k \hat{J}_i^k| \Delta a$$

$$= \left| \sum_{j=1}^n [\beta_j^k (A_j^k - \hat{A}_j^k) + (\beta_j^k - \hat{\beta}_j^k) \hat{A}_j^k] \Delta x \right|$$

$$+ \sum_{i=1}^m |\nu_i^k (J_i^k - \hat{J}_i^k) + (\nu_i^k - \hat{\nu}_i^k) \hat{J}_i^k| \Delta a$$

$$\leq \sum_{j=1}^n \beta_j^k |v_j^k| \Delta x + \sum_{j=1}^n |\beta_Q(x_j, t_k, \overline{Q}^k)(Q^k - \hat{Q}^k)||\hat{A}_j^k| \Delta x + \sum_{i=1}^m \nu_i^k |u_i^k| \Delta a$$

$$+ \sum_{i=1}^m |\nu_P(a_i, t_k, \overline{P}^k)(P^k - \hat{P}^k)||\hat{J}_i^k| \Delta a$$

$$\leq \max_j \beta_j^k \|v^k\|_1 + \sup_{(x,t,Q) \in \mathbb{D}_3} |\beta_Q||\hat{A}^k| \|Q^k - \hat{Q}^k\| + \max_i \nu_i^k \|u^k\|_1$$

$$+ \sup_{(a,t,P) \in \mathbb{D}_4} |\nu_P||\hat{J}^k\|_1 \|P^k - \hat{P}^k\|,$$

where $\mathbb{D}_4 = [0, a_{\max}] \times [0, T] \times [0, M_1]$, $\overline{P}^k$ is between $P^k$ and $\hat{P}^k$, $\overline{Q}^k$ is between $Q^k$ and $\hat{Q}^k$.

Note that

$$|P^k - \hat{P}^k| = \left| \sum_{i=1}^m (J_i^k - \hat{J}_i^k) \Delta a \right| \leq \sum_{i=1}^m |u_i^k| \Delta a = \|u^k\|_1,$$

$$|Q^k - \hat{Q}^k| = \left| \sum_{j=1}^n (A_j^k - \hat{A}_j^k) \Delta x \right| \leq \sum_{j=1}^n |v_j^k| \Delta x = \|v^k\|_1.$$

Thus, by assumptions (H1), (H4) and Lemma 1.3.1, there exist positive constants $c_9$
and $c_{10}$ such that

$$|u_0^k| + \sum_{i=1}^m |\nu_i^k J_i^k - \hat{\nu}_i^k \hat{J}_i^k| \Delta a \leq c_0 \|v^k\|_1 + c_{10} \|u^k\|_1.$$ 

Applying the above inequality to (1.5.2), we get

$$\|u^{k+1}\|_1 \leq \|u^k\|_1 - \Delta t \|u_m^k\| + \Delta t (c_9 \|v^k\|_1 + c_{10} \|u^k\|_1). \quad (1.5.3)$$

On the other hand, using the second equation of (1.5.1) and (H7), we obtain

$$|v_{j+1}^{k+1}| = \left| \frac{\Delta t}{\Delta x} [g_{j-1}^k (A_{j-1}^k - \hat{A}_{j-1}^k) + (g_{j-1}^k - \hat{g}_{j-1}^k) \hat{A}_{j-1}^k] + v_j^k \right. \right.$$

$$- \frac{\Delta t}{\Delta x} \left[ g_j^k (A_j^k - \hat{A}_j^k) + (g_j^k - \hat{g}_j^k) \hat{A}_j^k \right] - \Delta t \left[ \mu_j^k (A_j^k - \hat{A}_j^k) + (\mu_j^k - \hat{\mu}_j^k) \hat{A}_j^k \right] \right.$$ 

$$= \left| \left( 1 - \frac{\Delta t}{\Delta x} g_j^k - \Delta t \mu_j^k \right) v_j^k + \frac{\Delta t}{\Delta x} g_{j-1}^k v_{j-1}^k + \frac{\Delta t}{\Delta x} (g_{j-1}^k - \hat{g}_{j-1}^k) \hat{A}_{j-1}^k \right.$$ 

$$- \frac{\Delta t}{\Delta x} (g_j^k - \hat{g}_j^k) \hat{A}_j^k - \Delta t (\mu_j^k - \hat{\mu}_j^k) \hat{A}_j^k \right| \right.$$ 

$$\leq \left( 1 - \frac{\Delta t}{\Delta x} g_j^k - \Delta t \mu_j^k \right) |v_j^k| + \frac{\Delta t}{\Delta x} g_{j-1}^k |v_{j-1}^k| + \Delta t |D_{\Delta x} ((g_j^k - \hat{g}_j^k) \hat{A}_j^k)| \right.$$ 

$$+ \Delta t |\mu_j^k - \hat{\mu}_j^k||\hat{A}_j^k|. \quad (1.5.4)$$

Multiplying the above inequality by $\Delta x$, summing over the indices $j = 2, 3, \ldots, n$ and noticing that $\hat{g}_n^k = 0$, we get

$$\sum_{j=2}^n |v_{j+1}^{k+1}| \Delta x \leq \sum_{j=2}^n (1 - \Delta t \mu_j^k)|v_j^k| \Delta x + \Delta t g_1^k |v_1^k|$$

$$+ \Delta t \sum_{j=2}^n \left[ |D_{\Delta x} ((g_j^k - \hat{g}_j^k) \hat{A}_j^k)| + |\mu_j^k - \hat{\mu}_j^k||\hat{A}_j^k| \right] \Delta x. \quad (1.5.4)$$
For $j = 1$, by the second and fourth equations of (1.5.1) and (H7) we find

\[
|v_{x_1}^{k+1}| = \left| \frac{\Delta t}{\Delta x} u_{m}^{k} + v_{1}^{k} - \frac{\Delta t}{\Delta x} \left[ g_{x}^{k}(A_{1}^{k} - \hat{A}_{1}^{k}) + (g_{x}^{k} - \hat{g}_{x}^{k}) \hat{A}_{1}^{k} \right] \\
- \Delta t \left[ \mu_{1}^{k}(A_{1}^{k} - \hat{A}_{1}^{k}) + (\mu_{1}^{k} - \hat{\mu}_{1}^{k}) \hat{A}_{1}^{k} \right] \right|
\]

\[
= \left| \frac{\Delta t}{\Delta x} u_{m}^{k} + \left( 1 - \frac{\Delta t}{\Delta x} g_{x}^{k} - \Delta t \mu_{1}^{k} \right) v_{1}^{k} - \frac{\Delta t}{\Delta x} \left( g_{x}^{k} - \hat{g}_{x}^{k} \right) \hat{A}_{1}^{k} - \Delta t (\mu_{1}^{k} - \hat{\mu}_{1}^{k}) \hat{A}_{1}^{k} \right|
\]

\[
\leq \frac{\Delta t}{\Delta x} |u_{m}^{k}| + \left( 1 - \frac{\Delta t}{\Delta x} g_{x}^{k} - \Delta t \mu_{1}^{k} \right) |v_{1}^{k}| + \frac{\Delta t}{\Delta x} |g_{x}^{k} - \hat{g}_{x}^{k}| |\hat{A}_{1}^{k}| + \Delta t |\mu_{1}^{k} - \hat{\mu}_{1}^{k}| |\hat{A}_{1}^{k}|.
\]

Thus,

\[
|v_{x_1}^{k+1}| \Delta x \leq \Delta t |u_{m}^{k}| + (1 - \Delta t \mu_{1}^{k}) |v_{1}^{k}| \Delta x - \Delta t g_{x}^{k} |v_{1}^{k}| + \Delta t |g_{x}^{k} - \hat{g}_{x}^{k}| |\hat{A}_{1}^{k}| + \Delta t |\mu_{1}^{k} - \hat{\mu}_{1}^{k}| |\hat{A}_{1}^{k}| \Delta x.
\]

Adding (1.5.4) and (1.5.5), we get

\[
\|v^{k+1}\|_{1} \leq \|v^{k}\|_{1} + \Delta t |u_{m}^{k}| + \Delta t \left[ \sum_{j=2}^{n} |D_{\Delta x} ((g_{x}^{j} - \hat{g}_{x}^{j}) \hat{A}_{j}^{k})| \Delta x \right.
\]

\[
+ \sum_{j=1}^{n} |\mu_{j}^{k} - \hat{\mu}_{j}^{k}| |\hat{A}_{j}^{k}| \Delta x + |g_{x}^{k} - \hat{g}_{x}^{k}| |\hat{A}_{1}^{k}| \right].
\]

Moreover,

\[
\sum_{j=2}^{n} |D_{\Delta x} ((g_{x}^{j} - \hat{g}_{x}^{j}) \hat{A}_{j}^{k})| \Delta x + \sum_{j=1}^{n} |\mu_{j}^{k} - \hat{\mu}_{j}^{k}| |\hat{A}_{j}^{k}| \Delta x + |g_{x}^{k} - \hat{g}_{x}^{k}| |\hat{A}_{1}^{k}|
\]

\[
= \sum_{j=2}^{n} |D_{\Delta x} (g_{x}^{j} - \hat{g}_{x}^{j}) \hat{A}_{j}^{k}| \Delta x + (g_{x}^{j-1} - \hat{g}_{x}^{j-1}) D_{\Delta x} (\hat{A}_{j}^{k})| \Delta x
\]

\[
+ \sum_{j=1}^{n} |\mu_{j}^{k} - \hat{\mu}_{j}^{k}| |\hat{A}_{j}^{k}| \Delta x + |g_{x}^{k} - \hat{g}_{x}^{k}| |\hat{A}_{1}^{k}|
\]

\[
= \sum_{j=2}^{n} |D_{\Delta x} [g_{Q}(x_{j}, t_{k}, \bar{Q}_{1}^{k})(Q_{x}^{k} - \hat{Q}_{x}^{k})] \hat{A}_{j}^{k} + g_{Q}(x_{j-1}, t_{k}, \bar{Q}_{2}^{k})(Q_{x}^{k} - \hat{Q}_{x}^{k}) D_{\Delta x} (\hat{A}_{j}^{k})| \Delta x
\]

\[
+ \sum_{j=1}^{n} |\mu_{Q}(x_{j}, t_{k}, \bar{Q}_{1}^{k})(Q_{x}^{k} - \hat{Q}_{x}^{k})| |\hat{A}_{j}^{k}| \Delta x + |g_{Q}(x_{1}, t_{k}, \bar{Q}_{2}^{k})(Q_{x}^{k} - \hat{Q}_{x}^{k})| |\hat{A}_{1}^{k}|
\]

\[
\leq \left[ \omega_{3} |\hat{A}_{1}^{k}|_{1} + \sup_{(x,t,Q) \in D_{3}} |g_{Q}| |D_{\Delta x} (\hat{A}_{1}^{k})|_{1} \right.
\]

\[
+ \sup_{(x,t,Q) \in D_{3}} |\mu_{Q}| |\hat{A}_{1}^{k}|_{1} + \sup_{(x,t,Q) \in D_{3}} |g_{Q}| |\hat{A}_{1}^{k}|_{\infty} \right] |Q_{x}^{k} - \hat{Q}_{x}^{k}|,
\]
where $\overline{Q}_1^k, \overline{Q}_2^k, \tilde{Q}_1^k, \tilde{Q}_2^k$ are between $Q^k$ and $\hat{Q}^k$, and $\omega_3 = \sup_{(x,t,Q) \in D_3} |g(x,t,Q)|$.

Therefore, by Lemmas 1.3.1, 1.3.2, 1.3.4 and assumptions (H2)-(H3), there exists a positive constant $c_{11}$ such that

$$n \sum_{j=2}^{n} |D_{\Delta x}((g_j^k - \hat{g}_j^k)A_j^k)| \Delta x + \sum_{j=1}^{n} |\mu_j^k - \hat{\mu}_j^k| |\hat{A}_j^k| \Delta x + |g_1^k - \hat{g}_1^k| |\hat{A}_1^k| \leq c_{11} |Q^k - \hat{Q}^k|. \quad (1.5.7)$$

Applying the above inequality to (1.5.6) and noticing that $|Q^k - \hat{Q}^k| \leq \|v^k\|_1$, we have

$$\|v^{k+1}\|_1 \leq \|v^k\|_1 + \Delta t |u_{m_n}^k| + \Delta t c_{11} \|v^k\|_1. \quad (1.5.8)$$

Adding (1.5.8) to (1.5.3) we arrive at

$$\|u^{k+1}\|_1 + \|v^{k+1}\|_1 \leq (1 + \Delta t c_{10}) \|u^k\|_1 + (1 + \Delta t c_9 + \Delta t c_{11}) \|v^k\|_1.$$

Setting $\sigma = c_9 + c_{10} + c_{11}$, we establish the result.

Next, we prove that the BV solution defined in Lemma 1.4.1 and Theorem 1.4.1 is unique.

**Theorem 1.5.2.** Suppose that $\{J, A\}$ and $\{\hat{J}, \hat{A}\}$ are bounded variation weak solutions of problem (1.1.1) corresponding to the initial conditions $\{J^0, A^0\}$ and $\{\hat{J}^0, \hat{A}^0\}$, respectively, then there exist positive constant $\rho$ and $\lambda$ such that

$$\|J(\cdot, t) - \hat{J}(\cdot, t)\|_1 + \|A(\cdot, t) - \hat{A}(\cdot, t)\|_1 \leq \rho e^{\lambda t} [\|J(\cdot, 0) - \hat{J}(\cdot, 0)\|_1 + \|A(\cdot, 0) - \hat{A}(\cdot, 0)\|_1].$$

**Proof.** Assume that $P, Q$ and $B$ are given Lipschitz continuous functions and consider
the following initial-boundary value problem:

\[
\begin{align*}
J_t + J_a + \nu(a, t, P(t))J &= 0, \quad (a, t) \in (0, a_{\text{max}}] \times (0, T], \\
A_t + (g(x, t, Q(t))A)_x + \mu(x, t, Q(t))A &= 0, \quad (x, t) \in (x_{\text{min}}, x_{\text{max}}] \times (0, T], \\
J(0, t) &= B(t), \quad t \in (0, T], \\
g(x_{\text{min}}, t, Q(t))A(x_{\text{min}}, t) &= J(a_{\text{max}}, t), \quad t \in (0, T], \\
J(a, 0) &= J^0(a), \quad a \in [0, a_{\text{max}}], \\
A(x, 0) &= A^0(x), \quad x \in [x_{\text{min}}, x_{\text{max}}].
\end{align*}
\]

(1.5.9)

Since (1.5.9) is a linear problem with local boundary conditions, it has a unique weak solution. Actually, a weak solution can be defined as a limit of the finite difference approximation with the given numbers \(P^k = P(t_k), Q^k = Q(t_k)\) and \(B^k = B(t_k)\), and the uniqueness can be established by using similar techniques as in [17]. In addition, as in the proof of Theorem 1.5.1, we can show that if \(\{J^k_i, A^k_j\}\) and \(\{\hat{J}^k_i, \hat{A}^k_j\}\) are solutions of the difference scheme corresponding to given functions \(\{P^k, Q^k, B^k\}\) and \(\{\hat{P}^k, \hat{Q}^k, \hat{B}^k\}\), respectively, then there exist positive constants \(c_{12}, c_{13}\) such that

\[
\begin{align*}
\|u^{k+1}\|_1 + \|v^{k+1}\|_1 &\leq (1 + c_{12}\Delta t)(\|u^{k}\|_1 + \|v^{k}\|_1) + [c_{13}(|P^k - \hat{P}^k| + |Q^k - \hat{Q}^k|) + |B^k - \hat{B}^k|]\Delta t, \\
\end{align*}
\]

(1.5.10)

where \(u^k = J^k - \hat{J}^k, v^k = A^k - \hat{A}^k\).

In fact, here \(J^0_k = B(t_k) = B^k, \hat{J}^0_k = \hat{B}(t_k) = \hat{B}^k, u^0_k = B^k - \hat{B}^k\), so by (1.5.2) we have

\[
\|u^{k+1}\|_1 \leq \|u^k\|_1 + \Delta t|B^k - \hat{B}^k| - \Delta t|u_m^k| + \Delta t \sum_{i=1}^{m} |\nu_i^k J_i^k - \hat{\nu}_i^k \hat{J}_i^k| \Delta a.
\]

Moreover, we have

\[
\sum_{i=1}^{m} \left| \nu_i^k J_i^k - \hat{\nu}_i^k \hat{J}_i^k \right| \Delta a \leq \max_i (\nu_i^k) \|u^k\|_1 + \sup_{(a, t, P) \in \mathbb{D}_4} |\nu_P| \|\hat{J}^k\|_1 |P^k - \hat{P}^k|.
\]
Thus,
\[
\|u^{k+1}\|_1 \leq (1 + \max_i (\nu_i^k) \Delta t)\|u^k\|_1 + \Delta t|B^k - \hat{B}^k| - \Delta t|u^m_k|
\]
\[
+ \sup_{(a,t,P) \in \mathcal{D}_4} |\nu_P|\|\hat{J}^k\|_1 |P^k - \hat{P}^k| \Delta t. 
\]  
(1.5.11)

On the other hand, from (1.5.6)-(1.5.7) we find
\[
\|v^{k+1}\|_1 \leq \|v^k\|_1 + \Delta t|u^m_k| + \Delta t c_{11} (Q^k - \hat{Q}^k).
\]  
(1.5.12)

Adding (1.11) and (1.12), and letting \(c_{12} = \max_i |\nu_i^k|, c_{13} = \sup_{(a,t,P) \in \mathcal{D}_4} |\nu_P|\|\hat{J}^k\|_1 + c_{11}\), we obtain (1.10). Furthermore, (1.10) is equivalent to
\[
\|u^k\|_1 + \|v^k\|_1 \leq (1 + c_{12} \Delta t)^k (\|u^0\|_1 + \|v^0\|_1)
\]
\[
+ \sum_{r=0}^{k-1} (1 + c_{12} \Delta t)^r \left[ c_{13} (|P^{k-1-r} - \hat{P}^{k-1-r}| + |Q^{k-1-r} - \hat{Q}^{k-1-r}| + |B^{k-1-r} - \hat{B}^{k-1-r}|) \right] \Delta t. 
\]  
(1.5.13)

Hence,
\[
\|u^k\|_1 + \|v^k\|_1 \leq (1 + c_{12} \Delta t)^k \left[ \|u^0\|_1 + \|v^0\|_1 + \sum_{r=0}^{k-1} \left( c_{13} (|P^{k-1-r} - \hat{P}^{k-1-r}| + |Q^{k-1-r} - \hat{Q}^{k-1-r}| + |B^{k-1-r} - \hat{B}^{k-1-r}|) \right) \Delta t \right]. 
\]  
(1.5.14)

Now, from Theorem 1.4.1 we can take the limit in (1.13) to obtain
\[
\|u(t)\|_1 + \|v(t)\|_1 \leq e^{c_{12} T} \left[ \|u(0)\|_1 + \|v(0)\|_1 + \int_0^T \left( c_{13} (|P(s) - \hat{P}(s)| + |Q(s) - \hat{Q}(s)| + |B(s) - \hat{B}(s)|) \right) ds \right],
\]  
(1.5.14)

where \(u(t) = J(\cdot , t) - \hat{J}(\cdot , t), v(t) = A(\cdot , t) - \hat{A}(\cdot , t), \{J(\cdot , t), A(\cdot , t)\}\) and
\(\{\hat{J}(\cdot , t), \hat{A}(\cdot , t)\}\) are the unique solutions of (1.5.9) with any set of given functions
\(\{P(t), Q(t), B(t)\}\) and \(\{\hat{P}(t), \hat{Q}(t), \hat{B}(t)\}\), respectively.

We then apply the estimate given in (1.14) for the corresponding solutions of
(1.5.9) with two specific sets of functions \(\{P(t), Q(t), B(t)\}\) and \(\{\hat{P}(t), \hat{Q}(t), \hat{B}(t)\}\)

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which are constructed using the limits obtained in Lemma 1.4.1 as follows:

\[
P(t) = \int_0^{a_{\max}} J(a, t) \, da, \quad \hat{P}(t) = \int_0^{a_{\max}} \hat{J}(a, t) \, da, \\
Q(t) = \int_{x_{\min}}^{x_{\max}} A(x, t) \, dx, \quad \hat{Q}(t) = \int_{x_{\min}}^{x_{\max}} \hat{A}(x, t) \, dx, \\
B(t) = \int_{x_{\min}}^{x_{\max}} \beta(x, t, Q(t)) A(x, t) \, dx, \quad \hat{B}(t) = \int_{x_{\min}}^{x_{\max}} \beta(x, t, \hat{Q}(t)) \hat{A}(x, t) \, dx.
\]

Thus, we get

\[
|P(s) - \hat{P}(s)| = \int_0^{a_{\max}} \left| J(a, s) - \hat{J}(a, s) \right| \, da \leq \int_0^{a_{\max}} |u(a, s)| \, da = \|u(s)\|_1,
\]
\[
|Q(s) - \hat{Q}(s)| = \int_{x_{\min}}^{x_{\max}} \left| A(x, s) - \hat{A}(x, s) \right| \, dx \leq \int_{x_{\min}}^{x_{\max}} |v(x, s)| \, dx = \|v(s)\|_1
\]

and

\[
|B(s) - \hat{B}(s)| \\
\leq \int_{x_{\min}}^{x_{\max}} \left| \beta(x, s, Q(s)) [A(x, s) - \hat{A}(x, s)] + [\beta(x, s, Q(s)) - \beta(x, s, \hat{Q}(s))] \hat{A}(x, s) \right| \, dx \\
= \int_{x_{\min}}^{x_{\max}} \left| \beta(x, s, Q(s)) [A(x, s) - \hat{A}(x, s)] + \beta_Q(x, s, Q(s)) [Q(s) - \hat{Q}(s)] \hat{A}(x, s) \right| \, dx \\
\leq \sup_{(x, t, Q) \in \mathbb{D}_2} \beta \|v(s)\|_1 + \sup_{(x, t, Q) \in \mathbb{D}_3} \|\beta_Q\| \|v(s)\|_1 \|\hat{A}\| \mathcal{L}^{\infty((x_{\min}, x_{\max}) \times (0, T))} (x_{\max} - x_{\min}).
\]

Hence,

\[
\int_0^t \left[ c_{13} (|P(s) - \hat{P}(s)| + |Q(s) - \hat{Q}(s)|) + |B(s) - \hat{B}(s)| \right] \, ds \\
\leq \int_0^t \left[ c_{13} (\|u(s)\|_1 + \|v(s)\|_1) \right. \\
+ \left. \left( \sup_{(x, t, Q) \in \mathbb{D}_2} \beta \right) \sup_{(x, t, Q) \in \mathbb{D}_3} \|\beta_Q\| \|\hat{A}\| \mathcal{L}^{\infty((x_{\min}, x_{\max}) \times (0, T))} (x_{\max} - x_{\min}) \|v(s)\|_1 \right] \, ds \\
\leq c_{14} \int_0^t [\|u(s)\|_1 + \|v(s)\|_1] \, ds,
\]
where \( c_{14} = c_{12} + \sup_{(x,t,Q) \in \mathbb{D}_2} \beta + \sup_{(x,t,Q) \in \mathbb{D}_3} |\beta_Q| ||\hat{A}||_{L^\infty((x_{\min},x_{\max}) \times (0,T))}(x_{\max} - x_{\min})\). Therefore,

\[
\|u(t)\|_1 + \|v(t)\|_1 \leq e^{c_{12}T} \left[ \|u(0)\|_1 + \|v(0)\|_1 + c_{14} \int_0^t (\|u(s)\|_1 + \|v(s)\|_1) ds \right].
\]

Using Grownwall’s inequality, we find

\[
\|u(t)\|_1 + \|v(t)\|_1 \leq \exp\{c_{12}T + c_{14}e^{c_{12}T}t\}(\|u(0)\|_1 + \|v(0)\|_1).
\]

Letting \( \rho = e^{c_{12}T}, \lambda = c_{14}e^{c_{12}T} \), we obtain

\[
\|J(\cdot,t) - \hat{J}(\cdot,t)\|_1 + \|A(\cdot,t) - \hat{A}(\cdot,t)\|_1 \leq \rho e^{\lambda t} \left[ \|J(\cdot,0) - \hat{J}(\cdot,0)\|_1 + \|A(\cdot,0) - \hat{A}(\cdot,0)\|_1 \right].
\]
REFERENCES


In this chapter\(^1\), we derive several stochastic models from a deterministic population model that describes the dynamics of age-structured juveniles coupled with size-structured adults. Numerical simulation results of the stochastic models are compared with the solution of the deterministic model. These models are then used to understand the effect of demographic stochasticity on the dynamics of an urban green tree frog (*Hyla cinerea*) population.

2.1 Introduction

The effects of stochastic factors, including demographic and environmental stochasticity, on the dynamics of many populations are well documented (e.g., [12, 16]). In the past few years, researchers have devoted serious efforts to the development of continuous (discrete)-time Markov chain population models, individual-based models and continuous Itô stochastic differential equation models. The analyses of these models have contributed new insights into the effect of randomness on population dynamics [4, 8, 9, 12, 16].

The focus of this chapter is on the development of stochastic models that describe the dynamics of a juvenile population structured by age and coupled to an

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\(^1\)The results of this chapter have been published by *Journal of Biological Dynamics* [5 (2011), 64-83].
adult population structured by size. Here we are interested in demographic stochasticity. Our deterministic skeleton model is given by the following system of partial differential equations which we recently developed in [2] to describe the dynamics of an amphibian population:

\[
\begin{align*}
\frac{\partial J(a,t)}{\partial t} &+ \frac{\partial J(a,t)}{\partial a} + \nu(a,t,P(t))J(a,t) = 0 \\
J(0,t) &= \int_{x_{\min}}^{x_{\max}} \beta(x,t,Q(t))A(x,t)dx \\
J(a,0) &= J_0(a) \\
\frac{\partial A(x,t)}{\partial t} &+ \frac{\partial (g(x,t,Q(t))A(x,t))}{\partial x} + \mu(x,t,Q(t))A(x,t) = 0 \\
g(x_{\min},t,Q(t))A(x_{\min},t) &= J(a_{\max},t) \\
A(x,0) &= A_0(x),
\end{align*}
\]

where \(0 \leq a \leq a_{\max}, \ x_{\min} \leq x \leq x_{\max} \) and \(t > 0\). The density of juveniles of age \(a\) at time \(t\) is given by \(J(a,t)\), and \(A(x,t)\) is the density of adults having size \(x\) at time \(t\). \(P(t) = \int_0^{a_{\max}} J(a,t)da\) and \(Q(t) = \int_{x_{\min}}^{x_{\max}} A(x,t)dx\) are the number of juveniles and adults at time \(t\), respectively. The maximum age of juveniles is given by \(a_{\max}\), while \(x_{\min}\) and \(x_{\max}\) are the minimum and maximum sizes of an adult, respectively. The function \(\nu(a,t,P)\) denotes the mortality rate of a juvenile of age \(a\) at time \(t\) which depends on the number of juveniles \(P\). The function \(\mu(x,t,Q)\) represents the mortality rate of an adult of size \(x\) at time \(t\) which depends on the number of adults \(Q\). In model (2.1.1), it is assumed that there is no competition between juveniles and adults, which is typical of amphibian populations since juveniles (tadpoles) live in water and adults (frogs) live on land. The function \(g(x,t,Q)\) is the growth rate of an adult of size \(x\) at time \(t\) with adult population level \(Q\), and \(\beta(x,t,Q)\) represents the reproduction rate of
an adult of size \( x \) at time \( t \) with adult population level \( Q \). Notice that the first boundary condition represents total reproduction of juveniles of age 0 by adults, and the second boundary condition states that juveniles of the maximum age \( a_{\text{max}} \) metamorphose into adults of the minimum size \( x_{\text{min}} \).

We point out that model (2.1.1), which is an extension of the model developed and studied in [1], was investigated in [2]. Therein, we developed an explicit finite difference approximation to model (2.1.1), proved the existence and uniqueness of a weak solution, and established the convergence of the finite difference approximation to the unique weak solution.

Although many stochastic analogues of deterministic difference/differential equation population models have been developed [4, 5, 9, 12, 13, 16], very few stochastic models for continuous size-structured populations are available in the literature. In fact, to the best of our knowledge, there is only one such stochastic model which describes the dynamics of a single species structured by age and size [10]. One of goals of this chapter is to extend the model developed in [7] to the juvenile-adult population model described above and to construct a computational scheme for solving this model. The derivation of a discrete Markov chain model (DMCM) and the system of stochastic partial differential equations (SSPDE) is based on the method developed in [6, 7, 8] which runs roughly as follows: we divide the juveniles and adults into different age and size classes, the changes of the population level of each class are assumed to occur randomly due to the randomness in reproduction, growth and mortality. A DMCM is then developed by considering the
changes that occur over a small time interval with probabilities obtained from the vital
rates in the deterministic model (2.1.1). This DMCM results in an Itô system as the
time interval decreases to zero. Letting the age intervals and size intervals shrink to
zero, an SSPDE for the juvenile-adult model is derived.

We note that the SSPDE formulation offers a number of interesting advantages
over discrete modeling approaches such as Markov chains, continuous time jump
processes, or stochastic ordinary differential equations. In particular, the SSPDE
provides a structure that unifies these three models as different types of
approximations. For example, the finite difference solution of the SSPDE often leads to
Markov chain approximation, while method-of-lines approximations yield stochastic
ordinary differential equations.

We organize this chapter as follows. In Section 2.2, we derive a DMCM based
on the deterministic model (2.1.1). In Section 2.3, an SSPDE corresponding to the
partial differential equation system (2.1.1) is constructed. In Section 2.4, we formulate
an individual-based model (IBM) with general birth distribution from the
deterministic model (2.1.1). In Section 2.5, the stochastic models are numerically
solved and compared with the solution of the deterministic model. In Section 2.6, the
SSPDE and the IBM are used to understand the effect of demographic stochasticity on
the dynamics of an urban Green Tree Frog (*Hyla Cinerea*) population. Finally, we
make some concluding remarks in Section 2.7.
2.2 A discrete Markov chain model (DMCM)

We derive a DMCM by discretizing the age and size categories and assuming that each possible transition involves a single individual at a rate obtained from the deterministic model (2.1.1). Consider the changes of juvenile and adult population level over a small time interval. To find these changes, we divide the juveniles into $M$ age intervals $[a_{i-1}, a_i]$ for $i = 1, 2, \cdots, M$, where $a_i = i\Delta a$, $\Delta a = a_{\text{max}}/M$, and $P_i(t)$ represents the number of juveniles between age $a_{i-1}$ and age $a_i$ at time $t$. The possible changes of $P_i(t), i > 1$ and their probabilities are given in Table 2.1, and the possible changes of $P_1(t)$ and their probabilities are listed in Table 2.2 when births are added to the first age class. Similarly, we divide adults into $N$ size intervals $[x_{j-1}, x_j]$ for $j = 1, 2, \cdots, N$ where $x_j = x_{\text{min}} + j\Delta x$, $\Delta x = (x_{\text{max}} - x_{\text{min}})/N$, and $Q_j(t)$ is the number of adults between size $x_{j-1}$ and size $x_j$ at time $t$. The possible changes of $Q_j(t), j > 1$ and their probabilities are listed in Table 2.3, and the possible changes of $Q_1(t)$ and their probabilities are listed in Table 2.4 when the last age class of juveniles is added to the first size class of adults (metamorphosis).

Table 2.1. The possible changes of juvenile population level $P_i(t), i > 1$ and their probabilities over sufficiently small time interval $\Delta t$.

<table>
<thead>
<tr>
<th>Possible Change $\Delta P_i$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$P_{i-1}\Delta t/\Delta a$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$P_i\Delta t/\Delta a$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$\nu_i P_i \Delta t$</td>
</tr>
</tbody>
</table>
Table 2.2. The possible changes of juvenile population level $P_1(t)$ and their probabilities over sufficiently small time interval $\Delta t$.

<table>
<thead>
<tr>
<th>Possible Change $\Delta P_1$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\sum_{j=1}^{N} \beta_j Q_j \Delta t$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$P_1 \Delta t / \Delta a$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$\nu_1 P_1 \Delta t$</td>
</tr>
</tbody>
</table>

Table 2.3. The possible changes of adult population level $Q_j(t), j > 1$ and their probabilities over sufficiently small time interval $\Delta t$.

