The canonical fractional Galois ideal at $s = 0$

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Abstract

Stickelberger ideals are known, in certain circumstances, to provide annihilators for class-groups of number fields. Defined in terms of certain values of L-functions, Stickelberger ideals are thus examples of a general phenomenon sought after in arithmetic geometry in which analytically defined objects pass information on algebraically defined objects. However, they do not give all relations in the class-group in general, and are often zero. In this thesis, we study the recently defined fractional Galois ideal of Snaith associated to an abelian extension of number fields, which, by using leading coefficients of L-functions rather than values, is hoped to improve on the annihilator relations provided by the Stickelberger ideal.

We describe a general relationship of the fractional Galois ideal with the conjectural Stark elements, which, should they exist, will be connected to class-groups via the theory of Euler systems. This relationship will be examined explicitly in some cyclotomic situations to illustrate that we do indeed obtain more annihilators in this way, later being combined with results from Iwasawa theory to show that a limit of the fractional Galois ideals in a $\mathbb{Z}_p$-extension gives rise to Fitting ideals of limits of p-parts of class-groups.

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This thesis is dedicated to my family.

Contents

Chapter 1

Introduction

1.1 The analytic class number formula

The motivating principle of this thesis can be taken to be the desire to generalize the analytic class number formula (see (1.1.1) below) in a Galois module-theoretic way. This formula, usually stated in terms of the leading coefficient (residue, in this case) $\zeta_L^*(1)$ of the Dedekind ζ-function $\zeta_L(s)$ of a number field L at $s = 1$, relates the orders of the class-group and group of roots of unity of L via $\zeta_L^*(1)$ and a quantity R_L, the Dirichlet regulator of L. For our purposes, it will be more convenient and natural to use the version obtained by means of the functional equation for $\zeta_L(s)$. This incarnation of the formula involves instead the leading coefficient $\zeta_L^*(0)$ of $\zeta_L(s)$ at $s = 0$, and says

$$\frac{\zeta_L^*(0)}{R_L} = -\frac{|\mathrm{Cl}(L)|}{|\mu(L)|}, \tag{1.1.1}$$

where $\mathrm{Cl}(L)$ is the class-group of L and $\mu(L)$ is the group of roots of unity in L.

What we take from (1.1.1) is that it makes a connection between an analytic object, the Dedekind ζ-function, and two important groups associated with L: $\mathrm{Cl}(L)$ and $\mu(L)$. Even before the development of K-theory this was interesting, but we have since been able to view these groups as the torsion subgroups of K_0 and K_1 respectively of the ring of integers $\mathcal{O}_L$ in L. We will say more about the general K-theoretic setting in Section 1.2.3, although in this thesis it will serve only as motivation.

For the time being, let us give a name to the $\mathbb{Z}$-submodule of $\mathbb{R}$ generated by $\frac{\zeta_L^*(0)}{R_L}$; we shall call it $\mathcal{J}(L)$. Then straight off (1.1.1) tells us that $\mathcal{J}(L)$ in fact lies in $\mathbb{Q}$. However, it further tells us that

$$\mathrm{ann}_{\mathbb{Z}}(\mu(L))\mathcal{J}(L) \subseteq \mathrm{ann}_{\mathbb{Z}}(\mathrm{Cl}(L)). \tag{1.1.2}$$

Although the annihilator ideal of a module does not determine the isomorphism class of that module, it nevertheless carries significant information, hence the interest in an inclusion like (1.1.2).

1.1.1 Formulation in terms of Galois structure

The more structure with which we endow $\mathrm{Cl}(L)$ and $\mu(L)$, the more information one is potentially capable of extracting, and so it is with this in mind that we wish to view these objects not just as $\mathbb{Z}$-modules, but as *Galois* modules, that is, modules over the integral group-ring of a Galois group. This will of course be a generalization of the above situation, since we can always take that Galois group to be trivial, and (1.1.2) will serve as a model for this generalization. In fact, $\mathcal{J}(L)$ is the first example of a "fractional Galois ideal at $s = 0$", although the "Galois" aspect does not play a role in this case because we have neglected any kind of Galois action.

So, let us suppose we have a subfield K of L such that the extension L/K is Galois, with Galois group G say. $\mathrm{Cl}(L)$ and $\mu(L)$ come with natural G-actions and we consider them as $\mathbb{Z}[G]$-modules.

Question 1.1.1 *Is there a way of constructing non-trivial elements $\alpha \in \mathbb{Q}[G]$ such that*

$$\mathrm{ann}_{\mathbb{Z}[G]}(\mu(L))\alpha \subseteq \mathrm{ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(L))?$$

We could instead have viewed (1.1.1) in terms of the *Fitting ideal* of $\mathrm{Cl}(L)$ (see Section 2.2), observing that it tells us

$$\mathrm{ann}_{\mathbb{Z}}(\mu(L))\mathcal{J}(L) = \mathrm{Fitt}_{\mathbb{Z}}(\mathrm{Cl}(L)).$$

Therefore when G is abelian (so that $\mathbb{Z}[G]$ is a commutative ring and Fitting ideals for $\mathbb{Z}[G]$-modules are defined), we could ask further:

Question 1.1.2 *Is there an ideal A of $\mathbb{Z}[G]$ constructed by analytic methods such that*

$$\mathrm{ann}_{\mathbb{Z}[G]}(\mu(L))A = \mathrm{Fitt}_{\mathbb{Z}[G]}(\mathrm{Cl}(L))?$$

1.2 Evidence and conjecture

1.2.1 Stickelberger elements

Given a pair $(L/K, S)$ where L/K is an abelian extension of number fields and S a finite set of places of K containing the infinite ones, we can define an element $\theta_{L/K,S} \in \mathbb{C}[G]$, where $G = \mathrm{Gal}(L/K)$, in terms of values of L-functions for the pair $(L/K, S)$. L-functions, which are meromorphic functions of the complex plane, contain deep information about the given extension of number fields. They will be defined and discussed in Section 2.3.1, but for the time being we remark simply that we have such a function for each character χ of G, and that it will be denoted $L_{L/K,S}(s, \chi)$ where s is a complex variable. The Stickelberger element is then defined as follows:

Definition 1.2.1

$$\theta_{L/K,S} = \sum_{\chi \in \widehat{G}} L_{L/K,S}(0, \bar{\chi}) e_{\chi},$$

where $e_\chi \in \mathbb{C}[G]$ is the idempotent associated to χ. ($L_{L/K,S}(s, \chi)$ is indeed analytic at 0 and hence has a true value there.)

On a first glance, one might believe that $\theta_{L/K,S}$ could have non-rational complex coefficients. In fact, one has the following theorem, due to work of Siegel on partial ζ-functions in [37]. (Partial ζ-functions will be discussed in Section 2.3.5.)

Theorem 1.2.2 *The Stickelberger element* $\theta_{L/K,S}$ *lies in* $\mathbb{Q}[G]$.

In fact, when K is totally real there is a much stronger result of Deligne and Ribet in [13]:

Theorem 1.2.3 *If* K *is totally real and* S *contains the places which ramify in* L/K*, then*

$$\mathrm{ann}_{\mathbb{Z}[G]}(\mu(L))\theta_{L/K,S} \subseteq \mathbb{Z}[G].$$

1.2.2 $\mathbb{Q}$ as base-field

Let us consider the case when $K = \mathbb{Q}$ and S is the set consisting exactly of the infinite place of $\mathbb{Q}$ and the places which ramify in $L/\mathbb{Q}$. The next result is classical in nature, first proven in essence by Stickelberger in 1890, although the outward appearance of his version would have been somewhat different from the more modern one, which we have opted to provide here. (His version of $\theta_{L/K,S}$ differed slightly from the one given here and was not defined in terms of L-function values.) We emphasize that $L/\mathbb{Q}$ is an arbitrary abelian extension of number fields with Galois group G.

Theorem 1.2.4 *Any* $\mathbb{Z}[G]$*-multiple of* $\theta_{L/\mathbb{Q},S}$ *having integral coefficients annihilates* $\mathrm{Cl}(L)$*.*

A proof of Theorem 1.2.4 can be found in [45, Section 6.2], although it is then necessary to make the transition from the form of the result found there (akin to what Stickelberger proved) to the form above. This transition can be found in the proof of the relevant case of the Brumer–Stark Conjecture found in [29, Ch.15].

We now see, by combining Theorems 1.2.3 and 1.2.4, that

$$\mathrm{ann}_{\mathbb{Z}[G]}(\mu(L))\theta_{L/\mathbb{Q},S} \subseteq \mathrm{ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(L)), \tag{1.2.1}$$

and hence also that

$$\mathrm{ann}_{\mathbb{Z}[G]}(\mu(L))\theta_{L/\mathbb{Q},S} \subseteq \mathrm{ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{L,S})) \tag{1.2.2}$$

since $\mathrm{Cl}(\mathcal{O}_{L,S})$ is a quotient of $\mathrm{Cl}(L)$. However, (1.2.1) and (1.2.2) are limited in their usefulness, since we see readily from Definition 1.2.1 that $\theta_{L/\mathbb{Q},S} = 0$ whenever all of the L-functions $L_{L/\mathbb{Q},S}(s,\chi)$ vanish at $s = 0$. This certainly can happen; for example, when L is totally real. Our attempt to overcome this problem will involve constructing group-ring elements using the *leading coefficients* of the L-functions at $s = 0$, rather than their actual values. This construction will take place in Chapter 4.

1.2.3 The K-theoretic context

Notwithstanding what has been said in Section 1.2.2 concerning the vanishing (in some cases) of the Stickelberger element, it is still interesting to ask the following question: Suppose we are once again given an arbitrary abelian extension L/K of number fields with Galois group G and a finite set S of places of K containing the infinite places and the ones which ramify in L/K. Do we have

$$\mathrm{ann}_{\mathbb{Z}[G]}(\mu(L))\theta_{L/K,S} \subseteq \mathrm{ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_{L,S}))? \tag{1.2.3}$$

This is the conjecture known as Brumer's Conjecture, and it is discussed in more detail in [27, Section 4] for example.

In observing that one has

$$\begin{aligned} \mathrm{Cl}(\mathcal{O}_{L,S}) &= \mathrm{tors}(K_0(\mathcal{O}_{L,S})) \\ \mu(L) &= \mathrm{tors}(K_1(\mathcal{O}_{L,S})), \end{aligned}$$

it became natural to conjecture analogues of (1.2.3) involving the higher algebraic K-groups of $\mathcal{O}_{L,S}$.

Definition 1.2.5 *Let L/K and S be given as above. Then for $k < 0$, define the Stickelberger element $\theta_{L/K,S}(k) \in \mathbb{C}[G]$ by*

$$\theta_{L/K,S}(k) = \sum_{\chi \in \widehat{G}} L_{L/K,S}(k, \chi) e_{\bar{\chi}}.$$

Remark. We point out that, as in Section 1.2.1, there is no need to assume that the set S contains the ramified places in order to define $\theta_{L/K,S}(k)$.

We now come to a higher-dimensional analogue of the Brumer Conjecture, posed when K is totally real, L is totally real or is a CM-field, and S contains the places which ramify in L/K; see [38, Ch.7]. Whereas (1.2.3) asks about L-function values at $s = 0$ (through $\theta_{L/K,S}$), the following concerns values at a negative integer k.

Conjecture 1.2.6 *For any integer $k < 0$,*

$$\mathrm{ann}_{\mathbb{Z}[G]}(\mathrm{tors}(K_{1-2k}(\mathcal{O}_{L,S})))\theta_{L/K,S}(k) \subseteq \mathrm{ann}_{\mathbb{Z}[G]}(K_{-2k}(\mathcal{O}_{L,S})).$$

We observe that we do not need to take the torsion in $K_{-2k}(\mathcal{O}_{L,S})$ for $k < 0$, because these groups are already torsion. The odd K-groups need not be torsion, however.

1.2.4 Higher fractional ideals

As with (1.2.2), in which $\theta_{L/K,S}$ can be zero, the same problem could arise in Conjecture 1.2.6. Now, under the assumption that L/K satisfies the higher Stark Conjectures, Snaith shows in [40] how one can attach to L/K a family $\{\mathcal{J}^k(L/K)\}_{k \in \mathbb{Z}_{<0}}$ of $\mathbb{Z}[G]$-submodules of $\mathbb{Q}[G]$, hoped to appear in a generalization of Conjecture 1.2.6. Namely,

Conjecture 1.2.7 *If L/K is an abelian extension of number fields with Galois group G and S contains the places which ramify in L/K, then for each odd* *prime p and each $k < 0$,*

$$\mathrm{ann}_{\mathbb{Z}_p[G]}(\mathrm{tors}(K_{1-2k}(\mathcal{O}_{L,S})) \otimes_{\mathbb{Z}} \mathbb{Z}_p)\mathcal{J}^k(L/K) \cap \mathbb{Z}_p[G] \subseteq \mathrm{ann}_{\mathbb{Z}_p[G]}(K_{-2k}(\mathcal{O}_{L,S}) \otimes_{\mathbb{Z}} \mathbb{Z}_p).$$

What is more, $\mathcal{J}^k(L/K)$ will contain non-trivial elements even when $\theta_{L/K,S}(k)$ is zero. Conjecture 1.2.7 is formulated in [40, Section 5.1], and evidence for it is given in [40, Section 6] assuming the Lichtenbaum–Quillen Conjecture (which relates étale cohomology to K-theory).

1.3 Return to L-functions at $s = 0$

1.3.1 Iwasawa theory and Stark elements

We now return our attention to the relationship of the behaviour of L-functions at the integer $k = 0$ with class-groups. An important area of study for this relationship is Iwasawa theory, which approaches the problem by considering not just isolated extensions of number fields, but all the subextensions at once of some given infinite extension whose Galois group G is a compact p-adic Lie group. An advantage of such an idea is that one can make use of properties of modules over the completed group-ring of G which are not exhibited by modules over group-rings of finite Galois groups. For example, a classical theorem of Iwasawa – proven largely by algebraic means – states that when G is isomorphic to $\mathbb{Z}_p$, the growth of the orders of the Sylow p-subgroups of the class-groups in the given extension is determined, at least sufficiently far up the $\mathbb{Z}_p$-extension, by just three integers. The precise statement and proof can be found in [45, Chapter 13].

It was an idea of Iwasawa himself that p-adic L-functions, meromorphic functions of $\mathbb{C}_p$ obtained by interpolating p-adically the values of Artin L-functions at negative integers, should carry strong information about the limit Cl_∞ of p-parts of class-groups in a $\mathbb{Z}_p$-extension. The conjecture he posed became known as the Main Conjecture of Iwasawa theory, and was proven by Mazur and Wiles in [23], with Wiles later proving a generalization in [46]. A version suitable for our needs is formulated precisely in Section 6.3.2. We observe, however, that p-adic L-functions only tell us about the so-called *minus* part of Cl_∞, i.e. the part on which complex conjugation acts by -1. One might ask the question: what analytic object would tell us about both the minus *and* plus parts for complex conjugation?

There have since been further generalizations of the Main Conjecture posed, all taking the form

Certain p-adic L-functions are closely related to a limit of Selmer groups,

a Selmer group being an algebraically defined object containing deep arithmetic information. Class-groups are, essentially, examples of Selmer groups.

Some of these conjectures have been proven, and one of the major approaches in tackling them is the consideration of *Euler systems*. In principle, an Euler system is a collection of Galois cohomology classes satisfying some coherence property, and in terms of which bounds on the orders of associated Selmer groups can be formed. However, as discussed in [34, Chapter 8], an important aspect of the story is the expected relationship of Euler systems not just with Selmer groups, but with p-adic L-functions as well, a relationship which has been verified in a number of cases. Examples of Euler systems which have led to proofs of main conjectures and related statements are given in Table 1.3.1 below.

The idea that leading coefficients of L-functions can be expressed in terms of special elements (like cyclotomic units, elliptic units) has been formalized by the "integral" versions of Stark's Conjectures; see [33] and [26] for very general formulations, and [41] and [43] for formulations in an important special case. Furthermore, [32] and [26, Section 4] show how these *Stark elements*, as they are

Table 1.3.1: Examples of Euler systems.

	Euler system	Used in
(i)	Cyclotomic units	Thaine [44] and Rubin [30], to construct annihilators of class-groups of real abelian extensions of $\mathbb{Q}$ from annihilators of units modulo cyclotomic units
(ii)	Elliptic units	Rubin [31], to prove a main conjecture for elliptic curves with complex multiplication
(iii)	Heegner points	Kolyvagin [19, 20], to bound the orders of Selmer groups of elliptic curves
(iv)	Beilinson elements	Kato [17], to prove one direction of a main conjecture for elliptic curves in a large class of cases

sometimes called, give rise to Euler systems leading to a link with class-groups. That Stark-type elements can give rise to annihilators of arithmetic objects like Selmer groups is also discussed in [6] as part of the general framework of Burns and Flach's Equivariant Tamagawa Number Conjecture; see in particular [6, Theorem 5.5]. This points to the Stark elements being a stepping stone between the analytic domain of L-functions and the algebraic one of class-groups. Although the Stark elements are only conjectural in general, we shall see in Section 3.6.1 a number of situations in which they are known to exist.

1.3.2 The fractional Galois ideal at $s = 0$

The central object of this thesis is an invariant $\mathcal{J}(L/K, S)$ associated to a pair $(L/K, S)$ where L/K is an abelian extension of number fields satisfying Stark's Conjecture – see Chapter 3 for a formulation of this conjecture – and S is a finite set of places of K containing the infinite ones. Based on Snaith's higher fractional ideals $\mathcal{J}^k(L/K)$, it is a finitely generated $\mathbb{Z}[G]$-submodule of $\mathbb{Q}[G]$, $G = \mathrm{Gal}(L/K)$, defined in terms of leading coefficients of L-functions at $s = 0$. We will provide evidence for $\mathcal{J}(L/K, S)$ improving on the role of the Stickelberger element $\theta_{L/K,S}$ which, although contributing annihilators for class-groups, is often zero, as discussed in Section 1.2.2.

The strength of $\mathcal{J}(L/K, S)$ over $\theta_{L/K,S}$, or rather the module $\mathbb{Z}[G]\theta_{L/K,S}$ it generates, is that $\mathcal{J}(L/K, S)$ is never zero. We will see in Proposition 4.2.4 and equation (4.2.1) a natural way to decompose $\mathcal{J}(L/K, S)$ according to the orders of vanishing of L-functions at 0. The part corresponding to L-functions which are non-zero at 0 will be exactly $\mathbb{Z}[G]\theta_{L/K,S}$. However, we will exhibit in Theorem 4.3.3 a very general relationship between the various parts of $\mathcal{J}(L/K, S)$ and the Stark elements discussed above. More precisely, Stark elements all come with a *rank*, a non-negative integer r, and the part of $\mathcal{J}(L/K, S)$ corresponding to L-functions with order of vanishing r at 0 will be closely tied in with the rank r Stark elements, for any $r \geq 0$.

This relationship in the case $r = 1$, together with a result of Rubin, will allow us to demonstrate explicitly for certain cyclotomic fields the role of the rank 1 part of the fractional ideal in providing annihilators for the plus part of the class-group; see Proposition 6.2.1. Combined instead with a theorem of Cornacchia and Greither [12] on the Fitting ideals of class-groups, and the classical Main Conjecture of Iwasawa theory, we will be able to show how a limit of the $\mathcal{J}(L/\mathbb{Q}, S)$ "twisted" by certain characters is equal to the Fitting ideal of the corresponding eigencomponent of Cl_∞; see Theorem 6.3.1. Since this will be proven for both even and odd characters (characters that act by $+1$ and -1 respectively), we will have found an object analytic in nature which gives information on both the plus and minus parts of Cl_∞.

As remarked above, the proof of Theorem 6.3.1 uses a result of Cornacchia and Greither comparing the Fitting ideals of units modulo cyclotomic units and class-groups, stating that these Fitting ideals are equal. In fact, the proof of Theorem 6.3.1 only uses a form of this result obtained by tensoring with $\mathbb{Z}_p$, and in Section 6.4 we give a new proof of one direction of this: that the Fitting ideal of units modulo cyclotomic units (tensored with $\mathbb{Z}_p$) is contained in that of the class-group (tensored with $\mathbb{Z}_p$). The basis of the proof of this statement, which is Proposition 6.4.1, is to show that the information tied up in the determinant (of a certain complex) concerning limits of cyclotomic units and limits of class-groups in an extension of number fields can be "descended" to give the required information on the individual class-groups themselves. The contribution here is the descent, following [39], and not the calculation of the determinant in question, which was done by Burns and Greither in [8].

Chapter 2

Preliminary material

2.1 Representation theory

We include this section because it contains statements which, although basic, will be referred to many times. We take as read the principal statements of the representation theory (over $\mathbb{C}$) of finite groups; namely that

- to each representation is associated a character
- two representations are isomorphic if and only if their characters are equal
- the irreducible characters form an orthonormal basis (with respect to a suitable Hermitian product) for the space of class functions on a finite group G
- the character table of G exhibits row and column orthogonality.

We record here a selection of corollaries, although we do this only in the abelian situation, which is sufficient for this thesis.

2.1.1 The isomorphism φ_G

Let G be a finite abelian group, $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$ its character group and $R(G)$ its representation ring, and let $\mathbb{Q}^c$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

Row and column orthogonality of the character table of G show that the ring homomorphism

$$\begin{aligned} \varphi_G : \mathrm{Map}(\widehat{G}, \mathbb{C}) &\to \mathbb{C}[G] \\ h &\mapsto \sum_{\chi \in \widehat{G}} h(\chi) e_\chi, \end{aligned} \tag{2.1.1}$$

where $e_\chi = \frac{1}{|G|} \sum_{\sigma \in G} \bar{\chi}(\sigma)\sigma$ is the idempotent in $\mathbb{C}[G]$ associated to χ, is an isomorphism. If we extend a character $\chi \in \widehat{G}$ linearly to $\mathbb{C}[G]$, then the inverse of φ_G is given by sending a group-ring element α to the map $\chi \mapsto \chi(\alpha)$.

We observe that the group of units in $\mathrm{Map}(\widehat{G}, \mathbb{C})$ is $\mathrm{Map}(\widehat{G}, \mathbb{C}^\times)$, so that φ_G induces a group isomorphism $\mathrm{Map}(\widehat{G}, \mathbb{C}^\times) \to \mathbb{C}[G]^\times$. It is also clear that φ_G restricts to a ring isomorphism $\mathrm{Map}(\widehat{G}, \mathbb{Q}^c) \to \mathbb{Q}^c[G]$.

Now, the Galois group $G_\mathbb{Q} = \mathrm{Gal}(\mathbb{Q}^c/\mathbb{Q})$ acts on $\mathrm{Map}(\widehat{G}, \mathbb{Q}^c)$ as follows: Firstly, $G_\mathbb{Q}$ acts on $\widehat{G}$ (on the left) by $\chi^\delta = \delta \circ \chi$ for $\delta \in G_\mathbb{Q}$ and $\chi \in \widehat{G}$. Then if $h \in \mathrm{Map}(G, \mathbb{Q}^c)$, δh is the map defined by $(\delta h)(\chi) = \delta(h(\chi^{\delta^{-1}}))$. On the other hand, $G_\mathbb{Q}$ acts on $\mathbb{Q}^c[G]$ by acting on coefficients. It is straightforward to check that the isomorphism φ_G respects these actions.

The above is summarized in the following lemma:

Lemma 2.1.1 *The ring isomorphism* $\varphi_G : \mathrm{Map}(\widehat{G}, \mathbb{C}) \to \mathbb{C}[G]$ *defined by (2.1.1) restricts to give isomorphisms (of groups and rings as appropriate)*

$$\begin{aligned} \mathrm{Map}(\widehat{G}, \mathbb{C}^\times) &\to \mathbb{C}[G]^\times \\ \mathrm{Map}(\widehat{G}, \mathbb{Q}^c) &\to \mathbb{Q}^c[G] \\ \mathrm{Map}_{G_\mathbb{Q}}(\widehat{G}, \mathbb{Q}^c) &\to \mathbb{Q}[G]. \end{aligned}$$

Since the representation ring $R(G)$ of G is generated freely as a group by $\widehat{G}$, we can identify $\mathrm{Map}(\widehat{G}, \mathbb{C}^\times)$ with $\mathrm{Hom}(R(G), \mathbb{C}^\times)$. Then the main point to draw from the above is

Lemma 2.1.2 $\varphi_G : \mathrm{Hom}(R(G), \mathbb{C}^\times) \to \mathbb{C}[G]^\times$ *restricts to an isomorphism* $\mathrm{Hom}_{G_\mathbb{Q}}(R(G), (\mathbb{Q}^c)^\times) \to \mathbb{Q}[G]^\times$.

