Linear Algebra II (MATH 225): Solutions to the Practice Problems – v 1.12

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1. (a) If $v \in V$, then

$$0v = (0+0)v = 0v + 0v \tag{1}$$

by distributivity. If we let w be the additive inverse of $0v$, so that $0v + w = \mathbf{0}$, then (1) gives

$$\begin{aligned} 0v + w &= (0v + 0v) + w \\ \text{i.e.,} \quad \mathbf{0} &= 0v + (0v + w) \\ &= 0v + \mathbf{0} \\ &= 0v, \end{aligned}$$

as required.

(b) If $a \in \mathbb{R}$ and $v \in V$, then

$$\begin{aligned} (-a)v + av &= (-a + a)v \\ &= 0v \\ &= \mathbf{0} \end{aligned}$$

by part (a). Thus, $(-a)v$ is the additive inverse of av, which is to say $(-a)v = -(av)$.

2. We wish to find $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$\begin{aligned} 2x^2 + 7x + 10 &= a_1(x+1)^2 + a_2(x+1) + a_3 \cdot 1 \\ &= a_1(x^2 + 2x + 1) + a_2(x+1) + a_3 \\ &= a_1 x^2 + (2a_1 + a_2)x + (a_1 + a_2 + a_3). \end{aligned}$$

Hence, equating coefficients of the powers of x, we are to solve the equations

$$\begin{aligned} a_1 \qquad\quad &= 2 \\ 2a_1 + a_2 \qquad &= 7 \\ a_1 + a_2 + a_3 &= 10 \end{aligned}$$

We may read off straight away that the solution is $a_1 = 2$, $a_2 = 3$, $a_3 = 5$. Thus, $p = 2p_1 + 3p_2 + 5p_3$.

3. We show that in fact f is not a linear combination of g and h. Suppose it were, i.e., that there were $a, b \in \mathbb{R}$ such that $f = ag + bh$. Then $f(x) = ag(x) + bh(x)$ for all $x \in \mathbb{R}$, i.e.,

$$x^3 = a(x+1) + b\ln(x^2+1) \quad \text{for all } x \in \mathbb{R}.$$

In this equation, choose $x = 0, 1, -1$ in turn:

$$\begin{aligned} 0 &= a \quad (\text{when } x = 0) & (2)\\ 1 &= 2a + b\ln(2) \quad (\text{when } x = 1) & (3)\\ -1 &= b\ln(2) \quad (\text{when } x = -1) & (4) \end{aligned}$$

Equations (2)–(4) have no common solution for a and b. Indeed, because the first equation states that $a = 0$, the second two state, respectively, $1 = b\ln(2)$ and $-1 = b\ln(2)$, a contradiction.

4. The nth term of s is

$$\begin{aligned} (n+1)^3 &= n^3 + 3n^2 + 3n + 1\\ &= (n^3 + 1) + 3n(n+1), \end{aligned}$$

so $s = t + 3u$.

5. Observe that

$$A^2 = \begin{pmatrix} 1 & 2\\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2\\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10\\ 15 & 22 \end{pmatrix},$$

so we are to solve

$$\begin{pmatrix} 7 & 10\\ 15 & 22 \end{pmatrix} = c\begin{pmatrix} 1 & 2\\ 3 & 4 \end{pmatrix} + d\begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c+d & 2c\\ 3c & 4c+d \end{pmatrix}$$

for $c, d \in \mathbb{R}$. Equating corresponding entries, we obtain the four equations

$$\begin{aligned} c + d &= 7\\ 2c \quad &= 10\\ 3c \quad &= 15\\ 4c + d &= 22 \end{aligned}$$

The system has a unique solution, namely $c = 5$, $d = 2$, so $A^2 = 5A + 2I$.

6. Because A has trace zero, we may write it as

$$A = \begin{pmatrix} a & b\\ c & -a \end{pmatrix}.$$

Squaring A, we obtain

$$A^2 = \begin{pmatrix} a & b\\ c & -a \end{pmatrix}\begin{pmatrix} a & b\\ c & -a \end{pmatrix} = \begin{pmatrix} a^2+bc & 0\\ 0 & a^2+bc \end{pmatrix} = \begin{pmatrix} \lambda & 0\\ 0 & \lambda \end{pmatrix},$$

where $\lambda = a^2 + bc = -\det(A)$, a non-zero integer. (Remember that A is assumed to have integer entries and to be invertible.) Hence,

$$A^4 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}^2 = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix} = nI,$$

where $n = \lambda^2$, a positive integer.

7. We have the well-known trigonometric identity $\cos(2x) = 2\cos^2(x) - 1$, i.e., $f(x) = 2g(x) - h(x)$. This being true for all $x \in \mathbb{R}$, we see that $f = 2g - h$ as functions.

8. The function f is not a linear combination of g and h. We prove this by contradiction. Suppose that $f = ag + bh$ for some $a, b \in \mathbb{R}$. That is, $f(x) = ag(x) + bh(x)$ for all $x \in \mathbb{R}$, which is to say

$$\sin(2x) = a\cos(x) + b\sin(x) \quad \text{for all } x \in \mathbb{R}.$$

In this equation, choose $x = 0, \pi/2, \pi/4$ in turn:

$$\begin{aligned} 0 &= a \quad (\text{when } x = 0) && (5) \\ 0 &= b \quad (\text{when } x = \pi/2) && (6) \\ 1 &= (a+b)/\sqrt{2} \quad (\text{when } x = \pi/4) && (7) \end{aligned}$$

Equations (5)–(7) have no common solution for a and b, since the first two state that $a = b = 0$, while the third implies that $a + b \neq 0$. This gives us the desired contradiction.

9. All the axioms except (i) and (vi) hold. Let us show this by going through the eight axioms in turn.

(i) This axiom (associativity) does not hold. For example,

$$\begin{aligned} & (1 \oplus 2) \oplus 3 = |1-2| \oplus 3 = 1 \oplus 3 = |1-3| = 2, \\ \text{while} \quad & 1 \oplus (2 \oplus 3) = 1 \oplus |2-3| = 1 \oplus 1 = |1-1| = 0. \end{aligned}$$

(ii) This axiom (commutativity) does hold: If $u, v \in \mathbb{R}_{\geq 0}$, then

$$u \oplus v = |u - v| = |v - u| = v \oplus u.$$

(iii) This axiom does hold: If $u \in \mathbb{R}_{\geq 0}$, then

$$u \oplus 0 = |u - 0| = |u| = u,$$

because $u \geq 0$ by assumption.

(iv) This axiom does hold: Let $u \in \mathbb{R}_{\geq 0}$, and let $v = u$. Then

$$u \oplus v = |u - v| = |u - u| = 0.$$

(v) This axiom does hold: If $u, v \in \mathbb{R}_{\geq 0}$ and $c \in \mathbb{R}$, then

$$\begin{aligned}
c \odot (u \oplus v) &= |c|(u \oplus v) \quad \text{by definition of } \odot \\
&= |c|\,|u - v| \quad \text{by definition of } \oplus \\
&= \big||c|(u - v)\big| \\
&= \big||c|u - |c|v\big| \\
&= (|c|u) \oplus (|c|v) \quad \text{by definition of } \oplus \\
&= (c \odot u) \oplus (c \odot v) \quad \text{by definition of } \odot.
\end{aligned}$$

(vi) This axiom does not hold. For example, let $c = d = 1 \in \mathbb{R}$, and let $u = 1 \in \mathbb{R}_{\geq 0}$. Then

$$(c + d) \odot u = (1 + 1) \odot 1 = 2 \odot 1 = |2|1 = 2,$$
$$\text{while} \quad (c \odot u) \oplus (d \odot u) = (1 \odot 1) \oplus (1 \odot 1) = |1 - 1| = 0.$$

(vii) This axiom does hold: If $c, d \in \mathbb{R}$ and $u \in \mathbb{R}_{\geq 0}$, then

$$\begin{aligned}
(cd) \odot u &= |cd|u \\
&= |c|\,|d|u \\
&= |c|(d \odot u) \\
&= c \odot (d \odot u).
\end{aligned}$$

(viii) This axiom does hold: If $u \in \mathbb{R}_{\geq 0}$, then $1 \odot u = |1|u = 1u = u$.

10. (a) B_1 is a subspace. Certainly it contains the zero polynomial, because the derivative of the zero polynomial is identically zero. Next, if $p'(a) = q'(a) = 0$, then $(p + q)'(a) = p'(a) + q'(a) = 0$, and if b is a scalar, $(bp)'(a) = b(p'(a)) = b \cdot 0 = 0$. Therefore, B_1 is closed under addition and scalar multiplication.

(b) B_2 is not a subspace. For example, the zero polynomial is not in B_2.

(c) B_3 is a subspace. One way to see this is to observe that B_3 is the set of constant polynomials, which is a subspace: The zero polynomial is constant, and adding or scaling a constant polynomial yields a constant polynomial. Alternatively, we may work directly with the derivative: If $p' = q' = 0$, then $(p + q)' = p' + q' = 0 + 0 = 0$, and if b is a scalar, then $(bp)' = bp' = b \cdot 0 = 0$.

(d) B_4 is not a subspace. For example, $x + 2 \in B_4$, but $\frac{1}{2}(x + 2) \notin B_4$, so B_4 is not closed under scalar multiplication.