<table>
<thead>
<tr>
<th>Possible Change $\Delta Q_j$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$g_{j-1} Q_{j-1} \Delta t / \Delta x$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$g_j Q_j \Delta t / \Delta x$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$\mu_j Q_j \Delta t$</td>
</tr>
</tbody>
</table>

Table 2.4. The possible changes of adult population level $Q_1(t)$ and their probabilities over sufficiently small time interval $\Delta t$.

<table>
<thead>
<tr>
<th>Possible Change $\Delta Q_1$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$P_M \Delta t / \Delta a$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$g_1 Q_1 \Delta t / \Delta x$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$\mu_1 Q_1 \Delta t$</td>
</tr>
</tbody>
</table>

In these tables, $\nu_i = \nu(a_i, t, P(t))$ is the mortality rate for juveniles of age class $i$, $\mu_j = \mu(x_j, t, Q(t))$ is the mortality rate for adults of size class $j$; while $g_j = g(x_j, t, Q(t))$ and $\beta_j = \beta(x_j, t, Q(t))$ are the growth rate and the reproduction rate of adults, respectively. These tables determine a DMCM for the system of $M$ sub-juveniles and $N$ sub-adults. The numerical results of the DMCM are obtained through Monte Carlo simulations and compared with that of the deterministic model (2.1.1) in Section 2.5.
The mean and variance of the changes play important roles in constructing a system of Itô stochastic differential equations. The mean can be easily calculated using the above tables:

\[
E(\Delta P_1) = \sum_{j=1}^{N} \beta_j Q_j \Delta t - \frac{P_1 \Delta t}{\Delta a} - \nu_1 P_1 \Delta t, \\
E(\Delta P_i) = \frac{P_{i-1} \Delta t}{\Delta a} - \frac{P_i \Delta t}{\Delta a} - \nu_i P_i \Delta t, \quad i = 2, 3, \cdots, M, \\
E(\Delta Q_1) = \frac{P_M \Delta t}{\Delta a} - \frac{g_1 Q_1 \Delta t}{\Delta x} - \mu_1 Q_1 \Delta t, \\
E(\Delta Q_j) = \frac{g_{j-1} Q_{j-1} \Delta t}{\Delta x} - \frac{g_j Q_j \Delta t}{\Delta x} - \mu_j Q_j \Delta t, \quad j = 2, 3, \cdots, N.
\]

In addition, if terms of order \((\Delta t)^2\) are neglected, the covariance matrix can be approximated by a \((M + N) \times (M + N)\) tridiagonal matrix \(V \Delta t\), where \(V = (v_{i,j})\) satisfies:

\[
v_{1,1} = \sum_{j=1}^{N} \beta_j Q_j + \frac{P_1}{\Delta a} + \nu_1 P_1, \\
v_{i,i} = \frac{P_{i-1}}{\Delta a} + \frac{P_i}{\Delta a} + \nu_i P_i, \quad i = 2, 3, \cdots, M, \\
v_{M+1,M+1} = \frac{P_M}{\Delta a} + \frac{g_1 Q_1}{\Delta x} + \mu_1 Q_1, \\
v_{M+j,M+j} = \frac{g_{j-1} Q_{j-1}}{\Delta x} + \frac{g_j Q_j}{\Delta x} + \mu_j Q_j, \quad j = 2, 3, \cdots, N, \\
v_{i,i+1} = v_{i+1,i} = -\frac{P_i}{\Delta a}, \quad i = 1, 2, \cdots, M - 1, \\
v_{M,M+1} = v_{M+1,M} = -\frac{P_M}{\Delta a}, \\
v_{M+j,M+j+1} = v_{M+j+1,M+j} = -\frac{g_j Q_j}{\Delta x}, \quad j = 1, 2, \cdots, N - 1.
\]

In the next section we use the DMCM to construct an SSPDE. First, the time interval is decreased to zero to obtain an Itô SDE system, and then the age and size intervals are decreased to zero to obtain an SSPDE.
2.3 Construction of stochastic differential and partial differential equation models

In [8] the authors introduced two modeling procedures to obtain an Itô SDE system from a discrete model such as the one presented above, and they showed that the following two systems of stochastic differential equations which result from these modeling procedures are equivalent:

\begin{align}
    d\vec{X}(t) &= \vec{f}(t, \vec{X}(t))dt + G(t, \vec{X}(t))d\vec{W}(t), \quad (2.3.1) \\
    d\vec{X}^*(t) &= \vec{f}(t, \vec{X}^*(t))dt + B(t, \vec{X}^*(t))d\vec{W}^*(t), \quad (2.3.2)
\end{align}

Here,

\[ \vec{f} : [0, \tau] \times \mathbb{R}^d \to \mathbb{R}^d, \quad G : [0, \tau] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}, \quad B : [0, \tau] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}, \]

\[ \vec{X}(t) = [X_1(t), \ldots, X_d(t)]^T, \quad \vec{X}^*(t) = [X_1^*(t), \ldots, X_d^*(t)]^T, \]

\[ \vec{W}(t) = [W_1(t), \ldots, W_m(t)]^T, \quad \vec{W}^*(t) = [W_1^*(t), \ldots, W_d^*(t)]^T, \]

where \( W_i, \ i = 1, \ldots, m \) and \( W_j^*, \ j = 1, \ldots, d \) are independent Wiener processes and \( m \geq d \). Matrices \( G \) and \( B \) are related through the \( d \times d \) variance matrix \( V \), where

\[ V(t, \vec{z}) = G(t, \vec{z})G^T(t, \vec{z}) \quad \text{and} \quad B(t, \vec{z}) = V^{1/2}(t, \vec{z}) \quad \text{for} \ \vec{z} \in \mathbb{R}^d. \]

In particular, the authors of [8] proved that the two systems of stochastic differential equations (2.3.1) and (2.3.2) possess the same probability distribution, and a sample path solution of one equation is a sample path solution of the second equation. In this, we focus on the modeling procedure that results in the SDE system (2.3.1). To this end, following [8] we let the random changes of each class of juveniles
due to reproduction, growth and mortality be approximated by independent normal
random variables $\zeta, \xi_i$ and $\eta_i$ for $i = 1, 2, \cdots, M$, respectively. Similarly, let the
random changes of each class of adults due to growth and mortality are approximated
using independent normal random variables $\xi_j$ and $\eta_j$ for $j = 1, 2, \cdots, N$, respectively.
The normal approximation may be justified by arguments involving the Central Limit
Theorem or by normal approximations to Poisson random variables [8]. Then, the
DMCM for small but fixed $\Delta t$ is approximated by

$$
\Delta P_i = E(\Delta P_i) + \sqrt{\frac{P_i \Delta t}{\Delta a}} \xi_{i-1} - \sqrt{\nu_i P_i \Delta t \eta_i},
$$

(2.3.3)

for $i = 2, 3, \cdots, M$ with

$$
\Delta P_1 = E(\Delta P_1) + \sqrt{\sum_{j=1}^{N} \beta_j Q_j \Delta t \xi_j} - \sqrt{\nu_1 P_1 \Delta t \eta_1},
$$

(2.3.4)

and

$$
\Delta Q_j = E(\Delta Q_j) + \sqrt{\frac{g_j Q_j \Delta t}{\Delta x}} \xi_{j-1} - \sqrt{\mu_j Q_j \Delta t \eta_j},
$$

(2.3.5)

for $j = 2, 3, \cdots, N$ with

$$
\Delta Q_1 = E(\Delta Q_1) + \sqrt{\frac{P_M \Delta t}{\Delta a}} \xi_M - \sqrt{\mu_1 Q_1 \Delta t \eta_1}.
$$

(2.3.6)

For small $\Delta t$, the discrete stochastic model (2.3.3)-(2.3.6) is an Euler-Maruyama
approximation [17] to the following Itô system:

$$
dP_i(t) = -\frac{P_i - P_{i-1}}{\Delta a} dt - \nu_i P_i dt + \sqrt{\frac{P_i \Delta t}{\Delta a}} dW_{i-1}(t)
- \sqrt{\nu_i P_i d\hat{W}_i(t)} - \sqrt{\nu_i P_i d\hat{W}_i(t)},
$$

(2.3.7)
for $i = 2, 3, \ldots, M$ with

$$dP_1(t) = \sum_{j=1}^{N} \beta_j Q_j dt - \frac{P_1}{\Delta a} dt - \nu_1 P_1 dt + \sqrt{\sum_{j=1}^{N} \beta_j Q_j dW(t)}$$

$$- \sqrt{\frac{P_1}{\Delta a}} dW_1(t) - \nu_1 P_1 d\hat{W}_1(t),$$

(2.3.8)

and

$$dQ_j(t) = -\frac{g_j Q_j - g_{j-1} Q_{j-1}}{\Delta x} dt - \mu_j Q_j dt + \sqrt{\frac{g_{j-1} Q_{j-1}}{\Delta x}} dW_{j-1}(t)$$

$$- \sqrt{\frac{g_j Q_j}{\Delta x}} dW_j(t) - \mu_j Q_j d\hat{W}_j(t),$$

(2.3.9)

for $j = 2, 3, \ldots, N$ with

$$dQ_1(t) = \frac{P_M}{\Delta a} dt - \frac{g_1 Q_1}{\Delta x} dt - \mu_1 Q_1 dt + \sqrt{\frac{P_M}{\Delta a}} dW_M(t)$$

$$- \sqrt{\frac{g_1 Q_1}{\Delta x}} dW_1(t) - \mu_1 Q_1 d\hat{W}_1(t).$$

(2.3.10)

Here $W_i(t)$, $W_j(t)$, $\hat{W}_i(t)$, $\hat{W}_j(t)$ and $W(t)$ are again independent Wiener processes for $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$. Notice that for small $\Delta t$, the system (2.3.7)-(2.3.10) has approximately the same mean and variance changes as the DMCM. Furthermore, if we let $G = (g_{i,j})$ be an $(M + N) \times (2M + 2N + 1)$ matrix whose nonzero elements satisfy

$$g_{1,1} = \sqrt{\sum_{j=1}^{N} \beta_j Q_j}, \; g_{1,2} = -\nu_1 P_1, \; g_{1,3} = -\sqrt{\frac{P_1}{\Delta a}},$$

$$g_{i,2i-1} = \sqrt{\frac{P_{i-1}}{\Delta a}}, \; g_{i,2i} = -\nu_i P_i, \; g_{i,2i+1} = -\sqrt{\frac{P_i}{\Delta a}}, \; i = 2, 3, \ldots, M,$$

$$g_{M+1,2M+1} = \sqrt{\frac{P_M}{\Delta a}}, \; g_{M+1,2M+2} = -\sqrt{\mu_1 Q_1}, \; g_{M+1,2M+3} = -\sqrt{\frac{g_1 Q_1}{\Delta x}},$$

$$g_{M+j,2M+2j-1} = \sqrt{\frac{g_{j-1} Q_{j-1}}{\Delta x}}, \; g_{M+j,2M+2j} = -\sqrt{\mu_j Q_j},$$

$$g_{M+j,2M+2j+2} = -\sqrt{\frac{g_j Q_j}{\Delta x}}, \; j = 2, 3, \ldots, N,$$
then it is easy to see that the matrix $G$ satisfies $GG^T = V$, where $V$ is the covariance matrix given in Section 2.2. It is worth noting that an advantage of this procedure is that it is generally easier to solve computationally than the other procedure which requires the computation of the square root of the covariance matrix $V$ (see [8]).

Next, as in [7] we introduce one-dimensional Wiener processes $W^*(t; a)$ and $W^*(t; x)$ parameterized by age $a$ and size $x$, respectively, such that $W^*(t; a)$ and $W^*(t; x)$ are independent Wiener process for each value of $a$ and $x$, respectively. Furthermore, two Brownian sheets $W(a, t)$ and $W(x, t)$ are applied, and then Equations (2.3.7)-(2.3.10) can be written as:

$$\frac{dP_i(t)}{dt} = - \frac{P_i - P_{i-1}}{\Delta a} - \nu_i P_i + \sqrt{\frac{P_{i-1}}{\Delta a}} \frac{\partial W^*(t; a_{i-1})}{\partial t}$$

$$- \sqrt{\frac{P_i}{\Delta a}} \frac{\partial W^*(t; a_i)}{\partial t} - \nu_i P_i \frac{1}{\sqrt{\Delta a}} \int_{a_{i-1}}^{a_i} \frac{\partial^2 W(a, t)}{\partial a \partial t} da,$$

for $i = 2, 3, \ldots, M$ with

$$\frac{dP_1(t)}{dt} = \sum_{j=1}^{N} \beta_j Q_j - \frac{P_1}{\Delta a} - \nu_1 P_1 + \sqrt{\sum_{j=1}^{N} \beta_j Q_j} \frac{dW(t)}{dt}$$

$$- \sqrt{\frac{P_1}{\Delta a}} \frac{\partial W^*(t; a_1)}{\partial t} - \nu_1 P_1 \frac{1}{\sqrt{\Delta a}} \int_{a_0}^{a_1} \frac{\partial^2 W(a, t)}{\partial a \partial t} da,$$

and

$$\frac{dQ_j(t)}{dt} = - \frac{g_j Q_j - g_{j-1} Q_{j-1}}{\Delta x} - \mu_j Q_j + \sqrt{\frac{g_{j-1} Q_{j-1}}{\Delta x}} \frac{\partial W^*(t; x_{j-1})}{\partial t}$$

$$- \sqrt{\frac{g_j Q_j}{\Delta x}} \frac{\partial W^*(t; x_j)}{\partial t} - \mu_j Q_j \frac{1}{\sqrt{\Delta x}} \int_{x_{j-1}}^{x_{j}} \frac{\partial^2 W(x, t)}{\partial x \partial t} dx,$$

(2.3.13)
for \( j = 2, 3, \ldots, N \) with

\[
\frac{dQ_j(t)}{dt} = \frac{P_M}{\Delta a} - \frac{g_1 Q_j}{\Delta x} - \mu_1 Q_j + \sqrt{\frac{P_M}{\Delta a}} \frac{\partial W^*(t, a_M)}{\partial t} - \sqrt{\frac{g_1}{\Delta x}} \frac{\partial W^*(t, x_1)}{\partial t} - \sqrt{\mu_1} \frac{1}{\sqrt{\Delta x}} \int_{x_0}^{x_1} \frac{\partial^2 W(x, t)}{\partial x \partial t} dx.
\] (2.3.14)

Letting \( P_i(t) = J(a_i, t) \Delta a, \ Q_j(t) = A(x_j, t) \Delta x \), following [7] and allowing the age interval \( \Delta a \) and the size interval \( \Delta x \) to approach zero in Equation (2.3.11)-(2.3.14), we have

\[
\frac{\partial J(a, t)}{\partial t} = - \frac{\partial J(a, t)}{\partial a} - \nu(a, t, P(t)) J(a, t) - \frac{\partial}{\partial a} \left( \sqrt{J(a, t)} \frac{\partial W^*(t; a)}{\partial t} \right) - \sqrt{\nu(a, t, P(t))} J(a, t) \frac{\partial^2 W(a, t)}{\partial a \partial t},
\] (2.3.15)

where Equation (2.3.12) reduces to

\[
J(0, t) + \sqrt{J(0, t)} \frac{\partial W^*(t; 0)}{\partial t} = \int_{x_{\min}}^{x_{\max}} \beta(x, t, Q(t)) A(x, t) dx + \sqrt{\int_{x_{\min}}^{x_{\max}} \beta(x, t, Q(t)) A(x, t) dx} \frac{dW(t)}{dt},
\] (2.3.16)

and

\[
\frac{\partial A(x, t)}{\partial t} = - \frac{\partial (g(x, t, Q(t)) A(x, t))}{\partial x} - \mu(x, t, Q(t)) A(x, t)
- \frac{\partial}{\partial x} \left( \sqrt{g(x, t, Q(t)) A(x, t)} \frac{\partial W^*(t; x)}{\partial t} \right)
- \sqrt{\mu(x, t, Q(t))} A(x, t) \frac{\partial^2 W(x, t)}{\partial x \partial t},
\] (2.3.17)

where Equation (2.3.14) reduces to

\[
g(x_{\min}, t, Q(t)) A(x_{\min}, t) + \sqrt{g(x_{\min}, t, Q(t)) A(x_{\min}, t)} \frac{\partial W^*(t; x_{\min})}{\partial t} = J(a_{\max}, t) + \sqrt{J(a_{\max}, t)} \frac{\partial W^*(t; a_{\max})}{\partial t}.
\] (2.3.18)
Here $W^*(t; a)$, $W^*(t; x)$ and $W(t)$ are independent Wiener processes, and $W(a, t)$, $W(x, t)$ are Brownian sheets. The initial conditions at time $t = 0$ are given by

$$J(a, 0) = J_0(a), \quad A(x, 0) = A_0(x), \quad (2.3.19)$$

and the number of juveniles and adults are given by

$$P(t) = \int_0^{a_{\max}} J(a, t) da, \quad Q(t) = \int_{x_{\min}}^{x_{\max}} A(x, t) dx.$$

Equations (2.3.15)-(2.3.18) together with the initial conditions (2.3.19) constitute an SSPDE based on the juvenile-adult model (2.1.1). Note that if the stochastic terms are set equal to zero, then system (2.3.15)-(2.3.18) is identical to system (2.1.1).

2.4 An individual-based model (IBM) with a general birth distribution

In this section, we formulate a discrete IBM for the deterministic model (2.1.1) with a general birth distribution ([13]). Let $P_i(t)$ for $i = 1, 2, \cdots, M$ and $Q_j(t)$ for $j = 1, 2, \cdots, N$ denote the random variables for the number of juveniles between age $a_{i-1}$ and age $a_i$ and the number of adults between size $x_{j-1}$ and size $x_j$, respectively at time $t = 0, \Delta t, 2\Delta t, \cdots$. For each fixed $t$, we define a birth distribution $\{b_{n,j}\}_{n=1}^{\infty}$ for $j = 1, 2, \cdots, N$ where $b_{n,j}$ is the probability of $n$ births produced by an adult between size $x_{j-1}$ and size $x_j$. Then $\{b_{n,j}\}$ has the following two properties:

$$\sum_{n=0}^{\infty} b_{n,j} = 1 \quad \text{and} \quad \sum_{n=0}^{\infty} nb_{n,j} = \beta_j < \infty.$$

Let $\{X_{j,r}\}_{r=1}^{Q_j}$ for $j = 1, 2, \cdots, N$ denote independent and identically distributed random variables, where $X_{j,r}$ represents the number of births produced by the $r$th
individual in the $j$th class of adults over the time interval $\Delta t$, then

$$E(X_{j,r}) = \sum_{n=0}^{\infty} nb_{n,j} \Delta t = \beta_j \Delta t.$$  

Similarly, we let $\{Y_{i,r}^{J}\}_{r=1}^{P_i}$ for $i = 1, 2, \ldots, M$ and $\{Y_{j,r}^{A}\}_{r=1}^{Q_j}$ for $j = 1, 2, \ldots, N$ denote independent and identically distributed random variables which model the mortality of juveniles and adults, respectively. Let $\{Z_{i,r}^{J}\}_{r=1}^{P_i}$ for $i = 1, 2, \ldots, M$ and $\{Z_{j,r}^{A}\}_{r=1}^{Q_j}$ for $j = 1, 2, \ldots, N$ denote independent and identically distributed random variables which model the movement of juveniles and adults from the current class to the next class due to increase in age and growth in size. Assume $\Delta t$ is sufficiently small and the random variables $\{Y_{i,r}^{J}\}, \{Y_{j,r}^{A}\}, \{Z_{i,r}^{J}\}$ and $\{Z_{j,r}^{A}\}$ have Bernoulli distributions with means

$$E(Y_{i,r}^{J}) = \nu_i \Delta t, \quad r = 1, 2, \ldots, P_i, \quad i = 1, 2, \ldots, M,  
E(Y_{j,r}^{A}) = \mu_j \Delta t, \quad r = 1, 2, \ldots, Q_j, \quad j = 1, 2, \ldots, N,  
E(Z_{i,r}^{J}) = \Delta t/\Delta a, \quad r = 1, 2, \ldots, P_i, \quad i = 1, 2, \ldots, M,  
E(Z_{j,r}^{A}) = g_j \Delta t/\Delta x, \quad r = 1, 2, \ldots, Q_j, \quad j = 1, 2, \ldots, N.$$

The discrete random variables for the number of juveniles and the number of adults are defined in terms of the previously defined random variables. Given the number of juveniles in each age class and the number of adults in each size class at time $t$, the corresponding population level at time $t + \Delta t$ satisfy

$$P_1(t + \Delta t) = P_1(t) + \sum_{j=1}^{N} \sum_{r=1}^{Q_j(t)} X_{j,r} - \sum_{r=1}^{P_1(t)} Y_{1,r}^{J} - \sum_{r=1}^{P_1(t)} Z_{1,r}^{J},  

P_i(t + \Delta t) = P_i(t) - \sum_{r=1}^{P_i(t)} Y_{i,r}^{J} + \sum_{r=1}^{P_{i-1}(t)} Z_{i-1,r}^{J}, \quad i = 2, 3, \ldots, M  

Q_1(t + \Delta t) = Q_1(t) - \sum_{r=1}^{P_{M}(t)} Z_{M,r}^{J},  

Q_j(t + \Delta t) = Q_j(t) - \sum_{r=1}^{P_{j-1}(t)} Z_{j-1,r}^{J}, \quad j = 2, 3, \ldots, N.  \quad (2.4.1)$$
for $t = 0, \Delta t, 2\Delta t, \cdots$. The conditional mean of the IBM (2.4.1) approximates the
deterministic model (2.1.1) (in fact it agrees with the difference approximation of the
deterministic model given in Equation (2.5.5)) and is given by

$$E(P_1(t + \Delta t)|P_1(t)) = P_1(t) + \frac{\sum_{j=1}^{N} Q_j(t) \beta_j(t) \Delta t - P_1(t) \nu_1(t) \Delta t}{\Delta a} \Delta t,$$

$$E(P_i(t + \Delta t)|P_i(t)) = P_i(t) - P_i(t) \nu_i(t) \Delta t - \frac{P_i(t) - P_{i-1}(t)}{\Delta a} \Delta t, \quad i = 2, 3, \cdots, M,$$

$$E(Q_1(t + \Delta t)|Q_1(t)) = Q_1(t) - Q_1(t) \mu_1(t) \Delta t - \frac{Q_1(t) g_1(t) \Delta t}{\Delta x} + \frac{P_M(t)}{\Delta a} \Delta t,$$

$$E(Q_j(t + \Delta t)|Q_j(t)) = Q_j(t) - Q_j(t) \mu_j(t) \Delta t - \frac{Q_j(t) g_j(t) - Q_{j-1}(t) g_{j-1}(t)}{\Delta x} \Delta t, \quad j = 2, 3, \cdots, N.$$
$A_j^k$ denote the difference approximation of $A(x_j, t_k)$. We discretize the stochastic partial differential equation system (2.3.15)-(2.3.18) using the following explicit difference approximation:

$$J_{i+1}^k = J_i^k - J_i^k - J_{i-1}^k \Delta t - \nu_i^k J_i^k \Delta t + \frac{1}{\Delta a} \sqrt{J_{i-1}^k \Delta t \xi_{i-1}^k} - \frac{1}{\Delta a} \sqrt{J_i^k \Delta t \xi_i^k} - \frac{\nu_i^k J_i^k \Delta t}{\Delta a} \xi_i^k \tag{2.5.1}$$

$$J_0^k + \sqrt{\frac{J_0^k \Delta t}{\Delta a} \xi_0^k} = \sum_{j=1}^N \beta_j^k A_j^k \Delta x + \sqrt{\sum_{j=1}^N \beta_j^k A_j^k \Delta x} \frac{\Delta t}{\Delta a} \bar{\eta}_k, \tag{2.5.2}$$

$$A_j^{k+1} = A_j^k - \frac{g_j^k A_j^k - g_{j-1}^k A_{j-1}^k}{\Delta x} \Delta t - \mu_j^k A_j^k \Delta t + \frac{1}{\Delta x} \sqrt{g_{j-1}^k A_{j-1}^k \Delta t \eta_{j-1}^k} \tag{2.5.3}$$

$$g_0^k A_0^k + \sqrt{\frac{g_0^k A_0^k}{\Delta t} \eta_0^k} = J_M^k + \sqrt{\frac{J_M^k}{\Delta t} \bar{\eta}_k}, \tag{2.5.4}$$

for $i = 1, \cdots, M$, $j = 1, 2, \cdots, N$ and $k = 0, 1, 2, \cdots$. Here $\xi_i^k, \xi_i^k, \xi_i^k$ and $\eta_j^k, \eta_j^k, \eta_j^k$ are independent normally distributed random variables with mean 0 and variance 1. It is easy to see that Equations (2.5.1) and (2.5.3) are Euler-Maruyama [17] schemes for the system of Itô stochastic differential equations shown in Equations (2.3.7) and (2.3.9).

In solving Equations (2.5.1)-(2.5.4), the values of $J_i^0, i = 1, 2, \cdots, M$ and $A_j^0, j = 1, 2, \cdots, N$ are calculated using the initial density of juveniles and adults, respectively. Thus, Equation (2.5.1) is solved for $i = 1, 2, \cdots, M$ and $k = 1, 2, \cdots$. In particular, when $i = 1$, the terms containing $J_0^k$ in Equation (2.5.1) are replaced by the
equivalent terms on the right-hand side of Equation (2.5.2). Equations (2.5.3) and (2.5.4) can be solved similarly.

Note that if all random terms in Equations (2.5.1)-(2.5.4) are set to zero, the stochastic difference approximation reduces to the following deterministic difference approximation

\[
\begin{align*}
J^{k+1}_i &= J^k_i - \frac{J^k_i - J^{k-1}_i}{\Delta a} \Delta t - \nu^k_i J^k_i \Delta t \\
J^k_0 &= \sum_{j=1}^{N} \beta^k_j A^k_j \Delta x \\
A^{k+1}_j &= A^k_j - \frac{g^k_j A^k_j - g^{k-1}_j A^{k-1}_j}{\Delta x} \Delta t - \mu^k_j A^k_j \Delta t \\
g^k_0 A^k_0 &= J^k_M.
\end{align*}
\]

System (2.5.5) is just the numerical scheme developed in [2] for solving the PDE system (2.1.1). In [2] we proved the convergence of the finite difference approximation (2.5.5) to the weak solution of the PDE deterministic model (2.1.1).

For our numerical examples in this section, we set \(M = N = 50\), select \(\Delta t = 0.002\) and solve the model on a time interval \(t \in [0, 2]\). We choose

\[
\begin{align*}
\nu(a,t,P) &= 0.1(a + 1) \exp(0.0005P), \quad \mu(x,t,Q) = 0.1 \exp(2x) \exp(0.0001Q), \\
g(x,t,Q) &= 1.5(1 - x) \exp(-0.01Q), \quad \beta(x,t,Q) = 3x \exp(-0.005Q)(\sin(2\pi t) + 1), \\
J(a,0) &= 10 \exp(- (2a - 0.125)^2) \quad \text{and} \quad A(x,0) = 200 \exp(- (2x - 0.25)^2). \quad \text{We let} \\
amax &= 1, \quad x_{\min} = 0 \quad \text{and} \quad x_{\max} = 1. \quad \text{The above choice of parameters is merely to test the accuracy of the derivation of the stochastic models and may not have any biological significance, with the exception that the birth rate is chosen to be periodic to represent seasonal populations (e.g., amphibians).}
\end{align*}
\]

With the above choice of parameters we numerically solve the DMCM, the stochastic numerical procedure (2.5.1)-(2.5.4) and the IBM (2.4.1). For the IBM
(2.4.1), we assume that $X_{j,r} \sim NB(m, p)$ such that $E(X_{j,r}) = mp/(1 - p) = \beta_j \Delta t$. Clearly, $p = \beta_j \Delta t/(m + \beta_j \Delta t)$. Since $V(X_{j,r}) = mp/(1 - p)^2 = \beta_j \Delta t + (\beta_j \Delta t)^2/m$, we choose $m = 1$ in our simulations to obtain large variance. We also round the number of juveniles and adults in each class to an integer value. We compare the mean of 1000 stochastic realizations for all the stochastic models with the numerical solution of the deterministic model (2.1.1). In Figure 2.1, we present the results for the juvenile density $J(a, t)$ and the adult density $A(x, t)$ from the deterministic model, the mean of 1000 Monte Carlo simulations, the mean of 1000 sample paths using the SSPDE, and the mean of 1000 sample paths using the IBM. In Figure 2.2, we graph the 95% population level confidence intervals for the Monte Carlo simulations, the SSPDE, the IBM and the population level of the deterministic model. These confidence intervals are computed in the following manner: at any fixed time level $t_k$ we exclude the highest and lowest 2.5% of the stochastic realizations for that time level. Thus, at each time level $t_k$ these intervals contain 95% of the stochastic realizations (i.e., 950 stochastic realizations).

In Figure 2.3 we present histograms which describe the distribution of the 1000 sample paths over the number of juveniles and adults for the Monte Carlo simulations, the SSPDE, and the IBM at time $t = 2$; while histograms describing the distribution of the 1000 sample paths over the average juvenile age and the average adult size at time $t = 2$ are shown in Figure 2.4. It is clear from the numerical results that the mean of 1000 sample paths for the DMCM, the SSPDE and the IBM agree with the deterministic solution; an indication of the proper derivations of the stochastic models.
2.6 Application to a green tree frog population

In what follows, we utilize the stochastic models to understand the effects of demographic stochasticity on the dynamics of an urban Green Tree Frog (GTF) (*Hyla cinerea*) population. In recent years we have been monitoring a GTF population with a capture-mark-recapture (CMR) method. Using a hypergeometrical statistical method we were able to obtain weekly population estimates for this population during the breeding seasons in 2004-2007 (see [18, 20, 21]). Throughout our discussion, we assume that the time unit is 1 year. We choose the time step to be $\Delta t = 1/250$, and we set $a_{\text{max}} = 5/52$, as it takes about 5 weeks for a tadpole to metamorphose into a frog [11, 14, 15, 19]. We let $x_{\text{min}} = 1.5$ cm, $x_{\text{max}} = 6$ cm, as observed in the study site [18].

Using a least squares approach we fit the model (2.1.1) to the statistical population estimates to obtain the following estimates for the model parameters [3]:

$$g(x) = 3(1 - (x - 1.5)/(6 - 1.5)), \quad \nu(a, P) = 0.021479P \exp(1.6777a),$$

$$\mu(Q) = 2.8776 + 0.004562Q, \quad \beta(x, t) = 0 \text{ if } 0 \leq x \leq 3 \text{ and } \beta(x, t) = 17658b(t) \text{ if } x > 3 \text{ with }$$

$$b(t) = \begin{cases} 
\exp(-300(t - 0.4)^2) + \exp(-300(t - 0.15)^2), & \text{if } 0 \leq t < 0.45, \\
\exp(-300(t - 1.25)^2) + \exp(-300(t - 1.4)^2), & \text{if } 1 \leq t < 1.45, \\
\exp(-300(t - 2)^2) + \exp(-300(t - 2.15)^2), & \text{if } 1.75 \leq t < 2.2, \\
\exp(-300(t - 3.1)^2) + \exp(-300(t - 3.25)^2), & \text{if } 2.85 \leq t < 3.3, \\
\exp(-300(t - 4.1)^2) + \exp(-300(t - 4.25)^2), & \text{if } 3.85 \leq t \leq 4, \\
0, & \text{else.}
\end{cases}$$

The above birth rate indicates that the breeding season may change by a few weeks from year to year (i.e., it may begin a few weeks earlier or later); as already observed from our field calling data.

Since the breeding season begins without tadpoles, we assume that the initial
density of juveniles is $J(a, 0) = 0$. As for the initial density of adults it is given by $A(x, 0) = 717.6038 \exp(-0.75x)$.