2.1.2 Determinants of $\mathbb{C}[G]$-endomorphisms

Maschke's Theorem (for any finite group G and any field F whose characteristic does not divide $|G|$) says that in any $F[G]$-module, all submodules are direct summands. Hence every $F[G]$-module is projective. This allows us, when G is abelian, to define the determinant of an endomorphism of any finitely generated $F[G]$-module.

In fact, this can be done more generally. Let R be a commutative ring, M a finitely generated projective R-module, and $h \in \mathrm{End}_R(M)$. We arbitrarily choose a finitely generated R-module N such that $M \oplus N$ is free, and extend h to $h \oplus 1 : M \oplus N \to M \oplus N$. Then we simply define $\det_R(h)$ to be $\det_R(h \oplus 1)$. This is independent of the choice of N. We note that, as usual, $\det_R(h_1 \circ h_2) = \det_R(h_1)\det_R(h_2)$.

So, take G to be a finite abelian group. We will be interested in the following situation: Let V, W be finitely generated $\mathbb{C}[G]$-modules and $h \in \mathrm{End}_{\mathbb{C}[G]}(W)$. Then one obtains a $\mathbb{C}$-linear map

$$\begin{aligned} h_{V,W} : \mathrm{Hom}_{\mathbb{C}[G]}(V, W) &\to \mathrm{Hom}_{\mathbb{C}[G]}(V, W) \\ \phi &\mapsto h \circ \phi. \end{aligned} \tag{2.1.2}$$

Proposition 2.1.3 *Let W be a finitely generated $\mathbb{C}[G]$-module, $h \in \mathrm{End}_{\mathbb{C}[G]}(W)$, and V an irreducible $\mathbb{C}[G]$-module with character χ. Then the determinant of the $\mathbb{C}$-linear map $h_{V,W}$ is $\chi(\det_{\mathbb{C}[G]}(h))$.*

Proof. We may reduce to the case that W is free. Indeed, choose W' finitely generated such that $W \oplus W'$ is free. We extend h to $h \oplus 1 : W \oplus W' \to W \oplus W'$, and note that, by definition, $\det_{\mathbb{C}[G]}(h) = \det_{\mathbb{C}[G]}(h \oplus 1)$. Introduce the notation $H(U) = \mathrm{Hom}_{\mathbb{C}[G]}(V, U)$ for a $\mathbb{C}[G]$-module U. Then from the commutativity of

$$\begin{array}{ccc} H(W \oplus W') & \xrightarrow{h_{V,W \oplus W'}} & H(W \oplus W') \\ \downarrow & & \downarrow \\ H(W) \oplus H(W') & \xrightarrow{h_{V,W} \oplus 1} & H(W) \oplus H(W') \end{array}$$

we see that $\det_{\mathbb{C}}((h \oplus 1)_{V,W\oplus W'}) = \det_{\mathbb{C}}(h_{V,W})$. This completes the reduction step.

Now assume that W is free as a $\mathbb{C}[G]$-module, and let $\{\beta_1, \ldots, \beta_n\}$ be a basis (over $\mathbb{C}[G]$). Of course, any homomorphism in $\mathrm{Hom}_{\mathbb{C}[G]}(V, W)$ has image in $e_\chi W$, and since $\{e_\chi\beta_1, \ldots, e_\chi\beta_n\}$ is a $\mathbb{C}$-basis for $e_\chi W$, we can find a $\mathbb{C}$-basis for $\mathrm{Hom}_{\mathbb{C}[G]}(V, W)$ as follows: Choose $v_0 \in V \setminus \{0\}$ and for $i = 1, \ldots, n$ define a $\mathbb{C}$-linear map $\eta_i : V \to W$ by $\eta_i(v_0) = e_\chi\beta_i$. Then the η_i are $\mathbb{C}[G]$-module homomorphisms, and $\{\eta_1, \ldots, \eta_n\}$ is a $\mathbb{C}$-basis for $\mathrm{Hom}_{\mathbb{C}[G]}(V, W)$.

We let $A = (a_{i,j})$ be the matrix representing h with respect to this basis. A simple calculation shows that for $j = 1, \ldots, n$, $h_{V,W}(\eta_j) = \chi(a_{1,j})\eta_1 + \cdots + \chi(a_{n,j})\eta_n$, and so the matrix representing $h_{V,W}$ with respect to the η_i is $\chi(A)$. The lemma follows. ∎

In the situation where an endomorphism of a $\mathbb{C}[G]$-module is given by multiplying by some fixed element of $\mathbb{C}[G]$, the determinant of that endomorphism can be found easily, as the following lemma shows.

Lemma 2.1.4 *Let G be a finite abelian group, M a finitely generated $\mathbb{C}[G]$-module with character χ_M, and $\alpha \in \mathbb{C}[G]$. Then the $\mathbb{C}[G]$-determinant of the multiplication-by-α map $[\alpha] : M \to M$ is equal to*

$$\sum_{\chi \in \widehat{G}} \alpha^{\langle \chi, \chi_M \rangle} e_\chi,$$

where we understand α^0 to be 1 always (including the case $\alpha = 0$).

Proof. We may assume M is the direct sum $\bigoplus_{\chi\in\widehat{G}}(e_\chi\mathbb{C}[G])^{r_\chi}$ for some non-negative integers r_χ. Of course, r_χ is just $\langle \chi, \chi_M \rangle$. Letting $r = \max_\chi\{r_\chi\}$ and $N = \bigoplus_{\chi\in\widehat{G}}(e_\chi\mathbb{C}[G])^{r-r_\chi}$, we see that $M \oplus N$ is free of rank r. Now choose the natural $\mathbb{C}[G]$ basis for $M \oplus N$ to work out the determinant of $[\alpha] \oplus 1 : M \oplus N \to M \oplus N$. ∎

We also observe the following lemma, which will be applied later in the case $R = \mathbb{C}[G]$ with G finite abelian. The proof is straightforward, and omitted.

Lemma 2.1.5 *Let R be a commutative ring, M a finitely generated projective R-module and $e \in R$ an idempotent. Then eM is also projective, and if $\alpha \in \mathrm{End}_R(M)$,*

$$e\det_R(\alpha) = e\det_R(\alpha|_{eM}).$$

2.1.3 Rank idempotents

Let G be a finite abelian group and M a finitely generated $\mathbb{C}[G]$-module. If $\langle \cdot, \cdot \rangle_G$ denotes the canonical Hermitian product on the space of class functions of G, then we write

$$r_M(\chi) = \langle \chi, \chi_M \rangle$$

for a character $\chi \in \widehat{G}$, where χ_M is the character of M. We will call $r_M(\chi)$ the rank of χ in M.

Remark. It is important to empasize that the rank of a character does not mean the dimension of the underlying vector space.

In subsequent chapters, we will often want to study the part of a module corresponding to the characters of some fixed rank, and this can be achieved with the aid of what will be called *rank idempotents.*

Definition 2.1.6 *For each $r \geq 0$, define $e_M[r] \in \mathbb{C}[G]$ by*

$$e_M[r] = \sum_{\substack{\chi \in \widehat{G} \\ r_M(\chi) = r}} e_\chi,$$

and call it the rth rank idempotent for M.

In our applications, the rank idempotents we will make use of will, in fact, have rational coefficients.

2.2 Fitting ideals

In this section, R can be any commutative ring with identity $1 \neq 0$. Given a finitely presentable R-module M, the Fitting ideal of M (by which we mean the zeroth Fitting invariant as defined in [25]) is an ideal in R, given as follows: Choose positive integers a and b and an exact sequence

$$R^a \xrightarrow{\phi} R^b \to M \to 0.$$

Then $\mathrm{Fitt}_R(M)$ is the ideal generated by all $b \times b$ minors of any matrix representing ϕ. This definition is independent of any choices.

To give some feel for the role of the Fitting ideal, first note that it is always contained in the annihilator ideal; further, if M can be generated by n elements then $\mathrm{ann}_R(M)^n \subseteq \mathrm{Fitt}_R(M)$. For modules M of the form

$$R/I_1 \oplus \cdots \oplus R/I_m, \tag{2.2.1}$$

$\mathrm{Fitt}_R(M) = I_1 \cdots I_m$. Thus, in the case when $R = \mathbb{Z}$ and M is finite, we see that $\mathrm{Fitt}_{\mathbb{Z}}(M) = |M|\mathbb{Z}$. For some rings R, torsion modules of the shape in (2.2.1) can be infinite, though we might still want to measure their "size". Often, the Fitting ideal is the appropriate measure. In Iwasawa theory, inverse limits of p-parts of class-groups, for a prime p, are important objects. While they are not finite in general, they do have a Fitting ideal; we will say more about this in Section 6.3.

We make one more remark concerning the relationship between Fitting ideals and annihilator ideals which will help us later on. It is clear from the above that if M is a cyclic R-module, then $\mathrm{Fitt}_R(M) = \mathrm{ann}_R(M)$. However, in the case $R = \mathbb{Z}[G]$ for a finite cyclic group G, we can say more. Given a finite $\mathbb{Z}[G]$-module M, denote by $M^\vee$ the $\mathbb{Z}[G]$-module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, where the action of G is given by $(\sigma f)(m) = f(\sigma^{-1}m)$ for $\sigma \in G$, $f \in M^\vee$ and $m \in M$.

Proposition 2.2.1 *Let M be a finite $\mathbb{Z}[G]$-module (G finite cyclic). If either M or $M^\vee$ is cyclic over $\mathbb{Z}[G]$, then* $\mathrm{Fitt}_{\mathbb{Z}[G]}(M) = \mathrm{ann}_{\mathbb{Z}[G]}(M)$.

Proof. If M is cyclic, the conclusion is obvious from the above, as remarked. Now, Proposition 1 of [23, Appendix] implies that $\mathrm{Fitt}_{\mathbb{Z}[G]}(M) = \kappa(\mathrm{Fitt}_{\mathbb{Z}[G]}(M^\vee))$, where $\kappa : \mathbb{Z}[G] \to \mathbb{Z}[G]$ is the linear extension to $\mathbb{Z}[G]$ of the automorphism of G sending each element to its inverse. So, if $M^\vee$ is cyclic over $\mathbb{Z}[G]$,

$$\begin{aligned}\mathrm{Fitt}_{\mathbb{Z}[G]}(M) &= \kappa(\mathrm{Fitt}_{\mathbb{Z}[G]}(M^\vee)) \\ &= \kappa(\mathrm{ann}_{\mathbb{Z}[G]}(M^\vee)) \\ &= \mathrm{ann}_{\mathbb{Z}[G]}(M),\end{aligned}$$

the last equality being clear. ■

2.3 Artin L-functions

The theory of Artin L-functions (let alone L-functions in general) is far too large a body of knowledge for any account possible here to be representative of the whole. However, because Artin L-functions play such an important role in what is to follow, it is deemed correct to give a summary of the main points. A good introduction is [5].

Artin L-functions are attached to triples $(L/K, S, \chi)$ where L/K is a Galois extension of number fields, S is a finite set of places of K containing the infinite ones, and χ is a character of the Galois group G of L/K. They are meromorphic complex functions which are defined, in the first instance, for complex numbers having real part greater than 1.

The key idea rests in the fact that given a prime (by which we shall mean a finite place) $\mathfrak{P}$ of L one can associate to it canonically a class $\sigma_{\mathfrak{P}}$ in $G_{\mathfrak{P}}/I_{\mathfrak{P}}$, where $G_{\mathfrak{P}}$ and $I_{\mathfrak{P}}$ are (respectively) the decomposition and inertia groups of $\mathfrak{P}$. Indeed, we let $\sigma_{\mathfrak{P}}$ be the unique class mapping to the Frobenius automorphism of the extension $(\mathcal{O}_L/\mathfrak{P})/(\mathcal{O}_K/\mathfrak{p})$ of finite fields, where $\mathfrak{p}$ is the prime below $\mathfrak{P}$.

2.3.1 Definition of Artin L-functions

Suppose that V is a realization of a character χ of G. Any representative σ of the class $\sigma_{\mathfrak{P}}$ defines an automorphism of the vector space $V^{I_{\mathfrak{P}}}$ of $I_{\mathfrak{P}}$-fixed elements, by $v \mapsto \sigma v$, and the automorphism is independent of the choice of $\sigma \in \sigma_{\mathfrak{P}}$. Further, if $\mathfrak{P}$ and $\mathfrak{P}'$ both lie above the same prime $\mathfrak{p}$ of K, with $\mathfrak{P}' = \tau\mathfrak{P}$ say, then one can obtain a representative σ' for $\sigma_{\mathfrak{P}'}$ by conjugating a representative σ for $\sigma_{\mathfrak{P}}$ by τ, i.e. $\sigma' = \tau\sigma\tau^{-1}$. Consequently, the automorphisms of $V^{I_{\mathfrak{P}}}$ and $V^{I_{\mathfrak{P}'}}$ defined as above fit into a commutative diagram

$$\begin{array}{ccc} V^{I_{\mathfrak{P}}} & \longrightarrow & V^{I_{\mathfrak{P}}} \\ \downarrow & & \downarrow \\ V^{I_{\mathfrak{P}'}} & \longrightarrow & V^{I_{\mathfrak{P}'}}, \end{array}$$

where the two vertical maps are given by $v \mapsto \tau v$. Hence the characteristic polynomials $\mathrm{char}_{V,\mathfrak{P}}(T)$ and $\mathrm{char}_{V,\mathfrak{P}'}(T)$ of the horizontal maps agree. It is also clear that $\mathrm{char}_{V,\mathfrak{P}}$ does not depend upon the choice of realization V of χ. This allows us to define, for any prime $\mathfrak{p}$ of K, a polynomial $\mathrm{char}_{\chi,\mathfrak{p}}(T) \in \mathbb{C}[T]$ by setting

$$\mathrm{char}_{\chi,\mathfrak{p}}(T) = \mathrm{char}_{V,\mathfrak{P}}(T)$$

for any realization V of χ and any choice of $\mathfrak{P}|\mathfrak{p}$.

Remark. We use the following convention here for defining characteristic polynomials: If α is an endomorphism of a finite dimensional vector space V, its characteristic polynomial is

$$\mathrm{char}(T) = \det(\mathrm{id} - TA)$$

for any matrix A representing α.

We finally come to the definition of Artin L-functions.

Definition 2.3.1 *For $s \in \mathbb{C}$ having real part greater than 1, define*

$$L_{L/K,S}(s,\chi) = \prod_{\mathfrak{p} \notin S} (\mathrm{char}_{\chi,\mathfrak{p}}(\mathbf{N}\mathfrak{p}^{-s}))^{-1},$$

where the product runs over all primes of K not in S.

The product in the definition of $L_{L/K,S}(s,\chi)$ converges uniformly on compact subsets of $\{s \in \mathbb{C} \mid \Re(s) > 1\}$, and hence defines an analytic function on this domain. Further, $L_{L/K,S}(s,\chi)$ has a meromorphic continuation to $\mathbb{C}$, although this is not immediate.

2.3.2 Formal properties of Artin L-functions

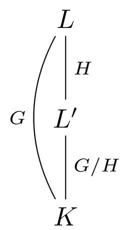
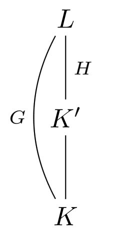
Good sources for the properties of Artin L-functions (which from now on will just be called L-functions unless otherwise stated) are [43, 22, 5]. The behaviour under addition, inflation and induction of characters is given in Table 2.3.1 below. In property (ii), H is a normal subgroup of G and L' its fixed field in L. In property (iii), H is any subgroup of G, K' its fixed field in L, and S' the set of places of K' above those in S.

There is a special L-function $\zeta_{L,S}$ associated to a number field L and a finite set S of places of L containing the infinite ones. It is defined to be the function $L_{L/L,S}(s,1)$, where 1 here is the trivial character of $\mathrm{Gal}(L/L)$, and is called the *Dedekind ζ-function* (relative to S) of L. If L/K is Galois, then once we observe that the character of $\mathrm{Gal}(L/K)$ induced by the trivial character of $\mathrm{Gal}(L/L)$ is the character of the regular representation of $\mathrm{Gal}(L/K)$, properties (i) and (iii) in the above table show that

$$\zeta_{L,S'}(s) = \prod_{\chi} L_{L/K,S}(s,\chi)^{d_\chi}, \tag{2.3.1}$$

where the notation means the following: S is any finite set of places of K containing the infinite ones and S' is the set of places of L above those in S; the product runs over all irreducible characters χ of $\mathrm{Gal}(L/K)$; d_χ is the dimension of χ.

Table 2.3.1: Behaviour of L-functions.

(i)	For any characters χ_1, χ_2 of G, $$L_{L/K,S}(s, \chi_1{+}\chi_2) = L_{L/K,S}(s, \chi_1)L_{L/K,S}(s, \chi_2).$$	
(ii)	If χ is a character of $\mathrm{Gal}(L'/K) = G/H$ and $\mathrm{Inf}(\chi)$ its inflation to G, then $$L_{L/K,S}(s, \mathrm{Inf}(\chi)) = L_{L'/K,S}(s, \chi).$$	
(iii)	If χ is a character of $\mathrm{Gal}(L/K') = H$ and $\mathrm{Ind}(\chi)$ its induction to G, then $$L_{L/K,S}(s, \mathrm{Ind}(\chi)) = L_{L/K',S'}(s, \chi).$$	

If $\mathfrak{a}$ is a non-zero ideal of $\mathcal{O}_L$, we shall write $(\mathfrak{a}, S) = 1$ if $\mathfrak{a}$ is coprime to every prime in S. Then the uniqueness of factorization of non-zero ideals into prime ideals shows that $\zeta_{L,S}$ can also be expressed in the following way:

Proposition 2.3.2 *For* $\Re(s) > 1$, $\zeta_{L,S}(s) = \sum_{(\mathfrak{a},S)=1} \mathbf{N}\mathfrak{a}^{-s}$.

2.3.3 Behaviour at $s = 0$

The behaviour of L-functions at the point $s = 0$ in the complex plane will be of much interest to us. Indeed, the construction of $\mathcal{J}(L/K, S)$ in Chapter 4 relies upon the truth of a conjecture concerning the leading coefficients of the Laurent expansions of L-functions at $s = 0$.

It has already been stated that L-functions are meromorphic on $\mathbb{C}$, but in fact they are known to be analytic at 0. Hence an L-function $L_{L/K,S}(s, \chi)$ has a Taylor series at $s = 0$, whose leading coefficient will be denoted $L^*_{L/K,S}(0, \chi)$. We will write $r_{L/K,S}(\chi)$ for the order of vanishing at 0, or simply $r(\chi)$ when the pair $(L/K, S)$ is understood, and call this the *rank* of χ. The relation of $r(\chi)$ to the ranks of characters defined in Section 2.1.3 will become clear after the next lemma.

$r(\chi)$ has a number of descriptions besides simply its definition as an order of vanishing of an L-function. To give as many of these descriptions as possible, we introduce $\mathbb{Z}[G]$-modules Y and X associated to a pair $(L/K, S)$ ($G = \mathrm{Gal}(L/K)$). These modules, particularly X, will play an important role in the main body of the thesis.

Definition 2.3.3 *We define Y and X as follows:*

(i) *Let Y be the free abelian group on the set S_L of places of L above those in S.*

(ii) *Let X be the kernel of the degree map* $\deg : Y \to \mathbb{Z}$ *which sums up the coefficients in an element of Y.*

G acts on Y by permuting places in S_L, and X is a $\mathbb{Z}[G]$-submodule of Y.

Now, suppose V is a representation of G with character χ. Then for a place v of K, the dimension of the subspace in V of elements fixed by the decomposition group G_w of $w|v$ is independent of the choice of w lying above v, and we call this dimension $d(V, v)$. Then we have the following lemma, which is [43, Prop.3.4 (Ch.I)].

Lemma 2.3.4 *If V is a representation of G with character χ, then $r(\chi)$ is equal to each of the following:*

- $\sum_{v \in S} d(V, v) - \dim_{\mathbb{C}} V^G$.
- $\langle \chi, \chi_X \rangle_G$, *where χ_X is the character of X and $\langle \cdot, \cdot \rangle_G$ is the usual inner product of characters.*
- $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(V^*, X \otimes_{\mathbb{Z}} \mathbb{C})$, *where V^* is the dual representation of V.*

2.3.4 Rank idempotents revisited

Lemma 2.3.4 shows that if $\chi \in \widehat{G}$, $r_{L/K,S}(\chi)$ is the rank of χ in $X \otimes_{\mathbb{Z}} \mathbb{C}$ in the sense of Section 2.1.3. In the future, the rank idempotents of Definition 2.1.6 that we will consider will be those for the module $X \otimes_{\mathbb{Z}} \mathbb{C}$, and will be denoted $e_{L/K,S}[r]$ or simply $e[r]$.

Note that since Galois conjugate characters have the same rank, the rank idempotents $e_{L/K,S}[r]$ have rational coefficients.

2.3.5 Partial ζ-functions

Let us assume in this section that L/K is an abelian extension of number fields with Galois group G, and let S be again a finite set of places of K containing the infinite ones. In Section 2.3.1, we saw that L-functions were associated to elements of $\widehat{G}$. We now look at a sort of dual to this, namely partial ζ-functions, which are associated to elements of G.

Definition 2.3.5 *For $\sigma \in G$, define*

$$\zeta_{L/K,S}(s,\sigma) = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \bar{\chi}(\sigma) L_{L/K,S}(s,\chi).$$

It follows from the orthogonality of the character table of G that

$$L_{L/K,S}(s,\chi) = \sum_{\sigma \in G} \chi(\sigma) \zeta_{L/K,S}(s,\sigma)$$

for all $\chi \in \widehat{G}$. Thus knowing all the L-functions $L_{L/K,S}(s,\chi)$ ($\chi \in \widehat{G}$) is equivalent to knowing all the partial ζ-functions $\zeta_{L/K,S}(s,\sigma)$ ($\sigma \in G$).

When S contains the places of K which ramify in L/K, the partial ζ-functions have a more meaningful description.

Proposition 2.3.6 *If S contains the places which ramify in L/K, then*

$$\zeta_{L/K,S}(s,\sigma) = \sum_{\substack{(\mathfrak{a},S)=1 \\ (\mathfrak{a},L/K)=\sigma}} \mathbf{N}\mathfrak{a}^{-s}$$

for $\Re(s) > 1$*, the sum running over non-zero ideals* $\mathfrak{a}$ *of* $\mathcal{O}_K$ *with* $(\mathfrak{a}, S) = 1$ *and* $(\mathfrak{a}, L/K) = \sigma$.

Proposition 2.3.6 justifies the use of the word *partial* in partial ζ-function, because we see (recall Proposition 2.3.2) that

$$\sum_{\sigma \in G} \zeta_{L/K,S}(\sigma, s) = \zeta_{K,S}(s).$$

Chapter 3

Stark's Conjecture

As motivation for introducing Stark's Conjecture, let us return our attention to the analytic class number formula for a number field L, and remind ourselves of the formulation involving the leading coefficient $\zeta_L^*(0)$ of the ζ-function $\zeta_L(s)$ of L at $s = 0$ (recall (1.1.1)):

$$\frac{\zeta_L^*(0)}{R_L} = -\frac{|\mathrm{Cl}(L)|}{|\mu(L)|}. \tag{3.0.1}$$

(We shorten the notation $\zeta_{L,S}$ to ζ_L when S is the set of infinite places of L.) In particular, it says that the quotient $\zeta_L^*(0)/R_L$ is rational; let us suppose for the moment that this is all we are interested in, rather than what the rational number actually is. Observe from (2.3.1) that

$$\zeta_L^*(0) = \prod_{\chi \text{ irred.}} L_{L/K}^*(0,\chi)^{d_\chi}.$$

In fact, Dirichlet's regulator R_L also splits up as a product over irreducible characters, with the factors being so-called *Stark regulators*. If L/K is abelian, we can combine these individual factors (the L-functions and the Stark regulators) into a unit of the group ring $\mathbb{C}[G]$. Does this element, as might be suggested by the analytic class number formula, actually lie in $\mathbb{Q}[G]^\times$? This is what Stark's Conjecture predicts in the abelian case of the conjecture. Of course, this needs to be made more precise, and we formulate that conjecture in Section 3.3 after first introducing some notation.