11. (a) B_1 is a subspace. Certainly it contains the zero sequence, because $0 = 2\cdot 0 - 3\cdot 0$. Next, if $\alpha = (a_n)_n$ and $\beta = (b_n)_n$ are in B_1 and c_n is the nth term in the sequence $\alpha + \beta$, then

$$\begin{aligned} c_n &= a_n + b_n \\ &= 2a_{n-1} - 3a_{n-2} + 2b_{n-1} - 3b_{n-2} \\ &= 2(a_{n-1} + b_{n-1}) - 3(a_{n-2} + b_{n-2}) \\ &= 2c_{n-1} - 3c_{n-2}, \end{aligned}$$

so $\alpha + \beta \in B_1$. Finally, if $\alpha = (a_n)_n \in B_1$, $\lambda \in \mathbb{R}$, and c_n is the nth term in the sequence $\lambda\alpha$, then

$$\begin{aligned} c_n &= \lambda a_n \\ &= \lambda(2a_{n-1} - 3a_{n-2}) \\ &= 2\lambda a_{n-1} - 3\lambda a_{n-2} \\ &= 2c_{n-1} - 3c_{n-2}, \end{aligned}$$

so $\lambda\alpha \in B_1$.

(b) B_2 is a subspace. It is clear that B_2 contains the zero sequence. Now suppose $\alpha = (a_n)_n$ and $\beta = (b_n)_n$ are in B_2, and let c_n be the nth term in the sequence $\alpha + \beta$. If 3 divides n, then $a_n = b_n = 0$, so $c_n = a_n + b_n = 0 + 0 = 0$. Further, if $\alpha = (a_n)_n \in B_2$, $\lambda \in \mathbb{R}$, and c_n is the nth term in the sequence $\lambda\alpha$, then $c_n = \lambda a_n = \lambda \cdot 0 = 0$ whenever 3 divides n.

(c) B_3 is not a subspace. For example, the sequence $\alpha = (a_n)_n$ where $a_n = n$ is in B_3, but $2\alpha = (0, 2, 4, 6, \ldots)$ is not. This shows simultaneously that B_3 is closed neither under addition nor under scalar multiplication. Alternatively, we may just observe that B_3 does not contain the zero sequence.

(d) B_4 is not a subspace. For example, consider the sequences $\alpha = (n)_n$ and $\beta = (-n)_n$. Then α and β are both in B_4, but $\alpha + \beta$ is the zero sequence, which is not in B_4. Or again, we may observe simply that B_4 does not contain the zero sequence. Alternatively, we may show that B_4 is not closed under scalar multiplication.

12. (a) B_1 is not a subspace of $\mathcal{P}_2$. The easiest way to see this is to observe that the zero polynomial is not in B_1. Indeed, if p is the zero polynomial, then $p + xp'$ is still the zero polynomial, which does not have degree 2.

(b) B_2 is a subspace of $M_2(\mathbb{R})$. If X is the zero 2×2 matrix, then XA is the zero 2×3 matrix by definition of matrix multiplication. Thus, B_2 contains the zero 2×2 matrix and is therefore non-empty. Now suppose that $X, Y \in B_2$. Then

$$(X + Y)A = XA + YA$$

$$
\begin{aligned}
&= 0 + 0 \\
&= 0,
\end{aligned}
$$

so $X + Y \in B_2$. Thus, B_2 is closed under addition. Finally, if $X \in B_2$ and $c \in \mathbb{R}$, then

$$
\begin{aligned}
(cX)A &= cXA \\
&= c0 \\
&= 0,
\end{aligned}
$$

so $cX \in B_2$, showing that B_2 is closed under scalar multiplication as well.

There is an alternative solution exploiting a sneaky observation about the subspace B_2: it is in fact just the zero subspace of $M_2(\mathbb{R})$. Indeed, suppose $X \in B_2$. Then because $XA = 0$, it follows from a standard property of matrix multiplication that $X\mathbf{w} = \mathbf{0}$ for each column $\mathbf{w}$ of A. In particular, considering the first two columns of A, we obtain

$$
\begin{aligned}
X \begin{pmatrix} 1 \\ 4 \end{pmatrix} &= \mathbf{0} \\
X \begin{pmatrix} 2 \\ 5 \end{pmatrix} &= \mathbf{0}.
\end{aligned}
$$

Now, let the columns of X be $\mathbf{u}$ and $\mathbf{v}$, so that $X = \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}$. Then the above two equations say, respectively,

$$
\begin{aligned}
\mathbf{u} + 4\mathbf{v} = \mathbf{0} \\
2\mathbf{u} + 5\mathbf{v} = \mathbf{0}
\end{aligned}
$$

Solving this system for $\mathbf{u}$ and $\mathbf{v}$ gives $\mathbf{u} = \mathbf{v} = \mathbf{0}$, so X is the zero matrix.

This alternative solution would not work for an arbitrary matrix $A \in M_{2,3}(\mathbb{R})$, so it is a good idea to understand the first solution.

(c) B_3 is not a subspace of $\mathcal{S}$. For example, it is not closed under addition. To see this, let $s = (1, 1, 1, 1, \ldots)$ and $t = (2, 4, 16, 256, \ldots)$. (Thus, t is the sequence whose zeroth term is 2, and in which each term after that is the square of the previous term.) Both s and t are in B_3, but $s + t = (3, 5, \ldots)$, and $5 \neq 3^2$. Thus, $s + t \notin B_3$, so B_3 is not closed under addition.

Actually, B_3 is not closed under scalar multiplication either. For example, the sequence $s = (1, 1, 1, 1, \ldots)$ is in B_3, but the sequence $2s = (2, 2, 2, 2, \ldots)$ is not, because $2^2 \neq 2$.

(d) B_4 is a subspace of $\mathcal{F}$. It is non-empty, because it contains the zero function, z. Indeed, $z(0) = 0 = z(1)$. Now suppose that $f, g \in B_4$. Then

$$
(f + g)(0) = f(0) + g(0) \quad \text{by definition of addition in } \mathcal{F}
$$

$$\begin{aligned} &= f(1) + g(1) \quad \text{because } f, g \in B_4 \\ &= (f+g)(1) \quad \text{by definition of addition again,} \end{aligned}$$

so $f + g \in B_4$. Finally, if $f \in \mathcal{F}$ and $c \in \mathbb{R}$, then

$$\begin{aligned} (cf)(0) &= cf(0) \quad \text{by definition of scalar multiplication in } \mathcal{F} \\ &= cf(1) \quad \text{because } f \in B_4 \\ &= (cf)(1) \quad \text{by definition of scalar multiplication again,} \end{aligned}$$

so $cf \in B_4$.

13. (a) A is closed under addition. If $(x_1, x_2, x_3), (y_1, y_2, y_3) \in A$, then x_1+y_1, x_2+y_2, and $x_3 + y_3$ are all positive. Further, because all terms in the expansion of the product $(x_1 + y_1)(x_2 + y_2)(x_3 + y_3)$ are positive, we have

$$(x_1 + y_1)(x_2 + y_2)(x_3 + y_3) > x_1x_2x_3 > 1,$$

so $(x_1 + y_1, x_2 + y_2, x_3 + y_3) \in A$.

(b) B is not closed under addition. For example, $(1, 1, 2)$ and $(-1, -1, 2)$ are in B, but their sum, $(0, 0, 4)$, is not.

14. (a) B_1 is not linearly independent, because

$$(2x + 1) - 2(3x + 2) + (4x + 3) = 0.$$

The set B_1 does not span $\mathcal{P}_2$, because, for example, x^2 is not in $\operatorname{Span}(B_1)$.

(b) B_2 is linearly independent: If $a, b, c \in \mathbb{R}$ and

$$ax + b(x + 2) + c(-x^2) = 0,$$

then

$$2b + (a + b)x - cx^2 = 0,$$

i.e.,

$$\begin{aligned} 2b &= 0 \\ a + b &= 0 \\ -c &= 0, \end{aligned}$$

so $a = b = c$.

B_2 spans $\mathcal{P}_2$. Once we have seen more theory, we will have at our disposal quick ways to show this. For now, let us do it the long way. Given $a_1x^2 + a_2x + a_3 \in \mathcal{P}_2$, we wish to decide whether there exist $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$a_1x^2 + a_2x + a_3 = c_1x + c_2(x + 2) + c_3(-x^2)$$

$$= \quad -c_3x^2 + (c_1 + c_2)x + 2c_2.$$

This amounts to solving the system

$$\begin{aligned} -c_3 &= a_1 \\ c_1 + \ \ c_2 \qquad &= a_2 \\ 2c_2 \qquad &= a_3 \end{aligned}$$

for $c_1, c_2, c_3 \in \mathbb{R}$, and we see that it has the solution

$$c_1 = a_2 - \frac{1}{2}a_3, \quad c_2 = \frac{1}{2}a_3, \quad c_3 = -a_1.$$

(c) B_3 is linearly independent, because the only scalars a, b such that $2a+bx^2 = 0$ (the zero polynomial) are $a = b = 0$. However, B_3 does not span $\mathcal{P}_2$, because there is no solution in $a, b \in \mathbb{R}$ to the equation $2a + bx^2 = x$ of polynomials.

(d) Let $p_1 = x + 2$, $p_2 = x - 1$, $p_3 = x^2$, and $p_4 = x^2 - 3$. Then B_4 is not linearly independent, because $p_1 - p_2 - p_3 + p_4 = 0$. (Or, once we have seen the notion of *dimension*, we may simply observe that 4 vectors in a 3-dimensional space cannot be linearly independent.)

B_4 does span $\mathcal{P}_2$: Given $a_1x^2 + a_2x + a_3 \in \mathcal{P}_2$, we wish to determine whether we may solve

$$\begin{aligned} a_1x^2 + a_2x + a_3 &= c_1(x + 2) + c_2(x - 1) + c_3x^2 + c_4(x^2 - 3) \\ &= (c_3 + c_4)x^2 + (c_1 + c_2)x + 2c_1 - c_2 - 3c_4 \end{aligned}$$

for $c_1, c_2, c_3, c_4 \in \mathbb{R}$. The system to solve is

$$\begin{aligned} c_3 + \ \ c_4 &= a_1 \\ c_1 + c_2 \qquad\qquad &= a_2 \\ 2c_1 - c_2 \qquad - 3c_4 &= a_3 \end{aligned}$$

All we are interested in is *whether* the system has a solution for any given a_1, a_2, a_3, not what the solutions are if so. For that, we need only decide whether the coefficient matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & -3 \end{pmatrix}$$

has a pivot in every row of a row-echelon form. The following row-echelon form shows that indeed there is a pivot in every row:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

15. (a) Consider $A = \{f \in \mathcal{F} \mid f(0) \in \mathbb{Z}\}$. Then A is closed under addition, for if $f, g \in A$, then

$$(f+g)(0) = f(0) + g(0) \in \mathbb{Z},$$

because the sum of two integers is an integer. However, A is not closed under scalar multiplication. For example, the function $f(x) = x + 1$ is in A, but $\frac{1}{2}f$ is not because its value at 0 is $1/2$.