Using the above parameters, we compare the numerical results for the solution of the deterministic model (2.5.5), 1000 sample paths obtained from the stochastic difference scheme (2.5.1)-(2.5.4) and 1000 stochastic realizations obtained from the IBM (2.4.1). We assume that $X_{j,r} \sim NB(1, \beta_j \Delta t/(1 + \beta_j \Delta t))$ when simulating the IBM (2.4.1). In Figure 2.5, we graph the 95% population level confidence intervals for the SSPDE, the IBM and the population level of the deterministic model. Summarized in Table 2.5 are the results of the standard deviation of population level at $t = 0.25$ years (a point in time at which large number of juveniles is present and hence larger variance is observed in this stage), at $t = 3.4$ years (a point in time at which large number of adults is present) and $t = 4$ years (the final time) for the SSPDE and the IBM. From these results it can been seen that the standard deviation of the IBM is significantly larger than that of the SSPDE at the juvenile stage when the population level is high ($t = 0.25$). It is also slightly larger when the population level is low ($t = 4$). However, at the adult stage no significant difference in standard deviation is observed between the IBM and the SSPDE. Furthermore, frequency histograms of population level of tadpoles and frogs, average age of tadpoles and average size of frogs at $t = 0.25$ from the SSPDE and the IBM are compared in Figures 2.6-2.7.

We then use the SSPDE and the IBM to understand the effect of demographic stochasticity on the dynamics of the green tree frog population. Thus, we simulate these stochastic models for 30 years using the above parameters. Because we have no
Table 2.5. The standard deviation of population level at \( t = 0.25, t = 3.4 \) and \( t = 4 \).

<table>
<thead>
<tr>
<th>Time</th>
<th>Tadpoles</th>
<th>Frogs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SSPDE</td>
<td>IBM</td>
</tr>
<tr>
<td>( t = 0.25 )</td>
<td>372.42</td>
<td>641.51</td>
</tr>
<tr>
<td>( t = 3.4 )</td>
<td>11.95</td>
<td>16.38</td>
</tr>
<tr>
<td>( t = 4 )</td>
<td>69.71</td>
<td>76.58</td>
</tr>
</tbody>
</table>

Knowledge of what birth rates may be in future time, we extend the birth rate defined above to a four-year periodic function. We compute 1000 sample paths for the SSPDE, 1000 sample paths for the IBM using negative binomial birth distribution

\[ (X_{j,r} \sim NB(1, \beta_j \Delta t/(1 + \beta_j \Delta t))) \]

and 1000 sample paths using Poisson birth distribution \( (X_{j,r} \sim \text{Pois}(\beta_j \Delta t)) \). We compute the maximum and minimum population level for the 1000 sample paths of each of these models and report the results in Table 2.6. This table shows that the lowest and highest frog population level reached are 29 and 720, respectively, and they occur when using negative binomial birth distribution in the IBM. Similarly, the lowest and highest juvenile population level occurring with the negative binomial birth distribution in the IBM are given by 0 and 9224, respectively. Note that when breeding ceases (after the breeding season ends) the tadpole population quickly goes to zero for the remainder of the year until the next breeding season starts; hence 0 is the minimum population level as shown in the table.

The absolute value of the difference of population level between a sample path of the SSPDE and the deterministic model is compared with that between a sample path of the IBM and the deterministic model in Figure 2.8. The figure shows that for this sample path the biggest difference between the IBM and the deterministic model
Table 2.6. The minimum and maximum population level over 30 years.

<table>
<thead>
<tr>
<th>Model</th>
<th>Tadpoles</th>
<th>Frogs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Minimum</td>
<td>Maximum</td>
</tr>
<tr>
<td>Deterministic</td>
<td>0</td>
<td>6657</td>
</tr>
<tr>
<td>SSPDE</td>
<td>0</td>
<td>7721</td>
</tr>
<tr>
<td>IBM NB $\left(1, \beta_k^k \Delta t\right)$</td>
<td>0</td>
<td>9224</td>
</tr>
<tr>
<td>Poisson</td>
<td>0</td>
<td>7682</td>
</tr>
</tbody>
</table>

is much larger than that between the SSPDE and the deterministic model for the juvenile stage; while at the adult (frog) stage the biggest difference between the IBM sample path and the deterministic model is about the same as that between the SSPDE and the deterministic model.

The above results show that even though the effects of demographic stochasticity on the dynamics of the population are significant, they are not strong enough to alter the persistence outcome for the deterministic model, at least over a period of 30 years.

2.7 Concluding remarks

In this we derived several stochastic models for a juvenile-adult population including an SSPDE which extends the one species model derived in [7]. We compared the numerical simulations of the stochastic models with the deterministic model (2.1.1). We also showed how to apply this method to an urban GTF population and how to obtain confidence intervals that can provide information about the persistence of the population. Furthermore, we showed how to use this model to obtain a distribution of the population over the size of adults (frogs). Our protocol of the
capture-mark-recapture field study includes measuring the length of all captured frogs [21]. In our future efforts we plan to compare the model predicted distribution over the size of frogs (Figure 2.7) with the distribution obtained from our field data.
Figure 2.1. The solution \( J(a, t), A(x, t) \) of deterministic model (first row). The mean of 1000 Monte Carlo simulations of DMCM (second row), 1000 sample paths of SSPDE (third row) and 1000 sample paths of the IBM (fourth row).
Figure 2.2. The population level of juveniles (left column) and adults (right column) with 95% confidence intervals for 1000 Monte Carlo simulations (first row), 1000 sample paths of the SSPDE (second row) and 1000 sample paths of the IBM (third row).
Figure 2.3. Frequency histograms of population level of juveniles (left column) and adults (right column) at time $t = 2$ for 1000 Monte Carlo simulations (first row), 1000 sample paths of the SSPDE (second row) and 1000 sample paths of the IBM (third row).
Figure 2.4. Frequency histograms of average age of juveniles (left column) and average size of adults (right column) at time $t = 2$ for 1000 Monte Carlo simulations (first row), 1000 sample paths of the SSPDE (second row) and 1000 sample paths of the IBM (third row).
Figure 2.5. The population level of tadpoles (left column) and frogs (right column) with 95% confidence intervals for 1000 sample paths of the SSPDE (first row) and 1000 sample paths of the IBM (second row).

Figure 2.6. Frequency histograms of population level of tadpoles (left column) and frogs (right column) at time $t = 0.25$ (years) for 1000 sample paths of the SSPDE (first row) and 1000 sample paths of the IBM (second row).
Figure 2.7. Frequency histograms of average age of tadpoles (left column) and average size of frogs (right column) at time $t = 0.25$ (years) for 1000 sample paths of the SSPDE (first row) and 1000 sample paths of the IBM (second row).

Figure 2.8. The absolute value of the difference of population level between a sample path of the SSPDE and the deterministic model (first row) and between a sample path of the IBM and the deterministic model (second row).
REFERENCES


In this chapter\textsuperscript{1}, we present an infinite-dimensional least-squares approach which compares a mathematical population model to the statistical population estimates obtained from the field data. The model is composed of nonlinear first order hyperbolic equations describing the dynamics of the amphibian population where individuals are divided into juveniles (tadpoles) and adults (frogs). To solve the least-squares problem, an explicit finite difference approximation is developed. Convergence results for the computed parameters are presented. Parameter estimates for the vital rates of juveniles and adults are obtained, and standard deviations for these estimates are computed. Numerical results for the model sensitivity with respect to these parameters are given. Finally, the above mentioned parameter estimates are used to illustrate the long-time behavior of the population under investigation.

\subsection{Introduction}

Major declines and even extinction of many amphibian populations have been reported around the world [19, 21, 25]. Causes attributed to such declines or extinction include habitat destruction, climate change, diseases, pollution and introduced species [11, 17]. There is now growing recognition of the need for long-term monitoring of amphibian

\textsuperscript{1}The results of this chapter have been accepted for publication.
populations to address questions related to the extent of such declines.

In 2004 authors of [4, 18] initiated a capture-mark-recapture (CMR) field sampling to understand the dynamics of an urban green tree frog (GTF) population (*Hyla cinerea*) in an area close to the campus of the University of Louisiana at Lafayette. Since then, every year they monitored this frog population from March through September or October. Such a monitoring period begins when the GTFs are just coming out of hibernation and before the breeding season, and it extends past the breeding season. Based on frog calling data collected by their group, the breeding season seems to begin in the middle of April through early May and end sometime in late July or early August. Each weekly CMR survey was conducted no earlier than 30 minutes after sunset. Search effort was recorded and varied over the season depending on the number of frogs that were being caught that evening. When few frogs were being caught, each section of the monitored location was searched for a minimum of 10 minutes. When many frogs were being caught, the search time was shortened to keep the maximum number of frogs caught below 40, a number that could be processed (checked for marks, marked, weighted, measured for length, etc.) before midnight in a given evening. A statistical method based on a generalization of the hypergeometric distribution was developed to derive estimates of population size for the weekly number of frogs and confidence intervals for such population sizes [18, 23]. This method was also used to obtain point and interval estimates for the 2004-2009 CMR data [5, 23, 24].

The purpose of this chapter is two fold: 1) To develop a least-squares approach to fit the model we developed in [6, 7] to the statistical population estimates obtained
from the 2004-2009 CMR data sets. 2) To use statistical estimates resulting from field data to estimate vital rates for juveniles and adults and then use these parameter estimates to understand possible long-time behavior of this population.

This chapter is organized as follows. In Section 3.2 we set up a least-squares problem that compares the output of the amphibian model developed in [6, 7] to given data on the adult population. In Section 3.3 we present a finite difference scheme for computing an approximate solution to this least-squares problem and provide convergence results for the computed parameters. In Section 3.4 we use the developed least-squares technique to compare the GTF model to the weekly estimates obtained from the CMR field data. We then utilize the computed parameters to understand the long-time behavior of the population. In Section 3.5 we numerically analyze the sensitivity of the model to the estimated model parameters and present a method to compute the standard deviations for these parameters. Finally, concluding remarks are made in Section 3.6.

3.2 The juvenile-adult model and the least-squares problem

Recently, we developed a model which describes the dynamics of an amphibian population divided into two groups 1) juveniles (tadpoles) and 2) adults (frogs) [6, 7]. We assume that juveniles are structured by their age while adults are structured by their size (since often in such population adults become sexually mature when they reach a certain length, e.g., see [22] for the green tree frogs). The model is given by the
following system of first order hyperbolic partial differential equations:

\[ J_t + J_a + \nu(a, t, P(t; \theta))J = 0, \quad (a, t) \in (0, a_{\text{max}}) \times (0, T), \]
\[ A_t + (g(x, t, Q(t; \theta))A)_x + \mu(x, t, Q(t; \theta))A = 0, \quad (x, t) \in (x_{\min}, x_{\max}) \times (0, T), \]
\[ J(0, t; \theta) = \int_{x_{\min}}^{x_{\max}} \beta(x, t, Q(t; \theta))A(x, t; \theta)dx, \quad t \in (0, T), \]
\[ g(x_{\min}, t, Q(t; \theta))A(x_{\min}, t; \theta) = J(a_{\text{max}}, t; \theta), \quad t \in (0, T), \]
\[ J(a, 0; \theta) = J^0(a), \quad a \in [0, a_{\text{max}}], \]
\[ A(x, 0; \theta) = A^0(x), \quad x \in [x_{\min}, x_{\max}]. \]

(3.2.1)

Here, \( \theta = (\nu, g, \mu, \beta) \) is the parameter to be identified. The function \( J(a, t; \theta) \) is the parameter-dependent density of juveniles of age \( a \) at time \( t \) and \( A(x, t; \theta) \) is the parameter-dependent density of adults having size \( x \) at time \( t \), \( a_{\text{max}} \) denotes the age at which a juvenile (tadpole) metamorphoses into a frog (\( a_{\text{max}} \) approximately equals five weeks for the green tree frog \([12, 15, 16, 20]\)), \( x_{\min} \) and \( x_{\max} \) denote the minimum size and the maximum size of a frog, respectively (1.5cm to 6cm for GTF \([18]\)).

\( P(t; \theta) = \int_0^{a_{\text{max}}} J(a, t; \theta)da \) and \( Q(t; \theta) = \int_{x_{\min}}^{x_{\max}} A(x, t; \theta)dx \) are the number of juveniles and adults at time \( t \), respectively. The function \( \nu(a, t, P) \) denotes the mortality rate of a juvenile of age \( a \) at time \( t \) which depends on the number of tadpoles \( P \) due to competition for resources. The function \( \mu \) represents the mortality rate of an adult of size \( x \), \( g \) represents the growth rate of an adult of size \( x \) and \( \beta \) represents the reproduction rate of an adult of size \( x \). The adult vital rates depend on \( t \) due to seasonality of such populations and depend on the number of adults (frogs) due to competition for resources. However, they do not depend on the number of juveniles, since juveniles live in water while adults live on land; thus they do not compete for resources.

The long-time behavior for a special case of the model (3.2.1) has been
investigated in [6]. In particular, conditions on the vital rates which lead to extinction or persistence of the population were established. In [7] we developed an explicit finite difference approximation to model (3.2.1), proved the existence and uniqueness of a weak solution, and established the convergence of this approximation to the unique weak solution.

By a weak solution to problem (3.2.1), we mean a set of functions 
\[(J, A) \in BV([0, a_{\text{max}}] \times [0, T]) \times BV([x_{\text{min}}, x_{\text{max}}] \times [0, T])\]  
satisfying
\[
\int_0^{a_{\text{max}}} J(a, t; \theta) \varphi(a, t) da - \int_0^{a_{\text{max}}} J(a, 0; \theta) \varphi(a, 0) da \\
= \int_0^t \int_0^{a_{\text{max}}} J(\varphi_s + \varphi_a - \nu \varphi) dads \\
+ \int_0^t \varphi(0, s) \int_{x_{\text{min}}}^{x_{\text{max}}} \beta(x, s, Q(s; \theta)) A(x, s; \theta) dx ds - \int_0^t \varphi(a_{\text{max}}, s) J(a_{\text{max}}, s; \theta) ds,
\]
\[
\int_{x_{\text{min}}}^{x_{\text{max}}} A(x, t; \theta) \psi(x, t) dx - \int_{x_{\text{min}}}^{x_{\text{max}}} A(x, 0; \theta) \psi(x, 0) dx \\
= \int_0^t \int_{x_{\text{min}}}^{x_{\text{max}}} A(\psi_s + g \psi_x - \mu \psi) dx ds + \int_0^t J(a_{\text{max}}, s; \theta) \psi(x_{\text{min}}, s) ds
\]  
(3.2.2) for \( t \in [0, T] \) and every test function \( \varphi \in C^1((0, a_{\text{max}}) \times (0, T)) \) and every test function \( \psi \in C^1((x_{\text{min}}, x_{\text{max}}) \times (0, T)) \).

For our theory we impose the following conditions on the initial data:

(H1) \( J^0 \in BV[0, a_{\text{max}}] \) and \( J^0(a) \geq 0 \).

(H2) \( A^0 \in BV[x_{\text{min}}, x_{\text{max}}] \) and \( A^0(x) \geq 0 \).

Remark 3.1. For the theoretical setup below we assume that the entire parameter \( \theta = (\nu, g, \mu, \beta) \) is unknown, so that the theory holds in the most general case. But often partial information is available for some (all) parameters and thus the problem is
reduced to a special case where we only need to identify the unknown information for that parameter. For example, assume that \( g, \mu, \beta \) are given and that \( \nu = \bar{\nu} \hat{\nu} \) where \( \hat{\nu} \) is also known (given) then we only need to identify \( \bar{\nu} \). To put even in simpler words, suppose all the parameters are given and \( \nu = \alpha (2 + \sin(t)) \exp(a)(1 + P) \) for an unknown constant \( \alpha \), then the identification problem reduces to a finite one-dimensional problem where the only unknown is the constant \( \alpha \).

Throughout the discussion we let \( D_1 = [0, a_{\text{max}}] \times [0, T] \times [0, \infty) \), \( D_2 = [x_{\text{min}}, x_{\text{max}}] \times [0, T] \times [0, \infty) \), \( D_3 = [0, T] \times [0, \infty) \). Then, we let \( B = C_b(D_1) \times C^1([x_{\text{min}}, x_{\text{max}}]; C_b(D_3)) \times C^0(D_2) \times C_b(D_2) \), where \( C_b(\Omega; X) \) denotes the space of bounded continuous functions from \( \Omega \) to \( X \). Recall that \( C_b(D_1) \) is a Banach space when endowed with the norm \( \| f_1 \|_{C_b(D_1)} = \sup_{y \in D_1} |f_1(y)| \). Similarly, this is true for the spaces \( C_b(D_2) \) and \( C_b(D_3) \) with respective supremum norms. Also, \( C^1([x_{\text{min}}, x_{\text{max}}]; C_b(D_3)) \) is a Banach space when endowed with the norm

\[
\| f_2 \|_{C^1([x_{\text{min}}, x_{\text{max}}]; C_b(D_3))} = \sup_{y \in [x_{\text{min}}, x_{\text{max}}]} (\| f_2(y) \|_{C_b(D_3)} + \| f'_2(y) \|_{C_b(D_3)}).
\]

Thus, \( B \) is a Banach space when endowed with the following norm:

\[
\| f \|_B = \max \{ \| f_1 \|_{C_b(D_1)}, \| f_2 \|_{C^1([x_{\text{min}}, x_{\text{max}}]; C_b(D_3))}, \| f_3 \|_{C_b(D_2)}, \| f_4 \|_{C_b(D_2)} \}.
\]

We let \( L \) and \( \omega \) be large positive constants and assume that our admissible parameter space \( \Theta \) is any compact subset of \( B \), such that every \( \theta = (\nu, g, \mu, \beta) \in \Theta \) satisfies (H3)-(H6) below. It is not difficult to check that (H3)-(H6) guarantee the convergence of the finite difference scheme in [7].
(H3) $\nu : \mathcal{D}_1 \to \mathbb{R}$ is a nonnegative Lipschitz continuous function with Lipschitz constant $L$. Furthermore, \[ \sup_{(a,t,P)\in \mathcal{D}_1} \nu(a,t,P) \leq \omega. \]

(H4) $g : \mathcal{D}_2 \to \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant $L$ and satisfies \[ \sup_{(x,t,Q)\in \mathcal{D}_2} g(x,t,Q) \leq \omega. \] Furthermore, $g(x,t,Q) > 0$ for $x \in [x_{\min}, x_{\max})$ and $g(x_{\max}, t, Q) = 0$, and $g_x(x, t, Q)$ is Lipschitz continuous with respect to $x$ and $Q$ for constant $L$.

(H5) $\mu : \mathcal{D}_2 \to \mathbb{R}$ is a nonnegative Lipschitz continuous function with Lipschitz constant $L$. Furthermore, \[ \sup_{(x,t,Q)\in \mathcal{D}_2} \mu(x,t,Q) \leq \omega. \]

(H6) $\beta : \mathcal{D}_2 \to \mathbb{R}$ is a nonnegative Lipschitz continuous function with Lipschitz constant $L$. Furthermore, \[ \sup_{(x,t,Q)\in \mathcal{D}_2} \beta(x,t,Q) \leq \omega. \]

For convenience of the reader, below we provide an example of a compact subset of $B$ which satisfy (H3)-(H6). Other examples of such compact sets of $B$ can be easily constructed. We let $\sigma$ be a small positive constant, $P_{\max}$ and $Q_{\max}$ be large positive constants and define

\[
\mathcal{B}_1 = \{ f \in C_b(\mathcal{D}_1) \mid 0 \leq f \leq \omega, |f(a,t,P) - \nu(\bar{a},\bar{t},\bar{P})| \leq L(|a - \bar{a}| + |t - \bar{t}| + |P - \bar{P}|), f(a,t,P) = f(a,t,P_{\max}) \text{ for } P \geq P_{\max}\},
\]

\[
\mathcal{B}_2 = \{ f \in C_b(\mathcal{D}_2) \mid 0 \leq f \leq \omega, |f(t,Q) - f(\bar{t},\bar{Q})| \leq L(|t - \bar{t}| + |Q - \bar{Q}|), f(t,Q) = f(t,Q_{\max}) \text{ for } Q \geq Q_{\max}\},
\]

and

\[
\mathcal{B}_3 = \{ f \in C_b(\mathcal{D}_2) \mid 0 \leq f \leq \omega, |f(x,t,Q) - f(\bar{x},\bar{t},\bar{Q})| \leq L(|x - \bar{x}| + |t - \bar{t}| + |Q - \bar{Q}|), f(a,t,Q) = f(a,t,Q_{\max}) \text{ for } Q \geq Q_{\max}\},
\]

\[
\mathcal{B}_4 = \{ f \in C^1([x_{\min}, x_{\max}]; \mathcal{B}_2) \mid \|f\|_{C^1([x_{\min}, x_{\max}]; C_b(\mathcal{D}_3))} \leq \omega, \|f'(x_1) - f'(x_2)\|_{C_b(\mathcal{D}_3)} \leq L|x_1 - x_2|, 
\]

\[
\sigma(x_{\max} - x) \leq f(x) \leq \omega \text{ for } x \in [x_{\min}, x_{\max}) \text{ and } f(x_{\max}) = 0\}.
\]
It follows from Arzela Ascoli’s theorem that $B_1$ is a compact subset of $C_b(\mathbb{D}_1)$, $B_2$ is a compact subset of $C_b(\mathbb{D}_3)$, $B_3$ is a compact subset of $C_b(\mathbb{D}_2)$ and $B_4$ is a compact subset of $C^1([x_{\text{min}}, x_{\text{max}}]; B_2)$. Thus, the admissible parameter space

$$\Theta = B_1 \times B_4 \times B_3 \times B_3$$

is a compact subset of $B$.

One of the main goals of this chapter is to develop a methodology for estimating parameters contained in (3.2.1) from data on number of frogs at different points in time. In particular, we are interested in the following problem: given observations $X_s$ which correspond to the number of frogs at time $t_s$, $s = 1, 2, \ldots, S$, find (compute) a parameter $\theta = (\nu, g, \beta, \mu) \in \Theta$ such that the least-squares cost functional

$$F(\theta) = \sum_{s=1}^{S} \left| \log(Q(t_s; \theta) + 1) - \log(X_s + 1) \right|^2$$

is minimized over the admissible parameter space $\Theta$. Here,

$$Q(t_s; \theta) = \int_{x_{\text{min}}}^{x_{\text{max}}} A(t_s, x; \theta) \, dx$$

is the number of adults (frogs) at time $t_s$. The reason for considering a logarithmically scaled least-squares functional is that the population we are interested in is seasonal, and its total number (as observed by the statistical population estimates provided in [5, 23, 24]) oscillates wildly during one year period.

### 3.3 Approximation scheme and convergence theory for parameter estimation

In order to solve the infinite dimensional minimization problem (3.2.3), we first need a method for solving the infinite dimensional juvenile-adult model (3.2.1). To this end, we utilize the finite-difference approximation scheme detailed below which we
developed in [7]. We divide the intervals $[0, a_{\text{max}}]$, $[x_{\text{min}}, x_{\text{max}}]$ and $[0, T]$ into $m$, $n$ and $l$ subintervals, respectively. The following notation will be used throughout this:

$\Delta a = \frac{a_{\text{max}}}{m}$, $\Delta x = \frac{(x_{\text{max}} - x_{\text{min}})}{n}$ and $\Delta t = \frac{T}{l}$ denote the age, size, and time mesh lengths, respectively. The mesh points are given by: $a_i = i\Delta a$, $i = 0, 1, \cdots, m$, $x_j = x_{\text{min}} + j\Delta x$, $j = 0, 1, \cdots, n$, $t_k = k\Delta t$, $k = 0, 1, \cdots, l$. We denote by $J_i^k(\theta)$, $A_j^k(\theta)$, $P^k(\theta)$ and $Q^k(\theta)$ the finite difference approximation of $J(a_i, t_k; \theta)$, $A(x_j, t_k; \theta)$, $P(t_k; \theta)$ and $Q(t_k; \theta)$, respectively, and let

$$\nu_i^k = \nu(a_i, t_k, P^k(\theta)), \quad g_j^k = g(x_j, t_k, Q^k(\theta)), \quad \mu_j^k = \mu(x_j, t_k, Q^k(\theta)), \quad \beta_j^k = \beta(x_j, t_k, Q^k(\theta)).$$

We define the difference operators

$$D_{\Delta a}(J_i^k) = \frac{J_i^k - J_{i-1}^k}{\Delta a}, \quad 1 \leq i \leq m, \quad D_{\Delta x}(A_j^k) = \frac{A_j^k - A_{j-1}^k}{\Delta x}, \quad 1 \leq j \leq n,$$

and denote the $\ell^1$, $\ell^\infty$ and $BV$ norms of $J^k$ and $A^k$ by

$$\|J^k\|_1 = \sum_{i=1}^m |J_i^k| \Delta a, \quad \|A^k\|_1 = \sum_{j=1}^n |A_j^k| \Delta x,$$

$$\|J^k\|_{\infty} = \max_{0 \leq i \leq m} |J_i^k|, \quad \|A^k\|_{\infty} = \max_{0 \leq j \leq n} |A_j^k|,$$

$$\|J^k\|_{BV} = \sum_{i=1}^m |D_{\Delta a}(J_i^k)| \Delta a, \quad \|A^k\|_{BV} = \sum_{j=1}^n |D_{\Delta x}(A_j^k)| \Delta x.$$
We then discretize the partial differential equation system (3.2.1) using the following explicit finite difference approximation

\[
\begin{align*}
\frac{J_i^{k+1}(\theta) - J_i^k(\theta)}{\Delta t} + \frac{J_i^k(\theta) - J_{i-1}^k(\theta)}{\Delta a} + \nu_i^k J_i^k(\theta) &= 0, \quad 0 \leq k \leq l - 1, \ 1 \leq i \leq m, \\
\frac{A_j^{k+1}(\theta) - A_j^k(\theta)}{\Delta t} + \frac{g_j^k A_j^k(\theta) - g_{j-1}^k A_{j-1}^k(\theta)}{\Delta x} + \mu_j^k A_j^k(\theta) &= 0, \quad 0 \leq k \leq l - 1, \ 1 \leq j \leq n, \\
J_0^{k+1}(\theta) &= \sum_{j=1}^{n} \beta_j^{k+1} A_j^{k+1}(\theta) \Delta x, \quad g_0^{k+1} A_0^{k+1}(\theta) = J_m^{k+1}(\theta), \quad 0 \leq k \leq l - 1, \\
P^{k+1}(\theta) &= \sum_{i=1}^{m} J_i^{k+1}(\theta) \Delta a, \quad Q^{k+1}(\theta) = \sum_{j=1}^{n} A_j^{k+1}(\theta) \Delta x, \quad 0 \leq k \leq l - 1
\end{align*}
\]

(3.3.1)

with the initial conditions

\[
\begin{align*}
J_0^0 &= J^0(0), \quad J_i^0 = \frac{1}{\Delta a} \int_{a(i-1)\Delta a}^{a(i)\Delta a} J^0(a) da, \quad i = 1, 2, \ldots, m, \\
A_0^0 &= A^0(0), \quad A_j^0 = \frac{1}{\Delta x} \int_{x(j-1)\Delta x}^{x(j)\Delta x} A^0(x) dx, \quad j = 1, 2, \ldots, n.
\end{align*}
\]

The following condition concerning \(\Delta t, \Delta a, \text{and} \Delta x\) is imposed throughout the :

\(\text{(H7) Assume that} \ \Delta t, \Delta a \text{and} \Delta x \text{are chosen such that} \)

\[\Delta t \left( \frac{1}{\Delta a} + \omega \right) \leq 1 \quad \text{and} \quad \omega \Delta t \left( \frac{1}{\Delta x} + 1 \right) \leq 1.\]

Equivalently, we can write (3.3.1) as the following system of linear equations:

\[
\begin{align*}
J_i^{k+1}(\theta) &= \frac{\Delta t}{\Delta a} J_{i-1}^k(\theta) + \left( 1 - \frac{\Delta t}{\Delta a} - \Delta t \nu_i^k \right) J_i^k(\theta), \quad 0 \leq k \leq l - 1, \ 1 \leq i \leq m, \\
A_j^{k+1}(\theta) &= \frac{\Delta t}{\Delta x} g_{j-1}^k A_{j-1}^k(\theta) + \left( 1 - \frac{\Delta t}{\Delta x} g_j^k - \Delta t \mu_j^k \right) A_j^k(\theta), \quad 0 \leq k \leq l - 1, \ 1 \leq j \leq n, \\
J_0^{k+1}(\theta) &= \sum_{j=1}^{n} \beta_j^{k+1} A_j^{k+1}(\theta) \Delta x, \quad g_0^{k+1} A_0^{k+1}(\theta) = J_m^{k+1}(\theta), \quad 0 \leq k \leq l - 1, \\
P^{k+1}(\theta) &= \sum_{i=1}^{m} J_i^{k+1}(\theta) \Delta a, \quad Q^{k+1}(\theta) = \sum_{j=1}^{n} A_j^{k+1}(\theta) \Delta x, \quad 0 \leq k \leq l - 1.
\end{align*}
\]

(3.3.2)
Under the assumptions (H1)-(H7), one can easily show that the system (3.3.2) has a unique solution satisfying

\[ [J_0^{k+1}, J_1^{k+1}, \ldots, J_m^{k+1}, A_0^{k+1}, A_1^{k+1}, \ldots, A_n^{k+1}] \geq \overrightarrow{0}, k = 0, 1, \ldots, l - 1. \]

The above approximation can be extended to a family of functions

\[ \{(U_{\Delta a, \Delta x, \Delta t}(a, t; \theta), V_{\Delta a, \Delta x, \Delta t}(x, t; \theta))\} \]

defined by

\[ U_{\Delta a, \Delta x, \Delta t}(a, t; \theta) = J_i^k(\theta), \quad V_{\Delta a, \Delta x, \Delta t}(x, t; \theta) = A_j^k(\theta) \]

for \((a, t) \in [a_{i-1}, a_i) \times [t_{k-1}, t_k)\) and \((x, t) \in [x_{j-1}, x_j) \times [t_{k-1}, t_k), i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, k = 1, 2, \ldots, l.\)

Since our parameter set is infinite dimensional, a finite-dimensional approximation of the parameter space is also necessary for computing minimizers. To this end, we consider the following finite-dimensional approximations of (3.2.3): Let

\[ Q_{\Delta a, \Delta x, \Delta t}(t_s) = \int_{x_{\min}}^{x_{\max}} V_{\Delta a, \Delta x, \Delta t}(x, t_s; \theta) dx \]

denote the finite difference approximation of the number of adults (frogs) and set

\[ F_{\Delta a, \Delta x, \Delta t}(\theta) = \sum_{s=1}^{S} |\log(Q_{\Delta a, \Delta x, \Delta t}(t_s; \theta) + 1) - \log(X_s + 1)|^2, \quad (3.3.3) \]

each of which is minimized over \(\Theta_M\), a compact finite-dimensional approximation of the parameter space \(\Theta\) in the sense that for each \(\theta \in \Theta\), there exists a sequence of \(\theta_M \in \Theta_M\) such that \(\theta_M \rightarrow \theta\) (in the topology of \(B\)) as \(M \rightarrow \infty\).

**Remark 3.2.** Note that if the compact parameter space \(\Theta\) is chosen to be finite dimensional, then the approximation space \(\Theta_M\) can be chosen equal to \(\Theta\).
To establish the convergence results for the parameter-estimation technique, we use an approach in the spirit of that used in [1, 2, 3], which is based on the abstract theory in [10]. In particular, we have the following theorem, the proof of which is given in the Appendix.

**Theorem 3.3.1.** Let \( \theta^r = (\nu^r, g^r, \mu^r, \beta^r) \) and suppose that \( \theta^r \to \theta \) in \( \Theta \) and \( \Delta a_r, \Delta x_r, \Delta t_r \to 0 \) as \( r \to \infty \). Let \( (U_{\Delta a_r, \Delta x_r, \Delta t_r}(a, t; \theta^r), V_{\Delta a_r, \Delta x_r, \Delta t_r}(x, t; \theta^r)) \) denote the solution of the finite difference scheme, and let \( (J(a, t; \theta), A(x, t, \theta)) \) be the unique weak solution of our problem with initial condition \( (J^0(a), A^0(x)) \) and parameter \( \theta \); then

\[
(U_{\Delta a_r, \Delta x_r, \Delta t_r}(\cdot, t; \theta^r), V_{\Delta a_r, \Delta x_r, \Delta t_r}(\cdot, t; \theta^r)) \to (J(\cdot, t; \theta), A(\cdot, t, \theta))
\]

in \( L^1(0, a_{\text{max}}) \times L^1(x_{\text{min}}, x_{\text{max}}) \) uniformly in \( t \in [0, T] \).