3.1 Dirichlet's regulator

We introduce the following notation, which will be used throughout the thesis. For any number field F and any finite set S of places of F containing the infinite ones, $\mathcal{O}_{F,S}$ will denote the ring of S-integers in F, i.e. the subring of F consisting of elements whose valuation at each prime of F not in S is non-negative. The group $\mathcal{O}_{F,S}^\times$ of units in $\mathcal{O}_{F,S}$ is therefore the subgroup of $F^\times$ of elements whose valuation at each prime of F not in S is zero.

We now fix a pair $(L/K, S)$ where L/K is a Galois extension of number fields with Galois group G and S is a finite set of places of K containing the infinite ones. By a slight abuse of notation, in this situation $\mathcal{O}_{L,S}$ will always mean $\mathcal{O}_{L,S_L}$. This should not cause any confusion. Dirichlet's regulator map is then the $\mathbb{R}$-linear map

$$\lambda : \mathcal{O}_{L,S}^\times \otimes_{\mathbb{Z}} \mathbb{R} \to X \otimes_{\mathbb{Z}} \mathbb{R}$$

defined by sending $u \otimes 1$ ($u \in \mathcal{O}_{L,S}^\times$) to $\sum_{w \in S_L} \log \|\sigma(u)\|_w w$. (Recall the definition of X in Definition 2.3.3.) Here, the valuation $\|\cdot\|_w$ is normalized as follows: If the place w corresponds to a prime ideal $\mathfrak{p}$ of L, then for $x \in L^\times$, $\|x\|_w = \mathbf{N}\mathfrak{p}^{-v_{\mathfrak{p}}(x)}$ where $v_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic valuation of L. If instead w comes from an embedding $\iota : L \to \mathbb{C}$, then

$$\|x\|_w = \begin{cases} |\iota(x)| & \text{if } \iota \text{ is real} \\ |\iota(x)|^2 & \text{if } \iota \text{ is complex,} \end{cases}$$

where $|\cdot|$ is the usual absolute value on $\mathbb{C}$.

An algebraic integer (in some number field) having absolute value 1 at all infinite places is necessarily a root of unity ([45, Lemma 1.6]), and so λ is injective. The fact that it is an isomorphism is exactly Dirichlet's Unit Theorem.

3.2 Stark's regulator

The construction of Stark's regulator stems from the observation that Dirichlet's regulator $\lambda : \mathcal{O}_{L,S}^{\times} \otimes_{\mathbb{Z}} \mathbb{R} \to X \otimes_{\mathbb{Z}} \mathbb{R}$ is not only an isomorphism of $\mathbb{R}$-vector spaces, but also an isomorphism of $\mathbb{R}[G]$-modules. Now, by the remark immediately after the proof of [36, Prop.33] (found in Section 12.1 there), we have:

Lemma 3.2.1 *For any finite group G, if M and N are finitely generated $\mathbb{Q}[G]$-modules such that $M \otimes_{\mathbb{Q}} \mathbb{C}$ and $N \otimes_{\mathbb{Q}} \mathbb{C}$ are isomorphic as $\mathbb{C}[G]$-modules, then M and N are isomorphic as $\mathbb{Q}[G]$-modules.*

Hence there is a $\mathbb{Q}[G]$-module isomorphism

$$f : \mathcal{O}_{L,S}^{\times} \otimes_{\mathbb{Z}} \mathbb{Q} \to X \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{3.2.1}$$

(We point out, however, that there is in general no canonical choice for f.) Then for a finitely generated $\mathbb{C}[G]$-module V, let R_V^f denote the determinant of the $\mathbb{C}$-linear map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{C}[G]}(V^*, X \otimes_{\mathbb{Z}} \mathbb{C}) & \to & \mathrm{Hom}_{\mathbb{C}[G]}(V^*, X \otimes_{\mathbb{Z}} \mathbb{C}) \\ \phi & \mapsto & \lambda \circ f^{-1} \circ \phi, \end{array}$$

where V^* is again the dual representation of V, and set

$$\mathcal{A}^f(V) = L_{L/K,S}^*(0, V^*)/R_{V^*}^f.$$

If V has character χ, we also write $R_\chi^f = R_V^f$ and $\mathcal{A}^f(\chi) = \mathcal{A}^f(V)$. We call the non-zero complex number R_χ^f the Stark f-regulator for χ. We note that $\mathcal{A}^f$ is an element of $\mathrm{Hom}(R(G), \mathbb{C}^{\times})$.

3.3 The conjecture

We now reproduce Stark's Conjecture as formulated in [43, Ch.I, Section 5]. In fact, the form here is due to Tate. We emphasize that there is no need to assume G is abelian in order to formulate the conjecture.

Conjecture 3.3.1 *Let χ be a (not-necessarily irreducible) character of G. Then*

(i) $\mathcal{A}^f(\chi) \in \mathbb{Q}(\chi)$, *and*

(ii) *for every* $\delta \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$, $\mathcal{A}^f(\chi)^\delta = \mathcal{A}^f(\chi^\delta)$.

Equivalently, the element $\mathcal{A}^f$ *of* $\mathrm{Hom}(R(G), \mathbb{C}^\times)$ *lies in the subgroup* $\mathrm{Hom}_{G_\mathbb{Q}}(R(G), (\mathbb{Q}^c)^\times)$.

Remark. The truth, or otherwise, of Conjecture 3.3.1 is shown in [43, Ch.I, Section 7] to be independent of the set S and the choice of $\mathbb{Q}[G]$-module isomorphism f. (Note however that $\mathcal{A}^f$ is not independent of f. It is not independent of S either, but we opt not to reflect the dependence on S in the notation.) Thus the truth of the conjecture is a property solely of the extension.

Stark's conjecture originated in the 1970s, and is known to hold whenever L is an abelian extension of either $\mathbb{Q}$ or an imaginary quadratic field, and is also true for characters taking rational values; see [43].

We point out a minor discrepancy between $\mathcal{A}^f(\chi)$ as defined here and the $A(\chi, f)$ as defined in [43, Conj.5.1 (Ch.I)]. Namely, $\mathcal{A}^f(\chi) = A(\bar{\chi}, f)^{-1}$. This clearly has no effect on the conjecture, and the form we have chosen is more convenient for our purposes.

3.3.1 The abelian case

As hinted at the beginning of Chapter 3, Stark's Conjecture for abelian extensions L/K has an equivalent form which is reminiscent of the rationality statement of the analytic class number formula, i.e. that $\zeta_L^*(0)/R_L \in \mathbb{Q}^\times$. Indeed, define elements $\mathcal{Z}_{L/K,S}$ and $\mathcal{R}^f_{L/K,S}$ of $\mathbb{C}[G]$ (for a choice of $\mathbb{Q}[G]$-module isomorphism f as in (3.2.1)) by

$$\begin{aligned} \mathcal{Z}_{L/K,S} &= \sum_{\chi\in\widehat{G}} L^*_{L/K,S}(0,\bar{\chi})e_\chi \\ \mathcal{R}^f_{L/K,S} &= \sum_{\chi\in\widehat{G}} R^f_{\bar{\chi}}e_\chi. \end{aligned}$$

Since the complex numbers $L^*_{L/K,S}(0,\chi)$ and R^f_χ are non-zero, $\mathcal{Z}_{L/K,S}$ and $\mathcal{R}^f_{L/K,S}$ are elements of $\mathbb{C}[G]^\times$. Further, because of the relations $L_{L/K,S}(0,\bar{\chi}) = \overline{L_{L/K,S}(0,\chi)}$ and $R^f_{\bar{\chi}} = \overline{R^f_\chi}$, $\mathcal{Z}_{L/K,S}$ and $\mathcal{R}^f_{L/K,S}$ in fact lie in $\mathbb{R}[G]^\times$, and we can consider the element

$$\mathcal{Z}_{L/K,S}(\mathcal{R}^f_{L/K,S})^{-1} \in \mathbb{R}[G]^\times.$$

Then in this situation, i.e. L/K abelian, Conjecture 3.3.1 is equivalent to

Conjecture 3.3.2 $\mathcal{Z}_{L/K,S}(\mathcal{R}^f_{L/K,S})^{-1} \in \mathbb{Q}[G]^\times$.

Indeed, $\mathcal{Z}_{L/K,S}(\mathcal{R}^f_{L/K,S})^{-1} = \varphi_G(\mathcal{A}^f)$ by definition, and Conjecture 3.3.1 says that the element $\mathcal{A}^f$ of $\mathrm{Hom}(R(G),\mathbb{C}^\times)$ actually lies in $\mathrm{Hom}_{G_\mathbb{Q}}(R(G),(\mathbb{Q}^c)^\times)$. Now use Lemma 2.1.2.

3.4 Towards integrality

We remind the reader that our aim is to relate leading coefficients of L-functions to class-groups, and so knowing a statement like Conjecture 3.3.2 will not be enough since as it stands it only says (loosely)

L-functions divided by regulators are rational.

It is the word "rational" that is the problem; what we would like is to know that leading coefficients of L-functions divided by regulators are equal to something precise, and this is where Stark elements come into the picture. Initially, these were conjectured by Stark to be S-units with a certain property. However, it was later realised that in general these special elements might live in higher exterior powers of S-units. Therefore, in order to even discuss the existence of these *higher rank* Stark elements, it is necessary to reformulate Stark's Conjecture.

If G is a finite abelian group and M a $\mathbb{Z}[G]$-module, then $\bigwedge^r M$ will mean $\bigwedge^r_{\mathbb{Z}[G]} M$ unless otherwise specified.

Definition 3.4.1 *If G is a finite abelian group, M_1 and M_2 are $\mathbb{Z}[G]$-modules and $\phi : M_1 \otimes_{\mathbb{Z}} \mathbb{C} \to M_2 \otimes_{\mathbb{Z}} \mathbb{C}$ is a homomorphism of $\mathbb{C}[G]$-modules, then for each $r \geq 0$ let $\phi^{(r)} : (\bigwedge^r M_1) \otimes_{\mathbb{Z}} \mathbb{C} \to (\bigwedge^r M_2) \otimes_{\mathbb{Z}} \mathbb{C}$ be the $\mathbb{C}[G]$-module homomorphism obtained by using the natural isomorphisms $(\bigwedge^r M_i) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigwedge^r_{\mathbb{C}[G]} (M_i \otimes_{\mathbb{Z}} \mathbb{C})$. We will use the same notation when $\mathbb{C}$ is replaced by any subring, eg $\mathbb{Q}$ or $\mathbb{R}$.*

For the remainder of the chapter, we fix an abelian extension of number fields L/K with Galois group G, and a finite set of places S of K containing the infinite ones.

Definition 3.4.2 *If $r \geq 0$, define $\theta_{L/K,S}[r] \in \mathbb{C}[G]$ by*

$$\theta_{L/K,S}[r] = \sum_{\substack{\chi \in \widehat{G} \\ r(\chi) = r}} L^*_{L/K,S}(0, \chi) e_{\bar{\chi}}.$$

Remark. It is important to point out here the distinction between the Stickelberger elements $\theta_{L/K,S}(k)$ defined in Chapter 1 for integers $k < 0$, and the elements $\theta_{L/K,S}[r]$ defined above for integers $r \geq 0$. The former are the usual higher Stickelberger elements defined in terms of L-function *values* at negative integers, while the latter are defined in terms of *leading coefficients* of L-functions strictly at the integer $k = 0$. We shall not refer to the $\theta_{L/K,S}(k)$ again.

Observe that $\theta_{L/K,S}[0]$ is the usual Stickelberger element $\theta_{L/K,S}$.

Proposition 3.4.3 *Let $r \geq 0$. Then the Stark Conjecture holds for the rank r characters $\chi \in \widehat{G}$ if and only if*

$$\theta_{L/K,S}[r] \bigwedge^r X \subseteq \mathbb{Q}\lambda^{(r)}(\bigwedge^r \mathcal{O}^\times_{L,S}). \tag{3.4.1}$$

Proof. See [33, Section 2.3]. ■

Given that $\bigwedge^r X$ is finitely generated, we observe from Proposition 3.4.3 that if L/K satisfies Stark's Conjecture, then there is some lattice Ω in $(\bigwedge^r \mathcal{O}_{L,S}^\times)\otimes_{\mathbb{Z}}\mathbb{Q}$ such that $\theta_{L/K,S}[r]\bigwedge^r X \subseteq \Omega$. In [33], Rubin conjectures what that lattice should be, provides evidence for this and explains why some other natural choices for Ω fail.

3.5 Rubin's lattice

As explained in [33, Section 1.2], if M is a $\mathbb{Z}[G]$-module (G an arbitrary finite abelian group), then for each $r \geq 0$ there is a well-defined homomorphism

$$\begin{array}{rcl} \bigwedge^r \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G]) & \to & \mathrm{Hom}_{\mathbb{Z}[G]}(\bigwedge^r M, \mathbb{Z}[G]) \\ \phi_1 \wedge \cdots \wedge \phi_r & \mapsto & (m_1 \wedge \cdots \wedge m_r \mapsto \det(\phi_i(m_j)))\,. \end{array}$$

By abuse of notation, we will denote the image of $\phi_1 \wedge \cdots \wedge \phi_r$ under this map by the same symbol, so that given $m_1, \ldots, m_r \in M$ we write simply

$$(\phi_1 \wedge \cdots \wedge \phi_r)(m_1 \wedge \cdots \wedge m_r) = \det(\phi_i(m_j)).$$

We will also extend the map $(\phi_1 \wedge \cdots \wedge \phi_r)(-)$ linearly to $(\bigwedge^r M) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}[G]$.

Definition 3.5.1 *For any $\mathbb{Z}[G]$-module M and any $r \geq 0$, define $\bigwedge_0^r M$ to be*

$$\{m \in (\textstyle\bigwedge^r M) \otimes_{\mathbb{Z}} \mathbb{Q} \mid (\phi_1 \wedge \cdots \wedge \phi_r)(m) \in \mathbb{Z}[G] \text{ for all } \phi_i \in \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])\}.$$

3.5.1 T-modified objects

Now let us return to our extension L/K. Rather than working in $\mathcal{O}_{L,S}^\times$ itself, we work inside a certain subgroup of finite index. Namely, choose a finite set of places T of K disjoint from S, and define

$$U_{S,T} = \{u \in \mathcal{O}_{L,S}^\times \mid u \equiv 1 \bmod \mathfrak{P} \text{ for all (finite) places } \mathfrak{P} \text{ of } L \text{ above } T\}.$$

To accompany this replacement of $\mathcal{O}_{L,S}^{\times}$ by $U_{S,T}$, we define also a T-modified L-function $L_{L/K,S,T}(s,\chi)$, for a character χ of the Galois group G, as follows:

$$L_{L/K,S,T}(s,\chi) = L_{L/K,S}(s,\chi) \prod_{\mathfrak{p}\in T} \mathrm{char}_{\chi,\mathfrak{p}}(\mathbf{N}\mathfrak{p}^{1-s}).$$

(Recall the definition of the polynomial $\mathrm{char}_{\chi,\mathfrak{p}}$ from Section 2.3.1.)

Definition 3.5.2 *For $r \geq 0$, define*

$$\theta_{L/K,S,T}[r] = \sum_{\substack{\chi\in\widehat{G} \\ r(\chi)=r}} L^*_{L/K,S,T}(0,\chi)e_{\bar{\chi}} \in \mathbb{C}[G].$$

3.5.2 The lattice $\Omega_{S,T,r}$

Definition 3.5.3 *Let $r \geq 0$. Then we define $\Omega_{S,T,r} \subseteq \bigwedge_0^r U_{S,T}$ to be*

$$\Omega_{S,T,r} = \{u \in \textstyle\bigwedge_0^r U_{S,T} \mid e_\chi u = 0 \text{ for all } \chi \in \widehat{G} \text{ with } r(\chi) \neq r\}.$$

3.6 The integral Stark Conjecture and Stark elements

We now come to a vast refinement of Stark's Conjecture, due to Rubin. Some hypotheses on the sets S and T (always assumed disjoint) are needed.

(**St**1) S contains the (infinite and) ramified places.

(**St**2) S contains at least r places which split completely in L/K.

(**St**3) S contains at least $r+1$ places.

(**St**4) $U_{S,T}$ is $\mathbb{Z}$-torsion free.

We note that (**St**4) forces T to be non-empty.

Since $U_{S,T}$ has maximal rank in $\mathcal{O}_{L,S}^{\times}$, the natural map

$$\iota_r : (\textstyle\bigwedge^r U_{S,T}) \otimes_{\mathbb{Z}} \mathbb{Q} \to (\bigwedge^r \mathcal{O}_{L,S}^{\times}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. The following is [33, Conjecture B].

Conjecture 3.6.1 (Rubin) *Choose $r \geq 0$ and assume the hypotheses* $(\mathbf{St}1)$ *to* $(\mathbf{St}4)$ *are satisfied. Then*

$$\theta_{L/K,S,T}[r]{\bigwedge}^r X \subseteq \lambda^{(r)} \circ \iota_r(\Omega_{S,T,r}). \tag{3.6.1}$$

Assuming this conjecture holds,

$$\theta_{L/K,S,T}[r]{\bigwedge}^r X = \lambda^{(r)} \circ \iota_r(\mathcal{E}_{S,T,r}) \tag{3.6.2}$$

for some $\mathbb{Z}[G]$-submodule $\mathcal{E}_{S,T,r}$ of $\Omega_{S,T,r}$.

Definition 3.6.2 *Suppose Conjecture 3.6.1 is satisfied. Then we call $\mathcal{E}_{S,T,r}$ the group of T-modified rank r Stark elements.*

We may wish to consider at once all of the Stark elements as T varies through all the permitted sets. Then it is possible to work inside $\Omega_{S,\emptyset,r}$, which we write simply $\Omega_{S,r}$. (Note that $\Omega_{S,T,r}$ is still defined even if T is empty.)

Definition 3.6.3 *Suppose Conjecture 3.6.1 holds for all T satisfying* $(\mathbf{St}4)$*. Then denote by $\mathcal{E}_{S,r} \subseteq \Omega_{S,r}$ the $\mathbb{Z}[G]$-submodule generated by all the modules $\iota_r(\mathcal{E}_{S,T,r})$.*

3.6.1 Evidence for Conjecture 3.6.1

Conjecture 3.6.1, or a slightly weaker form due to Popescu (see [26, Section 2.3]), has been verified in a number of situations. Rubin's Conjecture itself is verified in the following cases:

(i) $r = 0$

(ii) $K = \mathbb{Q}$, r arbitrary

(iii) K imaginary quadratic, r arbitrary

(iv) L/K any quadratic extension, r arbitrary

(v) S containing more than r places which split completely

(vi) A large class of multi-quadratic extensions when $r = 1$, with a partial result holding for all multi-quadratic extensions ($r = 1$ again).

(i) is an almost direct consequence of Deligne and Ribet [13]. (ii) is proven in [41] when L is real and in [10] when L is imaginary. (iii) can again be found in [41]. (iv) is treated in [33], where Rubin's Conjecture is first formulated, as is (v). For details of (vi), see [14].

Conjecture 3.6.1 also has the following base-change property (see [26, Theorem 2.3.2] and the paragraph immediately after):

> Suppose L/K satisfies Conjecture 3.6.1 with $r = 1$ and for some choices of S and T. Then the quadruple $(L/K', S_{K'}, T_{K'}, [K' : K])$ also satisfies the conjecture for any K' lying between K and L.

This is to say that if the rank 1 Stark elements exist for the triple $(L/K, S, T)$, then the rank $[K' : K]$ Stark elements exist for the triple $(L/K', S_{K'}, T_{K'})$.

We refer the reader to [26] for a formulation of Popescu's Conjecture; we mention here only that it is obtained by replacing the lattice $\Omega_{S,T,r}$ by a larger one. This weaker conjecture, as well as being true in (i)-(vi) above of course, also satisfies a stronger base-change property: Fixing $(L/K, S)$, if the rank 1 Stark elements exist for the triple $(L/K, S, T)$ for all T, then the rank $[K' : K]$ Stark elements exist for the triple $(L/K', S_{K'}, T')$ for all K' between K and L and all T'.

3.6.2 Structure of $e[r]\bigwedge^r X$

We always view the $\mathbb{Z}$-torsion free module $\bigwedge^r X$ as lying inside $(\bigwedge^r X) \otimes_{\mathbb{Z}} \mathbb{C}$. Still assuming hypotheses (**St**1) to (**St**4), [33, Lemma 2.6] provides a generator for $\theta_{L/K,S,T}[r]\bigwedge^r X$. Namely, if $w_1, \ldots, w_r$ are pairwise non-conjugate places in S_L having trivial decomposition group (possible by (**St**2)), and $w \in S_L$ is conjugate to none of the w_i (use (**St**3)), then

$$\mathbb{Z}[G]\theta_{L/K,S}[r]\mathbf{x} = \theta_{L/K,S}[r]\bigwedge^r X$$

where $\mathbf{x} = (w_1 - w) \wedge \cdots \wedge (w_r - w)$. Hence also

$$\mathbb{Z}[G]e[r]\mathbf{x} = e[r]{\bigwedge}^r X. \tag{3.6.3}$$

In fact, we can go further than this and give the relations that $\mathbf{x}$ satisfies. Indeed, one shows that $e[r]({\bigwedge}^r X) \otimes_{\mathbb{Z}} \mathbb{C} \cong e[r]\mathbb{C}[G]$, and hence if $\alpha \in \mathbb{C}[G]$ has $\alpha e[r]\mathbf{x} = 0$, then $\alpha e[r] = 0$.

3.6.3 Finiteness of $\Omega_{S,T,r}/\mathcal{E}_{S,T,r}$

Proposition 3.6.4 *$\mathcal{E}_{S,T,r}$ has finite index in $\Omega_{S,T,r}$, and $\mathcal{E}_{S,r}$ has finite index in $\Omega_{S,r}$.*

Proof. The second statement is immediate from the first. Now, the inclusions

$$|G|e[r]{\bigwedge}_0^r U_{S,T} \subseteq \Omega_{S,T,r} \subseteq e[r]{\bigwedge}_0^r U_{S,T}$$

show that $\mathrm{rk}_{\mathbb{Z}}(\Omega_{S,T,r}) = \mathrm{rk}_{\mathbb{Z}}(e[r]{\bigwedge}_0^r U_{S,T})$. However, the $\mathbb{Q}$-vector space that $e[r]{\bigwedge}_0^r U_{S,T}$ generates is isomorphic to $e[r](({\bigwedge}^r X) \otimes_{\mathbb{Z}} \mathbb{Q})$, so that

$$\mathrm{rk}_{\mathbb{Z}}(\Omega_{S,T,r}) = \dim_{\mathbb{Q}}(e[r](({\bigwedge}^r X) \otimes_{\mathbb{Z}} \mathbb{Q})). \tag{3.6.4}$$

On the other hand, $\mathrm{rk}_{\mathbb{Z}}(\mathcal{E}_{S,T,r}) = \mathrm{rk}_{\mathbb{Z}}(\theta_{L/K,S,T}[r]{\bigwedge}^r X)$, and this in turn is equal to

$$\begin{aligned} \dim_{\mathbb{Q}}(\mathbb{Q}[G]\theta_{L/K,S,T}[r]\mathbf{x}) &= \dim_{\mathbb{Q}}(\mathbb{Q}[G]e[r]\mathbf{x}) \\ &= \dim_{\mathbb{Q}}(e[r](({\bigwedge}^r X) \otimes_{\mathbb{Z}} \mathbb{Q})). \end{aligned}$$

Comparing this with (3.6.4), we have the proposition. ■

Chapter 4

The fractional Galois ideal

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4.1 A second reformulation of Stark's Conjecture

In [40], Snaith defined a fractional Galois ideal $\mathcal{J}^k_{L/K}$ in terms of leading coefficients of L-functions at $s = k$ for each integer $k \leq 0$. $\mathcal{J}^0_{L/K}$ was modified in [4] to allow for the consideration of S-truncated L-functions $L_{L/K,S}(s,\chi)$, where S may contain finite as well as infinite places. This modified version was shown in [4, Theorem 3.6] to be intimately related to the rank 1 Stark elements (the classical Stark units). In order to obtain similar relations with Stark elements of arbitrary rank, it is necessary to refine the definition of the fractional Galois ideal even further. This refinement leaves unchanged the part of the fractional Galois ideal concerned with characters of rank at most 1. In particular, any result in [4] concerning the fractional Galois ideal remains true for the new version. To procede, we will need an equivalent form of the abelian Stark Conjecture (Conjecture 3.3.1), more in the vein of (3.4.1).

