Assignment 4

1 Problem

Consider the initial-boundary-value problem (IBVP)

\[ \frac{\partial u}{\partial t} = \nabla^2 u, \quad \text{in} \quad \Omega = (0, 1) \times (0, 1), t > 0. \]

\[ u(x, 0, t) = 1, \quad u(x, 1, t) = 0, \quad \frac{\partial u}{\partial x}(0, y, t) = \frac{\partial u}{\partial x}(1, y, t) = 0, \quad t > 0 \]

\[ u(x, y, 0) = 0, \quad (x, y) \in (0, 1) \times (0, 1). \]

The following scheme is a consistent scheme for approximation of the solution of this problem (called the \( \theta \)-method)

\[ \frac{u_{h}^{n+1} - u_{h}^{n}}{k} = \theta \nabla^2_h u_{h}^{n+1} + (1 - \theta) \nabla^2_h u_{h}^{n}, \quad 0 \leq \theta \leq 1. \]

(i) Study the stability of this scheme in the one dimensional case (do not take into account the boundary conditions). How does the value of \( \theta \) affect the stability properties?

(ii) For which value of \( \theta \) is the scheme second order accurate in time? Use the scheme with this \( \theta \) to solve the problem on a grid 20 \( \times \) 20 (in space) integrating in time long enough so that the solution becomes time-independent (steady solution). This means that the difference between the solution on two subsequent steps should be very small. The criterion for ”very small” should be a little bit greater than the accuracy of your iterative linear solver (if you intend to use such solver for your linear solve). Using Maple or Matlab plot the isolines (level set lines) of this steady solution.

2 Problem

Consider the advection-diffusion IBVP

\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - Pe^{-1} \frac{\partial^2 u}{\partial x^2} = 0, \quad \text{in} \quad (0, 1), t > 0. \]

\[ u(t, 0) = 1, \quad \frac{\partial u}{\partial x}(t, 1) = 0, \quad u(0, x) = 0, \quad x \in (0, 1). \]

The approximation \( \delta_{x} u_{h, j}^{n} \approx \delta_{x}^{-1} u_{h, j}^{n} \) has a truncation error that contributes to a diffusion-like effect. Show that the implicit, centered-in-space scheme

\[ \delta_{t}^{-1} u_{h, j}^{n+1} + \delta_{x} u_{h, j}^{n+1} - Pe^{-1} \delta_{x}^2 u_{h, j}^{n+1} = 0 \]

also has a truncation error that contributes to a diffusion-like effect. Note, that the leading term in the error will also contribute to a dispersion-like effect.
**Hint:** Differentiate the PDE with respect to $t$ and $x$ to relate terms in the time truncation error to spatial derivatives.

Verify your conclusion computationally solving the problem numerically for $Pe = 10$ and time steps $k = 0.2, 0.1, 0.05$, and a fixed $h = 0.01$. You should not integrate too long in time since at certain time instant $t \approx 1$, the solution will contradict the right boundary condition (see the graph in the notes). The right boundary condition can be discretized by means of a backward difference scheme.