Since the logarithm function is continuous on \([1, \infty)\), as an immediate consequence of Theorem 3.3.1, we obtain the following:

**Corollary 3.1.** Let \( (U_{\Delta a_r, \Delta x_r, \Delta t_r}(a, t; \theta^r), V_{\Delta a_r, \Delta x_r, \Delta t_r}(x, t; \theta^r)) \) denote the numerical solution of (3.3.2) with parameter \( \theta^r \to \theta \) in \( \Theta \) and \( \Delta a_r, \Delta x_r, \Delta t_r \to 0 \) as \( r \to \infty \). Then

\[
F_{\Delta a_r, \Delta x_r, \Delta t_r}(\theta^r) \to F(\theta), \quad \text{as} \ r \to \infty.
\]

In the next theorem, we establish the continuity of the approximate cost functional in the parameter \( \theta \), so that the computational problem of finding the approximate minimizer is well posed (see the Appendix for the proof of this result).
Theorem 3.3.2. Let \( \Delta a, \Delta x \) and \( \Delta t \) be fixed. For each \( \theta \in \Theta \), let \((U_{\Delta a, \Delta x, \Delta t}(a, t; \theta), V_{\Delta a, \Delta x, \Delta t}(x, t; \theta))\) denote the solution of the finite difference scheme, and \( \theta^r \to \theta \) as \( r \to \infty \) in \( \Theta \); then
\[
(U_{\Delta a, \Delta x, \Delta t}(\cdot, t; \theta^r), V_{\Delta a, \Delta x, \Delta t}(\cdot, t; \theta^r)) \to (U_{\Delta a, \Delta x, \Delta t}(\cdot, t; \theta), V_{\Delta a, \Delta x, \Delta t}(\cdot, t; \theta)) \text{ as } r \to \infty \text{ in } L^1(0, a_{\text{max}}) \times L^1(x_{\text{min}}, x_{\text{max}}) \text{ uniformly in } t \in [0, T].
\]

Next we establish subsequential convergence of minimizers of the finite dimensional problem (3.3.3) to a minimizer of the infinite dimensional problem (3.2.3).

Theorem 3.3.3. Suppose that \( \Theta_M \) is a sequence of compact subset of \( \Theta \). Moreover, assume that for each \( \theta \in \Theta \), there exists a sequence of \( \theta_M \in \Theta_M \) such that \( \theta_M \to \theta \) as \( M \to \infty \). Then the functional \( F_{\Delta a, \Delta x, \Delta t} \) has a minimizer over \( \Theta_M \). Furthermore, if \( \theta^r_M \)
denotes a minimizer of \( F_{\Delta a, \Delta x, \Delta t} \) over \( \Theta_M \) and \( \Delta a_r, \Delta x_r, \Delta t_r \to 0 \), then any subsequence of \( \theta^r_M \) has a further subsequence which converges to a minimizer of \( F \).

Proof. The proof of this is a direct application of the abstract theory in [10], based on the convergence of \( F_{\Delta a_r, \Delta x_r, \Delta t_r}(\theta^r_r) \to F(\theta) \).

3.4 Fitting experiment

In this section, we use the least-squares approach developed above to estimate the vital rates of the urban GTF population modeled by (3.2.1). To this end, we regard the statistical population estimates for the number of frogs derived in [5, 23, 24] from the CMR data as the observations \( X_s, s = 1, 2 \cdots, S \) (in [5, 23, 24] authors derived population estimates for 136 weeks, i.e., \( S = 136 \)). We assume that \( t = 0 \) is January 1, 2004. The time unit is taken to be one year (i.e., \( 1/52 \) is one week), and the breeding
season begins around the middle of April and ends in early August as observed by our monitoring program [18]. Thus, for $(x, t) \in [1.5, 6] \times [0, 1]$, we let

\[
\beta(x, t) = \begin{cases} 
300 & 3 \leq x \leq 6, 0.3 \leq t \leq 0.6 \\
\frac{0.3 + \epsilon}{300(t + \epsilon - 0.3)} & 2.7 + t \leq x \leq 6, 0.3 - \epsilon \leq t < 0.3 \\
\frac{300(x - 3 + \epsilon)}{\epsilon(0.3 + \epsilon)} & 3 - \epsilon \leq x < 3, x - 2.7 < t < 3.6 - x \\
\frac{300(0.6 + \epsilon - t)}{\epsilon(0.3 + \epsilon)} & 3.6 - t \leq x \leq 6, 0.6 < t \leq 0.6 + \epsilon \\
0 & \text{else}
\end{cases}
\]

(where $\epsilon$ is a small positive number). Note that $\beta$ is Lipschitz continuous on $[1.5, 6] \times [0, 1]$. We then extend $\beta$ continuously and periodically over any time interval $[t, t + 1]$, $t = 1, 2, \ldots$ in a similar manner. Such reproduction function states that sexual maturity is reached around length 3 cm and the breeding season is between 0.3 ($0.3 \times 12 = 3.6$ month mid-April) and 0.6 ($0.6 \times 12 = 7.2$ month early-August).

Furthermore, the number of tadpoles produced by a sexually mature adult (assuming 1:1 ratio of males to females) in one year is 300, i.e., for each $x \in (3, 6], \int_0^1 \beta(x, t)dt = 300$ (see [4]). The growth rate of the frog stage is assumed to take the following standard form [13]:

\[
g(x, Q) = \alpha_1(6 - x).
\]

As for the mortality rate for the frog stage, we assume that it is given by the following form:

\[
\mu(x, t, Q) = \begin{cases} 
\phi(t)(1 + \alpha_0 Q) \exp(\alpha_7 x) & Q \leq Q_{\text{max}}, \\
\phi(t)(1 + \alpha_0 Q_{\text{max}}) \exp(\alpha_7 x) & Q > Q_{\text{max}},
\end{cases}
\]

where $\phi(t) = \sum_{i=0}^{3} \alpha_{i+2} \phi_i(t)$ and $\phi_i(t), 0 \leq i \leq 3$ are the well-known “hat” functions (see the Appendix). Note that we assume that the mortality rate depends linearly on
density (logistic type density-dependent mortality) and exponentially on the size of the frog. Because of the seasonal dynamics of these populations and the fact that they hibernate during winter, we also assume that the mortality rate explicitly depends on time and thus it can vary over time. We point out that besides the natural mortality, the above mortality can account for other loss terms including loss to predators and some immigration out of this pond to other nearby pond (about 1/3 mile). Such immigration has been documented by our CMR data, although the rate is low.

Because of the relative short duration for the tadpole stage, we assume for simplicity and for lack of other information that the mortality rate $\nu = \alpha_8$ is a constant. Here, $\alpha_1, \ldots, \alpha_8$ are the unknown constants to be estimated. We select appropriate positive constants $m_i$ and $M_i$, $i = 1, 2, \ldots, 8$, and assume for this numerical experiment that our unknown constant set

$$\mathcal{N} = \{\alpha = (\alpha_1, \ldots, \alpha_8) \in \mathbb{R}^8 | 0 < m_1 \leq \alpha_1 \leq M_1, \ 0 \leq \alpha_i \leq M_i, \ i = 2, \ldots, 8\}$$

which is a compact subset of $\mathbb{R}^8$. In other words, for this numerical experiment we choose our admissible parameter set

$$\Theta = \{\nu\} \times \{g\} \times \{\mu\} \times \{\beta\},$$

where $\nu$, $g$, $\mu$ and $\beta$ are as previously given in this section with $(\alpha_1, \ldots, \alpha_8) \in \mathcal{N}$.

Clearly, $\Theta$ is a compact subset of $B$ and hence all the general theory apply for this parameter set. Furthermore, since it is finite (eight) dimensional, the approximating parameter sets $\Theta_M$ can be chosen equal to $\Theta$.

We set $a_{\text{max}} = 5/52$ (five weeks), $x_{\text{min}} = 1.5\text{cm}$, $x_{\text{max}} = 6\text{cm}$, $\Delta a = 1/(52 \times 8)$,
$\Delta x = 1/400$ and $\Delta t = 1/(52 \times 40)$. Since our initial time is January and no tadpoles are present at this time, we assume that the initial density of juveniles is $J(a,0) = 0$.

To choose an initial density for frogs, as no CMR data available in January, we take as an approximation to the number of frogs at $t = 0$ the average of our CMR data of 2004 which results $Q(0) = 257.2$. As for the initial density of adults, we assume that it is an exponential distribution given by $A(x,0) = 615.96 \exp(-0.75x)$ which integrates to 257.2. Other distributions to $A(x,0)$ which integrate to $Q(0) = 257.2$ have been tested and the obtained results are similar.

To solve the minimization problem, we use the FORTRAN routine DBCLSF. This routine solves by double precision a nonlinear least-squares problem subject to bounds on the estimated parameters using a modified Levenberg-Marquardt algorithm and a finite-difference Jacobian. The resulting estimates for our least-squares problem (3.2.3) are given in Table 3.1.

Using the parameters $\nu, \mu, g, \beta$ determined by the values of $\alpha_i$ ($i = 1, \cdots, 8$) given in Table 3.1, we simulated our model (3.3.2) and compared the output to the statistical estimates in Figure 3.6. Clearly, the model output agrees with the population estimates resulting from the real data reasonably well for a period of six years.

We then use the resulting parameter estimates to present two possible scenarios of long-time behavior of this population. In the first scenario we assume that the mortality rate estimated over six years is periodic and thus future mortality ($t > 6$) is determined by this periodic function (see the left part of Figure 3.7). In the second
scenario we assume that the future mortality \((t > 6)\) is determined by the average of the estimated mortality function over six years (see the right part of Figure 3.7). Thus, we simulate our model for 30 years using each of these two scenarios. The results are presented in Figure 3.8 and Figure 3.9, respectively.

### 3.5 Sensitivity analysis and parameter standard deviation computation

We begin by numerically analyzing the sensitivity of the model to the estimated parameters \(\alpha_i, i = 1, \ldots, 8\). In particular, in Figure 3.10 we graph the relative sensitivity with respect to each parameter logarithmically scaled, i.e., we plot  
\[
\log\left(\frac{|Q_{\alpha_i}(t)/(1 + Q(t))| + 1}{|Q_{\alpha_i}(t)/(1 + Q(t))| + 1}\right) \text{ for } i = 1, \ldots, 8.
\]

Clearly, from these graphs one observes that the model is most sensitive to the coefficient of the density \(Q\), i.e., to the parameter \(\alpha_6\) and least sensitive to the tadpole natural mortality parameter \(\alpha_8\).

An estimate for the model parameters can also be accompanied by an estimate of uncertainty using standard regression formulations from statistics [14]. Thus, in the remaining part of this section, we present a statistically based method to compute the variance in the estimated model parameters \(\theta = (\alpha_1, \ldots, \alpha_8)\). To perform this analysis, we need to compute the sensitivity matrix

\[
Y(\theta) = \begin{bmatrix}
Q_{\alpha_1}(t_1; \theta) & \cdots & Q_{\alpha_8}(t_1; \theta) \\
1 + Q(t_1; \theta) & \cdots & 1 + Q(t_1; \theta) \\
Q_{\alpha_1}(t_2; \theta) & \cdots & Q_{\alpha_8}(t_2; \theta) \\
1 + Q(t_2; \theta) & \cdots & 1 + Q(t_2; \theta) \\
\cdots & \cdots & \cdots \\
Q_{\alpha_1}(t_S; \theta) & \cdots & Q_{\alpha_8}(t_S; \theta) \\
1 + Q(t_S; \theta) & \cdots & 1 + Q(t_S; \theta)
\end{bmatrix} . \tag{3.5.1}
\]

Note that we cannot compute \(Q(t; \theta), Q_{\alpha_i}(t; \theta), i = 1, \ldots, 8\), directly from our model.
Therefore, we use the difference scheme in Section 3.3 to obtain the following approximation of $Q(t; \theta)$:

$$\hat{Q}(t; \theta) = \int_{x_{\min}}^{x_{\max}} V_{\Delta \alpha, \Delta x, \Delta t}(x, t; \theta) \, dx.$$ 

Then, we use a forward difference approximation for the derivative $Q_{\alpha_i}(t; \theta)$, $i = 1, \ldots, 8$ given by

$$\hat{Q}_{\alpha_i}(t; \alpha_1, \ldots, \alpha_8) = \frac{1}{\Delta \alpha_i} \left( \hat{Q}(t; \alpha_1, \ldots, \alpha_i + \Delta \alpha_i, \ldots, \alpha_8) - \hat{Q}(t; \alpha_1, \ldots, \alpha_i, \ldots, \alpha_8) \right).$$

Substituting $\hat{Q}(t_s; \theta)$ and $\hat{Q}_{\alpha_i}(t_s; \theta)$, $s = 1, \ldots, S$, for $Q(t_s; \theta)$ and $Q_{\alpha_i}(t_s; \theta)$ in (3.5.1), respectively, we obtain the following approximation of $Y(\theta)$:

$$\hat{Y}(\theta) = \begin{bmatrix} \hat{Q}_{\alpha_1}(t_1; \theta) & \cdots & \hat{Q}_{\alpha_S}(t_1; \theta) \\ 1 + \hat{Q}(t_1; \theta) & \cdots & 1 + \hat{Q}(t_1; \theta) \\ \hat{Q}_{\alpha_1}(t_2; \theta) & \cdots & \hat{Q}_{\alpha_S}(t_2; \theta) \\ 1 + \hat{Q}(t_2; \theta) & \cdots & 1 + \hat{Q}(t_2; \theta) \\ \vdots & \ddots & \vdots \\ \hat{Q}_{\alpha_1}(t_S; \theta) & \cdots & \hat{Q}_{\alpha_S}(t_S; \theta) \\ 1 + \hat{Q}(t_S; \theta) & \cdots & 1 + \hat{Q}(t_S; \theta) \end{bmatrix}. $$

Under standard assumptions of classic nonlinear regression theory, we know that if $\hat{\epsilon}_s \sim \mathcal{N}(0, \sigma^2)$, where $\hat{\epsilon}_s$ is the difference between the observation and the model at time $t_s$, then the least squares estimator $\theta^*$ is expected to be asymptotically normally distributed. In particular, for large samples, we may assume

$$\theta^* \sim \mathcal{N}[\theta_0, \sigma^2 \{Y^T(\theta_0)Y(\theta_0)\}^{-1}], \quad (3.5.2)$$

where $\theta_0$ is the true vector of parameters and $\sigma^2 \{Y^T(\theta_0)Y(\theta_0)\}^{-1}$ is the true covariance matrix (see [14, chap. 2]).
Since $\theta_0$ and $\sigma^2$ are not available, we follow a standard statistical practice [3, 9]: substitute the computed estimator $\bar{\theta}^*$ for $\theta_0$ and approximate $\sigma^2$ by

\[
\hat{\sigma}^2 = \frac{1}{S - 8} \sum_{s=1}^{S} \left( \log \left( Q(t_s; \bar{\theta}^*) + 1 \right) - \log(X_s + 1) \right)^2
\]

in (3.5.2) to obtain the standard deviation for our estimates. In particular, if

\[
E = \hat{\sigma}^2 \{ \hat{Y}^T(\bar{\theta}^*) \hat{Y}(\bar{\theta}^*) \}^{-1} = \begin{bmatrix}
E_{1,1} & E_{1,2} & \ldots & E_{1,8} \\
E_{2,1} & E_{2,2} & \ldots & E_{2,8} \\
\vdots & \vdots & \ddots & \vdots \\
E_{8,1} & E_{8,2} & \ldots & E_{8,8}
\end{bmatrix},
\]

then we take $\sqrt{E_{ii}}$ to be the standard deviation for parameter $\alpha_i$, $i = 1, \ldots, 8$. Table 3.1 provides the standard deviation of $\alpha_i$ for the parameter estimation results obtained in the previous section.

**Table 3.1.** Parameter estimation values and corresponding standard deviation.

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_4$</th>
<th>$\alpha_5$</th>
<th>$\alpha_6$</th>
<th>$\alpha_7$</th>
<th>$\alpha_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimated Value</td>
<td>0.489</td>
<td>3.400</td>
<td>0.232</td>
<td>2.908</td>
<td>3.093</td>
<td>0.00343</td>
<td>0.00</td>
<td>28.185</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.0124</td>
<td>1.781</td>
<td>0.173</td>
<td>1.299</td>
<td>1.402</td>
<td>0.00180</td>
<td>0.181</td>
<td>2.996</td>
</tr>
</tbody>
</table>

3.6 **Concluding remarks**

We have fitted a juvenile-adult amphibian model to population estimates obtained from field data collected during the years 2004-2009. The resulting fit looks very promising. In particular, it captures the essential behavior that is seen from the data: high magnitude oscillatory behavior. This is due to the fact that the reproduction rates of such animals are enormously high and the death rate is also high. Furthermore, these animals are seasonal breeders breeding from April to August and then hibernating until the next season begins. Thus, during the breeding months sharp
increase in the population number is observed until the breeding season halts. During
the remaining months, the juvenile stage has zero density and the frogs mostly
hibernate, and hence the dynamics of this population is influenced by mortality only.
Thus, the population decreases during this part of the year.

Based on the two scenarios discussed in the previous section, the long-term
simulations of the model indicate that if the estimated mortality rates do not change
substantially, the population will persist (modula some serious environmental disaster).
This is consistent with the fact that this population has been observed in these urban
ponds since 1991 (i.e., for about 20 years). However, the population level may change
considerably during one year because of seasonality and high breeding rate. In
particular, the statistical estimates and the model simulations show that the
population can vary by two orders of magnitude (between the high and low points),
and some of the low points are as little as nine frogs. Thus, it is possible that
stochastic perturbations at these low points may result in population extinction. In the
future, we plan to investigate the effects of demographic and environmental
stochasticity on the dynamics of this population and combine stochastic models similar
to those developed in [8] with the parameter estimates obtained in this to derive a
probability for this population’s persistence or extinction.

Our field monitoring project has been very extensive for the past six years. In
particular, we have monitored the population (through a capture-mark-recapture
experiment) weekly during the breeding season of every year. This allowed for high
resolution data which served us well to derive a reasonable model that seems to predict
well the population behavior. These findings observed from data and corroborated by the model provide good suggestions for future monitoring projects of similar populations. In particular, as monitoring is costly, if budgets are limited, then to get a good assessment of the population dynamics any program should consider at least two annual observations/estimates of the population: 1) right before the breeding season begins and 2) right after it ends. This allows for obtaining population estimates at the low end of the annual cycle and at the high end of the cycle. This also provides the minimum needed information about the oscillation behavior of such populations.

Finally, we point out that the population estimates in Figure 1 indicate that during the years 2008-2009 the frog population is declining. A possible cause for the decline in the last two years is that the particular body of water in which we have been conducting the CMR experiment has gotten infested with an invasive grass (Salvinia). It is known that this grass can lead to lower oxygen levels in small bodies of water and thus could be inhibiting tadpoles growth and decreasing their survivorship rates. In the future we plan to investigate this hypothesis by modeling the interaction of the frog population with the Salvinia grass.
Figure 3.1. Comparison of population estimates derived from real data with our model output using estimated parameters.

Figure 3.2. The function \( \phi(t) \) over 30 years used in the two hypothetical scenarios.

Figure 3.3. Prediction of population level of frogs for 30 years using parameters estimated with \( \phi(t) \) defined by the first scenario.
**Figure 3.4.** Prediction of population level of frogs for 30 years using parameters estimated with $\phi(t)$ defined by the second scenario.

**Figure 3.5.** Relative sensitivity with respect to each parameters logarithmically scaled.

**Figure 3.6.** Comparison of population estimates derived from real data with our model output using estimated parameters.
Figure 3.7. The function $\phi(t)$ over 30 years used in the two hypothetical scenarios.

Figure 3.8. Prediction of population level of frogs for 30 years using parameters estimated with $\phi(t)$ defined by the first scenario.

Figure 3.9. Prediction of population level of frogs for 30 years using parameters estimated with $\phi(t)$ defined by the second scenario.
Figure 3.10. Relative sensitivity with respect to each parameters logarithmically scaled.
REFERENCES


*Population declines and priorities for amphibian, conservation in latin america,*

APPENDIX

Proof of Theorem 3.3.1: Define \((J^k_i, A^k_j) = (J_i^k(\theta^r), A_j^k(\theta^r))\). From the fact that \(\Theta\) is compact and from the results of [7], there exist positive constants \(c_1, c_2, c_3\) and \(c_4\) such that

\[
\|J^k_i\|_1 + \|A^k_j\|_1 < c_1, \quad \|J^k_i\|_\infty + \|A^k_j\|_\infty < c_2, \quad \|J^k_i\|_{BV} + \|A^k_j\|_{BV} < c_3
\]

and

\[
\sum_{i=1}^m \left| \frac{J^k_i - J^{k-1}_i}{\Delta t} \right| \Delta a_r + \sum_{j=1}^n \left| \frac{A^k_j - A^{k-1}_j}{\Delta t} \right| \Delta x_r \leq c_4(q - p),
\]

where \(q > p\). Thus, there exists \((\hat{J}(a,t), \hat{A}(x,t)) \in BV([0, a_{max}] \times [0, T]) \times BV([x_{min}, x_{max}] \times [0, T])\) such that

\[
(U_{\Delta a_r, \Delta x_r, \Delta t_r}(a, t; \theta^r), V_{\Delta a_r, \Delta x_r, \Delta t_r}(x, t; \theta^r)) \to (\hat{J}(a,t), \hat{A}(x,t)) \text{ in } L^1(0, a_{max}) \times L^1(x_{min}, x_{max}) \text{ uniformly in } t.
\]

Hence, from the uniqueness of bounded variation weak solutions stated in [7], we need only to show that \((\hat{J}(a,t), \hat{A}(x,t))\) is the weak solution corresponding to the parameter \(\theta\). To this end, we let

\(\varphi \in C^1((0, a_{max}) \times (0, T))\) and denote the finite difference approximations \(\varphi(a_i, t_k)\) by \(\varphi^k_i\). Multiplying the first equation of the difference scheme (3.3.2) by \(\varphi^{k+1}_i\), we have

\[
J^{k+1}_i \varphi^{k+1}_i = J^k_i \varphi^k_i + \frac{\Delta t}{\Delta a} \left( J^k_i - J^{k-1}_i \right) \varphi^{k+1}_i - \Delta t \nu^k_i J^k_i \varphi^k_i - J^k_i \varphi^{k+1}_i.
\]

Thus,

\[
J^{k+1}_i \varphi^{k+1}_i - J^k_i \varphi^k_i = J^k_i \left( \varphi^{k+1}_i - \varphi^k_i \right) + \frac{\Delta t}{\Delta a} \left[ J^k_i (\varphi^{k+1}_i - \varphi^{k+1}_{i-1}) + (J^k_i \varphi^{k+1}_{i-1} - J^k_i \varphi^k_i) \right] - \Delta t \nu^k_i J^k_i \varphi^k_i.
\]

Multiplying the above equation by \(\Delta a\), summing over \(k = 0, 1, \cdots, l - 1\) and
\[ i = 1, 2, \cdots, m \] and using the third equation of (3.3.2), we obtain

\[
\sum_{i=1}^{m} (J_i^{l,r} \phi_i^l - J_i^{0,r} \phi_i^0) \Delta a
\]

\[
= \sum_{k=0}^{l-1} \sum_{i=1}^{m} [J_i^{k,r} (\phi_i^k - \varphi_i^k) \Delta a + J_{i-1}^{k,r} (\varphi_i^k - \varphi_{i-1}^{k+1}) \Delta t - \nu_i^{k,r} J_i^{k,r} \phi_i^{k+1} \Delta a \Delta t]
\]

\[
+ \sum_{k=0}^{l-1} \left( J_0^{k,r} \varphi_0^k - J_m^{k,r} \varphi_m^k \right) \Delta t
\]

\[
= \sum_{k=0}^{l-1} \sum_{i=1}^{m} \left( J_i^{k,r} \varphi_i^k \frac{\Delta a}{\Delta t} + J_{i-1}^{k,r} \varphi_i^k \frac{\Delta t}{\Delta a} - \nu_i^{k,r} J_i^{k,r} \phi_i^{k+1} \right) \Delta a \Delta t
\]

\[
+ \sum_{k=0}^{l-1} \sum_{i=1}^{m} \left( \sum_{j=1}^{n} \beta_j^{k,r} A_j^{k,r} \Delta x \right) \Delta t - \sum_{k=0}^{l-1} J_m^{k,r} \varphi_m^{k+1} \Delta t.
\]

(3.6.1)

On the other hand, let \( \psi \in C^1((x_{\min}, x_{\max}) \times (0, T)) \) and denote the finite difference approximations \( \psi(x_j, t_k) \) by \( \psi^k_j \). Multiply the second equation of (3.3.2) by \( \psi^k_j \) to find

\[
A_j^{k+1,r} \psi_j^{k+1} = A_j^{k,r} \psi_j^k + \frac{\Delta t}{\Delta x} (g_j^{k,r} A_j^{k,r} - g_j^{k+1,r} A_j^{k+1,r}) \psi_j^{k+1} - \Delta t \mu_j^{k,r} A_j^{k,r} \psi_j^{k+1}.
\]

Hence,

\[
A_j^{k+1,r} \psi_j^{k+1} - A_j^{k,r} \psi_j^k = A_j^{k,r} (\psi_j^{k+1} - \psi_j^k) + \frac{\Delta t}{\Delta x} [g_j^{k,r} A_j^{k,r} (\psi_j^{k+1} - \psi_j^{k+1})]
\]

\[
+ (g_j^{k,r} A_j^{k+1,r} - g_j^{k+1,r} A_j^{k+1,r})] - \Delta t \mu_j^{k,r} A_j^{k,r} \psi_j^{k+1}.
\]

Multiplying the above equation by \( \Delta x \), summing over \( k = 0, 1, \cdots, l - 1 \) and
\( j = 1, 2, \cdots, n \), and using \( g_n^{k,r} = 0 \) and \( g_0^{k,r} A_0^{k,r} = J_m^{k,r} \), we have

\[
\sum_{j=1}^{n} (A_j^{k,r} u_j^r - A_j^{0,r} u_j^0) \Delta x = \sum_{k=0}^{l-1} \sum_{j=1}^{n} \left[ A_j^{k,r} (\psi_j^{k+1} - \psi_j^k) \Delta x + g_j^{k,r} A_{j-1}^{k,r} (\psi_j^{k+1} - \psi_j^{k-1}) \Delta t \right]
- \mu_j^{k,r} A_j^{k,r} \psi_j^{k+1} \Delta x \Delta t + \sum_{k=0}^{l-1} (g_0^{k,r} A_0^{k,r} \psi_0^k - g_n^{k,r} A_n^{k,r} \psi_n^{k+1}) \Delta t
\]

(3.6.2)

Since \( \theta^r \to \theta \) as \( r \to \infty \) in \( \Theta \), passing to the limit in (3.6.1) and (3.6.2) we find that

\((\hat{J}(a,t), \hat{A}(x,t))\) is the weak solution corresponding to the parameter \( \theta \). \( \square \)

**Proof of Theorem 3.3.2:** Define \((J_i^{k,r}, A_i^{k,r})\) and \((J_i^k, A_i^k)\) to be the solution of the finite difference scheme with parameter \( \theta^r \) and \( \theta \), respectively. Let

\[
u_i^{k,r} = \nu^r(a_i, t_k, P^k(\theta^r)), \nu_i^k = \nu(a_i, t_k, P^k) \]

and similar notations are used for the rest of the parameters.