4.1.1 The category $\mathbf{W}^{\mathbb{Q}}_{\mathbb{Z}[G]}$

Let G be a finite abelian group. Then we define a category $\mathbf{W}^{\mathbb{Q}}_{\mathbb{Z}[G]}$ as follows.

Objects: Finitely generated $\mathbb{Z}[G]$-modules M.

Morphisms $M_1 \to M_2$ **:** Elements of

$$\bigoplus_{r=0}^{\infty} \mathrm{Hom}_{\mathbb{Q}[G]}((\bigwedge\nolimits^r M_1) \otimes_{\mathbb{Z}} \mathbb{Q}, (\bigwedge\nolimits^r M_2) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

The sets of morphisms, endomorphisms, isomorphisms and automorphisms in $\mathbf{W}^{\mathbb{Q}}_{\mathbb{Z}[G]}$ will be denoted $\mathbf{Hom}^{\mathbb{Q}}_{\mathbb{Z}[G]}(-,-)$, $\mathbf{Isom}^{\mathbb{Q}}_{\mathbb{Z}[G]}(-,-)$, $\mathbf{End}^{\mathbb{Q}}_{\mathbb{Z}[G]}(-)$ and $\mathbf{Aut}^{\mathbb{Q}}_{\mathbb{Z}[G]}(-)$ resp.

4.1.2 The group-ring element $\mathcal{A}^{\mathbf{f}}$

Choose $\mathbf{f} = (f_0, f_1, f_2, \ldots) \in \mathbf{Isom}^{\mathbb{Q}}_{\mathbb{Z}[G]}(\mathcal{O}^{\times}_{L,S}, X)$. Then if $\chi \in \widehat{G}$ has realization V, we define $R^{\mathbf{f}}_{\chi} \in \mathbb{C}^{\times}$ to be the determinant of the $\mathbb{C}$-linear map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{C}[G]}(V^*, (\bigwedge^{r(\chi)} X) \otimes_{\mathbb{Z}} \mathbb{C}) & \to & \mathrm{Hom}_{\mathbb{C}[G]}(V^*, (\bigwedge^{r(\chi)} X) \otimes_{\mathbb{Z}} \mathbb{C}) \\ \phi & \mapsto & \lambda^{(r(\chi))} \circ f^{-1}_{r(\chi)} \circ \phi. \end{array}$$

Definition 4.1.1 *Having chosen* $\mathbf{f} \in \mathbf{Isom}^{\mathbb{Q}}_{\mathbb{Z}[G]}(\mathcal{O}^{\times}_{L,S}, X)$*, define* $\mathcal{A}^{\mathbf{f}} \in \mathbb{C}[G]^{\times}$ *by*

$$\mathcal{A}^{\mathbf{f}} = \sum_{\chi \in \widehat{G}} \frac{L^*_{L/K,S}(0,\chi)}{R^{\mathbf{f}}_{\chi}} e_{\bar{\chi}}.$$

Also, given $r \geq 0$*, define* $\mathcal{A}^{\mathbf{f}}[r] \in \mathbb{C}[G]$ *by*

$$\mathcal{A}^{\mathbf{f}}[r] = \sum_{\substack{\chi \in \widehat{G} \\ r(\chi)=r}} \frac{L^*_{L/K,S}(0,\chi)}{R^{\mathbf{f}}_{\chi}} e_{\bar{\chi}}.$$

Observe that $\mathcal{A}^{\mathbf{f}}$ in fact lies in $\mathbb{R}[G]^\times$. Note also that $\mathcal{A}^{\mathbf{f}} = \sum_{r=0}^{\infty} \mathcal{A}^{\mathbf{f}}[r]$ and $\mathcal{A}^{\mathbf{f}}[r] = e[r]\mathcal{A}^{\mathbf{f}}$.

Now, given an isomorphism $f : \mathcal{O}_{L,S}^\times \otimes_{\mathbb{Z}} \mathbb{Q} \to X \otimes_{\mathbb{Z}} \mathbb{Q}$, we can define $\bar{f} \in \mathbf{Isom}_{\mathbb{Z}[G]}^{\mathbb{Q}}(\mathcal{O}_{L,S}^\times, X)$ by taking exterior powers of f. To be explicit, $\bar{f} = (f^{(0)}, f^{(1)}, f^{(2)}, \ldots)$, remembering Definition 3.4.1. Recall that when we were setting up the formulation of Stark's Conjecture in Section 3.2 we defined R_χ^f and $\mathcal{A}^f$ (the latter of which we view as an element in $\mathbb{R}[G]^\times$), so it is natural to wonder how R_χ^f and $R_\chi^{\bar{f}}$ are related, and similarly $\mathcal{A}^f$ and $\mathcal{A}^{\bar{f}}$. In fact,

Lemma 4.1.2 *$R_\chi^f = R_\chi^{\bar{f}}$, and $\mathcal{A}^f = \mathcal{A}^{\bar{f}}$.*

Proof. The second equality follows from the first, which we prove as follows. Suppose $\chi \in \widehat{G}$ has rank r, and observe that $\lambda^{(r)} \circ (f^{(r)})^{-1} = (\lambda \circ f^{-1})^{(r)}$. Then

$$\begin{aligned} R_\chi^f &= \bar{\chi}(\det_{\mathbb{C}[G]}(\lambda \circ f^{-1})) \\ &= \bar{\chi}(\det_{\mathbb{C}[G]}(\lambda \circ f^{-1}|_{e[r]})) \qquad \text{(Lemma 2.1.5)} \\ &= \bar{\chi}(\det_{\mathbb{C}[G]}(\alpha)) \end{aligned}$$

where α extends $\lambda \circ f^{-1}|_{e[r]}$ to a $\mathbb{C}[G]$-module isomorphic to $\mathbb{C}[G]^r$. Then this is equal to

$$\begin{aligned} \bar{\chi}(\det_{\mathbb{C}[G]}(\alpha^{(r)}) &= \bar{\chi}(\det_{\mathbb{C}[G]}(\alpha^{(r)}|_{e[r]})) \\ &= \bar{\chi}(\det_{\mathbb{C}[G]}((\alpha|_{e[r]})^{(r)}) \\ &= \bar{\chi}(\det_{\mathbb{C}[G]}((\lambda \circ f^{-1})^{(r)}) \\ &= R_\chi^{\bar{f}}. \end{aligned}$$

■

Lemma 4.1.3 *Choose $\mathbf{f} = (f_0, f_1, f_2, \ldots) \in \mathbf{Isom}_{\mathbb{Z}[G]}^{\mathbb{Q}}(\mathcal{O}_{L,S}^\times, X)$ and suppose $\mathbf{h} = (h_0, h_1, h_2, \ldots) \in \mathbf{Aut}_{\mathbb{Z}[G]}^{\mathbb{Q}}(X)$. Then for each $r \geq 0$,*

$$\mathcal{A}^{\mathbf{h}\circ\mathbf{f}}[r] = \mathcal{A}^{\mathbf{f}}[r]\det_{\mathbb{Q}[G]}(h_r).$$

Proof. We use the following useful shorthand: If A is an abelian group and R a subring of $\mathbb{C}$, write RA for $A \otimes_{\mathbb{Z}} R$. Also, if V and W are finitely generated $\mathbb{C}[G]$-modules and h an endomorphism of W, denote by $h(V, W)$ the endomorphism of $\mathrm{Hom}_{\mathbb{C}[G]}(V, W)$ given by sending a homomorphism ψ to $h \circ \psi$.

Choose realizations V_χ for the characters $\chi \in \widehat{G}$. Then

$$R_\chi^{\mathbf{h}\circ\mathbf{f}} = R_\chi^{\mathbf{f}} \det_{\mathbb{C}}(h_r(V_{\bar{\chi}}, \mathbb{C}{\textstyle\bigwedge}^r X))^{-1}. \tag{4.1.1}$$

However, by Proposition 2.1.3,

$$\sum_{\substack{\chi \in \widehat{G} \\ r(\chi)=r}} \det_{\mathbb{C}}(h_r(V_\chi, \mathbb{C}{\textstyle\bigwedge}^r X))e_\chi = e[r]\det_{\mathbb{Q}[G]}(h_r).$$

Combining this with (4.1.1) finishes the proof. ∎

We see therefore that if $\mathbf{f} \in \mathbf{Isom}_{\mathbb{Z}[G]}^{\mathbb{Q}}(\mathcal{O}_{L,S}^\times, X)$ and $\mathbf{h} = (h_0, h_1, h_2, \ldots) \in \mathbf{Aut}_{\mathbb{Z}[G]}^{\mathbb{Q}}(X)$, then

$$\mathcal{A}^{\mathbf{h}\circ\mathbf{f}} = \mathcal{A}^{\mathbf{f}} \sum_{r=0}^{\infty} \det_{\mathbb{Q}[G]}(h_r)e[r]. \tag{4.1.2}$$

We are now ready to reformulate Stark's Conjecture in a more suitable form. We continue to assume that L/K is abelian.

Proposition 4.1.4 *Choose any $\mathbf{f} \in \mathbf{Isom}_{\mathbb{Z}[G]}^{\mathbb{Q}}(\mathcal{O}_{L,S}^\times, X)$. For any $r \geq 0$, Stark's Conjecture holds for all the rank r characters in $\widehat{G}$ if and only $\mathcal{A}^{\mathbf{f}}[r] \in \mathbb{Q}[G]$. The extension L/K satisfies Stark's Conjecture if and only if $\mathcal{A}^{\mathbf{f}} \in \mathbb{Q}[G]$.*

Proof. The second statement follows from the first being true for all r. Since, by Lemma 4.1.3, the rationality of $\mathcal{A}^{\mathbf{f}}[r]$ is independent of the choice of $\mathbf{f}$, we may assume that $\mathbf{f} = \bar{f}$ for some isomorphism $f : \mathcal{O}_{L,S}^\times \otimes_{\mathbb{Z}} \mathbb{Q} \to X \otimes_{\mathbb{Z}} \mathbb{Q}$. Now use Lemma 4.1.2. ∎

4.2 Definition of $\mathcal{J}(L/K, S)$

We have now prepared almost all that is necessary to define our fractional ideal $\mathcal{J}(L/K, S)$ for abelian extensions L/K satisfying Stark's Conjecture. We already have half of the definition, namely the invertible group-ring element $\mathcal{A}^{\mathbf{f}}$.

Definition 4.2.1 *Take* $\mathbf{f} \in \mathbf{Isom}^{\mathbb{Q}}_{\mathbb{Z}[G]}(\mathcal{O}^{\times}_{L,S}, X)$*. Then define* $\mathcal{I}^{\mathbf{f}}$ *to be the* $\mathbb{Z}[G]$*-submodule of* $\mathbb{Q}[G]$*-generated by*

$$\left\{ \sum_{r=0}^{\infty} \det_{\mathbb{Q}[G]}(\alpha_r) e[r] \mid \alpha_r \in \operatorname{End}_{\mathbb{Q}[G]}\left((\textstyle\bigwedge^r X) \otimes_{\mathbb{Z}} \mathbb{Q} \right), \right.$$
$$\left. \alpha_r \circ f_r \left(\textstyle\bigwedge_0^r \mathcal{O}^{\times}_{L,S} \right) \subseteq \textstyle\bigwedge^r X \text{ for all } r \right\}.$$

If $\mathbf{h} = (h_0, h_1, h_2, \ldots) \in \mathbf{Aut}^{\mathbb{Q}}_{\mathbb{Z}[G]}(X)$, then it is immediate that

$$\mathcal{I}^{\mathbf{h} \circ \mathbf{f}} = \left(\sum_{r=0}^{\infty} \det_{\mathbb{Q}[G]}(h_r)^{-1} e[r] \right) \mathcal{I}^{\mathbf{f}}.$$

Combining this with (4.1.2) and using Proposition 4.1.4, we see that

Proposition 4.2.2 *(i) The* $\mathbb{Z}[G]$*-submodule* $\mathcal{A}^{\mathbf{f}}\mathcal{I}^{\mathbf{f}}$ *of* $\mathbb{R}[G]$ *is independent of the choice of* $\mathbf{f} \in \mathbf{Isom}^{\mathbb{Q}}_{\mathbb{Z}[G]}(\mathcal{O}^{\times}_{L,S}, X)$*. (ii)* $\mathcal{A}^{\mathbf{f}}\mathcal{I}^{\mathbf{f}}$ *lies in* $\mathbb{Q}[G]$ *if and only if* L/K *satisfies the Stark Conjecture.*

This allows us to make the definition:

Definition 4.2.3 *Assume* L/K *satisfies the Stark Conjecture. Then define* $\mathcal{J}(L/K, S) = \mathcal{A}^{\mathbf{f}}\mathcal{I}^{\mathbf{f}}$ *for any* $\mathbf{f} \in \mathbf{Isom}^{\mathbb{Q}}_{\mathbb{Z}[G]}(\mathcal{O}^{\times}_{L,S}, X)$*. Being a finitely generated* $\mathbb{Z}[G]$*-submodule of* $\mathbb{Q}[G]$*, i.e. a fractional ideal in* $\mathbb{Q}[G]$*,* $\mathcal{J}(L/K, S)$ *will be called the fractional Galois ideal associated to the pair* $(L/K, S)$*.*

We note an almost immediate observation concerning $\mathcal{J}(L/K, S)$, namely its decomposition according to the rank idempotents $e[r]$.

Proposition 4.2.4 *For each* $r \geq 0$*,* $e[r]\mathcal{J}(L/K, S) \subseteq \mathcal{J}(L/K, S)$*.*

Proof. For any finitely generated $\mathbb{C}[G]$-module M and any $t \geq 0$, $e_M[t]\bigwedge^t_{\mathbb{C}[G]} M \cong e_M[t]\mathbb{C}[G]$, so that the zero endomorphism of $e_M[t]\bigwedge^t_{\mathbb{C}[G]} M$ has determinant $1 - e_M[t]$ over $\mathbb{C}[G]$. In particular, if, for some $t \geq 0$, β is the zero endomorphism of $(\bigwedge^t X) \otimes_{\mathbb{Z}} \mathbb{Q}$ (which certainly satisfies the integrality condition in the definition of $\mathcal{I}^{\mathbf{f}}$), then $\det_{\mathbb{Q}[G]}(\beta)e[t] = (1 - e[t])e[t] = 0$.

So, take maps α_t as in the definition of $\mathcal{I}^{\mathbf{f}}$, and let β_t be α_r if $t = r$ and the zero endomorphism of $(\bigwedge^t X) \otimes_{\mathbb{Z}} \mathbb{Q}$ otherwise. It is now clear from the above that

$$e[r] \sum_{t=0}^{\infty} \det_{\mathbb{Q}[G]}(\alpha_t)e[t] = \sum_{t=0}^{\infty} \det_{\mathbb{Q}[G]}(\beta_t)e[t],$$

which is again in $\mathcal{I}^{\mathbf{f}}$. ■

As a consequence of Proposition 4.2.4, $\mathcal{J}(L/K, S)$ decomposes as

$$\mathcal{J}(L/K, S) = \bigoplus_{r=0}^{\infty} e[r]\mathcal{J}(L/K, S). \tag{4.2.1}$$

We will turn in Section 4.3 to a general relationship of each of the direct summands $e[r]\mathcal{J}(L/K, S)$ with the Stark elements of Section 3.6. First, however, let us make a straightforward observation concerning $e[0]\mathcal{J}(L/K, S)$ and the Stickelberger element $\theta_{L/K,S}$.

Proposition 4.2.5 $e[0]\mathcal{J}(L/K, S) = \mathbb{Z}[G]\theta_{L/K,S}$.

Proof. Choose $\mathbf{f} = (f_0, f_1, f_2, \ldots) \in \mathbf{Isom}^{\mathbb{Q}}_{\mathbb{Z}[G]}(\mathcal{O}^{\times}_{L,S}, X)$ such that f_0 is the identity map $\mathbb{Q}[G] \to \mathbb{Q}[G]$. Then with f_0 chosen this way, $R^{\mathbf{f}}_{\chi} = 1$ for all χ of rank 0 and so $\mathcal{A}^{\mathbf{f}}[0] = \theta_{L/K,S}$. Bearing in mind the equality $\theta_{L/K,S} = \theta_{L/K,S}e[0]$, it remains to show that $e[0]\mathcal{I}^{\mathbf{f}} = e[0]\mathbb{Z}[G]$.

In the definition of $\mathcal{I}^{\mathbf{f}}$, the integrality condition on an endomorphism α_0 of $(\bigwedge^0 X) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[G]$ now becomes that $\alpha_0(\mathbb{Z}[G]) \subseteq \mathbb{Z}[G]$, because of our choice of f_0. Identifying endomorphisms of $\mathbb{Q}[G]$ with elements of $\mathbb{Q}[G]$, we see then that $e[0]\mathcal{I}^{\mathbf{f}}$ is generated by elements $e[0]\alpha$ where $\alpha \in \mathbb{Q}[G]$ and satisfies $\alpha\mathbb{Z}[G] \subseteq \mathbb{Z}[G]$. This simply says that $e[0]\mathcal{I}^{\mathbf{f}} = e[0]\mathbb{Z}[G]$. ■

4.3 $\mathcal{J}(L/K, S)$ and Stark elements

The fractional Galois ideal $\mathcal{J}(L/K, S)$ is related to the Stark elements via annihilators of $\mu(L)$.

Definition 4.3.1 *Let T be a finite set of places of K not containing any ramifying in L/K. Then define*

$$D_{L/K,T} = \prod_{\mathfrak{p}\in T}(1 - \mathbf{N}\mathfrak{p}\mathrm{Frob}_{\mathfrak{p}}^{-1}) \in \mathbb{Z}[G].$$

The next lemma is proven in [10].

Lemma 4.3.2 *The elements $D_{L/K,T}$, as T runs through all sets satisfying* (**St**4)*, generate* $\mathrm{ann}_{\mathbb{Z}[G]}(\mu(L))$ *over* $\mathbb{Z}[G]$.

The main result of this chapter is the following:

Theorem 4.3.3 *Assume L/K satisfies the Stark Conjecture, and let S be a finite set of places of K containing the infinite ones. Fix $r \geq 0$ and T a finite set of places of K disjoint from S and such that the hypotheses* (**St**1) *to* (**St**4) *hold. Then assuming the rank r Stark elements $\mathcal{E}_{S,T,r}$ of Definition 3.6.2 exist,*

$$e[r]D_{L/K,T}\mathcal{J}(L/K, S) \subseteq e[r]\mathrm{ann}_{\mathbb{Z}[G]}(\Omega_{S,T,r}/\mathcal{E}_{S,T,r}). \qquad (4.3.1)$$

If in fact $\mathcal{E}_{S,T,r}$ exists for all T satisfying (**St**4)*, then*

$$e[r]\mathrm{ann}_{\mathbb{Z}[G]}(\mu(L))\mathcal{J}(L/K, S) \subseteq e[r]\mathrm{ann}_{\mathbb{Z}[G]}(\Omega_{S,r}/\mathcal{E}_{S,r}). \qquad (4.3.2)$$

Proof. Granted Lemma 4.3.2, the inclusion (4.3.2) is proven in exactly the same way as (4.3.1), which we turn to now. Choose $\mathbf{x}$ as in Section 3.6.2, so that $e[r]\mathbf{x}$ generates $e[r]\bigwedge^r X$ over $\mathbb{Z}[G]$. Let $\epsilon \in \Omega_{S,T,r}$ be the unique element such that $\lambda^{(r)} \circ \iota_r(\epsilon) = \theta_{L/K,S,T}[r]\mathbf{x}$ (recall (3.6.1)). Then ϵ generates $\mathcal{E}_{S,T,r}$ over $\mathbb{Z}[G]$, and $\iota_r(\epsilon)$ generates $e[r](\bigwedge^r \mathcal{O}_{L,S}^{\times}) \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}[G]$ and satisfies the same relation as that by $\mathbf{x}$ described in Section 3.6.2. Therefore there is a unique $\mathbb{Q}[G]$-module homomorphism

$$e[r](\bigwedge^r \mathcal{O}_{L,S}^\times) \otimes_{\mathbb{Z}} \mathbb{Q} \to e[r](\bigwedge^r X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

sending $\iota_r(\epsilon)$ to $e[r]\mathbf{x}$, and it is necessarily an isomorphism. We extend this arbitrarily to an isomorphism

$$f_r : (\bigwedge^r \mathcal{O}_{L,S}^\times) \otimes_{\mathbb{Z}} \mathbb{Q} \to (\bigwedge^r X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Choosing arbitrary isomorphisms $f_t : (\bigwedge^t \mathcal{O}_{L,S}^\times) \otimes_{\mathbb{Z}} \mathbb{Q} \to (\bigwedge^t X) \otimes_{\mathbb{Z}} \mathbb{Q}$ for $t \neq r$, let $\mathbf{f} = (f_0, f_1, f_2, \ldots)$ be the resulting element of $\mathbf{Isom}_{\mathbb{Z}[G]}^{\mathbb{Q}}(\mathcal{O}_{L,S}^\times, X)$.

Now, if $\chi \in \widehat{G}$ has rank r, then $\lambda^{(r)} \circ f_r^{-1}|_{e_\chi}$ is multiplication by $R_{\bar{\chi}}^{\mathbf{f}}$, so

$$\begin{aligned} R_{\bar{\chi}}^{\mathbf{f}} e_\chi \mathbf{x} &= \lambda^{(r)} \circ f_r^{-1}(e_\chi \mathbf{x}) \\ &= e_\chi \lambda^{(r)} \circ \iota_r(\epsilon) \\ &= L_{L/K,S,T}^*(0, \bar{\chi}) e_\chi \mathbf{x}. \end{aligned}$$

Therefore $R_{\bar{\chi}}^{\mathbf{f}} e_\chi = L_{L/K,S,T}^*(0, \bar{\chi}) e_\chi$. Hence, for all rank r characters $\chi \in \widehat{G}$, $R_\chi^{\mathbf{f}} = L_{L/K,S,T}^*(0, \chi)$ and so

$$e[r] D_{L/K,T} \mathcal{A}^{\mathbf{f}} = e[r].$$

It remains to show that $e[r]\mathcal{I}^{\mathbf{f}}$ is contained in the right-hand side of (4.3.1). So, for each $t \geq 0$, take $\alpha_t \in \mathrm{End}_{\mathbb{Q}[G]}((\bigwedge^t X) \otimes_{\mathbb{Z}} \mathbb{Q})$ such that

$$\alpha_t \circ f_t(\bigwedge_0^t \mathcal{O}_{L,S}^\times) \subseteq \bigwedge^t X.$$

Of course, we only need to consider α_r. Now, $\alpha_r|_{e[r]}$ is multiplication by γ for some $\gamma \in \mathbb{Q}[G]$, and $e[r]\mathrm{det}_{\mathbb{Q}[G]}(\alpha_r) = e[r]\gamma$. Therefore if $u \in \Omega_{S,T,r}$,

$$\begin{aligned} f_r \circ \iota_r(\mathrm{det}_{\mathbb{Q}[G]}(\alpha_r) u) &= f_r(\iota_r(\gamma u)) \\ &= \gamma f_r(\iota_r(u)) \\ &= \alpha_r \circ f_r(\iota_r(u)) \\ &\in e[r] \bigwedge^r X \\ &= f_r \circ \iota_r(\mathcal{E}_{S,T,r}), \end{aligned}$$

so $\det_{\mathbb{Q}[G]}(\alpha_r)u \in \mathcal{E}_{S,T,r}$. This shows that

$$e[r]\mathcal{I}^{\mathbf{f}} \subseteq \{e[r]\beta \mid \beta \in \mathbb{Q}[G], \beta\Omega_{S,T,r} \subseteq \mathcal{E}_{S,T,r}\}.$$

Suppose, then, that we have a $\beta \in \mathbb{Q}[G]$ with $\beta\Omega_{S,T,r} \subseteq \mathcal{E}_{S,T,r}$. In particular, $\beta\epsilon \in \mathcal{E}_{S,T,r}$ and so is equal to $\beta'\epsilon$ for some $\beta' \in \mathbb{Z}[G]$. But then $e[r]\beta = e[r]\beta'$ (we can use Section 3.6.2 here), and $\beta'\Omega_{S,T,r} \subseteq \mathcal{E}_{S,T,r}$. Therefore

$$\begin{aligned}\{e[r]\beta \mid \beta \in \mathbb{Q}[G], \beta\Omega_{S,T,r} \subseteq \mathcal{E}_{S,T,r}\} &= \{e[r]\beta \mid \beta \in \mathbb{Z}[G], \beta\Omega_{S,T,r} \subseteq \mathcal{E}_{S,T,r}\} \\ &= e[r]\mathrm{ann}_{\mathbb{Z}[G]}(\Omega_{S,T,r}/\mathcal{E}_{S,T,r}).\end{aligned}$$

This completes the proof of the theorem. ■

To motivate Theorem 4.3.3, we reproduce a theorem of Büyükboduk on the relationship between Stark elements and class-groups. It is the main theorem of [9] and builds on ideas of Rubin [32] and Popescu [26].