(b) Let $B = \{f \in \mathcal{F} \mid f(0)f(1) = 0\}$. If $f \in B$ and $c \in \mathbb{R}$, then

$$\begin{aligned} (cf)(0) \cdot (cf)(1) &= cf(0) \cdot cf(1) \\ &= c^2 f(0)f(1) \\ &= c^2 \cdot 0 \\ &= 0, \end{aligned}$$

so $cf \in B$. Thus, B is closed under scalar multiplication. However, B is not closed under addition. To see this, consider the functions $f(x) = x$ and $g(x) = x - 1$. Both f and g are in B, but their sum, $h = f + g$, is not:

$$\begin{aligned} h(0)h(1) &= (f+g)(0) \cdot (f+g)(1) \\ &= (f(0)+g(0))(f(1)+g(1)) \\ &= g(0)f(1) \\ &= -1. \end{aligned}$$

16. First, note that

$$A^2 = \begin{pmatrix} 19 & 27 \\ 45 & 64 \end{pmatrix}.$$

Now, suppose that $c_1 I + c_2 A + c_3 A^2 = 0$, where $c_1, c_2, c_3 \in \mathbb{R}$, i.e.,

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} &= c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} + c_3 \begin{pmatrix} 19 & 27 \\ 45 & 64 \end{pmatrix} \\ &= \begin{pmatrix} c_1 + 2c_2 + 19c_3 & 3c_2 + 27c_3 \\ 5c_2 + 45c_3 & c_1 + 7c_2 + 64c_3 \end{pmatrix}. \end{aligned}$$

Then we obtain the linear system

$$\begin{aligned} c_1 + 2c_2 + 19c_3 &= 0 \\ 3c_2 + 27c_3 &= 0 \\ 5c_2 + 45c_3 &= 0 \\ c_1 + 7c_2 + 64c_3 &= 0 \end{aligned}$$

This system is represented by the matrix

$$\begin{pmatrix} 1 & 2 & 19 \\ 0 & 3 & 27 \\ 0 & 5 & 45 \\ 1 & 7 & 64 \end{pmatrix},$$

and after a few row operations, we find the row-echelon form

$$\begin{pmatrix} 1 & 2 & 19 \\ 0 & 1 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The system therefore has a free variable, so the equation $c_1 I + c_2 A + c_3 A^2 = 0$ has non-trivial solutions. Thus, I, A, A^2 are not linearly independent.

17. Suppose $c_1, c_2, c_3 \in \mathbb{R}$ satisfy $c_1 s + c_2 t + c_3 u = 0$, where 0 here means the zero sequence. The nth term of $c_1 s + c_2 t + c_3 u$ is $c_1(n+1) + c_2(2n+1) + c_3 n^2$, so we have

$$c_1(n+1) + c_2(2n+1) + c_3 n^2 = 0 \quad \text{for all integers } n \geq 0. \tag{8}$$

Taking $n = 0, 1, 2$ respectively in this equation gives

$$\begin{aligned} c_1 + c_2 \phantom{{}+4c_3} &= 0 \\ 2c_1 + 3c_2 + c_3 &= 0 \\ 3c_1 + 5c_2 + 4c_3 &= 0 \end{aligned}$$

This system is represented by the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 3 & 5 & 4 \end{pmatrix},$$

and a row-echelon form of this matrix is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

which has a pivot in every column. The system therefore has only the trivial solution $c_1 = c_2 = c_3 = 0$, so s, t, u are linearly independent.

A slightly different solution is as follows. Because the equation in (8) holds for infinitely many n, the polynomial $f(x) = c_1(x+1) + c_2(2x+1) + c_3 x^2$ must be the zero polynomial, so the coefficients of the powers of x in $f(x)$ are all zero. Rearranging, we have $f(x) = c_3 x^2 + (c_1 + 2c_2)x + c_1 + c_2$, so

$$c_3 = 0$$

$$
\begin{aligned}
c_1 + 2c_2 \quad &= 0 \\
c_1 + c_2 \quad &= 0
\end{aligned}
$$

This system has only the solution $c_1 = c_2 = c_3 = 0$.

18. (a) We calculate the first two derivatives of f as follows:

$$
\begin{aligned}
f'(x) &= 2axe^{ax^2} = 2axf(x) \\
f''(x) &= 2af(x) + 2axf'(x) = (2a + 4a^2x^2)f(x).
\end{aligned}
$$

Suppose, then, that $\lambda f + \mu f' + \nu f'' = 0$ for some $\lambda, \mu, \nu \in \mathbb{R}$. Then

$$
\begin{aligned}
0 &= \lambda f(x) + \mu \cdot 2axf(x) + \nu(2a + 4a^2x^2)f(x) \\
&= (4a^2\nu x^2 + 2a\mu x + 2a\nu + \lambda)f(x)
\end{aligned}
$$

for all $x \in \mathbb{R}$. Hence, because $f(x)$ is non-zero for all $x \in \mathbb{R}$,

$$4a^2\nu x^2 + 2a\mu x + 2a\nu + \lambda = 0 \quad \text{for all } x \in \mathbb{R}.$$

The only way the above polynomial function in x can equal the zero function is for all the coefficients to be zero, so $4a^2\nu = 2a\mu = 2a\nu + \lambda = 0$, and therefore $\lambda = \mu = \nu = 0$.

(b) Differentiation using the product rule gives

$$
\begin{aligned}
g''(x) &= -x\sin(x) + 2\cos(x) \\
g^{(4)}(x) &= x\sin(x) - 4\cos(x).
\end{aligned}
$$

Hence, $g + 2g'' + g^{(4)} = 0$, so $g, g'', g^{(4)}$ are linearly dependent, and the same is therefore true of $g, g', \ldots, g^{(4)}$.

19. (a) Suppose that $au + bv + cw = 0$, the zero sequence, for some $a, b, c \in \mathbb{R}$, i.e.,

$$4^n a + 2^n b + n^2 c = 0$$

for all integers $n \geq 0$. Taking $n = 0, 1, 2$ gives, respectively,

$$
\begin{aligned}
a + b &= 0 \\
4a + 2b + c &= 0 \\
16a + 4b + 4c &= 0.
\end{aligned}
$$

After multiplying the bottom row by $1/4$, we aim to show that the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 4 & 2 & 1 \\ 4 & 1 & 1 \end{pmatrix}$$

has a pivot in every column, and row reduction shows that this is the case. Therefore, u, v, w are linearly independent.

(b) Note that $\alpha = u - 4v$ and $\beta = u + 4v - 2w$. Hence, if $a\alpha + b\beta = 0$ for some $a, b \in \mathbb{R}$, then

$$\begin{aligned} 0 &= a(u - 4v) + b(u + 4v - 2w) \\ &= (a + b)u + (-4a + 4b)v - 2bw. \end{aligned}$$

By the linear independence of u, v, w we conclude that $a + b = -4a + 4b = -2b = 0$, so $a = b = 0$.

An alternative solution uses bases (I−6) and coordinate vectors (I−7). We may row-reduce the matrix of coordinate vectors of α and β with respect to the basis $\{u, v, w\}$ of $\mathrm{Span}(u, v, w)$. The matrix is

$$\begin{pmatrix} 1 & 1 \\ -4 & 4 \\ 0 & -2 \end{pmatrix},$$

and it has a pivot in both columns, so α and β are linearly independent.

(c) It helps to write γ_1 and γ_2 as

$$\gamma_1 = (4^n - n^2)_n \tag{9}$$
$$\gamma_2 = (2^n - \tfrac{1}{4}n^2)_n. \tag{10}$$

Now, we see that

$$\alpha + \beta = 2(4^n - n^2)_n = 2\gamma_1,$$

while

$$\begin{aligned} \beta - \alpha &= 2(2^{n+2} - n^2)_n \\ &= 8(2^n - \tfrac{1}{4}n^2)_n \\ &= 8\gamma_2. \end{aligned}$$

Thus, $\gamma_1 = \frac{1}{2}(\alpha + \beta)$ and $\gamma_2 = \frac{1}{8}(-\alpha + \beta)$.

Another answer uses the observation that $\gamma_1 = u - w$ and $\gamma_2 = v - \frac{1}{4}w$, by (9) and (10) respectively. In light of this observation, we look to express $u - w$ and $v - \frac{1}{4}w$ as linear combinations of $\alpha = u - 4v$ and $\beta = u + 4v - 2w$. We could eyeball the solution, or find it systematically by expressing the last two columns of the following matrix as linear combinations of the first two columns:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ -4 & 4 & 0 & 1 \\ 0 & -2 & -1 & -1/4 \end{pmatrix}.$$

Row-reducing this matrix, we arrive at the reduced row-echelon form

$$\begin{pmatrix} 1 & 0 & 1/2 & -1/8 \\ 0 & 1 & 1/2 & 1/8 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so $\gamma_1 = \frac{1}{2}\alpha + \frac{1}{2}\beta$ and $\gamma_2 = -\frac{1}{8}\alpha + \frac{1}{8}\beta$.