Using the first equation of (3.6.3) and (H7), we have

\[
|u_i^{k+1,r}| \leq \frac{\Delta t}{\Delta a} |u_i^{k,r}| + \left(1 - \frac{\Delta t}{\Delta a}\right) |u_i^{k,r}| + \Delta t |\nu_i^{k,r} J_i^{k,r} - \nu_i^k J_i^k|, \quad i = 1, 2, \cdots, m.
\]
On the other hand, using the second equation of (3.6.3) and (H7), we obtain

\[
\|u^{k+1,r}\|_1 \leq \|u^{k,r}\|_1 + \Delta t \left( |u_0^{k,r}| - |u_m^{k,r}| + \sum_{i=1}^m |\nu_i^{k,r} J_i^{k,r} - \nu_i^{k} J_i^k| \right) \Delta a. 
\]

(3.6.4)

Furthermore, we have

\[
|u_0^{k,r}| + \sum_{i=1}^m |\nu_i^{k,r} J_i^{k,r} - \nu_i^{k} J_i^k| \leq \sum_{j=1}^n (\beta_j^{k,r} A_j^{k,r} - \beta_j^k A_j^k) \Delta x + \sum_{j=1}^m |\beta_j^{k,r} (A_j^{k,r} - A_j^k) + (\beta_j^{k,r} - \beta_j^k) A_j^k| \Delta x \\
+ \sum_{i=1}^m |\nu_i^{k,r} (J_i^{k,r} - J_i^k) + (\nu_i^{k,r} - \nu_i^k) J_i^k| \Delta a \\
\leq \sup_j \beta_j^{k,r} \|v^{k,r}\|_1 + \sup_j |\beta_j^{k,r} - \beta_j^k| \|A^k\|_1 + \sup_i \|\nu_i^{k,r} \|_1 \|v^{k,r}\|_1 + \sup_i |\nu_i^{k,r} - \nu_i^k| \|J^k\|_1. 
\]

Applying the above inequality to (3.6.4) and using (H3), (H6) we get

\[
\|u^{k+1,r}\|_1 \leq \|u^{k,r}\|_1 - \Delta t |u_0^{k,r}| + \Delta t \left( \omega(\|u^{k,r}\|_1 + \|v^{k,r}\|_1) + \sup_j |\beta_j^{k,r} - \beta_j^k| \|A^k\|_1 + \|\nu_i^{k,r} \|_1 \|v^{k,r}\|_1 + \sup_i |\nu_i^{k,r} - \nu_i^k| \|J^k\|_1 \right). 
\]

(3.6.5)

On the other hand, using the second equation of (3.6.3) and (H7), we obtain

\[
|v_j^{k+1,r}| = \frac{\Delta t}{\Delta x} \left[ g_j^{k,r} (A_j^{k+1} - A_j^{k}) + (g_j^{k,r} - g_j^k) A_j^k \right] - \frac{\Delta t}{\Delta x} \left[ g_j^{k,r} (A_j^{k,r} - A_j^k) + (g_j^{k,r} - g_j^k) A_j^k \right] - \Delta t \left[ \mu_j^{k,r} (A_j^{k,r} - A_j^k) + (\mu_j^{k,r} - \mu_j^k) A_j^k \right] \\
= \left( 1 - \frac{\Delta t}{\Delta x} g_j^{k,r} - \Delta t \mu_j^{k,r} \right) v_j^{k,r} + \frac{\Delta t}{\Delta x} g_j^{k,r} v_j^{k,r} - \frac{\Delta t}{\Delta x} (g_j^{k,r} - g_j^k) A_j^k \\
- \frac{\Delta t}{\Delta x} (g_j^{k,r} - g_j^k) A_j^k - \Delta t (\mu_j^{k,r} - \mu_j^k) A_j^k \\
\leq \left( 1 - \frac{\Delta t}{\Delta x} g_j^{k,r} - \Delta t \mu_j^{k,r} \right) |v_j^{k,r}| + \frac{\Delta t}{\Delta x} g_j^{k,r} |v_j^{k,r}| + \Delta t |D_{\Delta x} (g_j^{k,r} - g_j^k) A_j^k| \\
+ \Delta t |\mu_j^{k,r} - \mu_j^k||A_j^k|. 
\]
Multiplying the above inequality by $\Delta x$, summing over the indices $j = 2, 3, \cdots, n$ and noticing that $g_n^{k,r} = 0$, we get
\[
\sum_{j=2}^{n} |v_j^{k+1,r}| \Delta x \leq \sum_{j=2}^{n} (1 - \Delta t \mu_j^{k,r}) |v_j^{k,r}| \Delta x + \Delta t g_1^{k,r} |v_1^{k,r}| + \Delta t \sum_{j=2}^{n} \left[ |D_{\Delta x}((g_j^{k,r} - g_j^{k})A_j^{k})| + |\mu_j^{k,r} - \mu_j^{k}||A_j^{k}| \right] \Delta x.
\]
(3.6.6)

For $j = 1$, by the second and fourth equations of (3.6.3) and (H7) we find
\[
|v_1^{k+1,r}| = \left| \frac{\Delta t}{\Delta x} u_m^{k,r} + v_1^{k,r} - \frac{\Delta t}{\Delta x} \left[ g_1^{k,r}(A_1^{k,r} - A_1^{k}) + (g_1^{k,r} - g_1^{k}) A_1^{k} \right] - \Delta t \left[ \mu_1^{k,r}(A_1^{k,r} - A_1) + (\mu_1^{k,r} - \mu_1^{k}) A_1^{k} \right] \right|
\]
\[
= \left| \frac{\Delta t}{\Delta x} u_m^{k,r} + \left( 1 - \frac{\Delta t}{\Delta x} g_1^{k,r} - \Delta t \mu_1^{k,r} \right) v_1^{k,r} - \frac{\Delta t}{\Delta x} \left( g_1^{k,r} - g_1^{k} \right) A_1^{k} \right|
\]
\[
\leq \frac{\Delta t}{\Delta x} |u_m^{k,r}| + \left( 1 - \frac{\Delta t}{\Delta x} g_1^{k,r} - \Delta t \mu_1^{k,r} \right) |v_1^{k,r}| + \frac{\Delta t}{\Delta x} \left( g_1^{k,r} - g_1^{k} \right) |A_1^{k}|
\]
\[
+ \Delta t \left| \mu_1^{k,r} - \mu_1^{k} \right| |A_1^{k}|.
\]
Thus,
\[
|v_1^{k+1,r}| \Delta x \leq \Delta t |u_m^{k,r}| + (1 - \Delta t \mu_1^{k,r}) |v_1^{k,r}| \Delta x - \Delta t g_1^{k,r} |v_1^{k,r}| + \Delta t |g_1^{k,r} - g_1^{k}||A_1^{k}| \Delta x.
\]
(3.6.7)

Adding (3.6.6) and (3.6.7), we get
\[
||v^{k+1,r}||_1 \leq ||v^{k,r}||_1 + \Delta t |u_m^{k,r}| + \Delta t \left[ \sum_{j=2}^{n} |D_{\Delta x}((g_j^{k,r} - g_j^{k})A_j^{k})| \Delta x \right.
\]
\[
+ \sum_{j=1}^{n} \left| \mu_j^{k,r} - \mu_j^{k} \right| A_j^{k} \Delta x + |g_1^{k,r} - g_1^{k}||A_1^{k}| \Delta x \right].
\]
(3.6.8)

Moreover,
\[
\sum_{j=2}^{n} |D_{\Delta x}((g_j^{k,r} - g_j^{k})A_j^{k})| \Delta x + \sum_{j=1}^{n} |\mu_j^{k,r} - \mu_j^{k}||A_j^{k}| \Delta x + |g_1^{k,r} - g_1^{k}||A_1^{k}|
\]
\[
= \sum_{j=2}^{n} |D_{\Delta x}(g_j^{k,r} - g_j^{k})A_j^{k} + (g_j^{k,r} - g_{j-1}^{k})D_{\Delta x}(A_j^{k})| \Delta x + \sum_{j=1}^{n} |\mu_j^{k,r} - \mu_j^{k}||A_j^{k}| \Delta x
\]
\[
+ |g_1^{k,r} - g_1^{k}||A_1^{k}| \leq \sup_j |D_{\Delta x}(g_j^{k,r} - g_j^{k})||A_k^{k}||_1 + \sup_j |g_j^{k,r} - g_{j-1}^{k}||A_j^{k}||_B V + \sup_j |\mu_j^{k,r} - \mu_j^{k}||A_j^{k}||_1
\]
\[
+ |g_1^{k,r} - g_1^{k}||A_1^{k}||_\infty.
\]

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Furthermore, straightforward computations yield

\[ \|u^{k+1,r}\|_1 + \|v^{k+1,r}\|_1 \]

\[ \leq \|u^{k,r}\|_1 + \|v^{k,r}\|_1 + \Delta t \left[ \omega(\|v^{k,r}\|_1 + \|u^{k,r}\|_1) + \left( \sup_j |\beta^k_{j,r} - \beta^k_j| \\
+ \sup_j |\mu^k_{j,r} - \mu^k_j| + \sup_j |D_{\Delta x}(g_j^{k,r} - g_j^k)| \right) \|A^k\|_1 + \sup_i |\nu^{k,r}_i - \nu^k_i| \|J^k\|_1 \right. \\
\left. + \sup_j |g_j^{k,r} - g_j^{k-1}| \|A^k\|_{BV} + |g_1^{k,r} - g_1^k| \|A^k\|_{\infty} \right]. \]

Noting that

\[ |\beta^k_{j,r} - \beta^k_j| \leq |\beta^r(x_j, t_k, Q^{k,r}) - \beta^r(x_j, t_k, Q^k)| + |\beta^r(x_j, t_k, Q^k) - \beta(x_j, t_k, Q^k)|, \]

and

\[ |Q^{k,r} - Q^k| = \left| \sum_{j=1}^n (A_j^{k,r} - A_j^k) \Delta x \right| \leq \sum_{j=1}^n |\nu^{k,r}_j| \Delta x = \|v^{k,r}\|_1, \]

by assumption (H3)-(H6) and the results of [7], we have

\[ \sup_j |\beta^k_{j,r} - \beta^k_j| \leq L\|v^{k,r}\|_1 + \sup_j |\beta^r(x_j, t_k, Q^k) - \beta(x_j, t_k, Q^k)|, \]

\[ \sup_j |\mu^k_{j,r} - \mu^k_j| \leq L\|v^{k,r}\|_1 + \sup_j |\mu^r(x_j, t_k, Q^k) - \mu(x_j, t_k, Q^k)|, \]

\[ \sup_i |\nu^{k,r}_i - \nu^k_i| \leq L\|v^{k,r}\|_1 + \sup_i |\nu^r(a_i, t_k, P^k) - \nu(a_i, t_k, P^k)|, \]

\[ \sup_j |g_j^{k,r} - g_j^k| \leq L\|v^{k,r}\|_1 + \sup_j |g^r(x_j, t_k, Q^k) - g(x_j, t_k, Q^k)|. \]

Furthermore, straightforward computations yield

\[ |D_{\Delta x}(g_j^{k,r} - g_j^k)| \]

\[ = \left| \frac{1}{\Delta x} \int_0^1 \frac{d}{d\tau} \left( g^r(\tau x_j + (1 - \tau)x_{j-1}, t_k, Q^{k,r}) - g(\tau x_j + (1 - \tau)x_{j-1}, t_k, Q^k) \right)d\tau \right| \]

\[ = \int_0^1 g_\tau^r(\tau x_j + (1 - \tau)x_{j-1}, t_k, Q^{k,r})d\tau - \int_0^1 g_\tau(\tau x_j + (1 - \tau)x_{j-1}, t_k, Q^k)d\tau \]

\[ \leq \int_0^1 |g_\tau^r(\tau x_j + (1 - \tau)x_{j-1}, t_k, Q^{k,r}) - g_\tau^r(\tau x_j + (1 - \tau)x_{j-1}, t_k, Q^k)|d\tau \]

\[ + \int_0^1 |g_\tau(\tau x_j + (1 - \tau)x_{j-1}, t_k, Q^k) - g_\tau(\tau x_j + (1 - \tau)x_{j-1}, t_k, Q^k)|d\tau. \]

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Hence, from (H4) and the results of [7] we obtain

\[ \sup_j |D_{\Delta x}^i (g_j^{k,r} - g_j^k)| \leq L \|v^{k,r}\|_1 + \sup_j \int_0^1 |g_x^r(\bar{x}_j, t_k, Q^k) - g_x(\bar{x}_j, t_k, Q^k)|dx, \]

where \( \bar{x}_j = \tau x_j + (1 - \tau)x_{j-1}. \) Set

\[ \delta_k = \omega + 3L \|A^k\|_1 + L \|J^k\|_1 + L(\|A^k\|_{BV} + \|A^k\|_{\infty}). \]

and

\[ \rho_{k,r} = \|A^k\|_1 \left( \sup_j |\beta^r(x_j, t_k, Q^k) - \beta(x_j, t_k, Q^k)| + \sup_j |\mu^r(x_j, t_k, Q^k) - \mu(x_j, t_k, Q^k)| 
+ \sup_j \int_0^1 |g_x^r(\bar{x}_j, t_k, Q^k) - g_x(\bar{x}_j, t_k, Q^k)|dx 
+ \sup_i |\nu^r(a_i, t_k, P^k) - \nu(a_i, t_k, P^k)| \|J^k\|_1 
+ \sup_j |g^r(x_j, t_k, Q^k) - g(x_j, t_k, Q^k)| (\|A^k\|_{BV} + \|A^k\|_{\infty}). \]

Then we have

\[ \|u^{k+1,r}\|_1 + \|v^{k+1,r}\|_1 \leq \|u^{k,r}\|_1 + \|v^{k,r}\|_1 + \Delta t[\delta_k(\|u^{k,r}\|_1 + \|v^{k,r}\|_1) + \rho_{k,r}]. \]

Since for each \( k, \rho_{k,r} \to 0 \) as \( r \to \infty, \) the desired result easily follows from this inequality. □

The following are the "hat" functions used in Section 3.4:

\[ \phi_0(t) = \begin{cases} 1 - t/2, & 0 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases} \]
\[ \phi_1(t) = \begin{cases} t/2, & 0 \leq t \leq 2 \\ 2 - t/2, & 2 \leq t \leq 4 \\ 0, & \text{otherwise} \end{cases} \]
\[ \phi_2(t) = \begin{cases} t/2 - 1, & 2 \leq t \leq 4 \\ 3 - t/2, & 4 \leq t \leq 6 \\ 0, & \text{otherwise} \end{cases} \]
\[ \phi_3(t) = \begin{cases} t/2 - 2, & 4 \leq t \leq 6 \\ 0, & \text{otherwise}. \end{cases} \]
CHAPTER 4

DIFFERENCE APPROXIMATION FOR AN AMPHIBIAN JUVENILE-ADULT DISPERSAL MODEL

In this chapter\(^1\), we consider an amphibian juvenile-adult population dispersing between ponds. We assume that juveniles (tadpoles) are structured by age and adults (frogs) are structured by size. This leads to a system of first order nonlocal hyperbolic equations. A finite difference approximation to this system is developed. Existence-uniqueness of the weak solution to the system is established and convergence of the finite difference approximation to the unique solution is proved.

4.1 Introduction

In this chapter, we consider the following system of first order hyperbolic equations which models an amphibian juvenile-adult population where individuals disperse between \(N\) separate ponds:

\[
\begin{align*}
J^I_t + J^I_a + \nu^I(a, t, P^I(t))J^I &= 0, \quad (a, t) \in (0, a_{\text{max}}) \times (0, T), \\
A^I_t + (g^I(x, t, Q^I(t))A^I)_x + \mu^I(x, t, Q^I(t))A^I &= \sum_{K=1, K \neq I}^N \tau_{KI}A^K - \sum_{K=1, K \neq I}^N \tau_{IK}A^I, \quad (x, t) \in (x_{\text{min}}, x_{\text{max}}) \times (0, T), \\
J^I(0, t) &= \int_{x_{\text{min}}}^{x_{\text{max}}} \beta^I(x, t, Q^I(t))A^I(x, t)dx, \quad t \in (0, T), \\
g^I(x_{\text{min}}, t, Q^I(t))A^I(x_{\text{min}}, t) &= J^I(a_{\text{max}}, t), \quad t \in (0, T), \\
J^I(a, 0) &= J^{I,0}(a), \quad a \in [0, a_{\text{max}}], \\
A^I(x, 0) &= A^{I,0}(x), \quad x \in [x_{\text{min}}, x_{\text{max}}].
\end{align*}
\]

\(^1\)The results of this chapter have been submitted for publication.
Here $J^I(a,t)$, $I = 1, 2, \cdots, N$, is the density of juveniles in the $I$th pond of age $a$ at time $t$, and $A^I(x,t)$ is the density of adults in the $I$th pond having size $x$ at time $t$. $a_{\text{max}}$ denotes the age at which a juvenile (tadpole) metamorphoses into an adult (frog), and $x_{\text{min}}$ and $x_{\text{max}}$ denote the minimum size and the maximum size of an adult, respectively. $P^I(t) = \int_0^{a_{\text{max}}} J^I(a,t)da$ is the total population of juveniles in the $I$th pond at time $t$, and $Q^I(t) = \int_{x_{\text{min}}}^{x_{\text{max}}} A^I(x,t)dx$ is the total population of adults in the $I$th pond at time $t$. The function $\nu^I$ represents the mortality rate of a juvenile in the $I$th pond, and $\mu^I$ represents the mortality rate of an adult in the $I$th pond. The function $\beta^I$ is the reproduction rate of an adult in the $I$th pond, and the function $g^I$ is the growth rate of an adult in the $I$th pond. The constant parameter $\tau_{KI}$ represents the dispersal rate of an adult from the $K$th pond to $I$th pond. In our model we assume that only adults (frogs) disperse between the separate ponds, which is typical of amphibian populations.

The above model extends the model we developed in [3] as it models the population in multi-ponds where the interaction takes place due to dispersal of adults between the ponds. In [3], we developed an explicit finite difference approximation scheme to the model therein, and we established the existence-uniqueness of the weak solution to the model and proved convergence of the finite difference approximation to the unique solution. In this chapter, we are also concerned with the existence and uniqueness of the weak solution of (4.1.1). For this purpose, we develop an implicit finite difference approximation for (4.1.1) in the spirit of the one initially used in [8, 13] for conservation laws and later extended to nonlocal first order hyperbolic
initial-boundary value problems arising in population ecology [1, 2, 4].

Many researchers have devoted considerable attention to studying spatially explicit continuous time models with dispersal. They considered either a discrete number of patches which results in a system of ordinary differential equations (or a system of integro-differential equations if time delays are included) (e.g. [9, 14, 15, 16, 17, 18, 19]) or a continuous spatial domain which leads to reaction-diffusion equations (e.g. [5, 6, 7, 10, 11, 12]). However, our system of nonlinear first order hyperbolic equations (4.1.1) differs from the models mentioned above as it combines continuous age and size structure with discrete spatial structure.

This chapter is organized as follows. In Section 4.2, we define a weak solution of (4.1.1) and develop an explicit finite difference approximation to the solution. In Section 4.3, we present certain estimates for this approximation. In Section 4.4, we establish the existence of a weak solution of (4.1.1). Finally in Section 4.5, we prove the uniqueness of the weak solution of (4.1.1).

4.2 Weak-solutioned and finite difference approximation

Throughout the discussion we let $\mathbb{D}_1 = [0, a_{\text{max}}] \times [0, T] \times [0, \infty)$, $\mathbb{D}_2 = [x_{\text{min}}, x_{\text{max}}] \times [0, T] \times [0, \infty)$, and $\omega_1, \omega_2$ be sufficiently large positive constants.

We assume that for any $I = 1, 2, \ldots, N$, the parameters in (4.1.1) satisfy the following assumptions:

(H1) $\nu^I(a, t, P^I)$ is a nonnegative bounded total variation function with respect to $a$ (uniformly in $t$ and $P^I$) and continuously differentiable with respect to $t$ and $P^I$. 

(H2) $g^I(x, t, Q^I)$ is twice continuously differentiable with respect to $x$ and continuously differentiable with respect to $t$ and $Q^I$, $g^I(x, t, Q^I) > 0$ for $x \in [x_{\min}, x_{\max})$ and $g^I(x_{\max}, t, Q^I) = 0$. Furthermore, $g^I_x$ is continuously differentiable with respect to $Q^I$.

(H3) $\mu^I(x, t, Q^I)$ is a nonnegative bounded total variation function with respect to $x$ (uniformly in $t$ and $Q^I$) and continuously differentiable with respect to $t$ and $Q^I$.

(H4) $\beta^I(x, t, Q^I)$ is a nonnegative bounded total variation function with respect to $x$ (uniformly in $t$ and $Q^I$) and continuously differentiable with respect to $t$ and $Q^I$. Furthermore, $\sup_{(x, t, Q^I) \in D} \beta^I(x, t, Q^I) \leq \omega_1$.

(H5) $J^{I,0} \in BV[0, a_{\max}]$ and $J^{I,0}(a) \geq 0$.

(H6) $A^{I,0} \in BV[x_{\min}, x_{\max}]$ and $A^{I,0}(x) \geq 0$.

(H7) For any sufficiently small $\delta > 0$,

$$\sup_{(x, t, Q^I) \in D_2} \left| \frac{g^I(x + \delta, t, Q^I) - g^I(x, t, Q^I)}{\delta} + \mu^I(x, t, Q^I) \right| \leq \omega_2.$$ 

Multiplying the first and second equations in (4.1.1) by $\varphi(a, t)$ and $\psi(x, t)$, respectively, and then formally integrating by parts and utilizing the initial and boundary conditions, we define a weak solution of (4.1.1) as follows:

**Definition 4.1.** A set of functions $J^1, A^1, \cdots, J^N, A^N$, where $J^I \times A^I \in BV([0, a_{\max}] \times [0, T]) \times BV([x_{\min}, x_{\max}] \times [0, T])$, $I = 1, 2, \cdots, N$, is called a
weak solution to problem (4.1.1) if this set satisfies the following:

\[
\begin{align*}
\int_0^{a_{\text{max}}} J^I(a,t)\varphi(a,t)\,da - \int_0^{a_{\text{max}}} J^{I,0}(a)\varphi(a,0)\,da \\
= \int_0^t \int_0^{a_{\text{max}}} J^I(\varphi_s + \varphi_a - \nu^I \varphi)\,dads \\
+ \int_0^t \varphi(0,s) \int_{x_{\text{min}}}^{x_{\text{max}}} \beta^I(x,s,Q^I(s))A^I(x,s)\,dxds \\
- \int_0^t J^I(a_{\text{max}},s)\varphi(a_{\text{max}},s)\,ds,
\end{align*}
\]

(4.2.1)

\[
\begin{align*}
\int_{x_{\text{min}}}^{x_{\text{max}}} A^I(x,t)\psi(x,t)\,dx - \int_{x_{\text{min}}}^{x_{\text{max}}} A^{I,0}(x)\psi(x,0)\,dx \\
= \int_0^t \int_{x_{\text{min}}}^{x_{\text{max}}} A^I(\psi_s + g^I \psi_x - \mu^I \psi)\,dxds + \int_0^t J^I(a_{\text{max}},s)\psi(x_{\text{min}},s)\,ds \\
+ \sum_{K=1,K\neq I}^N \tau_{IK} \int_0^t \int_{x_{\text{min}}}^{x_{\text{max}}} A^K \psi\,dxds - \sum_{K=1,K\neq I}^N \tau_{IK} \int_0^t \int_{x_{\text{min}}}^{x_{\text{max}}} A^I \psi\,dxds
\end{align*}
\]

for every test function \( \varphi \in C^1((0,a_{\text{max}}) \times (0,T)) \) and \( \psi \in C^1((x_{\text{min}},x_{\text{max}}) \times (0,T)) \) and \( t \in [0,T], I = 1,2,\cdots,N \).

We divide the intervals \([0,a_{\text{max}}], [x_{\text{min}},x_{\text{max}}] \) and \([0,T] \) into \( m, n \) and \( l \) subintervals, respectively. The following notations will be used throughout this chapter: \( \Delta a = a_{\text{max}}/m, \Delta x = (x_{\text{max}} - x_{\text{min}})/n \) and \( \Delta t = T/l \) denote the age, size, and time mesh lengths, respectively. The mesh points are given by: \( a_i = i\Delta a, i = 0,1,\cdots,m \), \( x_j = x_{\text{min}} + j\Delta x, j = 0,1,\cdots,n \), \( t_k = k\Delta t, k = 0,1,\cdots,l \). We denote by \( J^I_{i,k}, A^I_{j,k}, P^I_{j,k}, Q^I_{j,k} \) the finite difference approximation of \( J^I(a_i,t_k), A^I(x_j,t_k), P^I(t_k) \) and \( Q^I(t_k) \), respectively. We let

\[
\begin{align*}
\nu^I_{i,k} &= \nu^I(a_i,t_k,P^I_{i,k}), & g^I_{j,k} &= g^I(x_j,t_k,Q^I_{j,k}), \\
\mu^I_{j,k} &= \mu^I(x_j,t_k,Q^I_{j,k}), & \beta^I_{j,k} &= \beta^I(x_j,t_k,Q^I_{j,k}).
\end{align*}
\]
We define the difference operators

\[ D^{-\Delta a}(J_{i}^{I,k}) = \frac{J_{i}^{I,k} - J_{i-1}^{I,k}}{\Delta a}, 1 \leq i \leq m, \quad D^{-\Delta x}(A_{j}^{I,k}) = \frac{A_{j}^{I,k} - A_{j-1}^{I,k}}{\Delta x}, 1 \leq j \leq n, \]

and the \(\ell^1\) and \(\ell^\infty\) norms of \(J_{i}^{I,k}\) and \(A_{j}^{I,k}\) by

\[ \|J_{i}^{I,k}\|_1 = \sum_{i=1}^{m} |J_{i}^{I,k}| \Delta a, \quad \|A_{j}^{I,k}\|_1 = \sum_{j=1}^{n} |A_{j}^{I,k}| \Delta x, \]
\[ \|J_{i}^{I,k}\|_\infty = \max_{0 \leq i \leq m} |J_{i}^{I,k}|, \quad \|A_{j}^{I,k}\|_\infty = \max_{0 \leq j \leq n} |A_{j}^{I,k}|. \]

We then discretize the partial differential equation system (4.1.1) using the following finite difference approximation

\[ \frac{J_{i}^{I,k+1} - J_{i}^{I,k}}{\Delta t} + \frac{J_{i}^{I,k+1} - J_{i-1}^{I,k+1}}{\Delta a} + \nu_{i}^{I,k} J_{i}^{I,k+1} = 0, \quad 0 \leq k \leq l - 1, 1 \leq i \leq m, \]
\[ \frac{A_{j}^{I,k+1} - A_{j}^{I,k}}{\Delta t} + \frac{g_{j}^{I,k} A_{j}^{I,k+1} - g_{j-1}^{I,k} A_{j-1}^{I,k+1}}{\Delta x} + \mu_{j}^{I,k} A_{j}^{I,k+1} = 0, \quad 0 \leq k \leq l - 1, 1 \leq j \leq n, \]  
\[ J_{0}^{I,k+1} = \sum_{j=1}^{n} \beta_{j}^{I,k} A_{j}^{I,k} \Delta x, \quad g_{0}^{I,k} A_{0}^{I,k+1} = J_{m}^{I,k+1} = 0, \quad 0 \leq k \leq l - 1, \]
\[ P_{i}^{I,k+1} = \sum_{i=1}^{m} J_{i}^{I,k+1} \Delta a, \quad Q_{j}^{I,k+1} = \sum_{j=1}^{n} A_{j}^{I,k+1} \Delta x, \quad 0 \leq k \leq l - 1 \]

with the initial conditions

\[ J_{0}^{I,0} = J^{I,0}(0), \quad J_{i}^{I,0} = \frac{1}{\Delta a} \int_{(i-1)\Delta a}^{i\Delta a} J^{I,0}(a) \, da, \quad i = 1, 2, \ldots, m, \]
\[ A_{0}^{I,0} = A^{I,0}(0), \quad A_{j}^{I,0} = \frac{1}{\Delta x} \int_{(j-1)\Delta x}^{j\Delta x} A^{I,0}(x) \, dx, \quad j = 1, 2, \ldots, n. \]
We can equivalently write (4.2.2) as the following system of linear equations:

\[
\begin{align*}
(1 + \frac{\Delta t}{\Delta a} + \Delta t v_i^{l,k}) J_i^{l,k+1} &= J_i^{l,k} + \frac{\Delta t}{\Delta a} J_i^{l,k+1}, & 0 \leq k \leq l-1, 1 \leq i \leq m, \\
(1 + \frac{\Delta t}{\Delta x} g_j^{l,k} + \Delta t \mu_j^{l,k} + \Delta t \sum_{K=1, K \neq I}^N \tau_{IK} A_j^{l,k+1}) A_j^{l,k+1} &= \frac{\Delta t}{\Delta x} g_{j-1}^{l,k} A_{j-1}^{l,k+1} + A_j^{l,k} + \Delta t \sum_{K=1, K \neq I}^N \tau_{IK} A_j^{K,k}, & 0 \leq k \leq l-1, 1 \leq j \leq n, \\
J_0^{l,k+1} &= \sum_{j=1}^n \beta_j^{l,k} A_j^{l,k} \Delta x, & g_0^{l,k} A_0^{l,k+1} = J_m^{l,k+1}, & 0 \leq k \leq l-1, \\
P_i^{l,k+1} &= \sum_{i=1}^m J_i^{l,k+1} \Delta a, & Q_i^{l,k+1} &= \sum_{j=1}^n A_j^{l,k+1} \Delta x, & 0 \leq k \leq l-1.
\end{align*}
\]

(4.2.3)

**Lemma 4.2.1.** Under the assumptions on our parameters, the system (4.2.2) has a unique nonnegative solution for any choice of \(\Delta a, \Delta x\) and \(\Delta t\).

**Proof.** By assumptions (H5) and (H6) and initial conditions in (4.2.2), we have that

\( J_i^{l,0} \geq 0, A_j^{l,0} \geq 0, i = 0, 1, \ldots, m, j = 0, 1, \ldots, n, I = 1, \ldots, N. \)

Hence, from the third equation of (4.2.3), it is clear under the assumptions on our parameters that \( J_i^{l,1} \geq 0, I = 1, 2, \ldots, N. \)

Thus, by the first equation of (4.2.3) we get that \( J_i^{l,1} \geq 0, i = 1, 2, \ldots, m, I = 1, 2, \ldots, N. \)

Furthermore, using the fourth equation of (4.2.3), one can easily see that \( A_0^{l,1} \geq 0, I = 1, 2, \ldots, N. \)

Therefore, according to the second equation of (4.2.3) we find that \( A_j^{l,1} \geq 0, j = 1, 2, \ldots, n, I = 1, 2, \ldots, N. \)

In terms of the above facts of the case \( k = 1 \), by the similar observation, we can conclude that for \( k = 2, 3, \ldots, l, J_i^{l,k} \geq 0, A_j^{l,k} \geq 0, i = 0, 1, \ldots, m, j = 0, 1, \ldots, n, I = 1, 2, \ldots, N. \)

That is to say, the difference system (4.2.3) has a unique solution satisfying \( [J_0^{l,k}, J_1^{l,k}, \ldots, J_m^{l,k}, A_0^{l,k}, A_1^{l,k}, \ldots, A_n^{l,k}, A_i^{l,k}] \geq \overrightarrow{0}, k = 1, 2, \ldots, l, I = 1, 2, \ldots, N. \)

In other words, the system (4.2.2) has a unique nonnegative solution. \( \blacksquare \)
4.3 Estimates for the difference approximations

We first show that the difference approximation is bounded in $\ell^1$ norm.

**Lemma 4.3.1.** The following estimate holds:

$$
\sum_{l=1}^{N} (\|J^{l,k}\|_1 + \|A^{l,k}\|_1) \leq [1 + (\omega_1 + \theta) \Delta t]^k \sum_{l=1}^{N} (\|J^{l,0}\|_1 + \|A^{l,0}\|_1) \\
\leq [1 + (\omega_1 + \theta) \Delta t]^k \sum_{l=1}^{N} (\|J^{l,0}\|_1 + \|A^{l,0}\|_1) \equiv M_1,
$$

where $\theta = \max_{I=1, \ldots, N} \sum_{K=1}^{N} \tau_{IK}$.

**Proof.** Multiplying the first equation of (4.2.2) by $\Delta a$ and summing over $i = 1, 2, \cdots, m$, we have

$$
\frac{\|J^{I,k+1}\|_1 - \|J^{I,k}\|_1}{\Delta t} = J_0^{I,k+1} - J_m^{I,k+1} - \sum_{i=1}^{m} \nu_i^{I,k} J_i^{I,k+1} \Delta a.
$$

Treating the second equation of (4.2.2) similarly, and noticing that $g_n^{I,k} = 0$, we find

$$
\frac{\|A^{I,k+1}\|_1 - \|A^{I,k}\|_1}{\Delta t} = g_0^{I,k} A_0^{I,k+1} - \sum_{j=1}^{n} \mu_j^{I,k} A_j^{I,k+1} \Delta x + \sum_{K=1, K \neq I}^{N} \tau_{KI} \|A^{K,k}\|_1 - \sum_{K=1, K \neq I}^{N} \tau_{IK} \|A^{I,k+1}\|_1.
$$

Hence, using the boundary conditions given in the third and fourth equations of (4.2.2) and (H4), we get

$$
\frac{1}{\Delta t} [\|J^{I,k+1}\|_1 + \|A^{I,k+1}\|_1] - (\|J^{I,k}\|_1 + \|A^{I,k}\|_1) \\
= \sum_{j=1}^{n} \beta_j^{I,k} A_j^{I,k} \Delta x + \sum_{K=1, K \neq I}^{N} \tau_{KI} \|A^{K,k}\|_1 - \sum_{i=1}^{m} \nu_i^{I,k} J_i^{I,k+1} \Delta a \\
- \sum_{j=1}^{n} \mu_j^{I,k} A_j^{I,k+1} \Delta x - \sum_{K=1, K \neq I}^{N} \tau_{IK} \|A^{I,k+1}\|_1 \\
\leq \omega_1 \|A^{I,k}\|_1 + \sum_{K=1, K \neq I}^{N} \tau_{KI} \|A^{K,k}\|_1.
$$
Summing over $I = 1, 2, \cdots, N$ and by the fact that
\[ \sum_{I=1}^{N} \sum_{K=1, K \neq I}^{N} \tau_{KI} \| A^{K,k} \|_1 = \sum_{I=1}^{N} \sum_{K=1, K \neq I}^{N} \tau_{IK} \| A^{I,k} \|_1, \]
we have
\[ \sum_{I=1}^{N} (\| J^{I,k+1} \|_1 + \| A^{I,k+1} \|_1) \leq \sum_{I=1}^{N} (\| J^{I,k} \|_1 + \| A^{I,k} \|_1 + \omega_1 \Delta t \| A^{I,k} \|_1 + \Delta t \sum_{K=1, K \neq I}^{N} \tau_{IK} \| A^{I,k} \|_1) \leq \sum_{I=1}^{N} (1 + \omega_1 \Delta t + \Delta t \sum_{K=1, K \neq I}^{N} \tau_{IK})(\| J^{I,k} \|_1 + \| A^{I,k} \|_1) \leq [1 + (\omega_1 + \max_{I=1, \cdots, N} \sum_{K=1}^{N} \tau_{IK}) \Delta t] \sum_{I=1}^{N} (\| J^{I,k} \|_1 + \| A^{I,k} \|_1), \]
which implies the estimate.

We now let $\mathbb{D}_3 = [0, a_{\max}] \times [0, T] \times [0, M_1], \mathbb{D}_4 = [x_{\min}, x_{\max}] \times [0, T] \times [0, M_1]$. We then establish $\ell^\infty$ bound on the difference approximation.

**Lemma 4.3.2.** Assume that $\Delta t$ is chosen to satisfy $\omega_2 \Delta t < 1$, then we have the following estimates:

\[ \| J^{I,k} \|_\infty \leq \max \{ \| J^{I,0} \|_\infty, \omega_1 M_1 \}, \]

and

\[ \| A^{I,k} \|_\infty \leq \max \left\{ \frac{\max \{ \| J^{I,0} \|_\infty, \omega_1 M_1 \}}{\alpha}, \left( \frac{1 + \Theta (N-1) \Delta t}{1 - \omega_2 \Delta t} \right)^t \max_{1 \leq I \leq N} \| A^{I,0} \|_\infty \right\}, \]

where $\Theta = \max_{1 \leq K, I \leq N} \tau_{KI}$, and $\alpha \leq g^I(x_{\min}, t, Q^I)$ for $t \in [0, T]$ and $Q^I \in [0, M_1]$, $I = 1, 2, \cdots, N$. 

