Assignment 3

1 Problem

Consider the advection equation
\[ \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = 0, \quad v > 0. \] (1)

Beginning with the Taylor expansion
\[ u(t+k,x) = u(t,x) + k \frac{\partial u}{\partial t}(t,x) + k^2/2 \frac{\partial^2 u}{\partial t^2}(t,x) + O(k^3) \]

substitute the relationships
\[ \frac{\partial u}{\partial t} = -v \frac{\partial u}{\partial x} \]

and
\[ \frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} \]

To arrive at the Lax-Wendroff scheme:
\[ u_{i}^{n+1} = u_{i}^{n} - C/2(u_{i+1}^{n} - u_{i-1}^{n}) + C^2/2(u_{i+1}^{n} - 2u_{i}^{n} + u_{i-1}^{n}). \]

- When is the scheme stable?
  \textit{Hint:} Show that the von Neumann stability analysis yields an amplification factor satisfying \( A(\theta)^2 = 1 - 4(C^2 - C^4) \sin^4(\theta/2) \).

- What is the order of consistency in space and time?

2 Problem

Consider the initial-boundary value problem
\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \quad x > 0, \quad t > 0 \] (2)
\[ u(t,0) = 1 \quad \text{for} \quad t \geq 0, \quad u(0,x) = 0 \quad \text{for} \quad x > 0. \] (3)

Compute and plot approximate solutions to this problem at \( t = 5 \) using a truncated domain of length 10 in the \( x \)-direction on a grid of 100 finite-difference cells. At the right edge of the domain impose the artificial condition \( u(t,10) = 0 \). For computing these solutions use Lax-Wendroff scheme with Courant numbers \( C < 1, C = 1, C > 1 \).

\textit{Hint:} You should write a code to perform the computations.
3 Problem

Show that the Forward/central scheme for the advection equation (1):

\[ u_{i+1}^{n+1} = u_i^n - \frac{C}{2}(u_{i+1}^n - u_{i-1}^n) \]

is unconditionally unstable. Then consider its modification, called the Lax-Friedrichs scheme, in which the first term in the right hand side is approximated by an averaging operator:

\[ u_{i+1}^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{C}{2}(u_{i+1}^n - u_{i-1}^n). \]

Study its stability. Derive the consistency error for this scheme.