20. By adding the equations $\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{u}_2 = \mathbf{v}_1 - \mathbf{v}_2$, we obtain $\mathbf{v}_1 = \frac{1}{2}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2$, and hence $\mathbf{v}_2 = \frac{1}{2}\mathbf{u}_1 - \frac{1}{2}\mathbf{u}_2$. Therefore,

$$\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{u}_3 = \frac{5}{2}\mathbf{u}_1 - \frac{1}{2}\mathbf{u}_2 - \mathbf{u}_3.$$

Thus, every vector in the spanning set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is in $\mathrm{Span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, so $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is also a spanning set. Explicitly, if $c_1, c_2, c_3 \in \mathbb{R}$, then

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 &= c_1(\tfrac{1}{2}\mathbf{u}_1 + \tfrac{1}{2}\mathbf{u}_2) + c_2(\tfrac{1}{2}\mathbf{u}_1 - \tfrac{1}{2}\mathbf{u}_2) \\ &\quad + c_3(\tfrac{5}{2}\mathbf{u}_1 - \tfrac{1}{2}\mathbf{u}_2 - \mathbf{u}_3) \\ &= \frac{1}{2}(c_1 + c_2 + 5c_3)\mathbf{u}_1 + \frac{1}{2}(c_1 - c_2 - c_3)\mathbf{u}_2 - c_3\mathbf{u}_3. \end{aligned}$$

21. We shall answer part (b) first. The answer to part (a) will then be obvious.

Let N be a non-negative integer. Let us say that a sequence $t = (a_n)_n \in \mathcal{S}$ is *constant from* N if $a_n = a_N$ for all $n \geq N$. With this definition, we will show that a given sequence t lies in $\mathrm{Span}(X)$ if and only if there is some $N \geq 0$ such that t is constant from N.

Suppose first that $t \in \mathrm{Span}(X)$, i.e.,

$$c_1 s_{k_1} + \cdots + c_m s_{k_m} = t \tag{11}$$

for some non-negative integers $k_1 < k_2 < \cdots < k_m$ and some $c_1, \ldots, c_m \in \mathbb{R}$. If $N = k_m$, then each s_{k_i} in (11) is constant from N, because it has a 1 in its nth term for $n \geq N$. But then t, being a linear combination of the s_{k_i}, is also constant from N.

Conversely, suppose there is $N \geq 0$ such that $a_n = a_N$ when $n \geq N$. For $k = 0, \ldots, N$, let $\mathbf{u}_k$ be the vector in $\mathbb{R}^{N+1}$ whose $N+1$ entries are, in order, the first $N+1$ terms of s_k, i.e., terms 0 to N. The vectors $\mathbf{u}_0, \ldots, \mathbf{u}_N$ form a basis for $\mathbb{R}^{N+1}$, so we may solve the equation

$$c_0\mathbf{u}_0 + \cdots + c_N\mathbf{u}_N = (a_0, a_1, \ldots, a_N) \tag{12}$$

for $c_0, \ldots, c_N \in \mathbb{R}$. We claim that

$$c_0 s_0 + \cdots + c_N s_N = t, \tag{13}$$

that is, $b_n = a_n$ for all $n \geq 0$, where b_n is the nth term of $c_0 s_0 + \cdots + c_N s_N$. If $n \leq N$, then $b_n = a_n$ precisely because of the choice of $c_0, \ldots, c_N$ in (12). For the other b_n, note that each of the s_k appearing in (13) is constant from N, so $c_0 s_0 + \cdots + c_N s_N$ is constant from N. Therefore, if $n \geq N$, we have $b_n = b_N = a_N = a_n$.

Having thus characterized the sequences in $\text{Span}(X)$, we see immediately that $(1, 2, 3, 4, \ldots)$ is not in $\text{Span}(X)$, since there is no $N \geq 0$ such that $(1, 2, 3, 4, \ldots)$ is constant from N.

22. (a) We have $p \in V$ if and only if $p(1) = p(-1)$, if and only if

$$a_3 + a_2 + a_1 + a_0 = -a_3 + a_2 - a_1 + a_0,$$

if and only if $a_3 + a_1 = 0$.

(b) By part (a), a general polynomial in V takes the form

$$ax^3 + bx^2 - ax + c = a(x^3 - x) + bx^2 + c,$$

where $a, b, c \in \mathbb{R}$. Therefore, the polynomials

$$p_1 = x^3 - x, \quad p_2 = x^2, \quad p_3 = 1$$

span V. To show that p_1, p_2, p_3 are linearly independent, we suppose that $c_1 p_1 + c_2 p_2 + c_3 p_3 = 0$, the zero polynomial, i.e.,

$$c_1(x^3 - x) + c_2 x^2 + c_3 = 0,$$

which is to say

$$c_1 x^3 + c_2 x^2 - c_1 x + c_3 = 0.$$

Then we see immediately that $c_1 = c_2 = c_3 = 0$. Thus, $\{p_1, p_2, p_3\}$ is a basis for V. Because V has a basis consisting of 3 vectors, $\dim(V) = 3$.

23. (a) A general matrix in U takes the form

$$\begin{aligned} \begin{pmatrix} a & b \\ b & c \end{pmatrix} &= a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= aA_1 + cA_2 + bA_3, \end{aligned}$$

which is in $\text{Span}(A_1, A_2, A_3)$. Thus, $\mathcal{B}$ is a spanning set for U. For linear independence, we observe that if $c_1 A_1 + c_2 A_2 + c_3 A_3 = 0$, the zero matrix, then

$$\begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so $c_1 = c_2 = c_3 = 0$.

(b) If $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, then

$$\begin{aligned} X^{\mathrm{T}}X &= \begin{pmatrix} x & z \\ y & w \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} \\ &= \begin{pmatrix} x^2+z^2 & xy+zw \\ xy+zw & y^2+w^2 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{u}\cdot\mathbf{u} & \mathbf{u}\cdot\mathbf{v} \\ \mathbf{u}\cdot\mathbf{v} & \mathbf{v}\cdot\mathbf{v} \end{pmatrix} \\ &= (\mathbf{u}\cdot\mathbf{u})A_1 + (\mathbf{v}\cdot\mathbf{v})A_2 + (\mathbf{u}\cdot\mathbf{v})A_3, \end{aligned}$$

so

$$[X^{\mathrm{T}}X]_{\mathcal{B}} = \begin{pmatrix} \mathbf{u}\cdot\mathbf{u} \\ \mathbf{v}\cdot\mathbf{v} \\ \mathbf{u}\cdot\mathbf{v} \end{pmatrix}. \tag{14}$$

In fact, (14) holds for any matrix $X \in M_{m,2}(\mathbb{R})$ with columns $\mathbf{u}$ and $\mathbf{v}$; the matrix X need not be square. Indeed, we always have

$$X^{\mathrm{T}}X = \begin{pmatrix} \mathbf{u}\cdot\mathbf{u} & \mathbf{u}\cdot\mathbf{v} \\ \mathbf{u}\cdot\mathbf{v} & \mathbf{v}\cdot\mathbf{v} \end{pmatrix}$$

regardless of the number of rows in X.

24. (a) Suppose that $c_1f_1 + c_2f_2 + c_3f_3 = 0$, the zero function, where $c_1, c_2, c_3 \in \mathbb{R}$. Then

$$c_1\cos(2\pi x) + c_2\sin(2\pi x) + c_3x = 0 \quad \text{for all } x \in \mathbb{R}.$$

Taking $x = 0$ shows that $c_1 = 0$, and then taking $x = 1/2$ shows that $c_3 = 0$. We have only to take, for example, $x = 1/4$ to then see that $c_2 = 0$ as well. Thus, f_1, f_2, f_3 are linearly independent. Also, they span V by definition of V. Thus, $\{f_1, f_2, f_3\}$ is a basis for V.

(b) We are told that $g = 4f_1 - f_2 + f_3$, so

$$g(x) = 4\cos(2\pi x) - \sin(2\pi x) + 2x \quad \text{for all } x \in \mathbb{R}.$$

Hence,

$$g(3/8) = 4(-\sqrt{2}/2) - \sqrt{2}/2 + 3/4 = 3/4 - 5\sqrt{2}/2.$$

25. (a) The zero sequence is in U, because $0 = 0 + 0$. Next, suppose that $u = (x_n)_n$ and $v = (y_n)_n$ are in U. Then $u + v = (x_n + y_n)_n$, and

$$x_n + y_n \quad = \quad (x_{n-1} + x_{n-2}) + (y_{n-1} + y_{n-2})$$

$$= \quad (x_{n-1} + y_{n-1}) + (x_{n-2} + y_{n-2}),$$

so $u + v \in U$. Finally, suppose that $u = (x_n)_n \in U$ and $c \in \mathbb{R}$. Then $cu = (cx_n)_n$, and

$$\begin{aligned} cx_n &= c(x_{n-1} + x_{n-2}) \\ &= cx_{n-1} + cx_{n-2}, \end{aligned}$$

so $cu \in U$.

(b) For $n \geq 2$,

$$\begin{aligned} \alpha^n &= \alpha^{n-2}\alpha^2 \\ &= \alpha^{n-2}(\alpha + 1) \quad \text{by the hint} \\ &= \alpha^{n-1} + \alpha^{n-2}. \end{aligned}$$

Thus, $s \in U$. Exactly the same argument with β in place of α shows that $t \in U$.

(c) Suppose that $c_1 s + c_2 t = 0$, the zero sequence, i.e., $c_1\alpha^n + c_2\beta^n = 0$ for all non-negative integers n. Taking $n = 0$ and $n = 1$ gives

$$\begin{aligned} c_1 + \quad c_2 &= 0 \\ \alpha c_1 + \beta c_2 &= 0 \end{aligned}$$

This homogeneous system is represented by the square matrix $\begin{pmatrix} 1 & 1 \\ \alpha & \beta \end{pmatrix}$, which has determinant $\beta - \alpha = -\sqrt{5} \neq 0$, so the system has only the solution $c_1 = c_2 = 0$. Thus, s and t are linearly independent.