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Proof. If $J_{0}^{I,k+1} = \max_{0 \leq q \leq m} J_{q}^{I,k+1}$, then from the third equation of (4.2.3) and (H4) we get
\begin{equation}
J_{0}^{I,k+1} = \sum_{j=1}^{n} \beta_{j}^{I,k} A_{j}^{I,k} \Delta x \leq \omega_{1} \| A^{I,k} \|_{1} \leq \omega_{1} M_{1}.
\end{equation}
(4.3.1)

Otherwise, suppose that for some $1 \leq i \leq m$, $J_{i}^{I,k+1} = \max_{0 \leq q \leq m} J_{q}^{I,k+1}$, then from the first equation of (4.2.3) we have
\begin{equation}
(1 + \frac{\Delta t}{\Delta a} + \Delta t \nu_{i}^{I,k}) J_{i}^{I,k+1} - \frac{\Delta t}{\Delta a} J_{i-1}^{I,k+1} = J_{i}^{I,k}.
\end{equation}

Using the inequality $J_{i-1}^{I,k+1} \leq J_{i}^{I,k+1}$, we find
\begin{equation}
J_{i}^{I,k+1} \leq (1 + \Delta t \nu_{i}^{I,k}) J_{i}^{I,k+1} \leq J_{i}^{I,k} \leq \| J^{I,k} \|_{\infty}.
\end{equation}
(4.3.2)

A combination of (4.3.1) and (4.3.2) then yields
\begin{equation}
\| J^{I,k} \|_{\infty} \leq \max\{ \| J^{I,0} \|_{\infty}, \omega_{1} M_{1} \}.
\end{equation}

Similarly, if $A_{0}^{I,k+1} = \max_{0 \leq r \leq n} A_{r}^{I,k+1}$, then from the fourth equation of (4.2.3) we find
\begin{equation}
A_{0}^{I,k+1} \leq \frac{\| J^{I,k+1} \|_{\infty}}{\alpha} \leq \max\{ \| J^{I,0} \|_{\infty}, \omega_{1} M_{1} \}.
\end{equation}
(4.3.3)

Now, suppose $\max_{1 \leq L \leq N} \| A^{L,k+1} \|_{\infty} = \| A^{I,k+1} \|_{\infty}$ and $\max_{0 \leq r \leq n} A_{r}^{I,k+1}$ does not occur on the left boundary. For convenience, we assume that $A_{j}^{I,k+1} = \max_{0 \leq r \leq n} A_{r}^{I,k+1}$ for some $1 \leq j \leq n$, then from the second equation of (4.2.3) we get
\begin{equation}
\left( 1 + \Delta t \mu_{i}^{I,k} + \Delta t \sum_{K=1, K \neq I}^{N} \tau_{IK} \right) A_{j}^{I,k+1} + \frac{\Delta t}{\Delta x} \left( g_{j}^{I,k} A_{j}^{I,k+1} - g_{j-1}^{I,k} A_{j-1}^{I,k+1} \right) = A_{j}^{I,k} + \Delta t \sum_{K=1, K \neq I}^{N} \tau_{KI} A_{j}^{K,k}.
\end{equation}

Since $A_{j-1}^{I,k+1} \leq A_{j}^{I,k+1}$, we obtain
\[
\left( 1 + \Delta t\mu_j^{I,k} + \Delta t \sum_{K=1, K \neq I}^N \tau_{IK} + \Delta t \frac{g_j^{I,k} - g_{j-1}^{I,k}}{\Delta x} \right) A_j^{I,k+1} \leq A_j^{I,k} + \Delta t \sum_{K=1, K \neq I}^N \tau_{IK} A_j^{K,k}. \]

Hence, by (H7) we arrive at
\[
(1 - \omega_2 \Delta t) A_j^{I,k+1} \leq A_j^{I,k} + \Delta t \sum_{K=1, K \neq I}^N \tau_{IK} A_j^{K,k}. \]

Thus,
\[
(1 - \omega_2 \Delta t) \max_{1 \leq L \leq N} \|A^{L,k+1}\|_\infty \leq (1 + \Theta(N - 1)\Delta t) \max_{1 \leq L \leq N} \|A^{L,k}\|_\infty. \quad (4.3.4)
\]

A combination of (4.3.3) and (4.3.4) leads to the desired result.

The next two lemmas are necessary to show that the approximation \( J_i^{I,k} \) and \( A_j^{I,k} \) have bounded total variation.

**Lemma 4.3.3.** There exists a positive constant \( M_2 \) such that
\[
\left| \frac{J_0^{I,k+1} - J_0^{I,k}}{\Delta t} \right| \leq M_2, \quad k = 1, 2, \ldots, l - 1, I = 1, 2, \ldots, N.
\]

**Proof.** We have from the second and third equations of (4.2.2) that
\[
\frac{J_0^{I,k+1} - J_0^{I,k}}{\Delta t} = \frac{\Delta t}{\Delta t} \sum_{j=1}^n (\beta_j^{I,k} A_j^{I,k} - \beta_j^{I,k-1} A_j^{I,k-1}) \Delta x
\]
\[
= \frac{1}{\Delta t} \sum_{j=1}^n \beta_j^{I,k} (A_j^{I,k} - A_j^{I,k-1}) \Delta x + \frac{1}{\Delta t} \sum_{j=1}^n (\beta_j^{I,k} - \beta_j^{I,k-1}) A_j^{I,k-1} \Delta x
\]
\[
= \sum_{j=1}^n \beta_j^{I,k} \left( g_j^{I,k-1} A_{j-1}^{I,k} - g_j^{I,k-1} A_j^{I,k-1} \right) \Delta x + \sum_{j=1}^n \beta_j^{I,k} - \beta_j^{I,k-1}) A_j^{I,k-1} \Delta x
\]
\[
+ \sum_{K=1, K \neq I}^N \tau_{IK} A_j^{K,k} \Delta x - \sum_{K=1, K \neq I}^N \tau_{IK} A_j^{I,k} \Delta x
\]
\[
+ \frac{1}{\Delta t} \sum_{j=1}^n \left[ \beta_i^I(x_j, t_k, Q^{I,k}) \Delta t + \beta_i^I(x_j, t_k-1, \overline{Q}^{I,k})(Q^{I,k} - Q^{I,k-1}) \right] A_j^{I,k-1} \Delta x,
\]
where $\bar{t}_k \in [t_{k-1}, t_k]$, $Q^{I,k}$ is between $Q^{I,k-1}$ and $Q^{I,k}$.

Since $g_n^{I,k-1} = 0$, simple calculations yield

$$
\sum_{j=1}^n \beta_j^{I,k} (g_{j-1}^{I,k-1} A_{j-1}^{I,k} - g_j^{I,k-1} A_j^{I,k}) \\
= \beta_1^{I,k} g_0^{I,k-1} A_0^{I,k} - \beta_1^{I,k} g_1^{I,k-1} A_1^{I,k} + \sum_{j=2}^n \beta_j^{I,k} g_{j-1}^{I,k-1} A_{j-1}^{I,k} - \sum_{j=2}^{n-1} \beta_j^{I,k} g_j^{I,k-1} A_j^{I,k} \\
= \beta_1^{I,k} g_0^{I,k-1} A_0^{I,k} + \sum_{j=1}^{n-1} (\beta_{j+1}^{I,k} - \beta_j^{I,k}) g_j^{I,k-1} A_j^{I,k}.
$$

Hence,

$$
\frac{|J_0^{I,k+1} - J_0^{I,k}|}{\Delta t} \\
= \left| \beta_1^{I,k} g_0^{I,k-1} A_0^{I,k} + \sum_{j=1}^{n-1} (\beta_{j+1}^{I,k} - \beta_j^{I,k}) g_j^{I,k-1} A_j^{I,k} - \sum_{j=1}^n \beta_j^{I,k} g_j^{I,k-1} A_j^{I,k} \Delta x \\
+ \sum_{K=1,K \neq I}^N \tau_{KI} \sum_{j=1}^n \beta_j^{I,k} A_j^{K,k-1} \Delta x - \sum_{K=1,K \neq I}^N \tau_{IK} \sum_{j=1}^n \beta_j^{I,k} A_j^{I,k} \Delta x \\
+ \sum_{j=1}^n \left[ \beta_j^I(x_j, \bar{t}_k, Q^{I,k}) + \beta_{Q^I}(x_j, t_k, Q^{I,k}) \left( Q^{I,k} - Q^{I,k-1} \right) \right] A_j^{I,k-1} \Delta x \right| \\
\leq \sup_{(x,t,Q^I) \in D_2} (\beta_I g_I) \| A^{I,k} \|_\infty + \sup_{(x,t,Q^I) \in D_2} g_I \| A^{I,k} \|_\infty \sum_{j=1}^{n-1} |\beta_{j+1}^{I,k} - \beta_j^{I,k}| \\
+ \sup_{(x,t,Q^I) \in D_2} (\beta_I \mu_I) \| A^{I,k} \|_1 + \sum_{K=1,K \neq I}^N \tau_{KI} \sup_{(x,t,Q^I) \in D_2} \beta_I \| A^{K,k-1} \|_1 \\
+ \sum_{K=1,K \neq I}^N \tau_{IK} \sup_{(x,t,Q^I) \in D_2} \beta_I \| A^{I,k} \|_1 + \sup_{(x,t,Q^I) \in D_4} \beta_I \| A^{I,k-1} \|_1 \\
+ \sup_{(x,t,Q^I) \in D_4} |\beta_{Q^I}^I| \| A^{I,k-1} \|_1 \left| \frac{Q^{I,k} - Q^{I,k-1}}{\Delta t} \right| .
$$

Note that from (H4) it follows that there exists a $c_1 > 0$ such that

$$
\sum_{j=1}^n |\beta_{j+1}^{I,k} - \beta_j^{I,k}| \leq c_1. \quad (4.3.6)
$$
Furthermore,

\[
\frac{Q^{I,k} - Q^{I,k-1}}{\Delta t} = \frac{\sum_{j=1}^{n} (A^{I,k}_j - A^{I,k-1}_j) \Delta x}{\Delta t}
\]

\[
= \sum_{j=1}^{n} \left( g_{I,k-1,j} A^{I,k}_{j-1} - g_{I,k,j} A^{I,k}_j - \mu_{I,k-1,j} A^{I,k}_j \Delta x \right) + \sum_{K=1, K \neq I}^{N} \tau_{KI} A^{I,k}_{j,K} \Delta x - \sum_{K=1, K \neq I}^{N} \tau_{IK} A^{I,k}_j \Delta x
\]

\[
\leq \sup_{(x,t,Q^I) \in \mathbb{D}_2} g^I \| A^{I,k} \|_\infty + \sup_{(x,t,Q^I) \in \mathbb{D}_2} \mu^I \| A^{I,k} \|_1 + \sum_{K=1, K \neq I}^{N} \tau_{KI} \| A^{K,k-1} \|_1 + \sum_{K=1, K \neq I}^{N} \tau_{IK} \| A^{I,k} \|_1.
\]

Thus, by Lemmas 4.3.1-4.3.2 and (H2)-(H3), there exists a constant \( c_2 > 0 \) such that

\[
\left| \frac{Q^{I,k} - Q^{I,k-1}}{\Delta t} \right| \leq c_2.
\]  

(4.3.7)

Applying the bounds (4.3.6) and (4.3.7) to (4.3.5), we conclude that there exists a positive constant \( M_2 \) such that \( |(J^{I,k+1}_0 - J^{I,k}_0)/\Delta t| \leq M_2 \) for each \( k \) and \( I \).

Now we let \( \omega_3 \) be a positive constant such that \( \sup_{(x,t,Q^I) \in \mathbb{D}_4} |g^I_{x}(x,t,Q^I)| \leq \omega_3 \) and set \( \eta^{I,k}_j = D_{\Delta x}^{-}(A^{I,k}_j) \), we have the following result.

**Lemma 4.3.4.** There exists a positive constant \( M_3 \) such that

\[
\sum_{j=2}^{n} \left[ D^{-}_{\Delta x}(D^{-}_{\Delta x}(g_{I,k,j} A^{I,k+1}_j)) \Delta x + D^{-}_{\Delta x}(\mu_{I,k,j} A^{I,k+1}_j) \right] \text{sgn}(\eta^{I,k+1}_j) \Delta x
\]

\[
+ [D^{-}_{\Delta x}(g_{1,k} A^{I,k+1}_1) + \mu_{1,k} A^{I,k+1}_1] \text{sgn}(\eta^{I,k+1}_1) \geq -\omega_3 \| \eta^{I,k+1} \|_1 - M_3.
\]

**Proof.** We first consider the terms \( \sum_{j=2}^{n} D^{-}_{\Delta x}(D^{-}_{\Delta x}(g_{I,k,j} A^{I,k+1}_j)) \text{sgn}(\eta^{I,k+1}_j) \Delta x \) and
\( D_{\Delta x}(g_1^{I,k}A_1^{I,k+1})\text{sgn}(\eta_1^{I,k+1}). \) Straightforward computations give
\[
\sum_{j=2}^{n} D_{\Delta x}^{-}\left(\frac{g_j^{I,k}A_j^{I,k+1}-g_{j-1}^{I,k}A_{j-1}^{I,k+1}}{\Delta x}\right)\text{sgn}(\eta_j^{I,k+1})\Delta x
\]
\[
= \sum_{j=2}^{n} D_{\Delta x}^{-}\left(\frac{g_j^{I,k}A_j^{I,k+1}}{\Delta x}\right)\text{sgn}(\eta_j^{I,k+1})\Delta x
\]
\[
+ \sum_{j=2}^{n} D_{\Delta x}^{-}\left(\frac{g_j^{I,k}}{\Delta x}A_{j-1}^{I,k+1}\right)\text{sgn}(\eta_j^{I,k+1})\Delta x,
\]
and
\[
D_{\Delta x}(g_1^{I,k}A_1^{I,k+1})\text{sgn}(\eta_1^{I,k+1}) = D_{\Delta x}^{-}(g_1^{I,k})A_0^{I,k+1}\text{sgn}(\eta_1^{I,k+1}) + g_1^{I,k}\eta_1^{I,k+1}.
\]
Furthermore,
\[
\sum_{j=2}^{n} D_{\Delta x}^{-}\left(\frac{g_j^{I,k}A_j^{I,k+1}}{\Delta x}\right)\text{sgn}(\eta_j^{I,k+1})\Delta x + g_1^{I,k}\eta_1^{I,k+1}|
\]
\[
= \sum_{j=2}^{n} |\eta_j^{I,k+1}|D_{\Delta x}^{-}(g_j^{I,k})\Delta x + \sum_{j=2}^{n} g_j^{I,k}\frac{\eta_j^{I,k+1}-\eta_{j-1}^{I,k+1}}{\Delta x}\text{sgn}(\eta_j^{I,k+1})\Delta x + g_1^{I,k}\eta_1^{I,k+1}|
\]
\[
\geq \sum_{j=2}^{n} |\eta_j^{I,k+1}|D_{\Delta x}^{-}(g_j^{I,k})\Delta x + \sum_{j=2}^{n} g_j^{I,k}(|\eta_j^{I,k+1}| - |\eta_{j-1}^{I,k+1}|) + g_1^{I,k}\eta_1^{I,k+1}|
\]
\[
= \sum_{j=2}^{n} |\eta_j^{I,k+1}|(g_j^{I,k} - g_{j-1}^{I,k}) + \sum_{j=2}^{n} g_j^{I,k}(|\eta_j^{I,k+1}| - |\eta_{j-1}^{I,k+1}|) + g_1^{I,k}\eta_1^{I,k+1}|
\]
\[
= g_1^{I,k}\eta_1^{I,k+1} = 0.
\]
Hence,
\[
\sum_{j=2}^{n} D_{\Delta x}^{-}(D_{\Delta x}^{-}(g_j^{I,k}A_j^{I,k+1}))\text{sgn}(\eta_j^{I,k+1})\Delta x + D_{\Delta x}^{-}(g_1^{I,k}A_1^{I,k+1})\text{sgn}(\eta_1^{I,k+1})
\]
\[
\geq \sum_{j=2}^{n} D_{\Delta x}^{-}\left(\frac{g_j^{I,k} - g_{j-1}^{I,k}}{\Delta x}A_{j-1}^{I,k+1}\right)\text{sgn}(\eta_j^{I,k+1})\Delta x + D_{\Delta x}^{-}(g_1^{I,k})A_0^{I,k+1}\text{sgn}(\eta_1^{I,k+1})
\]
\[
= \sum_{j=2}^{n} D_{\Delta x}^{-}(D_{\Delta x}^{-}(g_j^{I,k}))A_j^{I,k+1}\text{sgn}(\eta_j^{I,k+1})\Delta x
\]
\[
+ \sum_{j=2}^{n} D_{\Delta x}^{-}(g_j^{I,k})D_{\Delta x}^{-}(A_j^{I,k+1})\text{sgn}(\eta_j^{I,k+1})\Delta x + D_{\Delta x}^{-}(g_1^{I,k})A_0^{I,k+1}\text{sgn}(\eta_1^{I,k+1})
\]
\[
\geq - \max_j |D_{\Delta x}^{-}(D_{\Delta x}^{-}(g_j^{I,k})))||A_j^{I,k+1}||_1 - \max_j |D_{\Delta x}^{-}(g_j^{I,k})||\eta_j^{I,k+1}||_1
\]
\[
- \max_j |D_{\Delta x}^{-}(g_j^{I,k})||A_j^{I,k+1}||_\infty.
\]

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We then consider other terms, simple calculations yield

\[
\sum_{j=2}^{n} D_{\Delta x}^{-}(\mu_{j}^{I,k} A_{j}^{I,k+1}) \text{sgn}(\eta_{j}^{I,k+1}) \Delta x + \mu_{1}^{I,k} A_{1}^{I,k+1} \text{sgn}(\eta_{1}^{I,k+1})
\]

\[
= \sum_{j=2}^{n} D_{\Delta x}^{-}(\mu_{j}^{I,k}) A_{j}^{I,k+1} \text{sgn}(\eta_{j}^{I,k+1}) \Delta x
\]

\[
+ \sum_{j=2}^{n} \mu_{j-1}^{I,k} D_{\Delta x}^{-}(A_{j}^{I,k+1}) \text{sgn}(\eta_{j}^{I,k+1}) \Delta x + \mu_{1}^{I,k} A_{1}^{I,k+1} \text{sgn}(\eta_{1}^{I,k+1})
\]

\[
\geq -\|A_{I,k+1}\|_{\infty} \sum_{j=2}^{n} |\mu_{j}^{I,k} - \mu_{j-1}^{I,k}| + \sum_{j=2}^{n} \mu_{j-1}^{I,k} |\eta_{j}^{I,k+1}| \Delta x - \max_{j} \mu_{j}^{I,k} \|A_{I,k+1}\|_{\infty}.
\]

(4.3.9)

Therefore, adding (4.3.8) and (4.3.9) and using (H2)-(H3) and Lemma 3.1-3.2, we obtain the desired result.

With the above Lemmas, we will show that approximations \(J_{i}^{I,k}\) and \(A_{j}^{I,k}\) have bounded total variation. The total variation bound plays an important role in establishing the sequential convergence of the difference approximation (4.2.2) to a weak solution of (4.1.1).

**Lemma 4.3.5.** Assume that \(\Delta t\) is chosen to satisfy \(\omega_{3} \Delta t < 1\). Then there exists a positive constant \(M_{4}\) such that

\[
\sum_{i=1}^{N} (\|D_{\Delta a}(J_{i}^{I,k})\|_{1} + \|D_{\Delta x}(A_{i}^{I,k})\|_{1}) \leq M_{4}.
\]

**Proof.** Set \(\xi_{i}^{I,k} = D_{\Delta a}(J_{i}^{I,k})\) and apply the operator \(D_{\Delta a}\) to the first equation of (4.2.2) to get

\[
\frac{\xi_{i}^{I,k+1} - \xi_{i}^{I,k}}{\Delta t} + \frac{\xi_{i}^{I,k+1} - \xi_{i-1}^{I,k+1}}{\Delta a} + D_{\Delta a}(\nu_{i}^{I,k} J_{i}^{I,k+1}) = 0, \quad 2 \leq i \leq m.
\]

Multiplying the above each equation by \(\text{sgn}(\xi_{i}^{I,k+1}) \Delta a\), and summing over

\(i = 2, 3, \cdots, m\), and noticing that

\[
\xi_{i}^{I,k} \text{sgn}(\xi_{i}^{I,k+1}) \leq |\xi_{i}^{I,k}|, \quad \xi_{i-1}^{I,k} \text{sgn}(\xi_{i}^{I,k+1}) \leq |\xi_{i-1}^{I,k}|,
\]

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we find
\[
\sum_{i=2}^{m} (|\xi_{i}^{I,k+1}| - |\xi_{i}^{I,k}|) \Delta t
\leq |\xi_{1}^{I,k+1}| - |\xi_{m}^{I,k+1}| + \sum_{i=2}^{m} |D_{\Delta a}(\nu_{i}^{I,k} J_{i}^{I,k+1})| \Delta a, \quad 2 \leq i \leq m.
\] (4.3.10)

For \(i = 1\), the first equation of (4.2.2) takes the form
\[
\frac{J_{1}^{I,k+1} - J_{1}^{I,k}}{\Delta t} + \frac{J_{0}^{I,k+1} - J_{0}^{I,k}}{\Delta a} + \nu_1^{I,k} J_{1}^{I,k+1} = 0.
\]

On the other hand,
\[
\frac{\xi_{1}^{I,k+1} - \xi_{1}^{I,k}}{\Delta t} = \frac{1}{\Delta t} \left( \frac{J_{1}^{I,k+1} - J_{0}^{I,k+1}}{\Delta a} - \frac{J_{1}^{I,k} - J_{0}^{I,k}}{\Delta a} \right)
= \frac{1}{\Delta a} \left( \frac{J_{1}^{I,k+1} - J_{1}^{I,k}}{\Delta t} - \frac{J_{0}^{I,k+1} - J_{0}^{I,k}}{\Delta t} \right).
\]

Hence,
\[
\frac{\xi_{1}^{I,k+1} - \xi_{1}^{I,k}}{\Delta t} = -\frac{1}{\Delta a} \left( \frac{J_{1}^{I,k+1} - J_{0}^{I,k+1}}{\Delta a} + \nu_1^{I,k} J_{1}^{I,k+1} + \frac{J_{0}^{I,k+1} - J_{0}^{I,k}}{\Delta t} \right).
\]

Multiplying the above equation by \(\text{sgn}(\xi_{1}^{I,k+1}) \Delta a\) and noticing that
\(\xi_{1}^{I,k} \text{sgn}(\xi_{1}^{I,k+1}) \leq |\xi_{1}^{I,k}|\), we have
\[
\frac{(|\xi_{1}^{I,k+1}| - |\xi_{1}^{I,k}|) \Delta a}{\Delta t} \leq -|\xi_{1}^{I,k+1}| + \nu_1^{I,k} J_{1}^{I,k+1} + \frac{|J_{0}^{I,k+1} - J_{0}^{I,k}|}{\Delta t}.
\] (4.3.11)

Adding (4.3.10) and (4.3.11), we have
\[
\frac{||\xi_{1}^{I,k+1}|| - ||\xi_{1}^{I,k}||}{\Delta t} \leq -|\xi_{1}^{I,k+1}| + \sum_{i=2}^{m} |D_{\Delta a}(\nu_{i}^{I,k} J_{i}^{I,k+1})| \Delta a + \nu_1^{I,k} J_{1}^{I,k+1}
+ \frac{|J_{0}^{I,k+1} - J_{0}^{I,k}|}{\Delta t}.
\]
Note that
\[
\sum_{i=2}^{m} |D_{\Delta a}^{-}(\nu_{i}^{I,k} J_{i}^{I,k+1})| \Delta a + \nu_{1}^{I,k} J_{1}^{I,k+1} \\
= \sum_{i=2}^{m} \left| (\nu_{i}^{I,k} - \nu_{i-1}^{I,k}) J_{i}^{I,k+1} + \nu_{i-1}^{I,k} (J_{i}^{I,k+1} - J_{i-1}^{I,k+1}) + \nu_{i}^{I,k} J_{i}^{I,k+1} \right| \\
\leq \sum_{i=2}^{m} |\nu_{i}^{I,k} - \nu_{i-1}^{I,k}| \| J^{I,k} \|_{\infty} + \max_{i} \nu_{i}^{I,k} \| \xi^{I,k+1} \|_{1} + \max_{i} \nu_{i}^{I,k} \| J^{I,k+1} \|_{\infty}.
\]

Therefore, by Lemma 3.2-3.3 and (H1), there exist positive constants $c_3$ and $c_4$ such that
\[
\sum_{i=2}^{m} |D_{\Delta a}^{-}(\nu_{i}^{I,k} J_{i}^{I,k+1})| \Delta a + \nu_{1}^{I,k} J_{1}^{I,k+1} + \left| \frac{J_{0}^{I,k+1} - J_{0}^{I,k}}{\Delta t} \right| \leq c_3 \| \xi^{I,k+1} \| + c_4.
\]

Thus,
\[
\frac{\| \xi^{I,k+1} \|_{1} - \| \xi^{I,k} \|_{1}}{\Delta t} \leq -|s_{m}^{I,k+1}| + c_3 \| \xi^{I,k+1} \| + c_4. \tag{4.3.12}
\]

Similarly, apply the operator $D_{\Delta x}^{-}$ to the second equation of (4.2.2) to get
\[
\frac{\eta_{j}^{I,k+1} - \eta_{j}^{I,k}}{\Delta t} + D_{\Delta x}^{-} \left( g_{j}^{I,k} A_{j}^{I,k+1} - g_{j-1}^{I,k} A_{j-1}^{I,k+1} \right) + D_{\Delta x}^{-} (\mu_{j}^{I,k} A_{j}^{I,k+1}) \\
= \sum_{K=1,K \neq I}^{N} \tau_{KI} \eta_{j}^{K,k} - \sum_{K=1,K \neq I}^{N} \tau_{IK} \eta_{j}^{I,k+1}, \quad 2 \leq j \leq n.
\]

Multiplying the above equation by $\text{sgn}(\eta_{j}^{I,k+1}) \Delta x$ and summing over $j = 2, 3, \cdots, n$
and noticing that $\text{sgn}(\eta_{j}^{I,k+1}) \eta_{j}^{I,k} \leq |\eta_{j}^{I,k}|$, we have
\[
\sum_{j=2}^{n} \left( |\eta_{j}^{I,k+1}| - |\eta_{j}^{I,k}| \right) \Delta x \\
\leq -\sum_{j=2}^{n} \left[ D_{\Delta x}^{-} (\Delta \eta_{j}^{I,k} A_{j}^{I,k+1}) + D_{\Delta x}^{-} (\mu_{j}^{I,k} A_{j}^{I,k+1}) \\
- \sum_{K=1,K \neq I}^{N} \tau_{KI} \eta_{j}^{K,k} + \sum_{K=1,K \neq I}^{N} \tau_{IK} \eta_{j}^{I,k+1} \right] \text{sgn}(\eta_{j}^{I,k+1}) \Delta x. \tag{4.3.13}
\]

If $j = 1$, the second equation of (4.2.2) takes the form
\[
\frac{\Delta t}{N} A_{1}^{I,k+1} - \frac{\Delta t}{N} A_{1}^{I,k} + \frac{\Delta t}{N} (g_{1}^{I,k} A_{1}^{I,k+1} - g_{0}^{I,k} A_{0}^{I,k+1}) + \mu_{1}^{I,k} A_{1}^{I,k+1} \\
= \sum_{K=1,K \neq I}^{N} \tau_{KI} A_{1}^{K,k} - \sum_{K=1,K \neq I}^{N} \tau_{IK} A_{1}^{I,k+1}.
\]

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On the other hand,
\[
\frac{\eta^{I,k+1}_1 - \eta^{I,k}_1}{\Delta t} = \frac{1}{\Delta t} \left( \frac{A^{I,k+1}_1 - A^{I,k+1}_0}{\Delta x} - \frac{A^{I,k}_1 - A^{I,k}_0}{\Delta t} \right) = \frac{1}{\Delta x} \left( \frac{A^{I,k+1}_1 - A^{I,k}_1}{\Delta t} - \frac{A^{I,k+1}_0 - A^{I,k}_0}{\Delta t} \right).
\]

Hence,
\[
\frac{\eta^{I,k+1}_1 - \eta^{I,k}_1}{\Delta t} = \frac{1}{\Delta x} \left[ -D_{\Delta x}(g^{I,k}_1 A^{I,k+1}_1) - \mu^{I,k}_1 A^{I,k+1}_1 + \sum_{K=1, K \neq I}^N \tau_{KI} A^{K,k}_1 - \sum_{K=1, K \neq I}^N \tau_{IK} A^{I,k+1}_1 - A^{I,k+1}_0 - A^{I,k}_0 \right] \Delta t.
\]

Multiplying the above equation by \(\text{sgn}(\eta^{I,k+1}_1) \Delta x\) and noticing that
\[\eta^{I,k}_1 \text{sgn}(\eta^{I,k+1}_1) \leq |\eta^{I,k}_1|,\]
we have
\[
\frac{|\eta^{I,k+1}_1| - |\eta^{I,k}_1|}{\Delta t} \Delta x \leq -\frac{\sum_{j=2}^N \left[ D_{\Delta x}^{-}(D_{\Delta x}^{-}(g^{I,k}_j A^{I,k+1}_1)) + D_{\Delta x}^{-}(\mu^{I,k}_j A^{I,k+1}_1) - \sum_{K=1, K \neq I}^N \tau_{KI} A^{K,k}_1 + \sum_{K=1, K \neq I}^N \tau_{IK} A^{I,k+1}_1 - A^{I,k+1}_0 - A^{I,k}_0 \right] \text{sgn}(\eta^{I,k+1}_1) \Delta x}{\Delta t}.
\]

Adding (4.3.13) and (4.3.14), we arrive at
\[
\frac{\|\eta^{I,k+1}_1\|_1 - \|\eta^{I,k}_1\|_1}{\Delta t} \leq -\frac{\sum_{j=2}^N \left[ D_{\Delta x}^{-}(D_{\Delta x}^{-}(g^{I,k}_j A^{I,k+1}_1)) + D_{\Delta x}^{-}(\mu^{I,k}_j A^{I,k+1}_1) - \sum_{K=1, K \neq I}^N \tau_{KI} A^{K,k}_1 + \sum_{K=1, K \neq I}^N \tau_{IK} A^{I,k+1}_1 - A^{I,k+1}_0 - A^{I,k}_0 \right] \text{sgn}(\eta^{I,k+1}_1) \Delta x}{\Delta t}.
\]
Thus, using Lemma 3.4 we obtain
\[
\frac{\|\eta^{I,k+1}\|_1 - \|\eta^{I,k}\|_1}{\Delta t} \\
\leq \omega_3 \|\eta^{I,k+1}\|_1 + M_3 + \sum_{K=1, K \neq I}^N \tau_{KI} \|\eta^{K,k}\|_1 + \sum_{K=1, K \neq I}^N \tau_{KI} A_{1}^{K,k} \\
+ \sum_{K=1, K \neq I}^N \tau_{IK} A_{1}^{I,k+1} + \left| \frac{A_{0}^{I,k+1} - A_{0}^{I,k}}{\Delta t} \right|.
\]

Summing over \( I = 1, 2, \ldots, N \) and by the fact that
\[
\sum_{I=1}^N \sum_{K=1, K \neq I}^N \tau_{KI} \|\eta^{K,k}\|_1 = \sum_{I=1}^N \sum_{K=1, K \neq I}^N \tau_{IK} \|\eta^{I,k}\|_1,
\]
we have
\[
\sum_{I=1}^N \frac{\|\eta^{I,k+1}\|_1 - \|\eta^{I,k}\|_1}{\Delta t} \\
\leq \omega_3 \sum_{I=1}^N \|\eta^{I,k+1}\|_1 + N M_3 + \sum_{I=1}^N \sum_{K=1, K \neq I}^N \tau_{IK} \|\eta^{I,k}\|_1 + \sum_{I=1}^N \sum_{K=1, K \neq I}^N \tau_{KI} \|A^{K,k}\|_\infty \\
+ \sum_{I=1}^N \sum_{K=1, K \neq I}^N \tau_{IK} \|A^{I,k+1}\|_\infty + \sum_{I=1}^N \left| \frac{A_{0}^{I,k+1} - A_{0}^{I,k}}{\Delta t} \right|.
\]

Therefore, by Lemma 3.2, we find that there exists a positive constant \( c_5 \) such that
\[
\sum_{I=1}^N \frac{\|\eta^{I,k+1}\|_1 - \|\eta^{I,k}\|_1}{\Delta t} \\
\leq \omega_3 \sum_{I=1}^N \|\eta^{I,k+1}\|_1 + \max_{I=1, \ldots, N} \sum_{K=1}^N \tau_{IK} \sum_{I=1}^N \|\eta^{I,k}\|_1 + c_5 + \sum_{I=1}^N \left| \frac{A_{0}^{I,k+1} - A_{0}^{I,k}}{\Delta t} \right|. \tag{4.3.15}
\]
By virtue of the first and fourth equations of (4.2.2), we get

\[
\frac{A_{I;k}^{I,k+1} - A_{I;k}^{I,k}}{\Delta t} = \frac{J_{m}^{I,k+1} / g_{0}^{I,k} - J_{m}^{I,k} / g_{0}^{I,k-1}}{\Delta t}
\]

\[
= \frac{(g_{0}^{I,k-1} - g_{0}^{I,k})J_{m}^{I,k+1} + g_{0}^{I,k}(J_{m}^{I,k+1} - J_{m}^{I,k})}{\Delta t g_{0}^{I,k} J_{m}^{I,k-1}}
\]

\[
= \frac{g^{I}(x_{\min}, t_{k-1}, Q^{I,k-1}) - g^{I}(x_{\min}, t_{k}, Q^{I,k})}{\Delta t} \frac{J_{m}^{I,k+1}}{g_{0}^{I,k}}
\]

\[
- \frac{1}{g_{0}^{I,k-1}} \left( \frac{J_{m}^{I,k+1} - J_{m}^{I,k-1}}{\Delta t} + \nu_{m}^{I,k} J_{m}^{I,k+1} \right)
\]

\[
\leq \frac{1}{g_{0}^{I,k-1}} \left( \frac{\xi_{m}^{I,k+1} + \nu_{m}^{I,k} J_{m}^{I,k+1}}{g_{0}^{I,k} g_{0}^{I,k-1}} \right)
\]

\[
\leq \left\| J_{m}^{I,k+1} \right\|_{\infty} \left[ \sup_{(x, t, Q')} \left| g^{I}_{l} \right| + \sup_{(x, t, Q')} \left| g^{I}_{Q'} \right| \left( \frac{Q^{I,k-1} - Q^{I,k}}{\Delta t} \right) \right]
\]

\[
+ \frac{1}{g_{0}^{I,k-1}} \left( \left\| \xi_{m}^{I,k+1} \right\| + \max_{i} \nu_{i}^{I,k} \left\| J_{m}^{I,k+1} \right\|_{\infty} \right),
\]

where \( \bar{t}_{k} \in [t_{k-1}, t_{k}] \), and \( \bar{Q}^{I,k} \) is between \( Q^{I,k-1} \) and \( Q^{I,k} \).