Theorem 4.3.4 (Büyükboduk) *Let K be a totally real number field and $\chi : \mathrm{Gal}(\bar{K}/K) \to \mathbb{Z}_p^\times$ an even character of order prime to p which is unramified at all primes above p. Then under some further technical assumptions and assuming the integral Stark Conjecture (Conjecture 3.6.1) holds,*

$$|\mathrm{Cl}(L)^\chi| = |{\textstyle\bigwedge}^r(\mathcal{O}_L^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\chi : \mathbb{Z}_p\epsilon_L^\chi|$$

where L is the fixed field in $\bar{K}$ of $\mathrm{Ker}(\chi)$ and ϵ_L^χ is a Stark element multiplied by the idempotent for χ. (See [9] for details.)

We also cite [6, Theorem 5.5], which, although a statement whose precise formulation would require more terminology than we can introduce here, is again in the vein of relating annihilators of an exterior power of units modulo special elements in the exterior power to class-groups, although in a more general setting.

4.4 The rank 1 case

The most widely studied Stark elements are those arising for characters of rank $r = 1$, in which case they are genuine S-units rather than elements in an exterior power. Consequently, they are called *Stark units*, the prefix S- habitually being dropped. In this section, we give a more tidy formulation of Theorem 4.3.3 in the case when $r = 1$; this is Proposition 4.4.6. In fact, it will be more convenient to use the definition of Stark units found in, for example, [43, Ch.IV] and [14].

4.4.1 The conjecture $\mathrm{St}(L/K, S)$

We define two $\mathbb{Z}[G]$-submodules of $\mathcal{O}_{L,S}^{\times}$ which the Stark units are going to lie in. Let $\mu = |\mu(L)|$ denote the number of roots of unity in L.

Definition 4.4.1 $U_{L/K}^{\mathrm{ab}} = \{u \in \mathcal{O}_{L,S}^{\times} \mid L(u^{1/\mu})/K \text{ is abelian}\}$.

Definition 4.4.2 *Suppose $v \in S$ splits completely in L/K. We define $U^{(v)}$ in two cases, namely:*

(a) $\#S \geq 3$: $U^{(v)} = \{u \in \mathcal{O}_{L,S}^{\times} \mid \|u\|_{w'} = 1 \text{ for all } w' \nmid v\}$.

(b) $\#S = 2$:

$$U^{(v)} = \{u \in \mathcal{O}_{L,S}^{\times} \mid \|u\|_{w'} = \|u\|_{w''} \text{ for all } w', w'' \in S_L \setminus \{w|v\}\}.$$

For a place w of L in S_L having trivial decomposition group in G, the following conjecture for the triple $(L/K, S, w)$ will be referred to as $\mathrm{St}(L/K, S, w)$. (It can be found in [43, Ch.IV].) It is shown in [33] to be equivalent to the $r = 1$ case of Conjecture 3.6.1 as T runs through all sets satisfying (**St**4).

Conjecture 4.4.3 *Assume* (**St**1), (**St**2) *and* (**St**3) *hold for $r = 1$, and let v be the place of k below w. Then there is $\epsilon \in U_{L/K}^{\mathrm{ab}} \cap U^{(v)}$ such that*

$$\log \|\epsilon\|_{\sigma w} = -\mu \zeta'_{L/K,S}(0, \sigma^{-1}) \text{ for all } \sigma \in G, \tag{4.4.1}$$

i.e.

$$L'_{L/K,S}(0,\chi) = -\frac{1}{\mu}\sum_{\sigma\in G}\bar{\chi}(\sigma)\log\|\epsilon\|_{\sigma w} \textit{ for all } \chi\in\widehat{G}, \tag{4.4.2}$$

i.e.

$$\sum_{\chi\in\widehat{G}} L'_{L/K,S}(0,\chi)e_{\bar{\chi}} = -\frac{1}{\mu}\sum_{\sigma\in G}\log\|\epsilon\|_{\sigma w}\sigma. \tag{4.4.3}$$

Remark. The equivalence of (4.4.1), (4.4.2) and (4.4.3) is nothing more than row and column orthogonality of the character table of G.

If an ϵ satisfying $\mathrm{St}(L/K,S,w)$ exists then it is necessarily unique up to a root of unity in L. We therefore see that the triple $(L/K,S,w)$ defines a class in $\mathcal{O}_{L,S}^{\times}/\mu(L)$, and in fact any element in this class satisfies the conjecture.

Definition 4.4.4 *Suppose we have ϵ satisfying* $\mathrm{St}(L/K,S,w)$*. Then the class $\epsilon\mu(L)$ will be denoted $\bar{\epsilon}(L/K,S,w)$, and its elements will be called the* Stark units *attached to w.*

The conjectures $\mathrm{St}(L/K,S,w)$, as w runs through all places in S_L having trivial decomposition group, are equivalent, and we call them all just $\mathrm{St}(L/K,S)$.

Stark units

Assume $\mathrm{St}(L/K,S)$ holds and let w be a place of L lying above S and having trivial decomposition group in G. Then the $\mathbb{Z}[G]$-submodule of $\mathcal{O}_{L,S}^{\times}$ generated by $\mu(L)$ and a Stark unit for w is independent of the choice of w, and we denote it $\mathcal{E} = \mathcal{E}_{L/K,S}$. It will be called the *group of Stark units* for the pair $(L/K,S)$.

Remark. The term *Stark unit* will be reserved for an element ϵ satisfying $\mathrm{St}(L/K,S,w)$, rather than for any element of the group of Stark units. The author hopes that this will not cause any confusion.

4.4.2 The assumption on $(L/K, S)$

We here discuss the assumption to be made on the pair $(L/K, S)$ (extra to the hypotheses (**St**1), (**St**2) and (**St**3) which are required for the formulation of $\mathrm{St}(L/K, S)$) in order to state and prove Proposition 4.4.6.

Proposition 4.4.5 *Assume* (**St**1), (**St**2) *and* (**St**3) *hold, with v splitting completely in L/K and $v' \in S \smallsetminus \{v\}$, and let $w|v$ and $w'|v'$. Suppose* $\mathrm{St}(L/K, S)$ *is true and let $\epsilon \in \bar{\epsilon}(L/K, S, w)$. Then the following are equivalent:*

(i) ϵ *generates* $\mathcal{O}_{L,S}^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}$ *freely over* $\mathbb{Q}[G]$.

(ii) $\mathcal{O}_{L,S}^{\times} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[G]$.

(iii) $w' - w$ *generates X freely over* $\mathbb{Z}[G]$.

(iv) $X \cong \mathbb{Z}[G]$.

(v) $r(\chi) = 1$ *for all* $\chi \in \widehat{G}$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (v) and (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are immediate from Lemma 2.3.4. To finish the proof, we show (v) $\Rightarrow$ (i) and (v) $\Rightarrow$ (iii). We do these simultaneously.

Assuming (v), we know first of all (from Lemma 2.3.4) that S has two elements, i.e. $S = \{v, v'\}$. But we also know from Lemma 2.3.4 that $X \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}[G]$ and so

$$\begin{aligned} |G| &= \mathrm{rk}(X) \\ &= \#\{\text{places above } v\} + \#\{\text{places above } v'\} - 1 \\ &= |G| + \#\{\text{places above } v'\} - 1, \end{aligned}$$

whence w' is the unique place of L above v'. Thus we see already that $w' - w$ necessarily generates X freely over $\mathbb{Z}[G]$, i.e. (iii) holds.

To continue, by definition of the regulator map λ,

$$\lambda(\epsilon) = \sum_{\sigma \in G} \log \|\epsilon\|_{\sigma w} \sigma w + \log \|\epsilon\|_{w'} w'.$$

Referring back to the statement of $\mathrm{St}(L/K,S,w)$ and noting that $L'_{L/K,S}(0,\chi) = L^*_{L/K,S}(0,\chi)$ for all $\chi \in \widehat{G}$, we see then that

$$\lambda(\epsilon) = -\mu \sum_{\chi \in \widehat{G}} L^*_{L/K,S}(0,\chi) e_{\bar{\chi}} w + \log \|\epsilon\|_{w'} w',$$

and so for $\sigma \in G$,

$$\lambda(\epsilon^\sigma) = -\mu \sum_{\chi \in \widehat{G}} \bar{\chi}(\sigma) L^*_{L/K,S}(0,\chi) e_{\bar{\chi}} w + \log \|\epsilon\|_{w'} w'. \tag{4.4.4}$$

Now, suppose we have $a_\sigma \in \mathbb{Q}$ for each $\sigma \in G$ such that $\sum_{\sigma \in G} \epsilon^\sigma \otimes a_\sigma = 0$ in $\mathcal{O}^\times_{L,S} \otimes_{\mathbb{Z}} \mathbb{Q}$. Applying λ to both sides, we find using (4.4.4) that $\sum_{\sigma \in G} a_\sigma \chi(\sigma) = 0$ for all $\chi \in \widehat{G}$, i.e. $a_\sigma = 0$ for all $\sigma \in G$. Since $\mathrm{rk}(\mathcal{O}^\times_{L,S}) = \mathrm{rk}(X) = |G|$, (i) holds. ■

Now, the assumption we make on $(L/K,S)$ is:

Assumption. (**St**1), (**St**2) and (**St**3) hold for S, and $r(\chi) = 1$ for all $\chi \in \widehat{G}$.

In particular $r(1) = 1$ so that $\#S = 2$, and so by [43, Ch.IV, Prop.3.10], $\mathrm{St}(L/K,S)$ holds automatically. Hence by Proposition 4.4.5, $\mathcal{O}^\times_{L,S} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $X \otimes_{\mathbb{Z}} \mathbb{Q}$ are rank 1 free $\mathbb{Q}[G]$-modules and we have natural choices for free generators.

4.4.3 Examples

In looking for examples of pairs $(L/K,S)$ satisfying the above assumption, it is perhaps more convenient to use its following form: S contains the infinite and ramified places and equals $\{v, v'\}$ where v splits completely and v' is non-split. We have the following examples: (We point out that if $(L/K,S)$ satisfies the assumption, then so does $(E/K,S)$ for any subextension E/K.)

(i) p an odd prime, $K = \mathbb{Q}$, $L = \mathbb{Q}(\zeta_{p^n})^+$, $S = \{\infty, p\}$.

(ii) p an odd prime, $K = \mathbb{Q}$, $L/\mathbb{Q}$ any finite subextension of the $\mathbb{Z}_p$-extension of $\mathbb{Q}$, $S = \{\infty, p\}$.

(iii) $p \equiv 3 \bmod 4$, prime, $K = \mathbb{Q}(\sqrt{-p})$, $L = \mathbb{Q}(\zeta_{p^r})$, $S = \{v, \mathfrak{p}\}$ where v is the infinite place of k and $\mathfrak{p}$ the unique place above p.

(iv) K any imaginary quadratic field, p a prime which is non-split in $K/\mathbb{Q}$, $L = EK$ where $E/\mathbb{Q}$ is a totally real abelian extension of p-power conductor such that p remains non-split in EF (eg if $[E : \mathbb{Q}]$ is odd), S consists of the infinite place of K and the unique place above p.

Examples of (iv) can be obtained by taking a quadratic imaginary field K and letting L be any finite subextension of the cyclotomic $\mathbb{Z}_p$-extension of K, where p is odd and non-split in K.

4.4.4 Description of $\mathcal{J}(L/K, S)$, rank 1 case

We emphasize that we proceed under the assumption (discussed in Section 4.4.2) that S satisfies (**St**1), (**St**2) and (**St**3) and $r(\chi) = 1$ for all $\chi \in \widehat{G}$. As mentioned, $\mathrm{St}(L/K, S)$ holds in this case and we choose v, v', w, w', ϵ as in Proposition 4.4.5.

Proposition 4.4.6 *Let $\mathcal{E}$ be the group of Stark units attached to $(L/K, S)$, i.e. the $\mathbb{Z}[G]$-submodule of $\mathcal{O}_{L,S}^\times$ generated by ϵ and the roots of unity in L. Then*

$$\mathcal{J}(L/K, S) = \frac{1}{\mu} \mathrm{ann}_{\mathbb{Z}[G]}(\mathcal{O}_{L,S}^\times / \mathcal{E}).$$

The proof of this proposition uses exactly the same argument as that of Theorem 4.3.3, and we will not reproduce it here. We only point out that in this situation we can go further by making the containment an equality.

Chapter 5

Cyclotomic examples

This chapter serves (i) to provide explicit examples of the fractional Galois ideal $\mathcal{J}(L/K, S)$ in situations which are tangible and well-known, and (ii) to illustrate by means of these examples the behaviour of $\mathcal{J}(L/K, S)$ under certain canonical changes of extension.

We fix for the whole section the following notation: p is an odd prime, n a positive integer, ζ a primitive p^nth root of unity in a fixed algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ and $L = \mathbb{Q}(\zeta)$. L^+ will denote the maximal totally real subfield of L, and we have the Galois groups $G = \mathrm{Gal}(L/\mathbb{Q})$ and $G^+ = \mathrm{Gal}(L^+/\mathbb{Q})$. S will be the set $\{\infty, p\}$ of places of $\mathbb{Q}$. We remark that the algebraic closure $\bar{\mathbb{Q}}$ should be thought of as being distinct from the algebraic closure $\mathbb{Q}^c$ of $\mathbb{Q}$ in $\mathbb{C}$ (as in

Section 2.1) in which characters take their values.

In all of the examples we will consider, the ranks of the irreducible characters of the Galois group are at most 1. This allows us to use the rank 1 version of Theorem 4.3.3, namely Proposition 4.4.6, to relate the fractional Galois ideals to Stark units, which always exist in the present situation.

5.1 $\mathbb{Q}(\zeta_{p^n})^+/\mathbb{Q}$

The following example of Stark units is worked out in [43, Ch.III, Section 5]. If w is the infinite place of L^+ arising from the embedding $\zeta+\zeta^{-1} \mapsto \exp(2\pi i/p^n)+\exp(-2\pi i/p^n)$, then $\bar{\epsilon}(L^+/\mathbb{Q}, S, w) = \{\pm(1-\zeta)(1-\zeta^{-1})\}$. Hence the group $\mathcal{E}^+$ of Stark units for the pair $(L^+/\mathbb{Q}, S)$ is generated over $\mathbb{Z}[G^+]$ by -1 and $\epsilon = (1-\zeta)(1-\zeta^{-1})$. Because this is an important example, we restate Proposition 4.4.6 in this special case:

Proposition 5.1.1 $\mathcal{J}(L^+/\mathbb{Q}, S) = \frac{1}{2}\mathrm{ann}_{\mathbb{Z}[G^+]}(\mathcal{O}_{L^+,S}^\times/\mathcal{E}^+)$.

We interpret Proposition 5.1.1 in terms of the *cyclotomic units* of L^+. These are defined for a general cyclotomic field and its maximal totally real subfield in [45, Section 8.1], and have the interesting property that they are of maximal rank in the group of units. Let $\mathfrak{C}$ denote the cyclotomic units in L^+. Then [45, Lemma 8.1] gives the following generators for $\mathfrak{C}$:

$$-1 \text{ and } \left\{\xi_a = \zeta^{(1-a)/2}\frac{1-\zeta^a}{1-\zeta} \mid 1 < a < \frac{1}{2}p^n, p \nmid a\right\}.$$

The equation

$$\xi_a^2 = \frac{(1-\zeta^a)(1-\zeta^{-a})}{(1-\zeta)(1-\zeta^{-1})}$$

shows that the cyclotomic units in L^+ are closely related to the Stark units.

We observe that there is of course an embedding of $\mathbb{Z}[G^+]$-modules

$$\mathcal{O}_{L^+}^\times/\mathcal{O}_{L^+}^\times \cap \mathcal{E}^+ \to \mathcal{O}_{L^+,S}^\times/\mathcal{E}^+. \tag{5.1.1}$$

Lemma 5.1.2 *The embedding* $\mathcal{O}_{L^+}^\times/\mathcal{O}_{L^+}^\times \cap \mathcal{E}^+ \rightarrow \mathcal{O}_{L^+,S}^\times/\mathcal{E}^+$ *is an isomorphism.*

Proof. This is immediate since ϵ generates the (unique) prime of L^+ above p. ■

Definition 5.1.3 *Let* $U(p) = \mathcal{O}_{L^+,S}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$ *and* $\mathcal{E}^+(p) = \mathcal{E}^+ \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Proposition 5.1.4

$$\begin{aligned} \mathcal{J}(L^+/\mathbb{Q}, S) \cap \mathbb{Z}[G^+] &= \operatorname{ann}_{\mathbb{Z}[G^+]}(\mathcal{O}_{L^+}^\times/\mathfrak{C}) \\ \textit{and } \mathcal{J}(L^+/\mathbb{Q}, S) &\subseteq \operatorname{ann}_{\mathbb{Z}_p[G^+]}(U(p)/\mathcal{E}^+(p)). \end{aligned}$$

Proof. The second statement follows immediately from Proposition 4.4.6. The first statement is shown as follows. From Proposition 4.4.6 and Lemma 5.1.2, we see

$$\mathcal{J}(L^+/\mathbb{Q}, S) = \frac{1}{2}\operatorname{ann}_{\mathbb{Z}[G^+]}(\lambda(\mathcal{O}_{L^+}^\times)/\lambda(\mathcal{O}_{L^+}^\times \cap \mathcal{E}^+)).$$

One sees by looking at bases that $\lambda(\mathcal{O}_{L^+}^\times \cap \mathcal{E}^+) = 2\lambda(\mathfrak{C})$. Then suppose α is an element of $\operatorname{ann}_{\mathbb{Z}[G^+]}(\mathcal{O}_{L^+}^\times/\mathfrak{C}) = \operatorname{ann}_{\mathbb{Z}[G^+]}(\lambda(\mathcal{O}_{L^+}^\times)/\lambda(\mathfrak{C}))$. Given $x \in \lambda(\mathcal{O}_{L^+}^\times)$, $\alpha x \in \lambda(\mathfrak{C})$ and so $2\alpha x \in 2\lambda(\mathfrak{C}) = \lambda(\mathcal{O}_{L^+}^\times \cap \mathcal{E}^+)$. Therefore $2\alpha \in \operatorname{ann}_{\mathbb{Z}[G^+]}(\lambda(\mathcal{O}_{L^+}^\times)/\lambda(\mathcal{O}_{L^+}^\times \cap \mathcal{E}^+))$, hence $\alpha \in \mathcal{J}(L^+/\mathbb{Q}, S) \cap \mathbb{Z}[G^+]$.

Conversely, suppose $\alpha \in \mathcal{J}(L^+/\mathbb{Q}, S) \cap \mathbb{Z}[G^+]$, and take $x \in \lambda(\mathcal{O}_{L^+}^\times)$. Writing $\alpha = \frac{1}{2}\beta$ with $\beta \in \operatorname{ann}_{\mathbb{Z}[G^+]}(\lambda(\mathcal{O}_{L^+}^\times)/\lambda(\mathcal{O}_{L^+}^\times \cap \mathcal{E}^+))$, we have

$$\beta x \in \lambda(\mathcal{O}_{L^+}^\times \cap \mathcal{E}^+) = 2\lambda(\mathfrak{C}).$$

Therefore $\alpha x = \frac{1}{2}\beta x \in \lambda(\mathfrak{C})$. Thus $\alpha \in \operatorname{ann}_{\mathbb{Z}[G^+]}(\lambda(\mathcal{O}_{L^+}^\times)/\lambda(\mathfrak{C}))$, which is $\operatorname{ann}_{\mathbb{Z}[G^+]}(\mathcal{O}_{L^+}^\times/\mathfrak{C})$. ■

5.2 $\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}$

Observe that the orders of vanishing of the L-functions $L_{L/\mathbb{Q},S}(s,\psi)$ are given by

$$r(\psi) = \begin{cases} 1 & \text{if } \psi \in \widehat{G} \text{ is even} \\ 0 & \text{if } \psi \in \widehat{G} \text{ is odd.} \end{cases}$$

This shows that the rank idempotents $e_{L/\mathbb{Q},S}[r] = e[r]$ in this case are given as follows: $e[0] = e_-$, $e[1] = e_+$ and $e[r] = 0$ for $r \geq 2$, where $e_+ = \frac{1}{2}(1+c)$ and $e_- = \frac{1}{2}(1-c)$ are the plus and minus idempotents for complex conjugation $c \in G$. Hence $X \otimes_{\mathbb{Z}} \mathbb{Q} \cong e_+\mathbb{Q}[G]$.

Proposition 5.2.1 $\mathcal{J}(L/\mathbb{Q},S) = \frac{1}{2}e_+\mathrm{ann}_{\mathbb{Z}[G]}(\mathcal{O}_{L,S}^\times/\mathcal{E}) \oplus \mathbb{Z}[G]\theta_{L/\mathbb{Q},S}$*, where* $\mathcal{E}$ *is the* $\mathbb{Z}[G]$*-submodule of* $\mathcal{O}_{L,S}^\times$ *generated by* $1-\zeta$*.*

Proof. By the decomposition in (4.2.1), we can break the proof up according to rank idempotents. Of course, $e_-\mathcal{J}(L/\mathbb{Q},S) = e[0]\mathcal{J}(L/\mathbb{Q},S) = \mathbb{Z}[G]\theta_{L/\mathbb{Q},S}$ by Proposition 4.2.5. We can find $e_+\mathcal{J}(L/\mathbb{Q},S)(= e[1]\mathcal{J}(L/\mathbb{Q},S))$ by applying Proposition 4.4.6, since the inflation property of L-functions, together with the fact that we are dealing with even characters, shows that we can work with the subextension $L^+/\mathbb{Q}$. ■

5.3 Comparison of $\mathcal{J}(L/\mathbb{Q},S)$ and $\mathcal{J}(L^+/\mathbb{Q},S)$.

Comparing the descriptions of $\mathcal{J}(L^+/\mathbb{Q},S)$ and $\mathcal{J}(L/\mathbb{Q},S)$ given in Propositions 5.1.1 and 5.2.1, we find

Proposition 5.3.1 *With notation as above,* $\mathcal{J}(L^+/\mathbb{Q},S)$ *is the image of* $\mathcal{J}(L/\mathbb{Q},S)$ *under the natural map* $\mathbb{Q}[G] \to \mathbb{Q}[G^+]$*.*

Proof. Given that the idempotents e_+ and e_- map to 1 and 0 resp., it remains to observe that $\mathcal{O}_{L^+,S}^\times/\mathcal{E}^+ \simeq \mathcal{O}_{L,S}^\times/\mathcal{E}$, which is straightforward. ■

It may appear a convenient coincidence that the images of e_+ and e_- in $\mathbb{Q}[G^+]$ are 1 and 0. However, this fact is explained by the following general property of rank idempotents, whose proof is straightforward:

Proposition 5.3.2 *Let E/F be any abelian extension of number fields, M/F any subextension, and S any finite set of places of F containing the infinite ones. Then the map $\mathbb{Q}[\mathrm{Gal}(E/F)] \to \mathbb{Q}[\mathrm{Gal}(M/F)]$ sends $e_{E/F,S}[r]$ to $e_{M/F,S}[r]$ for every $r \geq 0$.*

5.4 $\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}(\sqrt{-p})$, $p \equiv 3 \bmod 4$

Assume p is a prime congruent to 3 mod 4, so that $L = \mathbb{Q}(\zeta)$ contains the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-p})$, and let $H = \mathrm{Gal}(L/K)$. We let S_K be the set of places of K lying above those in S. Of course, S_K consists exactly of the infinite place of K and the unique place $\mathfrak{p}$ above p. Let w be the infinite place of L arising from the embedding $\zeta \mapsto \exp(2\pi i/p^n)$, and w^+ its restriction to the maximal real subfield L^+ of L.