26. (a) Suppose $c_1, c_2 \in \mathbb{R}$ satisfy $c_1 f + c_2 g = 0$, i.e., $c_1(e^{2x} + x) + c_2(e^x + 2x) = 0$ for all $x \in \mathbb{R}$. Taking $x = 0$ gives $c_1 + c_2 = 0$. Now take $x = \ln(2)$. (Actually, in this example, any $x \neq 0$ would do.) Then $c_1(4 + \ln(2)) + c_2(2 + 2\ln(2)) = 0$. The system

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1(4 + \ln(2)) + c_2(2 + 2\ln(2)) &= 0 \end{aligned}$$

is represented by a 2×2 matrix having determinant

$$\begin{vmatrix} 1 & 1 \\ 4 + \ln(2) & 2 + 2\ln(2) \end{vmatrix} = \ln(2) - 2 \neq 0,$$

so the system has the unique solution $c_1 = c_2 = 0$, as desired.

(b)

$$[h_1]_{\mathcal{E}} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad [h_2]_{\mathcal{E}} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

(c) The functions h_1, h_2 are linearly independent if and only if $[h_1]_{\mathcal{E}}, [h_2]_{\mathcal{E}}$ are, but

$$\begin{pmatrix} [h_1]_{\mathcal{E}} & [h_2]_{\mathcal{E}} \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

so $[h_1]_{\mathcal{E}}, [h_2]_{\mathcal{E}}$ are indeed linearly independent (pivot in each column). Further, h_1, h_2 span V because $[h_1]_{\mathcal{E}}, [h_2]_{\mathcal{E}}$ span $\mathbb{R}^2$:

$$\begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

(pivot in each row).

27. (a) We search for a basis $\mathcal{B}$ such that $S' \subseteq \mathcal{B} \subseteq S$, where $S' = \{\mathbf{v}_1, \mathbf{v}_2\}$ and S is the spanning set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ of $\mathbb{R}^4$. (The vectors $\mathbf{e}_1, \ldots, \mathbf{e}_4$ are the standard basis vectors in $\mathbb{R}^4$.) This amounts to row-reducing the matrix

$$\begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{pmatrix}.$$

The columns corresponding to the four vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_2$ are linearly independent and span the column space, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_2\}$ is a basis for $\mathbb{R}^4$ extending $\{\mathbf{v}_1, \mathbf{v}_2\}$.

(b) Let A_1, A_2 be the matrices in the given set, in that order, and let

$$A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\{A_1, \ldots, A_6\}$ is a spanning set for $M_2(\mathbb{R})$ containing the linearly independent set $\{A_1, A_2\}$, so it contains a basis for $M_2(\mathbb{R})$ extending the linearly independent set. To find one, we put the matrix of coordinate vectors with respect to the basis $\{A_3, A_4, A_5, A_6\}$ in row-echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

The columns corresponding to the four matrices A_1, A_2, A_3, A_4 are linearly independent and span the column space, so $\{A_1, A_2, A_3, A_4\}$ is a basis for $M_2(\mathbb{R})$ extending $\{A_1, A_2\}$.

28. (a) The set W is non-empty, because the zero 2×2 matrix is symmetric and has trace zero and is therefore in W. Next, if $A, B \in W$, then

$$(A + B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}} = A + B,$$

$$\text{and} \quad \mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B) = 0 + 0 = 0,$$

so $A + B \in W$. Finally, if $\lambda \in \mathbb{R}$, then

$$\begin{aligned} (\lambda A)^{\mathrm{T}} &= \lambda A^{\mathrm{T}} = \lambda A, \\ \text{and} \quad \mathrm{Tr}(\lambda A) &= \lambda\, \mathrm{Tr}(A) = \lambda \cdot 0 = 0, \end{aligned}$$

so $\lambda A \in W$.

(b) Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $A \in W$ if and only if $b = c$ and $a + d = 0$, if and only if

$$A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, W is the subspace spanned by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{15}$$

These matrices are linearly independent, because if

$$\begin{aligned} c_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ \text{i.e.,} \quad \begin{pmatrix} c_1 & c_2 \\ c_2 & -c_1 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

then $c_1 = c_2 = 0$. Therefore, the matrices in (15) form a basis for W. The space W has dimension 2, because a basis for it consists of two vectors.

29. Working with the basis $\mathcal{B} = \{x^3, x^2, x, 1\}$ of $\mathcal{P}_2$, we solve the equation $c_1[p_1]_{\mathcal{B}} + c_2[p_2]_{\mathcal{B}} + c_3[p_3]_{\mathcal{B}} + c_4[p_4]_{\mathcal{B}} = \mathbf{0}$ for $c_1, c_2, c_3, c_4 \in \mathbb{R}$. Arranging these coordinate vectors as columns in a matrix, we obtain the system

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 6 \\ -1 & -1 & 1 & 4 \\ 3 & 1 & -1 & -2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

There is no pivot in the final column, so p_1, p_2, p_3, p_4 are linearly dependent. Specifically, if the columns of the reduced row-echelon form are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, then $\mathbf{v}_4 = \mathbf{v}_1 - 2\mathbf{v}_2 + 3\mathbf{v}_3$, so the same is true of the columns of the matrix of coordinate vectors, i.e., $[p_4]_{\mathcal{B}} = [p_1]_{\mathcal{B}} - 2[p_2]_{\mathcal{B}} + 3[p_3]_{\mathcal{B}}$, and therefore $p_4 = p_1 - 2p_2 + 3p_3$. Alternatively, treating c_4 as free, we take $c_4 = -1$ and thus obtain $c_1 = 1$, $c_2 = -2$, and $c_3 = 3$.

30. (a) We row-reduce the matrix of coordinate vectors, with coordinates taken with respect to the basis

$$\left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right\},$$

i.e.,

$$\begin{pmatrix}1 & 1 & 2 & 1 & -1\\1 & 2 & 3 & 0 & 2\\1 & 3 & 4 & 1 & 1\\1 & -1 & 0 & 0 & -1\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 1 & 2 & 1 & -1\\0 & 1 & 1 & -1 & 3\\0 & 0 & 0 & 1 & -2\\0 & 0 & 0 & 0 & 0\end{pmatrix}.$$

The pivot columns are columns 1, 2, and 4, so the corresponding matrices in the set S form a basis for U:

$$\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}, \quad \begin{pmatrix}1 & 2\\3 & -1\end{pmatrix}, \quad \begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}.$$

(b) We represent the polynomials $p_1 = x^3 + x^2 - 1, p_2 = -x^3 + 2x + 1, p_3 = x^3 + 2x^2 + 2x - 1, p_4 = 2x^3 + x^2 + x - 2, p_5 = 4x^3 + 2x^2 - x - 4$ by their coordinate vectors with respect to the basis $\{x^3, x^2, x, 1\}$:

$$\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\\2\\1\end{pmatrix}, \begin{pmatrix}1\\2\\2\\-1\end{pmatrix}, \begin{pmatrix}2\\1\\1\\-2\end{pmatrix}, \begin{pmatrix}4\\2\\-1\\-4\end{pmatrix}.$$

By row-reducing the corresponding matrix, i.e.,

$$\begin{pmatrix}1 & -1 & 1 & 2 & 4\\1 & 0 & 2 & 1 & 2\\0 & 2 & 2 & 1 & -1\\-1 & 1 & -1 & -2 & -4\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & -1 & 1 & 2 & 4\\0 & 1 & 1 & -1 & -2\\0 & 0 & 0 & 3 & 3\\0 & 0 & 0 & 0 & 0\end{pmatrix},$$

we conclude that p_1, p_2, and p_4 form a basis for W. (Note from the row-echelon form above that $p_3, p_5 \in \operatorname{Span}(p_1, p_2, p_4)$.)

31. (a) We have the following coordinate vectors with respect to the basis $\mathcal{B} = \{f_1, f_2, f_3\}$ for the space $V = \operatorname{Span}(f_1, f_2, f_3)$:

$$[g_1]_{\mathcal{B}} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad [g_2]_{\mathcal{B}} = \begin{pmatrix}1\\-1\\1\end{pmatrix}, \quad [g_3]_{\mathcal{B}} = \begin{pmatrix}-3\\6\\-1\end{pmatrix}.$$

We put the matrix of coordinate vectors in row-echelon form:

$$\begin{pmatrix}1 & 1 & -3\\2 & -1 & 6\\3 & 1 & -1\end{pmatrix} \leftrightarrow \begin{pmatrix}1 & 1 & -3\\0 & 1 & -4\\0 & 0 & 0\end{pmatrix}.$$

The coordinate vectors are not linearly independent, because there is no pivot in the final column of the above row-echelon form, so g_1, g_2, g_3 are not linearly independent.

(b) The solution is the same as in part (a), except now we have the following matrix of coordinate vectors:

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ -2 & 1 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & 1 \\ 0 & 0 & 6 \end{pmatrix}.$$

This time, there is a pivot in every column, so the coordinate vectors are linearly independent, and so h_1, h_2, h_3 are linearly independent.

32. Let $p = a_3x^3 + a_2x^2 + a_1x + a_0 \in \mathcal{P}_3$. Then $p' = 3a_3x^3 + 2a_2x^2 + a_1$, so $p \in V$ if and only if

$$\begin{aligned} & 8a_3 + 4a_2 + 2a_1 + a_0 = 0 \\ \text{and} \quad & 12a_3 + 4a_2 + a_1 \phantom{{}+a_0} = 0 \end{aligned}$$

(The first equation says $p(2) = 0$, and the second says $p'(2) = 0$.) Putting the unknowns in the order a_0, a_1, a_2, a_3, we may represent this system of equations by the matrix

$$\begin{pmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 4 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & -4 & -16 \\ 0 & 1 & 4 & 12 \end{pmatrix},$$

from which we read off the general solution

$$\begin{aligned} a_0 &= 4\mu + 16\lambda \\ a_1 &= -4\mu - 12\lambda \\ a_2 &= \mu \\ a_3 &= \lambda, \end{aligned}$$

where $\lambda, \mu \in \mathbb{R}$ are free. Thus, the polynomials in V are those of the form

$$\begin{aligned} & \lambda x^3 + \mu x^2 - (12\lambda + 4\mu)x + 16\lambda + 4\mu \\ = \; & \lambda(x^3 - 12x + 16) + \mu(x^2 - 4x + 4) \end{aligned}$$

with $\lambda, \mu \in \mathbb{R}$. Hence, the polynomials $p_1 = x^3 - 12x + 16$ and $p_2 = x^2 - 4x + 4$ are in V, and every polynomial in V is a linear combination of p_1 and p_2. Further, p_1 and p_2 are linearly independent, for if $c_1(x^3 - 12x + 16) + c_2(x^2 - 4x + 4) = 0$, then

$$c_1x^3 + c_2x^2 - (12c_1 + 4c_2)x + 16c_1 + 4c_2 = 0,$$

so $c_1 = c_2 = 0$. Thus, $\{p_1, p_2\}$ is a basis for V.