Then, by (H1)-(H2) and Lemma 3.2 and (4.3.7), there exists a positive constant \( c_{6} \) such that

\[
\left| \frac{A_{0}^{I,k+1} - A_{0}^{I,k}}{\Delta t} \right| \leq c_{6} + \frac{\left\| \xi_{m}^{I,k+1} \right\|}{g_{0}^{I,k-1}}.
\]  

(4.3.16)

Applying (4.3.16) to (4.3.15) we get

\[
\sum_{I=1}^{N} \left\| \eta^{I,k+1} \right\|_{1} - \left\| \eta^{I,k} \right\|_{1} \leq \omega_{3} \sum_{I=1}^{N} \left\| \eta^{I,k+1} \right\|_{1} + \theta \sum_{I=1}^{N} \left\| \eta^{I,k} \right\|_{1} + c_{5} + c_{6} + \sum_{I=1}^{N} \frac{\left\| \xi_{m}^{I,k+1} \right\|}{g_{0}^{I,k-1}}.
\]  

(4.3.17)

Now dividing (4.3.12) by \( g_{0}^{I,k-1} \) we have

\[
\frac{\left\| \xi^{I,k+1} \right\|_{1} - \left\| \xi^{I,k} \right\|_{1}}{g_{0}^{I,k-1} \Delta t} \leq - \frac{\left\| \xi_{m}^{I,k+1} \right\|}{g_{0}^{I,k-1}} + c_{3} \left\| \xi^{I,k+1} \right\|_{1} + c_{4} \frac{\left\| \xi^{I,k} \right\|}{g_{0}^{I,k-1}}.
\]
Thus,

\[
\frac{\|\xi^{I,k+1}\|_1/g_0^{I,k-1} - \|\xi^{I,k}\|_1/g_0^{I,k-2}}{\Delta t} \\
\leq \frac{\|\xi^{I,k}\|_1 - \|\xi^{I,k+1}\|_1}{\Delta t g_0^{I,k-1}} - \frac{\|\xi^{I,k+1}\|_1}{g_0^{I,k-1}} - \frac{\|\xi^{I,k}\|_1}{g_0^{I,k-1}} + \frac{c_3\|\xi^{I,k+1}\|_1}{g_0^{I,k-1}} + \frac{c_4}{g_0^{I,k-1}}.
\]

Simple calculation gives

\[
\frac{1}{\Delta t} \left( \frac{1}{g_0^{I,k-1}} - \frac{1}{g_0^{I,k-2}} \right) \|\xi^{I,k}\|_1 - \frac{c_3\|\xi^{I,k+1}\|_1}{g_0^{I,k-1}} + \frac{c_4}{g_0^{I,k-1}}.
\]

where \(\bar{Q}^{I,k-1}\) is between \(Q^{I,k-2}\) and \(Q^{I,k-1}\).

Thus, by (H2), (4.3.7) and (4.3.12) which implies \(\|\xi^{I,k}\|_1\) is bounded, there exists a positive constant \(c_7\) such that

\[
\frac{\|\xi^{I,k+1}\|_1/g_0^{I,k-1} - \|\xi^{I,k}\|_1/g_0^{I,k-2}}{\Delta t} \leq c_7 - \frac{\|\xi^{I,k+1}\|_1}{g_0^{I,k-1}}.
\]

Summing over \(I = 1, 2, \ldots, N\), we have

\[
\sum_{I=1}^{N} \frac{\|\xi^{I,k+1}\|_1/g_0^{I,k-1} - \|\xi^{I,k}\|_1/g_0^{I,k-2}}{\Delta t} \leq Nc_7 - \sum_{I=1}^{N} \frac{\|\xi^{I,k+1}\|_1}{g_0^{I,k-1}}.
\] (4.3.18)

Therefore, letting \(\zeta^k = \sum_{I=1}^{N} (\|\xi^{I,k}\|_1/g_0^{I,k-2} + \|\eta^{I,k}\|_1)\) and adding (4.3.17) and (4.3.18), we obtain

\[
\frac{\zeta^{k+1} - \zeta^k}{\Delta t} \leq \omega_3\zeta^{k+1} + \theta\zeta^k + c_5 + c_6 + Nc_7.
\]

The result now easily follows from the above inequality.
The next result shows that the difference approximations satisfy a Lipschitz-type condition in $t$.

**Lemma 4.3.6.** Then exist positive constants $M_5$ and $M_6$ such that for any $q > p$, we have

$$
\sum_{i=1}^{m} \left| \frac{J_i^{I,q} - J_i^{I,p}}{\Delta t} \right| \Delta a \leq M_5(q - p), \quad \sum_{j=1}^{n} \left| \frac{A_j^{I,q} - A_j^{I,p}}{\Delta t} \right| \Delta x \leq M_6(q - p),
$$

for $I = 1, 2, \cdots, N$.

**Proof.** Summing the first equation of (4.2.2) over $i = 1, 2, \cdots, m$ and multiplying by $\Delta a$, we obtain

$$
\sum_{i=1}^{m} \left| \frac{J_i^{I,k+1} - J_i^{I,k}}{\Delta t} \right| \Delta a = \sum_{i=1}^{m} \left| \frac{J_i^{I,k+1} - J_i^{I,k+1}}{\Delta a} + \nu_i^{I,k} J_i^{I,k+1} \right| \Delta a \\
\leq \| D_{\Delta a} (J_i^{I,k+1}) \|_1 + \max_i \nu_i^{I,k} \| J_i^{I,k+1} \|_1.
$$

By Lemma 3.1, 3.5 and (H1), there exists a positive constant $M_5$ such that

$$
\sum_{i=1}^{m} \left| \frac{J_i^{I,k+1} - J_i^{I,k}}{\Delta t} \right| \Delta a \leq M_5.
$$

Hence,

$$
\sum_{i=1}^{m} \left| \frac{J_i^{I,q} - J_i^{I,p}}{\Delta t} \right| \Delta a \leq \sum_{k=p}^{q-1} \sum_{i=1}^{m} \left| \frac{J_i^{I,k+1} - J_i^{I,k}}{\Delta t} \right| \Delta a \leq M_5(q - p).
$$

Similarly, using the second equation of (4.2.2) and Lemma 3.1, 3.5 and assumptions
(H2), (H7), we get

\[
\sum_{j=1}^{n} \left| \frac{A_{j}^{I,k+1} - A_{j}^{I,k}}{\Delta t} \right| \Delta x = \sum_{j=1}^{n} \left| \frac{g_{j}^{I,k} A_{j}^{I,k+1} - g_{j-1}^{I,k} A_{j-1}^{I,k+1}}{\Delta x} + \mu_{j}^{I,k} A_{j}^{I,k+1} \right| \Delta x \\
- \sum_{K=1,K \neq I}^{N} \tau_{KI} A_{K}^{I,k} + \sum_{K=1,K \neq I}^{N} \tau_{IK} A_{j}^{I,k+1} \right| \Delta x \leq \omega_{2} \|A_{I}^{I,k+1}\|_{1} + \max_{j} g_{j-1}^{I,k} \|D_{\Delta x}(A_{I}^{I,k+1})\|_{1} \\
+ \sum_{K=1,K \neq I}^{N} \tau_{KI} \|A_{K}^{I,k}\|_{1} + \sum_{K=1,K \neq I}^{N} \tau_{IK} \|A_{I}^{I,k+1}\|_{1} \leq M_{6}.
\]

Thus,

\[
\sum_{j=1}^{n} \left| \frac{A_{j}^{I,q} - A_{j}^{I,p}}{\Delta t} \right| \Delta x \leq \sum_{k=p}^{q-1} \sum_{j=1}^{n} \left| \frac{A_{j}^{I,k+1} - A_{j}^{I,k}}{\Delta t} \right| \Delta x \leq M_{6}(q - p).
\]

4.4 Convergence of difference approximation and existence of a weak solution

Following [13] we define a family of functions \(\{U_{\Delta a,\Delta t}^{I}\}\) and \(\{V_{\Delta x,\Delta t}^{I}\}\) by

\[
U_{\Delta a,\Delta t}^{I}(a,t) = J_{I}^{I,k}, \quad V_{\Delta x,\Delta t}^{I}(x,t) = A_{j}^{I,k}
\]

for \(a \in [a_{i-1}, a_{i}], x \in [x_{j-1}, x_{j}], t \in [t_{k-1}, t_{k}], i = 1, \cdots, m, j = 1, \cdots, n, k = 1, \cdots, l\) and \(I = 1, 2, \cdots, N\). Then by Lemmas 3.1-3.6 the set of functions \(\{U_{\Delta a,\Delta t}^{I}\}, \{V_{\Delta x,\Delta t}^{I}\}\) is compact in the topology of \(L^{1}((0, a_{\text{max}}) \times (0, T)) \times L^{1}((x_{\text{min}}, x_{\text{max}}) \times (0, T))\) for \(I = 1, 2, \cdots, N\). Hence as in the proof of Lemma 16.7 on page 276 in [13], we have the following lemma.
Lemma 4.4.1. There exists a sequence of functions

\((\{U_{\Delta a,\Delta t}\}, \{V_{\Delta x,\Delta t}\}) \subset (\{U_{a,\Delta t}\}, \{V_{x,\Delta t}\})\) which converges to a function

\((J^I, A^I) \in BV([0,a_{\text{max}}] \times [0,T]) \times BV([x_{\text{min}},x_{\text{max}}] \times [0,T])\) for \(I = 1, 2, \ldots, N\), in the sense that for all \(t > 0\)

\[
\int_0^{a_{\text{max}}} U_{\Delta a,\Delta t}(a, t) - J^I(a, t) \, da \to 0,
\]

\[
\int_{x_{\text{min}}}^{x_{\text{max}}} V_{\Delta x,\Delta t}(x, t) - A^I(x, t) \, dx \to 0,
\]

\[
\int_0^T \int_0^{a_{\text{max}}} U_{\Delta a,\Delta t}(a, t) - J^I(a, t) \, da dt \to 0,
\]

\[
\int_0^T \int_{x_{\text{min}}}^{x_{\text{max}}} V_{\Delta x,\Delta t}(x, t) - A^I(x, t) \, dx dt \to 0,
\]

as \(\gamma \to \infty\) (i.e., \(\Delta a, \Delta x, \Delta t \to 0\)). Furthermore, there exist constants \(M_7\) and \(M_8\) (dependent on \(\|J^I,0\|_{BV[0,a_{\text{max}}]}\) and \(\|A^I,0\|_{BV[x_{\text{min}},x_{\text{max}}]}\)) such that the limit functions satisfy

\[
\|J^I\|_{BV([0,a_{\text{max}}] \times [0,T])} \leq M_7, \quad \|A^I\|_{BV([x_{\text{min}},x_{\text{max}}] \times [0,T])} \leq M_8.
\]

The next theorem shows that the set of limit functions

\(J^1(a, t), A^1(x, t), \ldots, J^N(a, t), A^N(x, t)\), constructed via our difference scheme is actually a weak solution of problem (4.1.1).

Theorem 4.4.1. The set of limit functions \(J^1(a, t), A^1(x, t), \ldots, J^N(a, t), A^N(x, t)\) defined in Lemma 4.4.1 is a weak solution of (4.1.1) and satisfies

\[
\sum_{I=1}^N [P^I(t) + Q^I(t)] \leq e^{(\omega_1+\theta)T} \sum_{I=1}^N (\|J^I,0\|_1 + \|A^I,0\|_1).
\]
and

\[
\|J^I\|_{\mathcal{L}^\infty((0,a_{\text{max}}) \times (0,T))} \leq \max \left\{ \|J^{I,0}\|_{\infty}, \omega_1 e^{(\omega_1 + \theta)T} \sum_{I=1}^{N} (\|J^{I,0}\|_1 + \|A^{I,0}\|_1) \right\},
\]

\[
\|A^I\|_{\mathcal{L}^\infty((x_{\text{min}},x_{\text{max}}) \times (0,T))} \leq \max \left\{ \max_{1 \leq I \leq N} \|J^{I,0}\|_{\infty}, \omega_1 e^{(\omega_1 + \theta)T} \sum_{I=1}^{N} (\|J^{I,0}\|_1 + \|A^{I,0}\|_1) \right\},
\]

\[
e^{(\Theta(N-1)-\omega_2)T} \max_{1 \leq I \leq N} \|A^{I,0}\|_{\infty} \right\}.
\]

**Proof.** Let \( \varphi \in C^1((0, a_{\text{max}}) \times (0, T)) \) and denote the finite difference approximations \( \varphi(a_i, t_k) \) by \( \varphi_i^k \). Multiplying the first equation of the difference scheme (4.2.2) by \( \varphi_i^{k+1} \), we have

\[
\frac{J_{i,k+1}^{I,k+1} \varphi_i^{k+1} - J_{i,k}^{I,k} \varphi_i^k}{\Delta t} = \frac{J_{i,k+1}^{I,k+1} \varphi_i^{k+1} - J_{i,k}^{I,k} \varphi_i^k}{\Delta t} + \frac{J_{i,k+1}^{I,k+1} \varphi_i^{k+1} - J_{i-1,k}^{I,k+1} \varphi_{i-1}^{k+1}}{\Delta a} - \frac{J_{i,k+1}^{I,k+1} \varphi_i^{k+1} - J_{i-1,k}^{I,k+1} \varphi_{i-1}^{k+1}}{\Delta a}.
\]

Multiplying the above equation by \( \Delta a \Delta t \), summing over \( k = 0, 1, \cdots, l - 1 \) and \( i = 1, 2, \cdots, m \) and using the third equation of (4.2.2), we obtain

\[
\sum_{i=1}^{m} (J_{i,k}^{I,k} \varphi_i^k - J_{i}^{I,0} \varphi_i^0) \Delta a
\]

\[
= \sum_{k=0}^{l-1} \sum_{i=1}^{m} \left( \frac{J_{i,k}^{I,k} \varphi_i^{k+1} - \varphi_i^k}{\Delta t} + \frac{J_{i-1,k}^{I,k} \varphi_{i-1}^{k+1} - \varphi_{i-1}^{k+1}}{\Delta a} - \nu_{i-1}^{I,k} J_{i-1,k}^{I,k+1} \varphi_i^{k+1} \right) \Delta a \Delta t
\]

\[
+ \sum_{k=0}^{l-1} \left( J_{0,k}^{I,k} \varphi_0^{k+1} - J_{m,k}^{I,k} \varphi_m^{k+1} \right) \Delta t
\]

\[
= \sum_{k=0}^{l-1} \sum_{i=1}^{m} \left( \frac{J_{i,k}^{I,k} \varphi_i^{k+1} - \varphi_i^k}{\Delta t} + \frac{J_{i-1,k}^{I,k} \varphi_{i-1}^{k+1} - \varphi_{i-1}^{k+1}}{\Delta a} - \nu_{i-1}^{I,k} J_{i-1,k}^{I,k+1} \varphi_i^{k+1} \right) \Delta a \Delta t
\]

\[
+ \sum_{k=0}^{l-1} \varphi_0^{k+1} \left( \sum_{j=1}^{n} \beta_j^{I,k} A_{j,k}^{I,k} \Delta x \right) \Delta t - \sum_{k=0}^{l-1} J_{m,k+1}^{I,k} \varphi_m^{k+1} \Delta t.
\]

On the other hand, let \( \psi \in C^1((x_{\text{min}}, x_{\text{max}}) \times (0, T)) \) and denote the finite difference approximations \( \psi(x_j, t_k) \) by \( \psi_j^k \). Multiply the second equation of (4.2.2) by \( \psi_j^{k+1} \) to find
\begin{align*}
&\frac{A_j^{k,1} \psi_j^{k+1} - A_j^{k} \psi_j^{k}}{\Delta t} - \frac{g_j^{k} A_j^{k,1} \psi_j^{k+1} - g_j^{k} A_j^{k} \psi_j^{k+1}}{\Delta x} \\
&\quad - g_j^{k} A_j^{k,1} \psi_j^{k+1} - \psi_j^{k+1} + \mu_j^{k} A_j^{k,1} \psi_j^{k+1} \\
&= \left( \sum_{K=1, K \neq I}^{N} \tau_{KI} A_j^{K,k} - \sum_{K=1, K \neq I}^{N} \tau_{IK} A_j^{I,k+1} \right) \psi_j^{k+1}.
\end{align*}

Multiplying the above equation by \( \Delta x \Delta t \), summing over \( k = 0, 1, \cdots, l - 1 \) and \( j = 1, 2, \cdots, n \) and using \( g_j^{I,k} = 0 \) and \( g_j^{0,k} A_j^{0,k+1} = J_m^{I,k+1} \), we have
\begin{align}
\sum_{j=1}^{n} (A_j^{I,j} \psi_j^{j} - A_j^{I,0} \psi_j^{0}) \Delta x \\
= \sum_{k=0}^{l-1} \sum_{j=1}^{n} \left( \frac{A_j^{I,k} \psi_j^{k+1} - \psi_j^{k}}{\Delta t} + \frac{g_j^{I,k} A_j^{I,k+1} \psi_j^{k+1} - \psi_j^{k+1}}{\Delta x} - \mu_j^{I,k} A_j^{I,k+1} \psi_j^{k+1} + \sum_{K=1, K \neq I}^{N} \tau_{KI} A_j^{K,k} \psi_j^{k+1} - \sum_{K=1, K \neq I}^{N} \tau_{IK} A_j^{I,k+1} \psi_j^{k+1} \right) \Delta x \Delta t \\
+ \sum_{k=0}^{l-1} J_m^{I,k+1} \psi_0^{k+1} \Delta t.
\end{align}

Using (4.4.1) and (4.4.2) and following an argument similar to that used in the proof of Lemma 16.9 on page 280 of [13] we obtain, by letting \( m, n, l \to \infty \), that the limit of the difference approximations defined in Lemma 4.1 is a weak solution of (4.1.1). Taking the limit in the bounds obtained in Lemmas 4.3.1-4.3.2, we get the bounds on \( P(t), Q(t), \|J(t)\|_{L^\infty((0,a_{\max}) \times (0,T))} \) and \( \|A(t)\|_{L^\infty((x_{\min},x_{\max}) \times (0,T))} \).

\section{Uniqueness of the weak solution}

The following theorem guarantees the continuous dependence of the solution
\[ \{J_i^{1,k}, A_j^{1,k}, \cdots, J_i^{N,k}, A_j^{N,k}\} \] and \[ \{\hat{J}_i^{1,k}, \hat{A}_j^{1,k}, \cdots, \hat{J}_i^{N,k}, \hat{A}_j^{N,k}\} \] of (4.2.2) with respect to the initial condition \[ \{J_i^{1,0}, A_j^{1,0}, \cdots, J_i^{N,0}, A_j^{N,0}\} \] and \[ \{\hat{J}_i^{1,0}, \hat{A}_j^{1,0}, \cdots, \hat{J}_i^{N,0}, \hat{A}_j^{N,0}\} \].
Theorem 4.5.1. Let \( \{J^{1,k}_i, A^{1,k}_j, \ldots, J^{N,k}_i, A^{N,k}_j\} \) and \( \{\hat{J}^{1,k}_i, \hat{A}^{1,k}_j, \ldots, \hat{J}^{N,k}_i, \hat{A}^{N,k}_j\} \) be the solutions of (4.2.2) corresponding to the initial conditions \( \{J^{1,0}_i, A^{1,0}_j, \ldots, J^{N,0}_i, A^{N,0}_j\} \) and \( \{\hat{J}^{1,0}_i, \hat{A}^{1,0}_j, \ldots, \hat{J}^{N,0}_i, \hat{A}^{N,0}_j\} \), respectively. Then there exists a positive constant \( \sigma \) such that

\[
\sum_{i=1}^{N} (\|J^{I,k+1}_i - \hat{J}^{I,k+1}_i\|_1 + \|A^{I,k+1}_i - \hat{A}^{I,k+1}_i\|_1) \\
\leq (1 + \sigma t) \sum_{i=1}^{N} (\|J^{I,k}_i - \hat{J}^{I,k}_i\|_1 + \|A^{I,k}_i - \hat{A}^{I,k}_i\|_1)
\]

for all \( k \geq 0 \).

Proof. Let \( u^{I,k}_i = J^{I,k}_i - \hat{J}^{I,k}_i, v^{I,k}_j = A^{I,k}_j - \hat{A}^{I,k}_j \) for \( i = 0, 1, \ldots, m, j = 0, 1, \ldots, n \), \( k = 0, 1, \ldots, l \) and \( I = 1, 2, \ldots, N \). Then \( u^{I,k}_i, v^{I,k}_j \) satisfy the following:

\[
\frac{u^{I,k+1}_i - u^{I,k}_i}{\Delta t} + D\Delta a(u^{I,k+1}_i) + \nu^{I,k}_i J^{I,k+1}_i - \hat{\nu}^{I,k}_i \hat{J}^{I,k+1}_i = 0,
\]

\[
\frac{v^{I,k+1}_j - v^{I,k}_j}{\Delta t} + D\Delta x(g^{I,k}_j A^{I,k+1}_j - \hat{g}^{I,k}_j \hat{A}^{I,k+1}_j) + \mu^{I,k}_j A^{I,k+1}_j - \hat{\mu}^{I,k}_j \hat{A}^{I,k+1}_j = \sum_{K=1,K\neq I}^{N} \tau^{K,k} v^{K,k+1}_j - \sum_{K=1,K\neq I}^{N} \tau^{K,k} v^{K,k+1}_j,
\]

where \( \hat{\nu}^{I,k}_i = \nu(a_i, t_k, \hat{P}^{I,k}_i) \) and similar notations are used for the rest of the parameters.

Multiplying the first equation of (4.5.1) by \( \text{sgn}(u^{I,k+1}_i) \Delta a \), noticing that \( u^{I,k}_i \text{sgn}(u^{I,k+1}_i) \leq |u^{I,k}_i| \) and summing over \( i = 1, 2, \ldots, m \), we find

\[
\frac{\|u^{I,k+1}\|_1 - \|u^{I,k}\|_1}{\Delta t} \leq - \sum_{i=1}^{m} \left[ D\Delta a(u^{I,k+1}_i) + \nu^{I,k}_i J^{I,k+1}_i - \hat{\nu}^{I,k}_i \hat{J}^{I,k+1}_i \right] \text{sgn}(u^{I,k+1}_i) \Delta a.
\]
Note that
\[ \sum_{i=1}^m D_{\Delta a}(u_i^{I,k+1}) \Delta a = \sum_{i=1}^m \frac{u_i^{I,k+1} - u_i^{I,k+1} \Delta a}{\Delta a} \geq \sum_{i=1}^m (|u_i^{I,k+1}| - |u_i^{I,k+1}|) = |u_m^{I,k+1}| - |u_0^{I,k+1}|, \]

and
\[ \sum_{i=1}^m (\nu_i^{I,k} \hat{J}_i^{I,k+1} - \hat{\nu}_i^{I,k} \hat{J}_i^{I,k+1}) \sigma(u_i^{I,k+1}) \Delta a = \sum_{i=1}^m (\nu_i^{I,k} \hat{J}_i^{I,k+1} - \nu_i^{I,k} \hat{J}_i^{I,k+1} + \nu_i^{I,k} \hat{J}_i^{I,k+1} - \hat{\nu}_i^{I,k} \hat{J}_i^{I,k+1}) \sigma(u_i^{I,k+1}) \Delta a = \sum_{i=1}^m \nu_i^{I,k} u_i^{I,k+1} + (\nu_i^{I,k} - \hat{\nu}_i^{I,k}) \hat{J}_i^{I,k+1} \sigma(u_i^{I,k+1}) \Delta a = \sum_{i=1}^m u_i^{I,k} |u_i^{I,k+1}| \Delta a + \sum_{i=1}^m (\nu_i^{I,k} - \hat{\nu}_i^{I,k}) \hat{J}_i^{I,k+1} \sigma(u_i^{I,k+1}) \Delta a. \]

Thus,
\[ \frac{\|u_i^{I,k+1}\|_1 - \|u_i^{I,k}\|_1}{\Delta t} \leq |u_0^{I,k+1}| - |u_m^{I,k+1}| - \sum_{i=1}^m \nu_i^{I,k} |u_i^{I,k+1}| \Delta a - \sum_{i=1}^m (\nu_i^{I,k} - \hat{\nu}_i^{I,k}) \hat{J}_i^{I,k+1} \sigma(u_i^{I,k+1}) \Delta a. \] \hspace{1cm} (4.5.2)

Furthermore,
\[ |u_0^{I,k+1}| - \sum_{i=1}^m (\nu_i^{I,k} - \hat{\nu}_i^{I,k}) \hat{J}_i^{I,k+1} \sigma(u_i^{I,k+1}) \Delta a \]
\[ \leq |u_0^{I,k+1}| + \sum_{i=1}^m |\nu_i^{I,k} - \hat{\nu}_i^{I,k}| |\hat{J}_i^{I,k+1}| \Delta a \]
\[ = \sum_{j=1}^n \left[ \beta_j^{I,k} (A_j^{I,k} - \hat{A}_j^{I,k}) + (\beta_j^{I,k} - \hat{\beta}_j^{I,k}) \hat{A}_j^{I,k} \right] \Delta x + \sum_{j=1}^m |\nu_j^{I,k} - \hat{\nu}_j^{I,k}| |\hat{J}_j^{I,k+1}| \Delta a \]
\[ \leq \sum_{j=1}^n \beta_j^{I,k} |v_j^{I,k}| |\Delta x| + \sum_{j=1}^n |\beta_j^{I,k} (x_j, t_k, \overline{Q}_j^{I,k}) (Q_j^{I,k} - \hat{Q}_j^{I,k})| |\hat{A}_j^{I,k}| \Delta x \]
\[ + \sum_{j=1}^m |\nu_j^{I,k} (a_i, t_k, \overline{P}_j^{I,k}) (P_j^{I,k} - \hat{P}_j^{I,k})| |\hat{J}_j^{I,k+1}| \Delta a \]
\[ \leq \max_j \beta_j^{I,k} \|v_j^{I,k}\|_1 + \sup_{(x, t, Q^i)} |\beta_j^{I,k} (x_j, t_k, \overline{Q}_j^{I,k}) (Q_j^{I,k} - \hat{Q}_j^{I,k})| + \sup_{(a, t, P^i)} \|v_j^{I,k} (a_i, t_k, \overline{P}_j^{I,k}) (P_j^{I,k} - \hat{P}_j^{I,k})| \]

where $\overline{P}_j^{I,k}$ is between $P_j^{I,k}$ and $\hat{P}_j^{I,k}$, $Q_j^{I,k}$ is between $Q_j^{I,k}$ and $\hat{Q}_j^{I,k}$.
Note that
\[|P^{I,k} - \hat{P}^{I,k}| = \left| \sum_{i=1}^{m} (J^{I,k}_i - \hat{J}^{I,k}_i) \Delta a \right| \leq \sum_{i=1}^{m} |u^{I,k}_i| \Delta a = \|u^{I,k}\|_1,\]
\[|Q^{I,k} - \hat{Q}^{I,k}| = \left| \sum_{j=1}^{n} (A^{I,k}_j - \hat{A}^{I,k}_j) \Delta x \right| \leq \sum_{j=1}^{n} |v^{I,k}_j| \Delta x = \|v^{I,k}\|_1.\]

Thus, by assumptions (H1), (H4) and Lemma 4.3.1, there exist positive constants \(c_8\) and \(c_9\) such that
\[|u^{I,k+1}_0| - \sum_{i=1}^{m} (v^{I,k}_i - \hat{v}^{I,k}_i) \hat{J}^{I,k+1}_i \text{sgn}(u^{I,k+1}_i) \Delta a \leq c_8 \|v^{I,k}\|_1 + c_9 \|u^{I,k}\|_1.\]

Applying the above inequality to (4.5.2), we get
\[
\frac{\|u^{I,k+1}\|_1 - \|u^{I,k}\|_1}{\Delta t} \leq c_8 \|v^{I,k}\|_1 + c_9 \|u^{I,k}\|_1 - |u^{I,k+1}_m|.
\]

On the other hand, multiplying the second equation of (4.5.1) by sgn\((v^{I,k+1}_j)\Delta x\), noticing that \(v^{I,k}_j\text{sgn}(v^{I,k+1}_j) \leq |v^{I,k}_j|\) and summing over \(j = 2, 3, \cdots, n\), we find
\[
\sum_{j=2}^{n} (|v^{I,k+1}_j| - |v^{I,k}_j|) \frac{\Delta t}{\Delta t} 
\leq -\sum_{j=2}^{n} \left[ D^{-\Delta x} (g^{I,k}_j A^{I,k+1}_j - \hat{g}^{I,k}_j \hat{A}^{I,k+1}_j) + \mu^{I,k}_j A^{I,k+1}_j - \hat{\mu}^{I,k}_j \hat{A}^{I,k+1}_j 
\right.
\]
\[
- \sum_{K=1,K \neq I}^{N} \tau_{Kj} v^{K,k}_j + \sum_{K=1,K \neq I}^{N} \tau_{IK} v^{I,k+1}_j \text{sgn}(v^{I,k+1}_j) \Delta x 
\leq -\sum_{j=2}^{n} \left[ D^{-\Delta x} (g^{I,k}_j A^{I,k+1}_j - \hat{g}^{I,k}_j \hat{A}^{I,k+1}_j) + \mu^{I,k}_j A^{I,k+1}_j - \hat{\mu}^{I,k}_j \hat{A}^{I,k+1}_j \right] \text{sgn}(v^{I,k+1}_j) \Delta x 
\]
\[
+ \sum_{K=1,K \neq I}^{N} \tau_{Kj} \sum_{j=2}^{n} |v^{K,k}_j| \Delta x - \sum_{K=1,K \neq I}^{N} \tau_{IK} \sum_{j=2}^{n} |v^{I,k+1}_j| \Delta x.
\]