Definition 5.4.1 *Define the element $\tilde{\theta}_{L/\mathbb{Q},S} \in \mathbb{Q}[H]$ by*

$$\tilde{\theta}_{L/\mathbb{Q},S} = \sum_{\sigma \in H} \zeta_{L/\mathbb{Q},S}(0,\sigma)\sigma^{-1}.$$

This "half Stickelberger element" is obtained from the usual Stickelberger element $\theta_{L/\mathbb{Q},S}$ by keeping only those terms corresponding to elements of the index two subgroup H of G.

Proposition 5.4.2 *Let $\tilde{\theta} = \tilde{\theta}_{L/\mathbb{Q},S}$ be as in Definition 5.4.1, and μ the number of roots of unity in L. Then $(1-\zeta)^{\mu\tilde{\theta}}$ is a Stark unit for the triple $(L/K, S_K, w)$.*

Proof. We show that $(1-\zeta)^{\mu\tilde{\theta}}$ satisfies (4.4.2). (This is sufficient because $\mathfrak{p}$ is totally ramified in L/K, so that $U^{(\infty)}$ is all of $\mathcal{O}_{L,S}^{\times}$.) Note that restriction gives an isomorphism $H \to G^+$. Then given $\chi \in \widehat{H}$, let χ' denote the corresponding character of G^+, and χ'' the character

$$G \to G^+ \xrightarrow{\chi'} \mathbb{C}^\times$$

of G. Also, let ρ be the non-trivial character of G extending the trivial character of H. If $\chi \in \widehat{H}$, then χ'' and $\rho\chi''$ are the two characters of G extending χ, and so by Frobenius reciprocity and the inflation and induction properties of L-functions,

$$\begin{aligned} L_{L/K,S_K}(s,\chi) &= L_{L/\mathbb{Q},S}(s,\chi'')L_{L/\mathbb{Q},S}(s,\rho\chi'') \\ &= L_{L^+/\mathbb{Q},S}(s,\chi')L_{L/\mathbb{Q},S}(s,\rho\chi''). \end{aligned}$$

Hence $L'_{L/K,S_K}(0,\chi) = L'_{L^+/\mathbb{Q},S}(0,\chi')L_{L/\mathbb{Q},S}(0,\rho\chi'')$. Of course, if $\epsilon \in \bar{\epsilon}(L^+/\mathbb{Q},S,w^+)$ then by Section 5.1 we have

$$\begin{aligned} L'_{L^+/\mathbb{Q},S}(0,\chi') &= -\frac{1}{2}\sum_{\tau \in G^+} \bar{\chi}'(\tau)\log\|\epsilon\|_{\tau w^+} \\ &= -\frac{1}{4}\sum_{\sigma \in H} \bar{\chi}(\sigma)\log\|\epsilon\|_{\sigma w} \\ &= -\frac{1}{2}\sum_{\sigma \in H} \bar{\chi}(\sigma)\log\|1-\zeta\|_{\sigma w}. \end{aligned}$$

Now, let $c \in G$ denote complex conjugation. One checks that if $\sigma \in G$ then $\zeta_{L/\mathbb{Q},S}(0,\sigma c) = -\zeta_{L/\mathbb{Q},S}(0,\sigma)$. (This is because L-functions of even characters of G vanish at 0.) Hence, using the decomposition $G = H\langle c\rangle$ and observing that $\rho\chi''$ is odd, we find

$$\begin{aligned} L_{L/\mathbb{Q},S}(0,\rho\chi'') &= 2\sum_{\sigma \in H} \zeta_{L/\mathbb{Q},S}(0,\sigma)\rho\chi''(\sigma) \\ &= 2\sum_{\sigma \in H} \zeta_{L/\mathbb{Q},S}(0,\sigma)\chi(\sigma). \end{aligned}$$

Putting this together, we obtain

$$\begin{aligned} L'_{L/K,S_K}(0,\chi) &= -\left(\sum_{\sigma\in H}\bar{\chi}(\sigma)\log\|1-\zeta\|_{\sigma w}\right)\left(\sum_{\tau\in H}\zeta_{L/\mathbb{Q},S}(0,\tau)\chi(\tau)\right) \\ &= -\sum_{\tau\in H}\zeta_{L/\mathbb{Q},S}(0,\tau)\sum_{\sigma\in H}\chi(\sigma^{-1}\tau)\log\|1-\zeta\|_{\sigma w} \\ &= -\sum_{\tau\in H}\zeta_{L/\mathbb{Q},S}(0,\tau)\sum_{\sigma\in H}\bar{\chi}(\sigma)\log\|(1-\zeta)^{\tau^{-1}}\|_{\sigma w} \\ &= -\frac{1}{\mu}\sum_{\sigma\in H}\bar{\chi}(\sigma)\log\|(1-\zeta)^{\mu\tilde{\theta}}\|_{\sigma w}. \end{aligned}$$

■

Using Proposition 4.4.6 and Proposition 5.4.2, we now have:

Proposition 5.4.3 *$\mathcal{J}(L/K,S_K) = \frac{1}{\mu}\mathrm{ann}_{\mathbb{Z}[H]}(\mathcal{O}_{L,S}^{\times}/\tilde{\mathcal{E}})$, where $\tilde{\mathcal{E}}$ is the $\mathbb{Z}[H]$-submodule of $\mathcal{O}_{L,S}^{\times}$ generated by ζ and $(1-\zeta)^{\mu\tilde{\theta}}$.*

5.5 Comparison of $\mathcal{J}(L/K,S_K)$ and $\mathcal{J}(L^+/\mathbb{Q},S)$

We continue with the notation of Section 5.4, and emphasise that $p\equiv 3 \bmod 4$. We continue to let $\mathcal{E}^+$ be the $\mathbb{Z}[G^+]$-submodule of $\mathcal{O}_{L^+,S}^{\times}$ generated by -1 and $\epsilon=(1-\zeta)(1-\zeta^{-1})$, and $\tilde{\mathcal{E}}$ the $\mathbb{Z}[H]$-submodule of $\mathcal{O}_{L,S}^{\times}$ generated by ζ and $(1-\zeta)^{\mu\tilde{\theta}}$.

Proposition 5.5.1 *Let $\Phi:\mathbb{Q}[H]\to\mathbb{Q}[G^+]$ be the canonical isomorphism. Then*

$$\Phi(\mathcal{J}(L/K,S_K)) = \Phi(2\tilde{\theta})\mathcal{J}(L^+/\mathbb{Q},S), \tag{5.5.1}$$

where $\tilde{\theta}=\tilde{\theta}_{L/\mathbb{Q},S}$ is the "half Stickelberger element" of Section 5.4. Equivalently,

$$\Phi(\mathrm{ann}_{\mathbb{Z}[H]}(\mathcal{O}_{L,S}^{\times}/\tilde{\mathcal{E}})) = \Phi(\mu\tilde{\theta})\mathrm{ann}_{\mathbb{Z}[H]}(\mathcal{O}_{L^+,S}^{\times}/\mathcal{E}^+), \tag{5.5.2}$$

where μ is again the number of roots of unity in L.

Before proving Proposition 5.5.1, we give a lemma concerning $\tilde{\theta}$.

Lemma 5.5.2 $\tilde{\theta}$ *is invertible in the group-ring* $\mathbb{Q}[H]$.

Remark. Observe that the full Stickelberger element $\theta_{L/\mathbb{Q},S}$ is *not* invertible, since $L_{L/\mathbb{Q},S}(0,\chi) = 0$ for even characters χ of G.

Proof. (Lemma 5.5.2) It is a simple matter to check that $(1-c)\tilde{\theta} = \theta_{L/\mathbb{Q},S}$, and hence for odd characters $\chi \in \widehat{G}$, $2\chi(\tilde{\theta}) = \chi(\theta_{L/\mathbb{Q},S}) \neq 0$. If $\chi \in \widehat{G}$ is instead even, then $\rho\chi$ is odd where $\rho \in \widehat{G}$ is again the odd extension to G of the trivial character of H. But as $\tilde{\theta} \in \mathbb{Q}[H]$, $\chi(\tilde{\theta}) = \rho\chi(\tilde{\theta})$, which we have just shown to be non-zero. ∎

Proof. (Proposition 5.5.1) The equivalence of (5.5.1) and (5.5.2) is just Proposition 4.4.6. The inclusion "$\supseteq$" in (5.5.2) is almost immediate when one recalls that $\mathcal{E}^+/\text{tors}$ is generated by $(1-\zeta)(1-\zeta^{-1})$ while $\tilde{\mathcal{E}}/\text{tors}$ is generated by $(1-\zeta)^{e\tilde{\theta}}$. The other inclusion is obtained by using that $\tilde{\theta}$ is invertible. ∎

5.6 Comparison of $\mathcal{J}(L/\mathbb{Q}, S)$ and $\mathcal{J}(L/K, S_K)$

We again assume $p \equiv 3 \bmod 4$, and continue with the above notation. The above work allows us to provide an example of the behaviour of the fractional ideal under passing to subgroups. It is akin to the base change for Stickelberger elements described in [35]. Following [35], given $\chi \in \widehat{G}$ we define a map $\chi[\cdot] : \mathbb{C}[G] \to \mathbb{C}[G]$ by

$$\chi\left[\sum_{\sigma\in G} a_\sigma \sigma\right] = \sum_{\sigma\in G} a_\sigma \chi(\sigma)\sigma.$$

$\chi[\cdot]$ is characterized uniquely by the property that $\psi(\chi[x]) = (\psi\chi)(x)$ for all $x \in \mathbb{C}[G]$ and all $\psi \in \widehat{G}$. Further, $\chi[\cdot]$ is a $\mathbb{C}$-algebra automorphism.

Let

$$\beta_H^G = \prod_{\substack{\chi \in \widehat{G} \smallsetminus \{1\} \\ \chi|_H = 1}} \chi[\theta_{L/\mathbb{Q},S}].$$

This is the element called $\beta(0)$ in [35]. As proven in [35, Prop.1], it satisfies

$$\theta_{L/K,S_K} = \beta_H^G \theta_{L/\mathbb{Q},S}.$$

(In fact, this is trivial in this case: both sides are easily seen to be zero. However, this equality holds in general and for higher Stickelberger elements.)

In our case, β_H^G takes a simple form.

Lemma 5.6.1 $\beta_H^G = (1+c)\tilde{\theta}$*, where* $c \in G$ *is complex conjugation.*

Proof. As in the proof of Proposition 5.4.2, let ρ be the unique non-trivial extension to G of the trivial character of H. Then

$$\begin{aligned} \beta_H^G &= \rho[\theta_{L/\mathbb{Q},S}] \\ &= \sum_{\sigma \in G} \zeta_{L/\mathbb{Q},S}(0,\sigma)\rho(\sigma^{-1})\sigma^{-1}. \end{aligned}$$

Using the decomposition $G = H\langle c \rangle$ and observing that $\rho(c) = -1$ and $\zeta_{L/\mathbb{Q},S}(0, c\sigma) = -\zeta_{L/\mathbb{Q},S}(0,\sigma)$ for all σ, we obtain the lemma. ∎

Proposition 5.6.2 *Let* $\pi_H : \mathbb{Q}[G] \to \mathbb{Q}[H]$ *be the ring homomorphism obtained by extending linearly the projection* $G \to H$ *arising from the decomposition* $G = H\langle c \rangle$*. Then*

$$\mathcal{J}(L/K, S_K) = \pi_H(\beta_H^G \mathcal{J}(L/\mathbb{Q}, S)).$$

Proof. Observe first that $\Phi : \mathbb{Q}[G] \to \mathbb{Q}[G^+]$ followed by the canonical isomorphism $\mathbb{Q}[G^+] \to \mathbb{Q}[H]$ is just π_H. We therefore see from Propositions 5.3.1 and 5.5.1 that $\mathcal{J}(L/K, S_K) = 2\tilde{\theta}\pi_H(\mathcal{J}(L/\mathbb{Q}, S))$. Now just use the fact that π_H is a ring homomorphism and the observation that $\pi_H(1+c) = 2$. ∎

Chapter 6

Connection with class-groups

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Having described in Chapter 4 how $\mathcal{J}(L/K, S)$ is tied in with Stark elements and given explicit examples in Chapter 5, it is now time to illustrate the role of the fractional Galois ideal in relation to class-groups. This will be done in the setting of Chapter 5 and will build on the explicit descriptions involving the cyclotomic units, the Stark units for cyclotomic extensions of $\mathbb{Q}$.

We carry out three tasks:

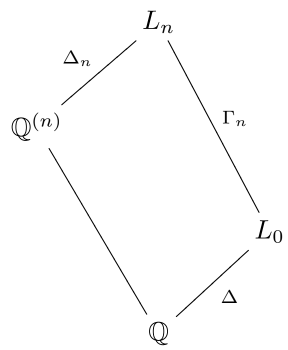
(i) to give a consequence of a result of Rubin, the consequence being that $\mathcal{J}(L/\mathbb{Q}, S)$ gives a positive answer (up to an explicit power of 2) to Question 1.1.1 for certain fields L,

(ii) to show that taking an inverse limit of certain fractional Galois ideals $\mathcal{J}(L/\mathbb{Q}, S)$ gives rise to the Fitting ideal of the inverse limit of p-parts of class-groups (in a precise way; see Theorem 6.3.1), and

(iii) to provide a new proof of a relationship between cyclotomic units and class-groups. This relationship is one direction of an equality of Fitting ideals given by Cornacchia and Greither in [12] and used in (ii).

6.1 Notation and setup

Assume p is an odd prime. Given $n \geq 0$, let L_n be the extension of $\mathbb{Q}$ obtained by adjoining the p^{n+1}th roots of unity in $\bar{\mathbb{Q}}$ and $\mathbb{Q}^{(n)}/\mathbb{Q}$ the degree p^n subextension of the $\mathbb{Z}_p$-extension $\mathbb{Q}^{(\infty)}$ of $\mathbb{Q}$. We then have the field diagram



in which $\mathbb{Q}^{(n)} \cap L_0 = \mathbb{Q}$ and $\mathbb{Q}^{(n)} L_0 = L_n$, so that the Galois group $G_n = \mathrm{Gal}(L_n/\mathbb{Q})$ is the internal direct product of Δ_n and Γ_n. S will denote the set of places $\{\infty, p\}$ of $\mathbb{Q}$.

By virtue of the natural isomorphism $\Delta_n \to \Delta$, characters of Δ_n correspond to characters of Δ. If $\delta \in \widehat{\Delta}$, we let δ_n denote the corresponding character in $\widehat{\Delta}_n$. Now, the idea is to view the group-ring $\mathbb{C}[G_n]$ as $\mathbb{C}[\Gamma_n][\Delta_n]$. In doing this, we can define a projection $\pi_n(\delta) : \mathbb{C}[G_n] \to \mathbb{C}[\Gamma_n]$ by extending δ_n linearly (over $\mathbb{C}[\Gamma_n]$).

Finally, fix an isomorphism $\nu : \mathbb{C}_p \to \mathbb{C}$ and let $\omega : \Delta \to \mathbb{C}^\times$ be the composition of the Teichmüller character $\Delta \to \mathbb{C}_p^\times$ with $\nu : \mathbb{C}_p^\times \to \mathbb{C}^\times$. Given $\delta \in \widehat{\Delta}$, δ^* will denote $\omega\delta^{-1}$.

6.2 A consequence of a result of Rubin

We make use of the fact, essentially [30, Theorem 2.2], that if $\phi : \mathcal{O}_{L_n^+,S}^\times \to \mathbb{Z}[G_n^+]$ is a $\mathbb{Z}[G_n^+]$-module homomorphism, then

$$4\phi(\mathcal{E}_n^+) \subseteq \mathrm{ann}_{\mathbb{Z}[G_n^+]}(\mathrm{Cl}(L_n^+)). \tag{6.2.1}$$

(We have used the second remark after [30, Theorem 1.3] to replace the units and cyclotomic units by $\mathcal{O}_{L_n^+,S}^\times$ and $\mathcal{E}_n^+$ resp.)

Proposition 6.2.1 $2^4\mathrm{ann}_{\mathbb{Z}[G_n]}(\mu(L_n))\mathcal{J}(L_n/\mathbb{Q},S) \subseteq \mathrm{ann}_{\mathbb{Z}[G_n]}(\mathrm{Cl}(L_n))$.

Proof. Using the fact that $\mathcal{E}_n^+/\pm 1$ is generated freely over $\mathbb{Z}[G_n^+]$ by $(1-\zeta_{p^{n+1}})(1-\zeta_{p^{n+1}}^{-1})$, we see easily from (6.2.1) that

$$4\mathrm{ann}_{\mathbb{Z}[G_n^+]}(\mathcal{O}_{L_n^+,S}^\times/\mathcal{E}_n^+) \subseteq \mathrm{ann}_{\mathbb{Z}[G_n^+]}(\mathrm{Cl}(L_n^+)). \tag{6.2.2}$$

Now, by Proposition 5.2.1 an element of the left-hand side in Proposition 6.2.1 looks like $2^4\beta(\frac{1}{2}e_+\alpha+\theta_{L/\mathbb{Q},S})$ with $\beta \in \mathrm{ann}_{\mathbb{Z}[G]}(\mu(L))$ and $\alpha \in \mathrm{ann}_{\mathbb{Z}[G]}(\mathcal{O}_{L,S}^\times/\mathcal{E})$. But $\beta\theta_{L/\mathbb{Q},S} \in \mathrm{ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(L))$ by Stickelberger's Theorem, and $\mathcal{O}_{L,S}^\times/\mathcal{E} \simeq \mathcal{O}_{L^+,S}^\times/\mathcal{E}^+$ so that by (6.2.2),

$$4(1+c)\alpha \in \mathrm{ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(L)).$$

(If $x \in \mathrm{Cl}(L)$, $(1+c)x$ can be identified with an element of $\mathrm{Cl}(L^+)$.) ■

6.3 Inverse limits of fractional Galois ideals

For ease of notation, we will set $\mathrm{Cl}_n = \mathrm{Cl}(L_n) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and let Cl_∞ be the limit of the Cl_n, viewed as a module over the Iwasawa algebra $\mathbb{Z}_p[\![G_\infty]\!]$. If $\delta \in \widehat{\Delta}$, then by abuse of notation we write e_δ for the idempotent in $\mathbb{Z}_p[\![G_\infty]\!]$ associated to the character of $\mathrm{Gal}(L_\infty/\mathbb{Q}^{(\infty)})$ corresponding to δ.

Theorem 6.3.1 *Let $\delta \in \widehat{\Delta}$. ($\delta$ may be even or odd.)*

$$\mathrm{Fitt}_{\mathbb{Z}_p[\![\Gamma_\infty]\!]}(e_{\delta^*}\mathrm{Cl}_\infty) = \begin{cases} \varprojlim_n \mathbb{Z}_p\pi_n(\delta^*)(\mathcal{J}(L_n/\mathbb{Q}, S)) & \text{if } \delta \neq 1 \\ \varprojlim_n \mathbb{Z}_p\pi_n(\delta^*)((1-(1+p)\sigma_n^{-1})\mathcal{J}(L_n/\mathbb{Q}, S)) & \text{if } \delta = 1 \end{cases}$$

where $\sigma_n = (1+p, L_n/\mathbb{Q})$.

In particular, we will show during the proof of Theorem 6.3.1, which will occupy the remainder of Section 6.3, that the limits given in the statement do indeed define ideals in $\mathbb{Z}_p[\![\Gamma_\infty]\!]$.

6.3.1 p-adic L-functions

p-adic L-functions will play a crucial role in half of the proof of Theorem 6.3.1. A full description of the theory of p-adic L-functions, including their definition in the non-commutative settings of, for example, [11] and [15], would take us further than is necessary. We content ourselves with the classical functions of Kubota and Leopoldt in [21] which interpolate p-adically Dirichlet L-functions at negative integers.

Kummer, with his famous congruences, observed an interesting arithmetic property of the values of the Riemann ζ-function $\zeta_\mathbb{Q}$ at negative integers. Namely, these values, which were already known to be rational, exhibit a sort of integrality and periodicity once viewed in the right way. Indeed, fixing a prime p, the integrality statement is that if $k \not\equiv 1 \bmod p-1$ is a negative odd integer, then $\zeta_\mathbb{Q}(k) \in \mathbb{Z}_p$, viewing $\mathbb{Q}$ inside $\mathbb{Q}_p$. The periodicity statement says that if k' is another such integer and is congruent to k mod $p-1$, then $\zeta_\mathbb{Q}(k)$ and $\zeta_\mathbb{Q}(k')$ define the same class in $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$.

These properties were summarised by Kubota and Leopoldt in the existence of a $\mathbb{C}_p$-valued continuous function on $\mathbb{Z}_p \smallsetminus \{1\}$ interpolating $\zeta_\mathbb{Q}$ at negative integers. In fact, they did this for arbitrary Dirichlet L-functions.

Theorem 6.3.2 (Kubota–Leopoldt) *Let $E/\mathbb{Q}$ be an abelian extension with Galois group G and let $\chi \in \widehat{G}$. Then there is a unique continuous $\mathbb{C}_p$-valued function $L_p(s, \chi)$ on $\mathbb{Z}_p \smallsetminus \{1\}$ such that*

$$L_p(1-n, \chi) = L(1-n, \chi\omega^{-n})\mathrm{char}_{\chi\omega^{-n},p}(p^{1-n})$$

for all integers $n \geq 1$.

In fact, when χ is non-trivial, $L_p(s, \chi)$ is defined at 1 as well. This theorem is only interesting for even characters χ, because when χ is odd the corresponding L-function vanishes at all negative integers, so that $L_p(s, \chi)$ is necessarily zero.

6.3.2 Algebraic p-adic L-functions

$L_p(s, \chi)$ has a more algebraic interpretation, namely as an element in the completed group-ring Λ of a certain $\mathbb{Z}_p$-extension (after having made a choice of topological generator for the Galois group of the extension). With this interpretation, Iwasawa was able to conjecture a remarkable connection of $L_p(s, \chi)$ with the Λ-module structure of the inverse limit of the (p-parts of the) class-groups in the $\mathbb{Z}_p$-extension. This connection was proven by Mazur and Wiles in [23], and later Wiles proved in [46] the more general statement for characters of abelian extensions of totally real fields.

For our purposes, given $\delta \in \widehat{\Delta}$ even, the power series giving the p-adic L-functions $L_p(s, \delta_n\psi)$ for $\psi \in \widehat{\Gamma}_n$ will satisfy the following proposition. We let $h_\delta(T) \in \mathbb{Z}_p[\![T]\!]$ be the power series $1 - \frac{1+p}{1+T}$ if $\delta = 1$ and 1 otherwise. The following is proven in, for example, [13].

Proposition 6.3.3 *Let $\delta \in \widehat{\Delta}$ be even. Then there is a unique power series $g_\delta(T) \in \mathbb{Z}_p[\![T]\!]$ such that*

$$L_p(s, \delta_n\psi) = \frac{g_\delta(\psi(\sigma_n)^{-1}(1+p)^s - 1)}{h_\delta(\psi(\sigma_n)^{-1}(1+p)^s - 1)}$$

for all $\psi \in \widehat{\Gamma}_n$. (Recall from the statement of Theorem 6.3.1 that σ_n is defined to be $(1+p, L_n/\mathbb{Q})$, visibly an element of Γ_n.)

Now, $\gamma = (\sigma_n)_n$ defines a topological generator for Γ_∞, and we choose to identify $\mathbb{Z}_p[\![\Gamma_\infty]\!]$ and $\mathbb{Z}_p[\![T]\!]$ via $\gamma \mapsto 1+T$. Under this identification, we denote by g_δ the element of $\mathbb{Z}_p[\![\Gamma_\infty]\!]$ corresponding to $g_\delta(T)$.

We reproduce here the case we need of the Main Conjecture as proven by Mazur and Wiles [23], opting not to provide the statement in its full generality in order to avoid introducing notation we will not need afterwards.