33. (a)

$$[B_1]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad [B_2]_{\mathcal{C}} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$[B_3]_{\mathcal{C}} = \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix} \qquad [B_4]_{\mathcal{C}} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$$

(b) We put the matrix of coordinate vectors in row-echelon form:

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 1 & -1 & 3 & -1 \\ 1 & 2 & -3 & 5 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

There is no pivot in the final row of this row-echelon form, so the coordinate vectors do not span $\mathbb{R}^3$, and so B_1, B_2, B_3, B_4 do not span V.

34. In both parts, we consider coordinates with respect to the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

of $M_2(\mathbb{R})$.

(a) The matrix of coordinate vectors of the elements of S_1 is

$$\begin{pmatrix} 1 & 1 & 1 & -1 & 1 \\ 2 & 3 & 1 & -1 & 3 \\ 1 & 3 & -1 & 2 & 2 \\ 2 & 3 & 1 & -1 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

There is a pivot in every row, so S_1 is a spanning set. However, the third column has no pivot, so S_1 is not linearly independent.

(b) The matrix of coordinate vectors of the elements of S_2 is

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & 3 \\ 2 & 5 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

There is a column without a pivot, so S_2 is not linearly independent. It is not a spanning set either, because not every row in the above row-echelon form has a pivot.

35.

$$\begin{pmatrix} w_1 & w_2 & w_3 & | & v_1 & v_2 & v_3 \end{pmatrix} = \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 11 & 5 & 4 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & -1 \end{array}\right)$$

$$\leftrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 2 & 1 \\ 0 & 1 & 0 & -3 & -1 & -1 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{array}\right),$$

so

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 4 & 2 & 1 \\ -3 & -1 & -1 \\ 2 & 1 & 1 \end{pmatrix}.$$

36. (a)

$$\begin{aligned} P_{\mathcal{B}\leftarrow\mathcal{E}} &= P_{\mathcal{E}\leftarrow\mathcal{B}}^{-1} \\ &= \begin{pmatrix} 3 & 8 \\ 1 & 2 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -1 & 4 \\ 1/2 & -3/2 \end{pmatrix}. \end{aligned}$$

For $P_{\mathcal{C}\leftarrow\mathcal{B}}$, we row reduce as follows:

$$\begin{pmatrix} P_{\mathcal{E}\leftarrow\mathcal{C}} & | & P_{\mathcal{E}\leftarrow\mathcal{B}} \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 1 & 3 & 8 \\ 1 & -1 & 1 & 2 \end{array}\right) \leftrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 2 & 5 \\ 0 & 1 & 1 & 3 \end{array}\right).$$

Thus,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}.$$

Finally,

$$\begin{aligned} P_{\mathcal{C}\leftarrow\mathcal{E}} &= P_{\mathcal{E}\leftarrow\mathcal{C}}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}. \end{aligned}$$

(b)

$$\begin{aligned} P_{\mathcal{C}\leftarrow\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{E}} &= \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 1/2 & -3/2 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \end{aligned}$$

$$= \quad P_{\mathcal{C}\leftarrow\mathcal{E}}.$$

37. (a) Observe that

$$\begin{aligned} p &= ax^2 + bx + c \\ p' &= 2ax + b \\ p'' &= 2a. \end{aligned}$$

The coordinate vectors of p, p', p'' with respect to the basis $\{x^2, x, 1\}$ of $\mathcal{P}_2$ are

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2a \\ b \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 2a \end{pmatrix},$$

which are linearly independent because $a \neq 0$. Therefore, these coordinate vectors necessarily form a basis for $\mathbb{R}^3$ because they are three linearly independent vectors in the 3-dimensional space $\mathbb{R}^3$. Hence, the polynomials p, p', p'' form a basis for $\mathcal{P}_2$.

(b) The coordinates of p, $p + p'$, and $p + p' + p''$ with respect to $\mathcal{B}$ are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(c) The matrix formed from the columns in part (b) has a pivot in every column and a pivot in every row:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, the coordinate vectors of p, $p + p'$, and $p + p' + p''$ with respect to $\mathcal{B}$ form a basis for $\mathbb{R}^3$, so $\{p, p + p', p + p' + p''\}$ is a basis for $\mathcal{P}_2$.

38. The vectors $v_1, \ldots, v_n$ form a basis for V if and only if the coordinate vectors $[v_1]_\mathcal{B}, \ldots, [v_n]_\mathcal{B}$ form a basis for $\mathbb{R}^n$, if and only if the columns of A form a basis for $\mathbb{R}^n$, if and only if A is invertible, if and only if $\det(A) \neq 0$.

Variations on this idea work. For example, $v_1, \ldots, v_n$ form a basis for V if and only if they are linearly independent (n vectors in an n-dimensional space), if and only if the coordinate vectors $[v_1]_\mathcal{B}, \ldots, [v_n]_\mathcal{B}$ are linearly independent, if and only if the columns of A are linearly independent, if and only if A is invertible (square matrix), if and only if $\det(A) \neq 0$.

39. Taking coordinate vectors of both sides of the equation $\mathbf{v}_1 = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$, where $c_1, c_2, c_3 \in \mathbb{R}$, we obtain

$$\begin{aligned} [\mathbf{v}_1]_{\mathcal{B}} &= c_1[\mathbf{u}_1]_{\mathcal{B}} + c_2[\mathbf{u}_2]_{\mathcal{B}} + c_3[\mathbf{u}_3]_{\mathcal{B}} \\ \text{i.e.,} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} &= c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 3 \\ -1 \\ -5 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ -1 \\ 4 \\ 9 \end{pmatrix}. \end{aligned}$$

We thus obtain a system of equations in c_1, c_2, c_3 represented by the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ -1 & 3 & -1 & 0 \\ 1 & -1 & 4 & 0 \\ 3 & -5 & 9 & 0 \end{array}\right) \leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

From the reduced row-echelon form, we see that the original equation has a solution, namely, $c_1 = 11$, $c_2 = 3$, and $c_3 = -2$, so $\mathbf{v}_1 = 11\mathbf{u}_1 + 3\mathbf{u}_2 - 2\mathbf{u}_3$.

40. (a) We work with the standard basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ to compute $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and $P_{\mathcal{E}\leftarrow\mathcal{C}}$. We row-reduce as follows:

$$\left(\begin{array}{cc|cc} 3 & 5 & 11 & 30 \\ 1 & 0 & 2 & 5 \end{array}\right) \leftrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 2 & 5 \\ 0 & 1 & 1 & 3 \end{array}\right).$$

Therefore,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}.$$

Similarly, the row reductions

$$\left(\begin{array}{cc|cc} 1 & 2 & 3 & 5 \\ 2 & -1 & 1 & 0 \end{array}\right) \leftrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

show that

$$P_{\mathcal{E}\leftarrow\mathcal{C}} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

(b)

$$\begin{aligned} P_{\mathcal{E}\leftarrow\mathcal{B}} &= P_{\mathcal{E}\leftarrow\mathcal{C}} P_{\mathcal{C}\leftarrow\mathcal{B}} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 8 \\ 4 & 11 \end{pmatrix}. \end{aligned}$$

(c)

$$\begin{aligned} P_{\mathcal{B}\leftarrow\mathcal{E}} &= P_{\mathcal{E}\leftarrow\mathcal{B}}^{-1} \\ &= \begin{pmatrix} 3 & 8 \\ 4 & 11 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 11 & -8 \\ -4 & 3 \end{pmatrix}. \end{aligned}$$

(d) By part (c),

$$\begin{aligned} \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_{\mathcal{B}} &= P_{\mathcal{B}\leftarrow\mathcal{E}} \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_{\mathcal{E}} \\ &= \begin{pmatrix} 11 & -8 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 11 \\ -4 \end{pmatrix}, \end{aligned}$$

so

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 11 \begin{pmatrix} 11 \\ 2 \end{pmatrix} - 4 \begin{pmatrix} 30 \\ 5 \end{pmatrix}.$$

41. We consider coordinates with respect to the basis $\mathcal{E} = \{x^2, x, 1\}$ of $\mathcal{P}_2$:

$$\begin{aligned} \left(P_{\mathcal{E}\leftarrow\mathcal{C}} \;\middle|\; P_{\mathcal{E}\leftarrow\mathcal{B}} \right) &= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 3 & 2 \\ 1 & 2 & 2 & 2 & 5 & 5 \\ 2 & 2 & 3 & 2 & 6 & 6 \end{array}\right) \\ &\leftrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & -2 & 0 & 2 \end{array}\right). \end{aligned}$$

The change-of-basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is equal to the 3×3 matrix to the right of the vertical line in the above reduced row-echelon form, i.e.,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 2 & 1 & -1 \\ 2 & 2 & 1 \\ -2 & 0 & 2 \end{pmatrix}.$$

42. (a) Let $\mathbf{u} = 2\mathbf{u}_1 + 3\mathbf{u}_2 + 5\mathbf{u}_3$. Then

$$\begin{aligned} [\mathbf{u}]_{\mathcal{C}} &= P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{u}]_{\mathcal{B}} \quad \text{by Prop. 10.1 in Section I} \\ &= \begin{pmatrix} 1 & -1 & 2 \\ -1 & 2 & -3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ -11 \\ 10 \end{pmatrix}. \end{aligned}$$

Thus, $\mathbf{u} = 9\mathbf{v}_1 - 11\mathbf{v}_2 + 10\mathbf{v}_3$.