(4.5.4)

For \(j = 1\), using the second and fourth equations of (4.5.1) we find
\[
\frac{v^{I,k+1}_1 - v^{I,k}_1}{\Delta t} = -D^{-\Delta x} (g^{I,k}_1 A^{I,k+1}_1 - \hat{g}^{I,k}_1 \hat{A}^{I,k+1}_1) - \mu^{I,k}_1 A^{I,k+1}_1 
\]
\[
+ \hat{\mu}^{I,k}_1 \hat{A}^{I,k+1}_1 + \sum_{K=1,K \neq I}^{N} \tau_{K1} v^{K,k}_1 - \sum_{K=1,K \neq I}^{N} \tau_{IK} v^{I,k+1}_1.
\]

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and using the fourth equation of (4.5.1) we have

\[
\frac{|v_{1}^{I,k+1}| - |v_{1}^{I,k}|}{\Delta t} \\
\leq \left[-D_{\Delta x}(g_{1} I_{k} A_{1}^{I,k+1} - \hat{g}_{1} I_{k} \hat{A}_{1}^{I,k+1}) - \mu_{1} I_{k} A_{1}^{I,k+1} + \hat{\mu}_{1} I_{k} \hat{A}_{1}^{I,k+1} \right. \\
+ \sum_{K=1,K \neq I}^{N} \tau_{K|I} v_{1}^{K,k} - \sum_{K=1,K \neq I}^{N} \tau_{I|K} v_{1}^{I,k+1} \bigg] \text{sgn}(v_{1}^{I,k+1}) \Delta x. \\
\leq |u_{m}^{I,k+1}| - (g_{1} I_{k} A_{1}^{I,k+1} - \hat{g}_{1} I_{k} \hat{A}_{1}^{I,k+1}) \text{sgn}(v_{1}^{I,k+1}) - (\mu_{1} I_{k} A_{1}^{I,k+1} - \hat{\mu}_{1} I_{k} \hat{A}_{1}^{I,k+1}) \text{sgn}(v_{1}^{I,k+1}) \Delta x. \\
\leq \sum_{K=1,K \neq I}^{N} \tau_{K|I} |v_{1}^{K,k}| \Delta x - \sum_{K=1,K \neq I}^{N} \tau_{I|K} |v_{1}^{I,k+1}| \Delta x. \\
\text{(4.5.5)}
\]

Adding (4.5.4) and (4.5.5) we have

\[
\frac{\|v_{1}^{I,k+1}\|_1 - \|v_{1}^{I,k}\|_1}{\Delta t} \\
\leq -\sum_{j=2}^{n} D_{\Delta x}(g_{j} I_{k} A_{j}^{I,k+1} - \hat{g}_{j} I_{k} \hat{A}_{j}^{I,k+1}) \text{sgn}(v_{j}^{I,k+1}) \Delta x - (g_{1} I_{k} A_{1}^{I,k+1} - \hat{g}_{1} I_{k} \hat{A}_{1}^{I,k+1}) \text{sgn}(v_{1}^{I,k+1}) \Delta x. \\
\text{(4.5.6)}
\]

Simple calculations give

\[
\sum_{j=2}^{n} D_{\Delta x}(g_{j} I_{k} A_{j}^{I,k+1} - \hat{g}_{j} I_{k} \hat{A}_{j}^{I,k+1}) \text{sgn}(v_{j}^{I,k+1}) \Delta x \\
= \sum_{j=2}^{n} D_{\Delta x} \left[ (g_{j} I_{k} A_{j}^{I,k+1} - \hat{g}_{j} I_{k} \hat{A}_{j}^{I,k+1}) + (g_{j} I_{k} - \hat{g}_{j} I_{k}) \hat{A}_{j}^{I,k+1} \right] \text{sgn}(v_{j}^{I,k+1}) \Delta x \\
= \sum_{j=2}^{n} \left[ D_{\Delta x}(g_{j} I_{k}) v_{j}^{I,k+1} + g_{j} I_{k} D_{\Delta x}(v_{j}^{I,k+1}) \right] \text{sgn}(v_{j}^{I,k+1}) \Delta x \\
+ \sum_{j=2}^{n} D_{\Delta x} \left[ (g_{j} I_{k} - \hat{g}_{j} I_{k}) \hat{A}_{j}^{I,k+1} \right] \text{sgn}(v_{j}^{I,k+1}) \Delta x,
\]
and
\[
(g^J_k A^J_{1,k+1} - \hat{g}^J_k \hat{A}^J_{1,k+1})\text{sgn}(v^J_{1,k+1}) \\
= \left[ g^J_k (A^J_{1,k+1} - \hat{A}^J_{1,k+1}) + (g^J_k - \hat{g}^J_k) \hat{A}^J_{1,k+1} \right] \text{sgn}(v^J_{1,k+1}) \\
= g^J_k |v^J_{1,k+1}| + (g^J_k - \hat{g}^J_k) \hat{A}^J_{1,k+1} \text{sgn}(v^J_{1,k+1}).
\]

Note that
\[
\sum_{j=2}^n \left[ D_{\Delta x}(g^J_j v^J_{1,k+1} + g^J_k D_{\Delta x}(v^J_{1,k+1}) \right] \text{sgn}(v^J_{j,k+1}) \Delta x + g^J_k |v^J_{1,k+1}| \\
\geq \sum_{j=2}^n (g^J_j - g^J_{j-1})|v^J_{1,k+1}| + \sum_{j=2}^n g^J_k (|v^J_{j,k+1}| - |v^J_{j-1,k+1}|) + g^J_k |v^J_{1,k+1}| \\
= g^J_n |v^J_{n,k+1}| = 0.
\]

Hence,
\[
\sum_{j=2}^n D_{\Delta x}(g^J_j A^J_{j,k+1} - \hat{g}^J_J \hat{A}^J_{j,k+1})\text{sgn}(v^J_{j,k+1}) \Delta x
+ (g^J_k A^J_{1,k+1} - \hat{g}^J_k \hat{A}^J_{1,k+1})\text{sgn}(v^J_{1,k+1}) \\
\geq -\sum_{j=2}^n \left| D_{\Delta x}(g^J_j - \hat{g}^J_J \hat{A}^J_{j,k+1}) \right| \Delta x - |g^J_k - \hat{g}^J_k| \hat{A}^J_{1,k+1} \\
= -\sum_{j=2}^n \left| D_{\Delta x}(g^J_j - \hat{g}^J_J \hat{A}^J_{j,k+1}) \hat{A}^J_{j,k+1} + (g^J_k - \hat{g}^J_k) D_{\Delta x}(\hat{A}^J_{j,k+1}) \right| \Delta x \\
- |g^J_k - \hat{g}^J_k| \hat{A}^J_{1,k+1} \\
= -\sum_{j=2}^n \left| D_{\Delta x}[g^J_Q(x_j, t_k, \tilde{Q}^J_{1,k})(Q^J - \hat{Q}^J_{1,k})] \hat{A}^J_{j,k+1} \\
+ g^J_Q(x_{j-1}, t_k, \tilde{Q}^J_{2,k})(Q^J - \hat{Q}^J_{1,k}) D_{\Delta x}(\hat{A}^J_{j,k+1}) \right| \Delta x \\
- |g^J_Q(x_1, t_k, \tilde{Q}^J_{1,k})(Q^J - \hat{Q}^J_{1,k})| \hat{A}^J_{1,k+1} \\
\geq \left( -\omega_4 \| \hat{A}^J_{1,k+1} \|_1 - \sup_{(x,t,Q^t)\in\mathbb{D}_4} |g^J_Q| \| D_{\Delta x}(\hat{A}^J_{1,k+1}) \|_1 \\
- \sup_{(x,t,Q^t)\in\mathbb{D}_4} |g^J_Q| \| \hat{A}^J_{1,k+1} \|_\infty \right) |Q^J - \hat{Q}^J_{1,k}|,
\]
where \(\tilde{Q}^J_{1,k}, \tilde{Q}^J_{2,k}, \hat{Q}^J_{1,k}\) are between \(Q^J_{1,k}\) and \(\hat{Q}^J_{1,k}\), and \(\omega_4 = \sup_{(x,t,Q^t)\in\mathbb{D}_4} |g^J_Q(x, t, Q^t)|\).

Therefore, by Lemmas 4.3.1, 4.3.2, 4.3.5 and assumption (H2), there exists a positive constant \(c_{10}\) such that
\[
\sum_{j=2}^n D_{\Delta x}(g^J_j A^J_{j,k+1} - \hat{g}^J_J \hat{A}^J_{j,k+1})\text{sgn}(v^J_{j,k+1}) \Delta x \\
+ (g^J_k A^J_{1,k+1} - \hat{g}^J_k \hat{A}^J_{1,k+1})\text{sgn}(v^J_{1,k+1}) \geq -c_{10}|Q^J - \hat{Q}^J_{1,k}|.
\]

(4.5.7)
Furthermore,
\[
\sum_{j=1}^{n} (\mu_j^{I,k} A_j^{I,k+1} - \hat{\mu}_j^{I,k} \hat{A}^{I,k+1}_j) \text{sgn}(v_j^{I,k+1}) \Delta x
\]
\[
= \sum_{j=1}^{n} \left[ \mu_j^{I,k} (A_j^{I,k+1} - \hat{A}^{I,k+1}_j) + (\mu_j^{I,k} - \hat{\mu}_j^{I,k}) \hat{A}^{I,k+1}_j \right] \text{sgn}(v_j^{I,k+1}) \Delta x
\]
\[
= \sum_{j=1}^{n} \mu_j^{I,k} |v_j^{I,k+1}| \Delta x + \sum_{j=1}^{n} (\mu_j^{I,k} - \hat{\mu}_j^{I,k}) \hat{A}^{I,k+1}_j \text{sgn}(v_j^{I,k+1}) \Delta x
\]
\[
\geq - \sum_{j=1}^{n} |\mu_j^{I,k} - \hat{\mu}_j^{I,k}| \hat{A}^{I,k+1}_j \Delta x
\]
\[
= - \sum_{j=1}^{n} |\mu_j^{I,k}(x_j, t_k, \bar{Q}^{I,k})(Q^{I,k} - \hat{Q}^{I,k})| \hat{A}^{I,k+1}_j \Delta x
\]
\[
\geq - \sup_{(x,t,Q^I) \in \mathcal{D}_4} |\mu_j^{I,k}| \|\hat{A}^{I,k+1}_j\|_1 |Q^{I,k} - \hat{Q}^{I,k}|.
\]
Thus, by lemma 3.1 and (H3), there exists a positive constant \( c_{11} \) such that
\[
\sum_{j=1}^{n} (\mu_j^{I,k} A_j^{I,k+1} - \hat{\mu}_j^{I,k} \hat{A}^{I,k+1}_j) \text{sgn}(v_j^{I,k+1}) \Delta x \geq -c_{11} |Q^{I,k} - \hat{Q}^{I,k}|.
\] (4.5.8)
Applying (4.5.7) and (4.5.8) to (4.5.6) and noticing that \(|Q^{I,k} - \hat{Q}^{I,k}| \leq \|v^{I,k}\|_1\), we have
\[
\frac{\|v^{I,k+1}\|_1 - \|v^{I,k}\|_1}{\Delta t} \leq (c_{10} + c_{11}) \|v^{I,k}\|_1 + \sum_{K=1, K \neq I}^{N} \tau_{KI} \|v^{K,k}\|_1 + |u_{m}^{I,k+1}|.
\] (4.5.9)
Adding (4.5.3) to (4.5.9) we arrive at
\[
\frac{\left(\|u^{I,k+1}\|_1 + \|v^{I,k+1}\|_1\right) - \left(\|u^{I,k}\|_1 + \|v^{I,k}\|_1\right)}{\Delta t}
\]
\[
\leq (c_8 + c_{10} + c_{11}) \|v^{I,k}\|_1 + c_9 \|u^{I,k}\|_1 + \sum_{K=1, K \neq I}^{N} \tau_{KI} \|v^{K,k}\|_1.
\]
Summing over \( I = 1, 2, \cdots, N \) and noticing that
\[
\sum_{I=1}^{N} \sum_{K=1, K \neq I}^{N} \tau_{KI} \|v^{K,k}\|_1 = \sum_{I=1}^{N} \sum_{K=1, K \neq I}^{N} \tau_{IK} \|v^{I,k}\|_1,
\]
we find
\[
\frac{\sum_{I=1}^{N} \left(\|u^{I,k+1}\|_1 + \|v^{I,k+1}\|_1\right) - \left(\|u^{I,k}\|_1 + \|v^{I,k}\|_1\right)}{\Delta t}
\]
\[
\leq (c_8 + c_{10} + c_{11}) \sum_{I=1}^{N} \|v^{I,k}\|_1 + c_9 \sum_{I=1}^{N} \|u^{I,k}\|_1 + \sum_{I=1}^{N} \sum_{K=1, K \neq I}^{N} \tau_{IK} \|v^{I,k}\|_1.
\]
Setting \( \sigma = c_8 + c_9 + c_{10} + c_{11} + \theta \), we establish the result.
Next, we prove that the $BV$ solution defined in Lemma 4.4.1 and Theorem 4.4.1 is unique.

**Theorem 4.5.2.** Suppose that \( \{J^1, A^1, \ldots, J^N, A^N\} \) and \( \{\hat{J}^1, \hat{A}^1, \ldots, \hat{J}^N, \hat{A}^N\} \) are bounded variation weak solutions of problem (4.1.1) corresponding to the initial conditions \( \{J^1, 0, A^1, 0, \ldots, J^N, 0, A^N, 0\} \) and \( \{\hat{J}^1, 0, \hat{A}^1, 0, \ldots, \hat{J}^N, 0, \hat{A}^N, 0\} \), respectively, then there exist positive constants \( \rho \) and \( \lambda \) such that

\[
\sum_{I=1}^{N} \left( \|J^I(\cdot, t) - \hat{J}^I(\cdot, t)\|_1 + \|A^I(\cdot, t) - \hat{A}^I(\cdot, t)\|_1 \right) \\
\leq \rho e^\lambda \sum_{I=1}^{N} \left( \|J^I(\cdot, 0) - \hat{J}^I(\cdot, 0)\|_1 + \|A^I(\cdot, 0) - \hat{A}^I(\cdot, 0)\|_1 \right).
\]

**Proof.** Assume that \( P^I, Q^I \) and \( B^I, I = 1, 2, \cdots, N \), are given Lipschitz continuous functions and consider the following initial-boundary value problem:

\[
J^I_t + J^I_a + \nu^I(a, t, P^I(t))J^I = 0, \quad (a, t) \in (0, a_{\text{max}}) \times (0, T), \\
A^I_t + (g^I(x, t, Q^I(t))A^I_x + \mu^I(x, t, Q^I(t))A^I \\
= \sum_{K=1, K \neq I}^{N} \tau_{KI}A^K - \sum_{K=1, K \neq I}^{N} \tau_{IK}A^I, \quad (x, t) \in (x_{\text{min}}, x_{\text{max}}) \times (0, T), \\
J^I(0, t) = B^I(t), \quad t \in (0, T), \\
g^I(x_{\text{min}}, t, Q^I(t))A^I(x_{\text{min}}, t) = J^I(a_{\text{max}}, t), \quad t \in (0, T), \\
J^I(a, 0) = J^{I, 0}(a), \quad a \in [0, a_{\text{max}}], \\
A^I(x, 0) = A^{I, 0}(x), \quad x \in [x_{\text{min}}, x_{\text{max}}].
\]

Since (4.5.10) is a linear problem with local boundary conditions, it has a unique weak solution. Actually, a weak solution can be defined as a limit of the finite difference approximation with the given numbers \( P^{I, k} = P^I(t_k), Q^{I, k} = Q^I(t_k) \) and \( B^{I, k} = B^I(t_k) \), \( I = 1, 2, \cdots, N \), and the uniqueness can be established by using similar techniques as in [13]. In addition, as in the proof of Theorem 4.5.1, we can show that if
\{J^{1,k}_i, A^{1,k}_j, \cdots, J^{N,k}_i, A^{N,k}_j\}$ and $\{\hat{J}^{1,k}_i, \hat{A}^{1,k}_j, \cdots, \hat{J}^{N,k}_i, \hat{A}^{N,k}_j\}$ are solutions of the difference scheme corresponding to given functions $\{P^{I,k}, Q^{I,k}, B^{I,k}\}$ and $\{\hat{P}^{I,k}, \hat{Q}^{I,k}, \hat{B}^{I,k}\}$, respectively, $I = 1, 2, \cdots, N$, then there exist positive constants $c_{12}$ such that

$$
\sum_{I=1}^{N} \left( \|u^{I,k+1}\|_1 + \|v^{I,k+1}\|_1 \right) 
\leq \sum_{I=1}^{N} \left[ (1 + \theta \Delta t) (\|u^{I,k}\|_1 + \|v^{I,k}\|_1) + (c_{12}(\|P^{I,k} - \hat{P}^{I,k}\|_1 + \|Q^{I,k} - \hat{Q}^{I,k}\|_1 + B^{I,k+1} - \hat{B}^{I,k+1})\Delta t) \right],
$$

(4.5.11)

where $u^{I,k} = J^{I,k} - \hat{J}^{I,k}, v^{I,k} = A^{I,k} - \hat{A}^{I,k}.$

In fact, here $J^{I,k}_0 = B^{I}(t_k) = B^{I,k}, \hat{J}^{I,k}_0 = \hat{B}^{I}(t_k) = \hat{B}^{I,k}, u^{I,k}_0 = B^{I,k} - \hat{B}^{I,k},$ so by (4.5.2) and Lemma 3.1 we have

$$
\frac{\|u^{I,k+1}\|_1 - \|u^{I,k}\|_1}{\Delta t} 
\leq \|B^{I,k+1} - \hat{B}^{I,k+1}\|_1 - \|u^{I,k+1}_{m}\|_1 + \sum_{i=1}^{m} |u^{I,k}_i - \hat{u}^{I,k}_i| |J^{I,k+1}\Delta a. 
\leq |B^{I,k+1} - \hat{B}^{I,k+1}| - \|u^{I,k+1}_{m}\|_1 + \sup_{(a, t, P^{I}) \in \mathbb{D}_3} \|v^{I}_{P^{I}}\|_1 \|P^{I,k} - \hat{P}^{I,k}\|_1, 
\leq |B^{I,k+1} - \hat{B}^{I,k+1}| - \|u^{I,k+1}_{m}\|_1 + M_1 \sup_{(a, t, P^{I}) \in \mathbb{D}_3} \|v^{I}_{P^{I}}\|_1 \|P^{I,k} - \hat{P}^{I,k}\|_1. 
$$

(4.5.12)

On the other hand, from (4.5.6)-(4.5.8) we find

$$
\frac{\|v^{I,k+1}\|_1 - \|v^{I,k}\|_1}{\Delta t} 
\leq \sum_{K=1, K \neq I}^{N} \tau_{K I} \|v^{K,k}\|_1 + \sup_{(a, t, P^{I}) \in \mathbb{D}_3} \|v^{I}_{P^{I}}\| + c_{10} + c_{11}, \quad \sum_{K=1, K \neq I}^{N} \tau_{K I} \|v^{K,k}\|_1 + (c_{10} + c_{11}) \|Q^{I,k} - \hat{Q}^{I,k}\|_1 + \|u^{I,k+1}_{m}\|_1. 
$$

(4.5.13)

Adding (4.5.12) and (4.5.13) and letting $c_{12} = M_1 \sup_{(a, t, P^{I}) \in \mathbb{D}_3} \|v^{I}_{P^{I}}\| + c_{10} + c_{11},$ we get

$$
\frac{\|u^{I,k+1}\|_1 - \|u^{I,k}\|_1}{\Delta t} - \frac{\|v^{I,k+1}\|_1 - \|v^{I,k}\|_1}{\Delta t} 
\leq \sum_{K=1, K \neq I}^{N} \tau_{K I} \|v^{K,k}\|_1 + c_{12}(\|P^{I,k} - \hat{P}^{I,k}\|_1 + \|Q^{I,k} - \hat{Q}^{I,k}\|_1 + |B^{I,k+1} - \hat{B}^{I,k+1}|. 
$$

Summing over $I = 1, 2, \cdots, N$ and noticing that

$$
\sum_{I=1}^{N} \sum_{K=1, K \neq I}^{N} \tau_{K I} \|v^{K,k}\|_1 = \sum_{I=1}^{N} \sum_{K=1, K \neq I}^{N} \tau_{I K} \|v^{I,k}\|_1, 
$$

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we have
\[
\sum_{I=1}^{N} \left( \frac{\|u^{I,k+1}\|_1 + \|v^{I,k+1}\|_1 - (\|u^{I,k}\|_1 + \|v^{I,k}\|_1)}{\Delta t} \right)
\leq \sum_{I=1}^{N} \left[ \sum_{K=1, K \neq I}^{N} \tau_{IK} \|v^{I,k}\|_1 + c_{12}(|P^{I,k} - \hat{P}^{I,k}| + |Q^{I,k} - \hat{Q}^{I,k}|) + |B^{I,k+1} - \hat{B}^{I,k+1}| \right].
\]
Thus, we obtain (4.5.11). Furthermore, (4.5.11) is equivalent to
\[
\sum_{I=1}^{N} (\|u^{I,k+1}\|_1 + \|v^{I,k+1}\|_1)
\leq (1 + \theta \Delta t)^k \sum_{I=1}^{N} \left[ \|u^{I,0}\|_1 + \|v^{I,0}\|_1 + \sum_{r=0}^{k-1} (1 + \theta \Delta t)^r \left( c_{12}(|P^{I,k-1-r} - \hat{P}^{I,k-1-r}| + |Q^{I,k-1-r} - \hat{Q}^{I,k-1-r}|) + |B^{I,k-r} - \hat{B}^{I,k-r}| \right) \right].
\]
Hence,
\[
\sum_{I=1}^{N} (\|u^{I,k+1}\|_1 + \|v^{I,k+1}\|_1)
\leq (1 + \theta \Delta t)^k \sum_{I=1}^{N} \left[ \|u^{I,0}\|_1 + \|v^{I,0}\|_1 + \sum_{r=0}^{k-1} \left( c_{12}(|P^{I,k-1-r} - \hat{P}^{I,k-1-r}| + |Q^{I,k-1-r} - \hat{Q}^{I,k-1-r}|) + |B^{I,k-r} - \hat{B}^{I,k-r}| \right) \Delta t \right].
\]
Now, from Theorem 4.2 we can take the limit in (4.5.14) to obtain
\[
\sum_{I=1}^{N} (\|u^{I}(t)\|_1 + \|v^{I}(t)\|_1)
\leq e^{\theta T} \sum_{I=1}^{N} \left[ \|u^{I}(0)\|_1 + \|v^{I}(0)\|_1 + \int_0^t \left( c_{12}(|P^{I}(s) - \hat{P}^{I}(s)| + |Q^{I}(s) - \hat{Q}^{I}(s)| + |B^{I}(s) - \hat{B}^{I}(s)|) \right) ds \right].
\]
where \(u^{I}(t) = J^{I}(\cdot, t) - \hat{J}(\cdot, t), v^{I}(t) = A^{I}(\cdot, t) - \hat{A}^{I}(\cdot, t),\)
\(\{J^{1}(\cdot, t), A^{1}(\cdot, t), \cdots, J^{N}(\cdot, t), A^{N}(\cdot, t)\}\) and \(\{\hat{J}^{1}(\cdot, t), \hat{A}^{1}(\cdot, t), \cdots, \hat{J}^{N}(\cdot, t), \hat{A}^{N}(\cdot, t)\}\) are the unique solutions of (4.5.10) with any set of given functions \(\{P^{I}(t), Q^{I}(t), B^{I}(t)\}\) and \(\{\hat{P}^{I}(t), \hat{Q}^{I}(t), \hat{B}^{I}(t)\}\), respectively, \(I = 1, 2, \cdots, N.\)

We then apply the estimate given in (4.5.14) for the corresponding solutions of (4.5.10) with two specific sets of functions \(\{P^{I}(t), Q^{I}(t), B^{I}(t)\}\) and
\{ \hat{P}^I(t), \hat{Q}^I(t), \hat{B}^I(t) \}, I = 1, 2, \ldots, N, \text{ which are constructed using the limits obtained}

in Lemma 4.4.1 as follows:

\begin{align*}
P^I(t) &= \int_0^{a_{\text{max}}} J^I(a, t) da, \quad \hat{P}^I(t) = \int_0^{a_{\text{max}}} J^I(a, t) da, \\
Q^I(t) &= \int_{x_{\text{min}}}^{x_{\text{max}}} A^I(x, t) dx, \quad \hat{Q}^I(t) = \int_{x_{\text{min}}}^{x_{\text{max}}} \hat{A}^I(x, t) dx, \\
B^I(t) &= \int_{x_{\text{min}}}^{x_{\text{max}}} \beta^I(x, t, Q^I(t)) A^I(x, t) dx, \\
\hat{B}^I(t) &= \int_{x_{\text{min}}}^{x_{\text{max}}} \beta^I(x, t, \hat{Q}^I(t)) \hat{A}^I(x, t) dx,
\end{align*}

Thus, we get

\begin{align*}
|P^I(s) - \hat{P}^I(s)| &= \int_0^{a_{\text{max}}} |J^I(a, s) - \hat{J}^I(a, s)| da \\
&\leq \int_0^{a_{\text{max}}} |u^I(a, s)| da = \|u^I(s)\|_1, \\
|Q^I(s) - \hat{Q}^I(s)| &= \int_{x_{\text{min}}}^{x_{\text{max}}} |A^I(x, s) - \hat{A}^I(x, s)| dx \\
&\leq \int_{x_{\text{min}}}^{x_{\text{max}}} |v^I(x, s)| dx = \|v^I(s)\|_1,
\end{align*}

and

\begin{align*}
|B^I(s) - \hat{B}^I(s)| &\leq \int_{x_{\text{min}}}^{x_{\text{max}}} \left| \beta^I(x, s, Q^I(s)) [A^I(x, s) - \hat{A}^I(x, s)] \\
&\quad + [\beta^I(x, s, Q^I(s)) - \beta^I(x, s, \hat{Q}^I(s))] \hat{A}^I(x, s) \right| dx \\
&= \int_{x_{\text{min}}}^{x_{\text{max}}} \left| \beta^I(x, s, Q^I(s)) [A^I(x, s) - \hat{A}^I(x, s)] \\
&\quad + \beta^I_t(x, s, \hat{Q}^I(s)) [Q^I(s) - \hat{Q}^I(s)] \hat{A}^I(x, s) \right| dx \\
&\leq \sup_{(x,t,Q^I)\in \mathcal{D}_2} \beta^I \|v^I(s)\|_1 \\
&\quad + \sup_{(x,t,Q^I)\in \mathcal{D}_4} |\beta^I_{Q^I}| \|v^I(s)\|_1 \|\hat{A}^I\|_{\mathcal{L}^\infty((x_{\text{min}},x_{\text{max}}) \times (0,T))} (x_{\text{max}} - x_{\text{min}}).
\end{align*}

Hence,

\begin{align*}
&\int_0^t \left[ c_{12} (|P^I(s) - \hat{P}^I(s)| + |Q^I(s) - \hat{Q}^I(s)|) + |B^I(s) - \hat{B}^I(s)| \right] ds \\
&\leq \int_0^t \left[ c_{12} (\|u^I(s)\|_1 + \|v^I(s)\|_1) + \left( \sup_{(x,t,Q^I)\in \mathcal{D}_2} \beta^I \\
&\quad + \sup_{(x,t,Q^I)\in \mathcal{D}_4} |\beta^I_{Q^I}| \|\hat{A}^I\|_{\mathcal{L}^\infty((x_{\text{min}},x_{\text{max}}) \times (0,T))} (x_{\text{max}} - x_{\text{min}}) \right) \|v^I(s)\|_1 \right] ds \\
&\leq c_{13} \int_0^t \left[ \|u^I(s)\|_1 + \|v^I(s)\|_1 \right] ds,
\end{align*}

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where $c_{13} = c_{12} + \sup_{(x,t,Q^I) \in \mathcal{D}_2} \beta^I + \sup_{(x,t,Q^I) \in \mathcal{D}_4} |\beta^I_{Q^I}||\hat{A}^I|_{L^\infty((x_{\min}, x_{\max}) \times (0,T))}(x_{\max} - x_{\min})$.

Therefore,

$$\sum_{I=1}^{N}(\|u^I(t)\|_1 + \|v^I(t)\|_1) \leq e^{\theta T} \sum_{I=1}^{N} \left[\|u^I(0)\|_1 + \|v^I(0)\|_1 + c_{13} \int_{0}^{t}(\|u^I(s)\|_1 + \|v^I(s)\|_1)ds\right].$$

Using Grownwall’s inequality, we find

$$\sum_{I=1}^{N}(\|u^I(t)\|_1 + \|v^I(t)\|_1) \leq \exp\{\theta T + c_{13}e^{\theta T}t\} \sum_{I=1}^{N}(\|u^I(0)\|_1 + \|v^I(0)\|_1).$$

Letting $\rho = e^{\theta T}$, $\lambda = c_{13}e^{\theta T}$, we obtain

$$\sum_{I=1}^{N}(\|J^I(\cdot,t) - \hat{J}^I(\cdot,t)\|_1 + \|A^I(\cdot,t) - \hat{A}^I(\cdot,t)\|_1) \leq \rho e^\lambda \sum_{I=1}^{N}(\|J^I(\cdot,0) - \hat{J}^I(\cdot,0)\|_1 + \|A^I(\cdot,0) - \hat{A}^I(\cdot,0)\|_1).$$
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Huang, Qihua. Bachelor of Science, Hubei Normal University, China, Spring 1997; Master of Science, Wuhan University, China, Spring 2006; Doctor of Philosophy, University of Louisiana at Lafayette, Summer 2011

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ABSTRACT

In this dissertation, we consider an amphibian population where individuals are divided into two groups: juveniles (tadpoles) and adults (frogs). We assume that juveniles are structured by age and adults are structured by size. Since juveniles (tadpoles) live in water and adults (frogs) live on land we assume that competition occurs within stage only. This leads to a system of nonlinear and nonlocal hyperbolic equations of first order.

In Chapter 1, we formulate the above system of nonlinear and nonlocal hyperbolic equations of first order. An explicit finite difference approximation to this partial differential equation system is developed. Existence and uniqueness of the weak solution to the model are established and convergence of the finite difference approximation to this unique solution is proved.

In Chapter 2, we derive several stochastic models from the deterministic population model developed in Chapter 1. Numerical simulation results of the stochastic models are compared with the solution of the deterministic model. These models are then used to understand the effect of demographic stochasticity on the dynamics of an urban green tree frog (Hyla cinerea) population.
In Chapter 3, we present an infinite-dimensional least-squares approach which compares a mathematical population model developed in Chapter 1 to the statistical population estimates obtained from the field data. To solve the least-squares problem, an explicit finite difference approximation is developed. Convergence results for the computed parameters are presented. Parameter estimates for the vital rates of juveniles and adults are obtained, and standard deviations for these estimates are computed. Numerical results for the model sensitivity with respect to these parameters are given. Finally, the above-mentioned parameter estimates are used to illustrate the long-time behavior of the population under investigation.

In Chapter 4, the deterministic model represented in Chapter 1 is extended to a dispersal model where individuals disperse between $N$ ponds. A implicit finite difference approximation to this model is developed. Existence-uniqueness of the weak solution to the model is established and convergence of the finite difference approximation to the unique solution is proved.
Qihua Huang was admitted to Hubei Normal University in China in September 1993 and received his Bachelor of Science in Mathematics Education in June 1997. After working as a instructor at Xianning College, he earned admittance to the graduate school at Wuhan University in September 2003 and received his degree of Master of Science in Fundamental Mathematics in June 2006. In August 2007, he joined the Ph.D. program in the Department of Mathematics at the University of Louisiana at Lafayette. He completed the requirements for the degree Doctor of Philosophy in July 2011.