Theorem 6.3.4 (Mazur–Wiles) *If $\delta \in \widehat{\Delta}$ is even, then we have the following equality of ideals in $\mathbb{Z}_p[\![\Gamma_\infty]\!]$:*

$$(g_\delta) = \mathrm{Fitt}_{\mathbb{Z}_p[\![\Gamma_\infty]\!]}(e_{\delta^*}\mathrm{Cl}_\infty).$$

6.3.3 Stickelberger elements

For ease of notation, θ_n will denote the Stickelberger element $\theta_{L_n/\mathbb{Q},S}$. Then given $\delta \in \widehat{\Delta}$, we set $\theta_n(\delta) = \pi_n(\delta)(\theta_n)$.

Proposition 6.3.5 $\theta_n(\delta) = \sum_{\tau\in\Gamma_n}\left(\sum_{\sigma\in\Delta_n}\zeta_{L_n/\mathbb{Q},S}(0,\sigma^{-1}\tau^{-1})\delta(\sigma)\right)\tau.$

Proof. This simply uses the description of θ_n in terms of partial ζ-functions. ■

It is straightforward to show that the field $\mathbb{Q}^{(n)}$ is totally real, so that Δ_n contains the complex conjugation element c of G_n. Therefore, since $\zeta_{L_n/\mathbb{Q},S}(0,c\sigma) = -\zeta_{L_n/\mathbb{Q},S}(0,\sigma)$ for all $\sigma \in G_n$, we see from Proposition 6.3.5 that $\theta_n(\delta^*) = 0$ when δ is odd (i.e. when δ^* is even).

Proposition 6.3.5 also tells us that $\theta_n(\delta) \in \mathbb{Q}_p[\Gamma_n]$ (recall that δ has order dividing $p-1$). Actually, we can say a lot more.

Proposition 6.3.6 *Given $\delta \in \widehat{\Delta}\smallsetminus\{1\}$, $\theta_n(\delta^*) \in \mathbb{Z}_p[\Gamma_n]$.*

Proof. We use some notation found in [45, Section 7.2], where most of the work for the proof of this proposition is done. Set $\xi_n(\delta) = \pi_n(\delta^*)(\theta_n - \frac{1}{2}N_{\mathbb{Q}}^{L_n})$, where $N_{\mathbb{Q}}^{L_n} \in \mathbb{Z}[G_n]$ is the sum of the elements of G_n. (This is not how the definition of $\xi_n(\delta)$ looked in [45], but it comes to exactly the same thing.) Then we see easily that

$$\theta_n(\delta^*) = \begin{cases} \frac{1}{2}|\Delta| \sum_{\tau \in \Gamma_n} \tau + \xi_n(1) & \text{if } \delta = 1 \\ \xi_n(\delta) & \text{otherwise.} \end{cases}$$

Now, [45, Prop.7.6(b)] says that when δ is even and non-trivial, $\xi_n(\delta) \in \mathbb{Z}_p[\Gamma_n]$, dealing with the even characters. But of course, when δ is odd, $\theta_n(\delta^*)$ is zero anyway. ■

6.3.4 Limit of fractional ideals

If n is a positive integer, $r_n : \mathbb{C}[G_n] \to \mathbb{C}[G_{n-1}]$ will denote the usual restriction map.

Lemma 6.3.7 *For any $\delta \in \widehat{\Delta}$ and any $n \geq 1$, the diagram*

$$\begin{array}{ccc} \mathbb{C}[G_n] & \xrightarrow{\pi_n(\delta)} & \mathbb{C}[\Gamma_n] \\ \downarrow r_n & & \downarrow r_n \\ \mathbb{C}[G_{n-1}] & \xrightarrow{\pi_{n-1}(\delta)} & \mathbb{C}[\Gamma_{n-1}] \end{array}$$

commutes.

Proof. All of the maps in the diagram are $\mathbb{C}$-algebra homomorphisms, so it is sufficient to check commutativity on elements of G_n. But

$$\begin{array}{ccc} \Delta_n & \longrightarrow & \Delta_{n-1} \\ & \searrow & \downarrow \\ & & \Delta \end{array}$$

commutes, so that $\delta_{n-1} \circ (r_n)|_{\Delta_n} = \delta_n$. The lemma follows.

■

The obvious corollary to this lemma is:

Lemma 6.3.8 *If we have subgroups $A_n \subseteq \mathbb{C}[G_n]$ forming a projective system with respect to the maps $\mathbb{C}[G_n] \to \mathbb{C}[G_{n-1}]$, then the $\pi_n(\delta)(A_n)$ form a projective system with respect to the maps $\mathbb{C}[\Gamma_n] \to \mathbb{C}[\Gamma_{n-1}]$.*

Lemma 6.3.9 *When $\delta \in \widehat{\Delta}$ is non-trivial, the $\mathbb{Z}_p$-submodule*

$$\mathbb{Z}_p \pi_n(\delta^*)(\mathcal{J}(L_n/\mathbb{Q}, S))$$

of $\mathbb{Q}_p[\Gamma_n]$ is an ideal in $\mathbb{Z}_p[\Gamma_n]$.

Proof. We use the description of $\mathcal{J}(L_n/\mathbb{Q}, S)$ found in [4, Prop.4.10], namely,

$$\mathcal{J}(L_n/\mathbb{Q}, S) = \frac{1}{2} e_+ \mathrm{ann}_{\mathbb{Z}[G_n]}(\mathcal{O}_{L_n,S}^\times / \mathcal{E}_n) \oplus \mathbb{Z}[G_n]\theta_n. \tag{6.3.1}$$

Of course, we only need to check that this set is mapped into $\mathbb{Z}_p[\Gamma_n]$ under $\pi_n(\delta)$, and this is clear from Proposition 6.3.6. ■

It is not difficult to see in this situation that the natural projections $\mathbb{Q}[G_n] \to \mathbb{Q}[G_{n-1}]$ map $\mathcal{J}(L_n/\mathbb{Q}, S)$ into $\mathcal{J}(L_{n-1}/\mathbb{Q}, S)$, so that the $\mathcal{J}(L_n/\mathbb{Q}, S)$ form a projective system. Hence, by Lemma 6.3.8, the $\mathbb{Z}_p \pi_n(\delta^*)(\mathcal{J}(L_n/\mathbb{Q}, S))$ form a projective system also, and by Lemma 6.3.9 its limit is an ideal in the Iwasawa algebra $\mathbb{Z}_p[\![\Gamma_\infty]\!]$.

We have been neglecting what happens when $\delta \in \widehat{\Delta}$ is the trivial character. The quotient $g_\delta(T)/h_\delta(T)$ defining $L_p(s, \delta)$ when $\delta = 1$ lies not in the power series ring $\mathbb{Z}_p[\![T]\!]$ but in its quotient field (analogous to the fact that the corresponding complex L-function has a pole at $s = 1$). To obtain an element of $\mathbb{Z}_p[\![T]\!]$ one needs to perform an operation (multiplying by $h_\delta(T)$) which is similar to cancelling out the pole of the complex L-function at $s = 1$. This operation must also be performed on the fractional Galois ideals in order to relate them to the p-adic L-function for the trivial character: if $\delta = 1$,

$$\varprojlim_n \mathbb{Z}_p \pi_n(\delta^*)((1 - (1+p)\sigma_n^{-1})\mathcal{J}(L_n/\mathbb{Q}, S))$$

is an ideal in $\mathbb{Z}_p[\![\Gamma_\infty]\!]$. (Note that the element of $\mathbb{Z}_p[\![T]\!]$ defined by the $1 - (1+p)\sigma_n^{-1}$ (using the isomorphism $\gamma \mapsto 1 + T$) is just $h_1(T)$.)

Proposition 6.3.10 *Let $\delta \in \widehat{\Delta}$ be even. Then we have the following equality of ideals in $\mathbb{Z}_p[\![\Gamma_\infty]\!]$:*

$$(g_\delta) = \begin{cases} \lim_{\leftarrow n} \mathbb{Z}_p \pi_n(\delta^*)(\mathcal{J}(L_n/\mathbb{Q}, S)) & \text{if } \delta \neq 1 \\ \lim_{\leftarrow n} \mathbb{Z}_p \pi_n(\delta^*)((1 - (1+p)\sigma_n^{-1})\mathcal{J}(L_n/\mathbb{Q}, S)) & \text{if } \delta = 1. \end{cases}$$

Proof. From (6.3.1), we see that $\mathbb{Z}_p \pi_n(\delta^*)(\mathcal{J}(L_n/\mathbb{Q}, S)) = \mathbb{Z}_p[\Gamma_n]\theta_n(\delta^*)$ when δ is even (i.e. δ^* is odd). A simple topological argument found in the Corollaire to [2, Prop.9] then shows that the limit of the right-hand side is the principal ideal in $\mathbb{Z}_p[\![\Gamma_\infty]\!]$ generated by the element $(\theta_n(\delta^*))_n$, assuming $\delta \neq 1$. We now appeal to Iwasawa's construction of p-adic L-functions in terms of Stickelberger elements, an exposition of which can be found in [45, Ch.7]. Specifically, we invoke [45, Theorem 7.10], which is exactly what is required to prove the equality with (g_δ). The case when $\delta = 1$ is done entirely similarly. ■

Using the classical Main Conjecture of Iwasawa theory as proven by Mazur and Wiles (Theorem 6.3.4 here), we now have the statement of Theorem 6.3.1 when δ is even.

6.3.5 Plus part of $\mathcal{J}(L_n/\mathbb{Q}, S)$

We now turn our attention to what happens when we take $\delta \in \widehat{\Delta}$ to be an *odd* character, so that, via the projection $\pi_n(\delta^*)$, we are working in the plus part of $\mathcal{J}(L_n/\mathbb{Q}, S)$, δ^* being even. It will be fruitless to turn to p-adic L-functions in this situation because, as remarked earlier, the p-adic L-functions associated to odd characters are identically zero. It is a conjecture of Vandiver that the corresponding parts $e_{\delta^*}\mathrm{Cl}_\infty$ of Cl_∞ are zero as well, but this is not yet proven. The fractional Galois ideals side-step this issue in the sense that their plus parts are equal to the Fitting ideals of units modulo cyclotomic units, and these Fitting ideals are known to be equal to those of the class-groups.

Lemma 6.3.11 *Let Δ be a finite abelian group, A an integral domain in which $|\Delta|$ is invertible and whose group of units contains an element of order equal to the exponent of Δ, and B a commutative ring containing A as a subring. For a character $\chi \in \mathrm{Hom}(\Delta, A^\times)$, let $e_\chi \in A[\Delta] \subseteq B[\Delta]$ be the associated idempotent and $\pi(\chi) : B[\Delta] \to B$ the ring homomorphism obtained by extending χ linearly over B. Then for any $B[\Delta]$-module M, we have*

(i) $\pi(\chi)(\mathrm{ann}_{B[\Delta]}(M)) = \mathrm{ann}_B(e_\chi M)$

(ii) $\pi(\chi)(\mathrm{Fitt}_{B[\Delta]}(M)) = \mathrm{Fitt}_B(e_\chi M)$ *if M is finitely presentable.*

Proof. (i) is straightforward. For (ii), we use the fact that $e_\chi M$ and $B \otimes_{B[\Delta]} M$ are isomorphic as B-modules, where we view B as a $B[\Delta]$-module via $\pi(\chi)$. Then $\mathrm{Fitt}_B(e_\chi M) = \mathrm{Fitt}_B(B \otimes_{B[\Delta]} M)$, and this is $\pi(\chi)(\mathrm{Fitt}_{B[\Delta]}(M))$ by right-exactness of the tensor product. ■

Lemma 6.3.12 *If $\delta \in \widehat{\Delta}$ is odd, then*

$$\mathbb{Z}_p \pi_n(\delta^*)(\mathcal{J}(L_n/\mathbb{Q}, S)) = \mathrm{Fitt}_{\mathbb{Z}_p[\Gamma_n]}(e_{\delta_n^*} \mathrm{Cl}_n).$$

Proof. Since δ^* is even, using (6.3.1) we see straightaway that

$$\mathbb{Z}_p \pi_n(\delta^*)(\mathcal{J}(L_n/\mathbb{Q}, S)) = \mathbb{Z}_p \pi_n(\delta^*)(\mathrm{ann}_{\mathbb{Z}[G_n]}(\mathcal{O}^\times_{L_n^+,S}/\mathcal{E}_n^+)).$$

Now, $\mathbb{Z}_p \mathrm{ann}_{\mathbb{Z}[G_n]}(\mathcal{O}^\times_{L_n^+,S}/\mathcal{E}_n^+) \subseteq \mathrm{ann}_{\mathbb{Z}_p[G_n]}((\mathcal{O}^\times_{L_n^+,S}/\mathcal{E}_n^+) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ but in fact the reverse inclusion holds as well: Take α in the right-hand side and choose a sequence α_i in $\mathrm{ann}_{\mathbb{Z}[G_n]}((\mathcal{O}^\times_{L_n^+,S}/\mathcal{E}_n^+) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ converging to α in $\mathbb{Z}_p[G_n]$. Choose also an integer d not divisible by p which annihilates the prime-to-p part of $\mathcal{O}^\times_{L_n^+,S}/\mathcal{E}_n^+$. Then $d\alpha_i$ is a sequence in $\mathrm{ann}_{\mathbb{Z}[G_n]}(\mathcal{O}^\times_{L_n^+,S}/\mathcal{E}_n^+)$ converging to $d\alpha$ in $\mathbb{Z}_p[G_n]$, but $\mathbb{Z}_p \mathrm{ann}_{\mathbb{Z}[G_n]}(\mathcal{O}^\times_{L_n^+,S}/\mathcal{E}_n^+)$ is closed in $\mathbb{Z}_p[G_n]$ and hence contains $d\alpha$. As $p \nmid d$, it contains α as well.

Thus $\mathbb{Z}_p \pi_n(\delta^*)(\mathcal{J}(L_n/\mathbb{Q}, S)) = \pi_n(\delta^*)(\mathrm{ann}_{\mathbb{Z}_p[G_n]}((\mathcal{O}^\times_{L_n^+,S}/\mathcal{E}_n^+) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$. Now, one can show that the dual of $(\mathcal{O}^\times_{L_n^+,S}/\mathcal{E}_n^+) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is cyclic over $\mathbb{Z}_p[G_n]$ (we will turn to this fact again in the proof Theorem 6.4.1); therefore, as G_n is cyclic,

$\mathrm{Fitt}_{\mathbb{Z}_p[G_n]}((\mathcal{O}^{\times}_{L_n^+,S}/\mathcal{E}_n^+) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ is the whole of the annihilator ideal. However, [12, Theorem 1] tells us (in particular), that

$$\mathrm{Fitt}_{\mathbb{Z}_p[G_n]}((\mathcal{O}^{\times}_{L_n^+,S}/\mathcal{E}_n^+) \otimes_{\mathbb{Z}} \mathbb{Z}_p) = \mathrm{Fitt}_{\mathbb{Z}_p[G_n]}(\mathrm{Cl}_n^+).$$

We can therefore finish the proof by applying part (ii) of Lemma 6.3.11 with $A = \mathbb{Z}_p$, $B = \mathbb{Z}_p[\Gamma_n]$, $\Delta = \Delta_n$ and $\chi = \delta_n^*$. ■

Observing that the norm maps $\mathrm{Cl}_{n+1} \to \mathrm{Cl}_n$ are surjective, taking Fitting ideals commutes with taking inverse limits in this situation (see, for example, [16, Theorem 2.1]) so that taking limits in Lemma 6.3.12 completes the proof of Theorem 6.3.1 for δ odd. Combined with the analogous result for δ even in Section 6.3.4, we have proven Theorem 6.3.1.

6.4 A new proof concerning Fitting ideals of class-groups

Let $\mathfrak{C}_n$ denote the group of cyclotomic units in $\mathcal{O}^{\times}_{L_n^+}$, the units in the maximal real subfield L^+ of L. In this section, we prove

Proposition 6.4.1 *For all $n \geq 0$,*

$$\mathrm{ann}_{\mathbb{Z}_p[G_n^+]}((\mathcal{O}^{\times}_{L_n^+}/\mathfrak{C}_n) \otimes_{\mathbb{Z}} \mathbb{Z}_p) \subseteq \mathrm{Fitt}_{\mathbb{Z}_p[G_n^+]}(\mathrm{Cl}_n).$$

As mentioned in Section 1, Snaith constructs (in [40]) $\mathbb{Z}[G]$-submodules $\mathcal{J}^r(L/K)$ of $\mathbb{Q}[G]$ for abelian extensions L/K satisfying the Stark Conjecture at $r < 0$. In [39], however, he had already constructed $\mathcal{J}^r(L/\mathbb{Q})$ when L is the maximal totally real subfield of a cyclotomic extension of $\mathbb{Q}$. Further, in the same paper he showed ([39, Theorem 1.8]) that the intersection of $\mathcal{J}^r(L/\mathbb{Q})$ with the p-adic group-ring $\mathbb{Z}_p[G]$ (p an odd prime) lies in the annihilator of a certain étale cohomology group when $r < 0$ is even. The methods used in the proof of [39, Theorem 1.8] work in the setting of the fractional ideal $\mathcal{J}(L/K,S)$

defined in Section 4.2 when $L = \mathbb{Q}$ and L is the maximal totally real subfield of a cyclotomic extension of $\mathbb{Q}$ having *p-power conductor*. In this case, the étale cohomology becomes the p-part of the class-group of L. This section will give an idea of the methods employed in the proof of [39, Theorem 1.8], but with emphasis on the $r = 0$ setting that we need.

6.4.1 Quick overview

The idea will be as follows. Fix an odd prime p and a positive integer $m \not\equiv 2 \bmod 4$ which is prime to p, and consider the fields $L_n^m = \mathbb{Q}(\zeta_{mp^{n+1}})$, $n \geq 0$. Of course, $L_n^m/\mathbb{Q}$ is abelian, and we denote its Galois group by G_n^m. Our interest will lie in the ring Λ_m obtained as the inverse limit $\lim_{\leftarrow n} \mathbb{Z}_p[G_n^m]$ of p-adic group rings, and in a complex C^m of Λ_m-modules. Eventually, the integer m will be taken to be 1, but much of the theory applies for any m, and so we work in more generality as much as possible. However, for the purposes of this introductory overview, let us take $m = 1$ now.

The cohomology of the Λ_1-complex C^1 is zero outside degrees 1 and 2, and if one tensors C^1 with one of the finite level group-rings $\mathbb{Z}_p[G_n^1]$, then the same is true of the resulting complex C_n^1 of $\mathbb{Z}_p[G_n^1]$-modules. For a suitable S, the cohomology groups in degrees 1 and 2 are resp. (almost) the quotient of the S-integers in L_n^1 by the cyclotomic units, and the p-part of the S-class-group of L_n^1. An entirely algebraic result of Snaith shows how one can form a relation involving the annihilators of these two cohomology groups and the determinant of C_n^1. Thanks to [8], the determinant of C^m (and so in particular C^1) is known, and a careful "descent" argument allows one to deduce from this the determinant of C_n^1. Taking plus parts for the action of complex conjugation then gives us the relation in Proposition 6.4.1.

6.4.2 The complex C^m

The complex C^m of Λ_m-modules is built from the étale complexes $\mathrm{R}\Gamma(\mathcal{O}_{L_n^m,S}, \mathbb{Z}_p(1))$, where $S = \{\infty\} \cup \{q|mp\}$. These complexes have cohomology groups $H^i_{\text{ét}}(\mathcal{O}_{L_n^m,S}, \mathbb{Z}_p(1))$ with the following descriptions:

Lemma 6.4.2 *$H^i_{\text{ét}}(\mathcal{O}_{L_n^m,S}, \mathbb{Z}_p(1)) = 0$ for $i \neq 1, 2$. There is a canonical isomorphism*

$$H^1_{\text{ét}}(\mathcal{O}_{L_n^m,S}, \mathbb{Z}_p(1)) \simeq \mathcal{O}^\times_{L_n^m,S} \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

of $\mathbb{Z}_p[G_n^m]$-modules and a canonical short exact sequence

$$0 \to \mathrm{Cl}(\mathcal{O}_{L_n^m,S}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^2_{\text{ét}}(\mathcal{O}_{L_n^m,S}, \mathbb{Z}_p(1)) \to X_n^m \to 0$$

where X_n^m is the kernel of the degree map on the free $\mathbb{Z}_p$-module on the finite places of L_n^m above S.

This lemma is [8, Lemma 3.2], where a proof can be found.

The inverse limit $\lim_{\leftarrow n} H^i_{\text{ét}}(\mathcal{O}_{L_n^m,S}, \mathbb{Z}_p(1))$ (with respect to the corestriction maps) actually arises as a cohomology group. Let Ω denote the Galois group of the maximal unramified-outside-S extension of $\mathbb{Q}$, and for $n \geq 0$ let Ω_n^m be the Galois group of the maximal unramified-outside-S extension of L_n^m. We take the inverse limit of the standard complexes $C^\bullet(\Omega, \mathrm{Ind}^{\Omega}_{\Omega_n^m}(\mathrm{Res}^{\Omega}_{\Omega_n^m}\mu_{p^{n+1}}))$, and call the resulting complex $\mathrm{R}\Gamma(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty)$. Then the cohomology $H^i(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty)$ of $\mathrm{R}\Gamma(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty)$ satisfies

$$H^i(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty) \simeq \lim_{\leftarrow n} H^i_{\text{ét}}(\mathcal{O}_{L_n^m,S}, \mathbb{Z}_p(1)).$$

$\mathrm{R}\Gamma(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty)$ is the complex we modify to obtain the promised complex C^m.

Proposition 6.4.3 *(i) $H^i_{\text{ét}}(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty) = 0$ for $i \neq 1, 2$.*

(ii) There is a canonical isomorphism $H^1_{\text{ét}}(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty) \simeq U^m_\infty$, where U^m_∞ is the limit $\lim_{\leftarrow n} \mathcal{O}^\times_{L_n^m,S} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ with respect to the norm maps.

(iii) There is a canonical short exact sequence

$$0 \to \mathrm{Cl}_\infty(p) \to H^2_{\text{ét}}(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty) \to X^m_\infty \to 0$$

where $\mathrm{Cl}_\infty(p) = \lim_{\leftarrow n} \mathrm{Cl}(\mathcal{O}_{L^m_n,S}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ *and* $X^m_\infty = \lim_{\leftarrow n} X^m_n$.

Proof. See [8, Prop. 5.1]. (The principle is to show that the statements in Lemma 6.4.2 pass to inverse limits over n.) ∎

6.4.3 The Λ_m-module $B^{m,+}$

Let B^m_n be the $\mathbb{Z}_p[G^m_n]$-module $\bigoplus_{\Sigma(L^m_n)} \mathbb{Z}_p$ where $\Sigma(L^m_n)$ is the set of embeddings of L^m_n into $\mathbb{C}$. With the action of G^m_n on B^m_n coming from the action on embeddings, B^m_n is a free rank one $\mathbb{Z}_p[G^m_n]$-module. The maps $B^m_{n+1} \to B^m_n$ make $(B^m_n)_n$ into a projective system, and the limit B^m is a free Λ_m-module on one generator. Generators of a certain form are given in [8, Section 5.1]. We pick one of this form and denote it v_m.

Now, the complex conjugation element c of Λ_m acts on B^m by permuting embeddings of the fields L^m_n, and since 2 is invertible in Λ_m (p is odd), B^m is the direct sum of the plus and minus parts for complex conjugation:

$$B^m = B^{m,+} \oplus B^{m,-}$$

where $B^{m,+} = e_+ B^m$ and $B^{m,-} = e_- B^m$ with

$$e_+ = \tfrac{1}{2}(1+c) \quad \text{and} \quad e_- = \tfrac{1}{2}(1-c)\ .$$

Note in particular that $B^{m,+}$ and $B^{m,-}$ are projective Λ_m-modules. Now, we wish to construct a cochain map $B^{m,+}[-1] \to \mathrm{R}\Gamma(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty)$. The following proposition shows that this is, in principle, the same as giving a homomorphism $B^{m,+} \to U^m_\infty$ of Λ_m-modules, where $U^m_\infty = \lim_{\leftarrow n} \mathcal{O}^\times_{L^m_n,S} \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Proposition 6.4.4 *There is a canonical isomorphism*

$$\mathrm{Hom}_{\mathcal{D}(\Lambda_m)}(B^{m,+}[-1], \mathrm{R}\Gamma(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty)) \simeq \mathrm{Hom}_{\Lambda_m}(B^{m,+}, U^m_\infty),$$

where $\mathcal{D}(\Lambda_m)$ is the derived category of the homotopy category of bounded complexes of Λ_m-modules.