(b) Let $\mathbf{v} = \mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3$. Then

$$\begin{aligned} [\mathbf{v}]_{\mathcal{B}} &= P_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{v}]_{\mathcal{C}} \quad \text{by Prop. 10.1 in Section I} \\ &= P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{v}]_{\mathcal{C}} \quad \text{by Prop. 10.2 in Section I} \\ &= \begin{pmatrix} 5 & 3 & -1 \\ -2 & -1 & 1 \\ -3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ -4 \end{pmatrix}. \end{aligned}$$

Thus, $\mathbf{v} = 8\mathbf{u}_1 - \mathbf{u}_2 - 4\mathbf{u}_3$.

43. (a) With respect to the basis $\mathcal{E} = \{f_1, f_2, f_3\}$ of V, the coordinates of the vectors in $\mathcal{B}$ are described by the matrix

$$\begin{pmatrix} -1 & -2 & 2 \\ -2 & -1 & 2 \\ -3 & 2 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

There is a pivot in every column (and every row, the matrix being square), so $\mathcal{B}$ is a basis for V.

(b) We consider the matrix

$$\begin{aligned} &\left([h_1]_{\mathcal{E}} \quad [h_2]_{\mathcal{E}} \quad [h_3]_{\mathcal{E}} \;\middle|\; [g_1]_{\mathcal{E}} \quad [g_2]_{\mathcal{E}} \quad [g_3]_{\mathcal{E}} \right) \\ = &\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & -1 & -2 & 2 \\ 1 & 2 & 1 & -2 & -1 & 2 \\ 2 & 3 & -1 & -3 & 2 & 0 \end{array} \right) \\ \leftrightarrow &\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right). \end{aligned} \tag{16}$$

Because we obtain the identity matrix to the left of the vertical line, the functions h_1, h_2, h_3 form a basis for V. But then, by the proposition in class on finding change-of-basis matrices via row reduction, the right-hand side of the reduced row-echelon form in (16) is $P_{\mathcal{C}\leftarrow\mathcal{B}}$, that is,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 0 & 2 & -1 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

44. Because V is spanned by four functions, $\dim(V) \le 4$. However, the fact given about the functions g_1, g_2, g_3, g_4 says precisely that the g_i are linearly independent, so $\dim(V) \ge 4$. Putting the two inequalities together gives $\dim(V) = 4$.

Next, f_1, f_2, f_3, f_4 are four functions spanning the 4-dimensional space V, so $\{f_1, f_2, f_3, f_4\}$ is a basis for V by Proposition 9.2 in Section I of the course notes. Similarly, g_1, g_2, g_3, g_4 are four linearly independent functions in the 4-dimensional space V, so $\{g_1, g_2, g_3, g_4\}$ is a basis for V.

45. Because every vector in V is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and because each $\mathbf{v}_i$ is in turn a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, it follows that every vector in V is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. That is, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ span V. But V has dimension 3, so $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ must be a basis for V by Proposition 9.2 in Section I of the course notes.

46. (a) The matrix of coordinates of $\mathbf{v}_1, \mathbf{v}_2$ with respect to $\mathcal{D}$ is

$$\begin{pmatrix} \frac{\sqrt{2}}{a+b} & \frac{1}{a-b} \\ \frac{1}{a+b} & \frac{\sqrt{2}}{a-b} \end{pmatrix},$$

which has determinant $1/(a^2 - b^2) \neq 0$, so $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$. Then, the matrix of coordinates of $\mathbf{u}_1, \mathbf{u}_2$ with respect to $\mathcal{C}$ is

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

which has determinant $a^2 - b^2 \neq 0$, so $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$ is also a basis for $\mathbb{R}^2$.

(b) Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^2$. Then the area in question is

$$\begin{aligned} |\det(P_{\mathcal{E}\leftarrow\mathcal{B}})| &= |\det(P_{\mathcal{E}\leftarrow\mathcal{D}})\det(P_{\mathcal{D}\leftarrow\mathcal{C}})\det(P_{\mathcal{C}\leftarrow\mathcal{B}})| \\ &= |\det(P_{\mathcal{E}\leftarrow\mathcal{D}})|\,|\det(P_{\mathcal{D}\leftarrow\mathcal{C}})|\,|\det(P_{\mathcal{C}\leftarrow\mathcal{B}})| \\ &= |\det(P_{\mathcal{E}\leftarrow\mathcal{D}})|\,\frac{1}{a^2-b^2}\,(a^2-b^2) \\ &= |\det(P_{\mathcal{E}\leftarrow\mathcal{D}})|, \end{aligned}$$

which is independent of a and b.

47. (a) Working with the basis $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$ of $\mathbb{R}^2$, we have

$$\begin{pmatrix} [\mathbf{u}_1]_{\mathcal{C}} & [\mathbf{u}_2]_{\mathcal{C}} \end{pmatrix} = \begin{pmatrix} x-1 & 5 \\ -5 & x-3 \end{pmatrix},$$

which has determinant

$$(x-1)(x-3) + 25 = x^2 - 4x + 28 = (x-2)^2 + 24 \neq 0.$$

The coordinate vectors therefore form a basis for $\mathbb{R}^2$, so $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for $\mathbb{R}^2$.

(b) Let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$, and let $\mathcal{E}$ be the standard basis of $\mathbb{R}^2$. Then

$$\begin{aligned}
|\det(P_{\mathcal{E}\leftarrow\mathcal{B}})| &= |\det(P_{\mathcal{E}\leftarrow\mathcal{C}}P_{\mathcal{C}\leftarrow\mathcal{B}})| \quad \text{by Prop. 10.2 in Sect. I} \\
&= |\det(P_{\mathcal{E}\leftarrow\mathcal{C}})|\,|\det(P_{\mathcal{C}\leftarrow\mathcal{B}})| \\
&= \frac{1}{2}((x-2)^2+24),
\end{aligned}$$

because we are told that $|\det(P_{\mathcal{E}\leftarrow\mathcal{C}})| = 1/2$, and we found in part (a) that $\det(P_{\mathcal{C}\leftarrow\mathcal{B}}) = (x-2)^2+24$.

(c) The expression $\frac{1}{2}((x-2)^2+24)$ is least when $x = 2$, and in this case the value of the expression is $24/2 = 12$.

48. Let $V = \operatorname{Span}(f_1, \ldots, f_n)$, which has basis $\mathcal{B} = \{f_1, \ldots, f_n\}$, because the f_j are assumed to be linearly independent. For a function $f = c_1f_1 + \cdots + c_nf_n \in V$, where $c_1, \ldots, c_n \in \mathbb{R}$, the condition $f'(x_i) = 0$ says

$$c_1f_1'(x_i) + \cdots + c_nf_n'(x_i) = 0.$$

Therefore, finding $c_1, \ldots, c_n \in \mathbb{R}$ satisfying the conditions $f'(x_1) = \cdots = f'(x_m) = 0$ is equivalent to solving

$$\begin{pmatrix} f_1'(x_1) & f_2'(x_1) & \cdots & f_n'(x_1) \\ f_1'(x_2) & f_2'(x_2) & \cdots & f_n'(x_2) \\ \vdots & \vdots & & \vdots \\ f_1'(x_m) & f_2'(x_m) & \cdots & f_n'(x_m) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0}.$$

Because $m < n$ (by assumption), this linear system has at least one non-zero solution. In fact, we may choose a non-zero solution $(c_1, \ldots, c_n)$ having rational entries, because the coefficient matrix has rational entries by another assumption given in the question. Hence, we may choose a positive integer d such that the numbers $a_i = dc_i$ $(i = 1, \ldots, n)$ are all integers. (Just clear denominators in $c_1, \ldots, c_n$.)

To summarize what we have done: $a_1, \ldots, a_n$ are integers (because we cleared denominators in the rational numbers $c_1, \ldots, c_n$), they are not all zero (because we chose $(c_1, \ldots, c_n)$ to not be the zero vector), and $(a_1f_1 + \cdots + a_nf_n)'(x_i) = 0$ for all i (because $(c_1f_1 + \cdots + c_nf_n)'(x_i) = 0$ for all i, and we simply scaled the c_i to get the a_i). Finally, because $f_1, \ldots, f_n$ are linearly independent and the a_i are not all zero, the function f is not the zero function.

49. (a) Let $c_1, c_2, c_3 \in \mathbb{R}$. Then

$$\frac{1}{(x^2+1)(x^2+2)(x^2+3)} = \frac{c_1}{x^2+1} + \frac{c_2}{x^2+2} + \frac{c_3}{x^2+3} \tag{17}$$

if and only if

$$\begin{aligned} 1 &= c_1(x^2+2)(x^2+3) + c_2(x^2+1)(x^2+3) + c_3(x^2+1)(x^2+2) \\ &= c_1(x^4+5x^2+6) + c_2(x^4+4x^2+3) + c_3(x^4+3x^2+2) \\ &= (c_1+c_2+c_3)x^4 + (5c_1+4c_2+3c_3)x^2 + (6c_1+3c_2+2c_1), \end{aligned}$$

if and only if

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 5c_1 + 4c_2 + 3c_2 &= 0 \\ 6c_1 + 3c_2 + 2c_1 &= 1 \end{aligned}$$

(To obtain this system, we have equated coefficients in the polynomials.) The system is represented by the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 5 & 4 & 3 & 0 \\ 6 & 3 & 2 & 1 \end{array}\right) \leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1/2 \end{array}\right). \tag{18}$$

From the reduced row-echelon form, we see that there is a unique solution, namely, $c_1 = 1/2$, $c_2 = -1$, $c_3 = 1/2$, so

$$g = \frac{1}{2}f_1 - f_2 + \frac{1}{2}f_3.$$

(b) *Solution 1:* We saw in our solution to part (a) that there is a unique solution (c_1, c_2, c_3) to the equation $g = c_1f_1 + c_2f_2 + c_3f_3$. If f_1, f_2, f_3 were linearly dependent, then there would be more than one solution to that equation, namely, $(c_1 + d_1, c_2 + d_2, c_3 + d_3)$ for any non-zero vector $(d_1, d_2, d_3) \in \mathbb{R}^3$ satisfying $d_1f_1 + d_2f_2 + d_3f_3 = 0$. Thus, f_1, f_2, f_3 are linearly independent.