Proof. Recall from Proposition 6.4.3 that $H^1_{\text{ét}}(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty) \simeq U^m_\infty$. Then the idea is as follows. Given a cochain map $\alpha : B^{m,+} \to \mathrm{R}\Gamma(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty)$, we obtain a Λ_m-module homomorphism $H^1(\alpha) : H^1(B^{m,+}[-1]) = B^{m,+} \to H^1_{\text{ét}}(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty)$. Conversely, given a Λ_m-module homomorphism $f : B^{m,+} \to H^1_{\text{ét}}(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty)$, we can lift it to a homomorphism

$$\tilde{\alpha} : B^{m,+} \to Z^1(\mathrm{R}\Gamma(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty))$$

since $B^{m,+}$ is projective. Clearly then α defined by

$$\alpha^i = \begin{cases} \tilde{\alpha} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

is a cochain map with $H^1(\alpha) = f$. ■

Consider Soulé's cyclotomic element

$$\epsilon_m = ((1 - \zeta_m^{p^{-n}} \zeta_{p^{n+1}})(1 - \zeta_m^{-p^{-n}} \zeta_{p^{n+1}}^{-1}))_n$$

in U^m_∞. Complex conjugation leaves ϵ_m fixed, so there is a homomorphism $B^{m,+} \to U^m_\infty$ such that $e_+ v_m \mapsto \epsilon_m$, and it is necessarily unique. By Proposition 6.4.4, this gives us a map $c_m : B^{m,+}[-1] \to \mathrm{R}\Gamma(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty)$ of complexes, and we denote its mapping cone by C^m.

Theorem 6.4.5 *The complex C^m of Λ_m-modules is acyclic outside degrees* 1 *and* 2*, and we have:*

(i) $H^1(C^m) \simeq U^m_\infty / \mathcal{E}_m$ where

$$\mathcal{E}_m = \mathrm{Im}(B^{m,+} \to U^m_\infty).$$

(ii) There is a canonical short exact sequence

$$0 \to \mathrm{Cl}_\infty(p) \to H^2(C^m) \to X^m_\infty \to 0.$$

Proof. Look at the long exact sequence in cohomology arising from the exact sequence

$$0 \to \mathrm{R}\Gamma(\mathbb{Z}_S, \mathbb{Z}_p(1)^m_\infty) \to C^m \to B^{m,+}[0] \to 0$$

of complexes, and use Proposition 6.4.3. ∎

6.4.4 Perfect complexes

If R is a ring, then a chain complex of R-modules is said to be *perfect* if it is bounded and the modules making up the complex are finitely generated and projective. Suppose we have an abelian group G and a prime p, and that we have a perfect complex $F_\bullet$ of $\mathbb{Z}_p[G]$-modules, all of whose homology groups are finite. Then we can form an isomorphism

$$\bigoplus_j F_{2j} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \bigoplus_j F_{2j+1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \tag{6.4.1}$$

using the exact sequences

$$0 \to B_i(F_\bullet) \to Z_i(F_\bullet) \to H_i(F_\bullet) \to 0 \tag{6.4.2}$$

and

$$0 \to Z_{i+1}(F_\bullet) \to F_{i+1} \to B_i(F_\bullet) \to 0. \tag{6.4.3}$$

One uses that (6.4.3) splits after tensoring with $\mathbb{Q}_p$, and (6.4.2) gives isomorphisms $B_i(F_\bullet) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong Z_i(F_\bullet) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. For details of how the isomorphism is constructed, see [39, Section 2].

The determinant of the isomorphism in (6.4.1) is known to be well-defined up to $\mathbb{Z}_p[G]^\times$, i.e. if different splittings are chosen then the determinant will change by an element of $\mathbb{Z}_p[G]^\times$ (see [42, Ch.15]). We denote the class of the isomorphism in $\mathbb{Q}_p[G]^\times/\mathbb{Z}_p[G]^\times$ by $\det(F_\bullet)$.

Remark. This works more generally. $\mathbb{Z}_p[G]$ and $\mathbb{Q}_p[G]$ can be replaced by any commutative rings $R \subseteq S$, and $F_\bullet$ by a perfect complex of R-modules which becomes exact on tensoring with S, though we must now assume that the projectives in $F_\bullet$ are in fact free. Then in the same way we obtain an element $\det(F_\bullet) \in S^\times/R^\times$. Further, the R-submodule of S that $\det(F_\bullet)^{-1}$ generates (note the inverse) is equal to $\mathrm{D}_R(F_\bullet)$, where D_R is the determinant functor introduced in [18].

The following proposition is [39, Cor. 2.11].

Proposition 6.4.6 *Let G be a finite abelian group and p a prime, and suppose that $F_\bullet$ is a perfect complex of $\mathbb{Z}_p[G]$-modules with finite homology in degrees 0 and 1 and zero homology elsewhere. Then if $\mathrm{Hom}(H_1(F_\bullet), \mathbb{Q}_p/\mathbb{Z}_p)$ is cyclic as a $\mathbb{Z}_p[G]$-module, we have the containment*

$$\det(F_\bullet)^{-1}\mathrm{ann}_{\mathbb{Z}_p[G]}(H_1(F_\bullet)) \subseteq \mathrm{ann}_{\mathbb{Z}_p[G]}(H_0(F_\bullet)). \tag{6.4.4}$$

If further the Sylow p-subgroup of G is cyclic, then the annihilator ideal in the right-hand side of (6.4.4) may be replaced by the Fitting ideal.

This is a special case of a more general theorem, namely [39, Theorem 2.4]. Proposition 6.4.6 as stated is sufficient for our purposes.

6.4.5 Application to étale cohomology

As explained in [39, p.563], after modifying C^m slightly if necessary, we may assume that it is a bounded complex of finitely generated free modules. Consider now the "finite level" complex $C_n^m = C^m \otimes_{\Lambda_m} \mathbb{Z}_p[G_n^m]$. Using [39, pp.573-575], one finds that the cohomology groups are described as follows: C_n^m is acyclic outside degrees 1 and 2, and there are exact sequences

$$0 \to (B_n^m)^+ \to \mathcal{O}_{L_n^m,S}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^1(C_n^m) \to 0 \tag{6.4.5}$$

$$0 \to \mathrm{Cl}(\mathcal{O}_{L_n^m,S}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^2(C_n^m) \to X_n^m \to 0, \tag{6.4.6}$$

where the map $(B_n^m)^+ \to \mathcal{O}_{L_n^m,S}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$ sends the image of $e_+ v_m$ in $(B_n^m)^+$ (which generates $(B_n^m)^+$ over $\mathbb{Z}_p[G_n^m]$) to $(1 - \zeta_m^{p^{-n}} \zeta_{p^{n+1}})(1 - \zeta_m^{-p^{-n}} \zeta_{p^{n+1}}^{-1})$.

Now let us take $m = 1$. Recalling the definition of X_n^m in Lemma 6.4.2, we see that $X_n^1 = 0$. Also, by [45, Ch.8] the map $(B_n^1)^+ \to \mathcal{O}_{L_n^1,S}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$ has finite cokernel. Therefore in order to apply Proposition 6.4.6, it would remain to check that $\mathrm{Hom}(H^1(C_n^1), \mathbb{Q}_p/\mathbb{Z}_p)$ is cyclic as a $\mathbb{Z}_p[G_n^1]$-module. In fact, as explained in [39, pp.575,576], this is the case if we replace C_n^1 by $C_n^{1,+}$, the complex obtained by taking plus parts for complex conjugation. Proposition 6.4.6 then says

$$\det(C_n^{1,+})^{-1} \mathrm{ann}_{\mathbb{Z}_p[G_1^{n,+}]}(U^{n,+}/\mathcal{E}^{n,+}) \subseteq \mathrm{Fitt}_{\mathbb{Z}_p[G_1^{n,+}]}(\mathrm{Cl}(\mathcal{O}_{L_1^{n,+},S}) \otimes_{\mathbb{Z}} \mathbb{Z}_p) \tag{6.4.7}$$

where $U^{n,+} = \mathcal{O}_{L_1^{n,+},S}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\mathcal{E}^{n,+}$ is the $\mathbb{Z}_p[G_1^{n,+}]$-submodule of $U^{n,+}$ generated by $(1 - \zeta_{p^{n+1}})(1 - \zeta_{p^{n+1}}^{-1})$.

6.4.6 Explicit descriptions of the determinants

For this section, we remove the assumption that $m = 1$. Let us first describe $\det(C^m)$. Since C^m becomes acyclic after tensoring with the total quotient ring $Q(\Lambda_m)$ ([8, Lemma 5.2]), we see from the remark before Proposition 6.4.6 that $\mathrm{D}_{\Lambda_m}(C^m) = \Lambda_m \det(C^m)^{-1}$. [8, Theorem 6.1] gives a basis for $\mathrm{D}_{\Lambda_m}(C^m)$, namely $e_+ + e_- g_m$ where g_m is an "equivariant Stickelberger element", obtained as a limit of Stickelberger elements. (See [8, Sec.5.2] for the definition of g_m, though note that what we call g_m is called $-g_m$ there.) Therefore $\det(C^m) = e_+ + e_- g_m \bmod \Lambda_m^\times$, and

$$\det(C^{m,+}) = 1 \bmod (\Lambda_m^+)^\times \tag{6.4.8}$$

where $\Lambda_m^+ = \lim_{\leftarrow n} \mathbb{Z}_p[G_m^{n,+}]$.

Now, if we had a natural map

$$Q(\Lambda_m^+)^\times/(\Lambda_m^+)^\times \to \mathbb{Q}_p[G_m^{n,+}]^\times/\mathbb{Z}_p[G_m^{n,+}]^\times, \tag{6.4.9}$$

then by the naturality of the construction of $\det(C^{m,+})$ and $\det(C_n^{m,+})$, the latter would be the image of the former under this map. However, the projection $\Lambda_m \to \mathbb{Z}_p[G_n^{m,+}]$ does not pass to total quotient rings and so the map in (6.4.9) does not exist.

6.4.7 Never-divisors-of-zero

This problem is solved by considering the set of so-called *never-divisors-of-zero* in Λ_m^+. This is defined to be the multiplicative subset of Λ_m^+ consisting of all those elements in Λ_m^+ whose image in $\mathbb{Z}_p[G_n^{m,+}]$ is a non-zero-divisor for all n. We let $\tilde{Q}(\Lambda_m^+)$ be the localization of Λ_m^+ at the never-divisors-of-zero, and observe that $\tilde{Q}(\Lambda_m^+)$ is a subring of $Q(\Lambda_m^+)$.

Let us return to the situation $m = 1$. It is a direct consequence of [28, Prop.4.4], [39, Prop.4.5] and the isomorphism $H^2(C^{1,+}) \simeq \mathrm{Cl}_p^{\infty,+}$ that $C^{1,+} \otimes_{\Lambda_1^+} \tilde{Q}(\Lambda_m^+)$ is exact. The naturality of the det construction in Section 6.4.4 then shows that $\det(C_n^{1,+})$ is the image of $\det(C^{1,+})$ under

$$\tilde{Q}(\Lambda_1^+)^\times/(\Lambda_1^+)^\times \to \mathbb{Q}_p[G_n^{1,+}]^\times/\mathbb{Z}_p[G_n^{1,+}]^\times.$$

Referring back to (6.4.8), we therefore see that $\det(C_n^{1,+}) = 1 \bmod \mathbb{Z}_p[G_n^{1,+}]^\times$.

6.4.8 The annihilator statement

Having found $\det(C_n^{1,+})$, we can conclude our description of the relationship between the fractional Galois ideal and class-groups in the present situtation. Since the unique prime of L_n^1 above p is principal (generated by $(1-\zeta_{p^{n+1}})(1-\zeta_{p^{n+1}}^{-1})$), [24, Prop.11.6] shows that $\mathrm{Cl}(L_1^{n,+}) = \mathrm{Cl}(\mathcal{O}_{L_1^{n,+},S})$, so (6.4.7) becomes

$$\mathrm{ann}_{\mathbb{Z}_p[G_1^{n,+}]}(U^{n,+}/\mathcal{E}^{n,+}) \subseteq \mathrm{Fitt}_{\mathbb{Z}_p[G_1^{n,+}]}(\mathrm{Cl}(L_1^{n,+}) \otimes_{\mathbb{Z}} \mathbb{Z}_p).$$

This completes the proof of Proposition 6.4.1, because of the isomorphism $U^{n,+}/\mathcal{E}^{n,+} \simeq (\mathcal{O}^{\times}_{L_n^+}/\mathfrak{C}_n) \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Remark. Although Vandiver's Conjecture predicts the p-part of $\mathrm{Cl}(L^+)$ to be trivial, the techniques used here are hoped to be applicable to more general fields.

Chapter 7

Further questions

In this final chapter, we will turn to some of the questions raised by the work carried out in the main body of the thesis, touching on whether the expected unification of the equivariant motivic phenomena brought about by the Equivariant Tamagawa Number Conjecture [7, Conj.4] can help the fractional Galois ideal find its proper place. We do not aim to provide conclusive answers; we intend simply to ask questions that are natural and reasonable.

7.1 The ETNC

I am grateful to David Burns for having made me aware of [6]. For brevity, many details and notational aspects have been condensed here.

The link between L-functions and class-groups is not an isolated phenomenon. For example, in Chapter 1 we discussed briefly analogous behaviour for higher K-groups. Also, the Selmer groups of elliptic curves are expected to be related in a similar way to values of L-functions of elliptic curves. These objects – class-groups, K-groups, Selmer groups of elliptic curves – all have the common property of arising, in essence, as cohomology groups of cochain complexes occurring naturally in arithmetic.

To formulate the Equivariant Tamagawa Number Conjecture (ETNC) in full generality would not be possible here. However, for a large class of motives, we can give an idea of what it is saying. Take a Galois extension L/K of number fields and suppose we have a motive M defined over K of the type discussed in [6, Section 4.3] (the examples above all come from such motives). We can form from this a motive M_L which has an action of $G = \mathrm{Gal}(L/K)$, and associated to M_L we have a canonical element ζ_{M_L} in the centre of $\mathbb{R}[G]$ defined in terms of special values of L-functions. On the other hand, one can also attach to the motive a cochain complex C_{M_L} (together with a regulator isomorphism $\lambda_{M_L} : H^0(C_{M_L}) \otimes_{\mathbb{Z}} \mathbb{R} \to H^1(C_{M_L}) \otimes_{\mathbb{Z}} \mathbb{R}$) that should contain arithmetic information on the motive.

We can compare ζ_{M_L} and C_{M_L} in a meaningful way by constructing from each an element of the relative K-group $K_0(\mathbb{Z}[G], \mathbb{R}[G])$. The ETNC then says that these elements are in fact equal, predicting a deep link between analytically defined and algebraically defined objects associated to the motive.

7.1.1 Generalizing fractional Galois ideals

An important consequence of the ETNC holding for a pair $(L/K, M)$, where M is a motive over K of the type required in the above discussion, is that the natural analogue of the Stark Conjecture for the leading coefficients of the associated L-functions holds. In this conjecture, the module X of Chapter 3 is replaced by $H^1(C_{M_L})$ and Dirichlet's regulator by λ_{M_L}.

Let us again assume that L/K is abelian (we will return to the non-abelian case in Section 7.3.1); then our first question is:

Can we construct a fractional Galois ideal $\mathcal{J}(L/K, M) \subseteq \mathbb{Q}[G]$ in an analogous way to Chapter 4?

Recalling the definition of $\mathcal{J}(L/K, S)$, we see that all that was required was a choice of lattice in $(\bigwedge^r \mathcal{O}_{L,S}^\times) \otimes_{\mathbb{Z}} \mathbb{Q}$ for each $r \geq 0$. (In that situation, the lattice chosen was the image of $\bigwedge_0^r \mathcal{O}_{L,S}^\times$.) Of course, in order to be able to relate $\mathcal{J}(L/K, S)$ to Stark elements using Theorem 4.3.3, it is important that the lattice be large enough to contain the Stark elements. As we saw, this is conjectured by Rubin to be the case for the lattices $\bigwedge_0^r \mathcal{O}_{L,S}^\times$. (This is taking the viewpoint that the Stark elements always exist *somewhere* in $(\bigwedge^r \mathcal{O}_{L,S}^\times) \otimes_{\mathbb{Z}} \mathbb{Q}$ so long as the Stark Conjecture holds, and it is a case of identifying what denominators can come in.)

In general, it might not be clear what lattices to take inside $(\bigwedge^r H^1(C_{M_L})) \otimes_{\mathbb{Z}} \mathbb{Q}$, so that a formulation of an integral Stark conjecture of the required form, let alone a proof, may not be forthcoming. However, in the case $r = 1$, [6, Remark 5.3] explains how the truth of the ETNC would give an explicit lattice in which the rank 1 Stark elements would lie. Consequently, $\mathcal{J}(L/K, M)$ could be defined so that at least $e[0]\mathcal{J}(L/K, M)$ and $e[1]\mathcal{J}(L/K, M)$ would be meaningful. Indeed, the proof of Theorem 4.3.3 should carry over almost without change, so that we ought to be able to construct from $\mathcal{J}(L/K, M)$ annihilators of an appropriate lattice modulo the rank 1 Stark elements.

7.2 Annihilating arithmetic objects

Let us observe that in this thesis, the underlying motive has been (tacitly) the Tate motive $M = h^0(\text{Spec } K)(0)$. In this case, $\text{tors}(H^1(C_{M_L})) = \text{Cl}(L)$, which we have been seeking to construct annihilators for. The natural question to ask is then

Does $\mathcal{J}(L/K, M)$ give rise to annihilators of $\text{tors}(H^1(C_{M_L}))$ *more generally?*

In the case of the Tate motive $M = h^0(\text{Spec } K)(k)$ with $k < 0$, where $\text{tors}(H^1(C_{M_L}))$ is the torsion in an even K-group of L, $\mathcal{J}(L/K, M)$ is essentially the fractional ideal $\mathcal{J}^k(L/K)$ defined in [40], where Snaith gave a positive answer to this question for $K = \mathbb{Q}$ assuming the Lichtenbaum–Quillen Conjecture.

In the case where M is the motive $h^1(E)(1)$ for an elliptic curve over $\mathbb{Q}$, $\mathrm{tors}(H^1(C_{M_L}))$ is closely related to the Shafarevich–Tate group (the dual of the torsion in the Selmer group) of $E_{/L}$, further highlighting the significance of $H^1(C_{M_L})$ generally.

Supposing we do indeed have the desired relationship between $\mathcal{J}(L/K, M)$ and Stark elements discussed in Section 7.1.1, the issue becomes one of constructing annihilators of $H^1(C_{M_L})$ from Stark elements. For evidence that such constructions are possible, we cite [6, Theorem 5.5], which provides sufficient conditions. We also recall the discussion in Section 1.3.1 concerning Rubin and Popescu's work on Euler systems arising from Stark elements, and draw attention to recent work of Büyükboduk [9] which removes certain restrictions which had been imposed on characters $\chi : \mathrm{Gal}(\bar{K}/K) \to \mathbb{Z}_p^\times$ for studying the corresponding parts of class-groups, where K is a totally real field.

7.2.1 Stark elements and splittings of localization sequences

We remark on an interesting idea of Banaszak in [1]. The aim of [1] is to show that for an even K-group of a number field L that is abelian over $\mathbb{Q}$, the corresponding (higher) Stickelberger element annihilates the group of divisible elements, which lies in the K-group of the ring of integers. The method makes use of an observation that a group-ring element α annihilates the group of divisible elements if and only if a splitting of a certain map δ in the localization sequence can be constructed so that δ followed by that splitting has the effect of multiplying by α. In the case where α is a higher Stickelberger element, an explicit splitting map is constructed in [1, Ch.IV, Section 1].

We remind the reader of the elements $\mathcal{A}^{\mathbf{f}}[r]$ defined in Section 4.1.2 for $r \geq 0$, and of the fact that $\mathbf{f}$ can be chosen such that $\mathcal{A}^{\mathbf{f}}[0] = \theta_{L/K,S}$. Is it possible to define splittings as above in the case $\alpha = \mathcal{A}^{\mathbf{f}}[r]$ for any $r \geq 0$, making use of the relationship in (3.6.2) between $\theta_{L/K,S}[r]$ and the rank r Stark elements? (Recall the definition of $\theta_{L/K,S}[r]$ in Definition 3.4.2, remembering that it means a Stickelberger element defined in terms of rth derivatives at $s = 0$, not a higher Stickelberger element defined at a negative integer.) The author is not aware of any such approach having been attempted before, but perhaps it could be fruitful as an alternative way of forming annihilators of class-groups from L-function derivatives via Stark elements.

7.3 Iwasawa theory

In Section 6.3 we saw how a limit of fractional Galois ideals in a $\mathbb{Z}_p$-extension gave rise to Fitting ideals of limits of class-groups. Can we expect similar statements to hold in greater generality? Although explicit examples were not given, the discussion in Section 4.4.3 and Proposition 4.4.6 show in particular that if F is an imaginary quadratic field, p is an odd prime not splitting in $F/\mathbb{Q}$, and F_n is the degree p^n subextension of the cyclotomic $\mathbb{Z}_p$-extension of F, then $\mathcal{J}(F_n/F, S)$ can be described precisely in terms of Stark units, where S consists of the infinite place of F and the unique place above p. Here the Stark units, which are the elliptic units in this situation, are well known by Iwasawa theory to be related to class-groups in a similar way to the case of cyclotomic units for the base field $\mathbb{Q}$. Could a limit of the $\mathcal{J}(F_n/F, S)$ therefore also be shown to give rise to a Fitting ideal of a limit of class-groups? One would need to take some care over what happens p-adically to the denominators in the $\mathcal{J}(F_n/F, S)$. One would hope, as in Section 6.3, that multiplying by annihilators of roots of unity would remove any denominators.

7.3.1 Non-commutative fractional Galois ideals

There is currently much interest in non-commutative Iwasawa theory, in which the Galois groups of the infinite Galois extensions under consideration may be non-abelian p-adic Lie groups. This situation arises, for example, if one considers the extension obtained by adjoining to the base field the coordinates of all points of p-power order on the elliptic curve. Is it possible, then, to define fractional Galois ideals for non-abelian Galois extensions? This question will be addressed in the paper in preparation [3], where non-commutative fractional Galois ideals are constructed for Tate motives $h^0(\mathrm{Spec}\ K)(k)$. [3] will show that

- (i) the natural maps on rational group-rings coming from quotient maps on groups map fractional Galois ideals into fractional Galois ideals,
- (ii) if the commutative fractional Galois ideals can be made integral (in a suitable way), then the same holds in the non-commutative case, and
- (iii) if the commutative fractional Galois ideals annihilate the corresponding arithmetic objects (which in this situation are K-groups in even degrees), then the same holds in the non-commutative case.

One of the important contributions of [11] to Iwasawa theory was to show how to define a characteristic ideal for a large class of finitely generated modules over completed group-rings of compact p-adic Lie groups. It is by means of characteristic ideals that one can compare the (conjectural) p-adic L-functions to limits of algebraic objects like class-groups and Shafarevich–Tate groups. (For example, in the simple case when the Galois group is the compact p-adic Lie group $\mathbb{Z}_p$, characteristic ideals are Fitting ideals, as in Section 6.3.) This prompts us to ask:

Can limits of fractional Galois ideals in (possibly non-abelian) p-adic Lie extensions be viewed as characteristic ideals of limits of suitable arithmetic objects?

If this were the case, then the place of the fractional Galois ideal in Iwasawa theory could be an interesting one. Indeed, its definition is not on the side of Selmer groups, and yet nor is it defined in the same way as p-adic L-functions – analytically defined p-adic L-functions (if they exist) are continuous p-adic functions which interpolate complex L-functions at integers, while algebraic p-adic L-functions (again, if they exist) are elements of localizations of K-groups which interpolate complex L-functions "algebraically" at Artin representations. An advantage brought about, should the above question have a positive answer, is that the fractional ideals would arguably be a more concrete object to study than characteristic ideals themselves, whose definition [11, (33)] is somewhat abstract. One would then hope that the connection with p-adic L-functions predicted by main conjectures in Iwasawa theory would be more explicit.

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