Solution 2: We consider the equation in (17) but with the left-hand side replaced by the zero function:

$$0 = \frac{c_1}{x^2+1} + \frac{c_2}{x^2+2} + \frac{c_3}{x^2+3}.$$

This time, the system of equations satisfied by c_1, c_2, c_3 is

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 5c_1 + 4c_2 + 3c_2 &= 0 \\ 6c_1 + 3c_2 + 2c_1 &= 0 \end{aligned} \tag{19}$$

This system is the same as before except that the numbers on the right are all zero. The augmented matrix representing this new system is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 5 & 4 & 3 & 0 \\ 6 & 3 & 2 & 0 \end{array}\right).$$

Again, this is the same augmented matrix as before apart from the numbers to the right of the vertical line. We therefore know that, if we were to put this matrix in row-echelon form, there would be a pivot in each of the first three columns, because that is what we found for the earlier augmented matrix. Hence, the new system in (19) has only the solution $c_1 = c_2 = c_3 = 0$.

(c) We saw in part (a) that $g_1 = \frac{1}{2}f_1 - f_2 + \frac{1}{2}f_3$, so $[g]_{\mathcal{B}} = (1/2, -1, 1/2)$.

50. (a)

$$\begin{aligned} f_1(x) &= \frac{(x^2+3)(x^2+5)}{p(x)} \\ &= \frac{x^4+8x^2+15}{p(x)} \\ &= g_1(x) + 8g_2(x) + 15g_3(x), \end{aligned}$$

$$\begin{aligned} f_2(x) &= \frac{(x^2+2)(x^2+5)}{p(x)} \\ &= \frac{x^4+7x^2+10}{p(x)} \\ &= g_1(x) + 7g_2(x) + 10g_3(x), \end{aligned}$$

$$\begin{aligned} f_3(x) &= \frac{(x^2+2)(x^2+3)}{p(x)} \\ &= \frac{x^4+5x^2+6}{p(x)} \\ &= g_1(x) + 5g_2(x) + 6g_3(x), \end{aligned}$$

so

$$[f_1]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 8 \\ 15 \end{pmatrix}, \quad [f_2]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 7 \\ 10 \end{pmatrix}, \quad [f_3]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}.$$

(b) The matrix of coordinate vectors of f_1, f_2, f_3 with respect to $\mathcal{C}$ is

$$\begin{pmatrix} 1 & 1 & 1 \\ 8 & 7 & 5 \\ 15 & 10 & 6 \end{pmatrix} = A,$$

which we are told is invertible, so f_1, f_2, f_3 are linearly independent.

(c) The functions f_1, f_2, f_3 are three linearly independent functions in the space V, which we are told has dimension 3. Therefore, by Proposition 9.2 in Section I of the course notes, $\{f_1, f_2, f_3\}$ is a basis for V. Alternatively, we may use the fact that the matrix of coordinate vectors, being an invertible matrix, has a pivot both in every column and in every row of a row-echelon form. Therefore, $\{f_1, f_2, f_3\}$ is both a linearly independent set and a spanning set for V by Proposition 8.1 in Section I, that is, $\{f_1, f_2, f_3\}$ is a basis for V.

(d)

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}^{-1} = A^{-1} = \begin{pmatrix} 4/3 & -2/3 & 1/3 \\ -9/2 & 3/2 & -1/2 \\ 25/6 & -5/6 & 1/6 \end{pmatrix},$$

so

$$\begin{aligned} g_1 &= \frac{4}{3}f_1 - \frac{9}{2}f_2 + \frac{25}{6}f_3 \\ g_2 &= -\frac{2}{3}f_1 + \frac{3}{2}f_2 - \frac{5}{6}f_3 \\ g_3 &= \frac{1}{3}f_1 - \frac{1}{2}f_2 + \frac{1}{6}f_3 \end{aligned}$$

51. (a) The map φ_1 is a linear transformation. If $A, B \in M_n(\mathbb{R})$, then

$$\begin{aligned} \varphi_1(A+B) &= \mathrm{Tr}((A+B)Y) \\ &= \mathrm{Tr}(AY+BY) \quad \text{by distributivity} \\ &= \mathrm{Tr}(AY) + \mathrm{Tr}(BY) \quad \text{by a property of the trace} \\ &= \varphi_1(A) + \varphi_1(B), \end{aligned}$$

so φ_1 respects addition. If $A \in M_n(\mathbb{R})$ and $c \in \mathbb{R}$, then

$$\begin{aligned} \varphi_1(cA) &= \mathrm{Tr}((cA)Y) \\ &= \mathrm{Tr}(cAY) \\ &= c\,\mathrm{Tr}(AY) \quad \text{by a property of the trace} \\ &= c\varphi_1(A), \end{aligned}$$

so φ_1 respects scalar multiplication.

(b) The map φ_2 is not a linear transformation. We show that φ_2 does not respect scalar multiplication. If $\mathbf{u} = (1,0,0)$, then

$$\begin{aligned} \varphi_2(2\mathbf{u}) &= \varphi_2(2,0,0) = (2^{n+1})_n, \\ \text{while} \quad 2\varphi_2(\mathbf{u}) &= 2\varphi_2(1,0,0) = 2(1)_n = (2)_n. \end{aligned}$$

Since $2^{n+1} \neq 2$ when $n \geq 1$, the sequences $(2^{n+1})_n$ and $(2)_n$ are not equal, so $\varphi_2(2\mathbf{u}) \neq 2\varphi_2(\mathbf{u})$.

(Alternatively, one may show that φ_2 does not respect addition.)

52. (a) The map φ_1 is a linear transformation. If

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

are in $\mathbb{R}^3$ and $c \in \mathbb{R}$, then

$$\begin{aligned} \varphi_1(\mathbf{u}+\mathbf{v}) &= \varphi_1 \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix} \\ &= (u_1+v_1)(x+1)^2 + (u_2+v_2)(x+1) + (u_3+v_3) \\ &= \big(u_1(x+1)^2 + u_2(x+1) + u_3\big) + \big(v_1(x+1)^2 + v_2(x+1) + v_3\big) \\ &= \varphi_1(\mathbf{u}) + \varphi_1(\mathbf{v}), \end{aligned}$$

and

$$\begin{aligned} \varphi_1(c\mathbf{u}) &= \varphi_1 \begin{pmatrix} cu_1 \\ cu_2 \\ cu_3 \end{pmatrix} \\ &= (cu_1)(x+1)^2 + (cu_2)(x+1) + (cu_3) \\ &= c\big(u_1(x+1)^2 + u_2(x+1) + u_3\big) \\ &= c\varphi_1(\mathbf{u}), \end{aligned}$$

so φ_1 respects both addition and scalar multiplication.

(b) The map φ_2 is not a linear transformation. For example, if $f, g \in \mathcal{F}$ are defined by $f(x) = 1$ and $g(x) = x$, then

$$\begin{aligned} \varphi_2(f+g) &= \begin{pmatrix} (f+g)(1)^2 & (f+g)(1)\cdot(f+g)(2) \\ (f+g)(1)\cdot(f+g)(2) & (f+g)(2)^2 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}, \end{aligned}$$

but

$$\varphi_2(f) + \varphi_2(g) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}.$$

(Alternatively, one may show that φ_2 does not respect scalar multiplication.)

53. (a) φ_1 is a linear transformation. Let $u = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $v = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ be in $\mathbb{R}^2$. Then

$$\begin{aligned} &\varphi_1(u+v) \\ = \quad &\varphi_1 \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= ((a_1 + b_1) - (a_2 + b_2))x^2 \\
&\quad + ((a_1 + b_1) + (a_2 + b_2))x + (2(a_1 + b_1) + 3(a_2 + b_2)) \\
&= (a_1 - a_2)x^2 + (a_1 + a_2)x + (2a_1 + 3a_2) \\
&\quad + (b_1 - b_2)x^2 + (b_1 + b_2)x + (2b_1 + 3b_2) \\
&= \varphi_1(u) + \varphi_1(v),
\end{aligned}
$$

so φ_1 respects addition. For scalar multiplication, take $u = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Then

$$
\begin{aligned}
\varphi_1(cu) &= \varphi_1 \begin{pmatrix} ca_1 \\ ca_2 \end{pmatrix} \\
&= (ca_1 - ca_2)x^2 + (ca_1 + ca_2)x + (2ca_1 + 3ca_2) \\
&= c((a_1 - a_2)x^2 + (a_1 + a_2)x + (2a_1 + 3a_2)) \\
&= c\varphi_1(u).
\end{aligned}
$$

(b) φ_2 is not a linear transformation. For example, it does not respect addition. To see this, observe that

$$\varphi_2(I + (-I)) = \varphi_2(0) = 0,$$

where I is the 3×3 identity matrix, while

$$\varphi_2(I) + \varphi_2(-I) = 0 + (-1) = -1.$$

(c) φ_3 is a linear transformation. Let $\alpha = (a_n)_n$ and $\beta = (b_n)_n$ be sequences. Then the nth term in $\alpha + \beta$ is $a_n + b_n$, so

$$
\begin{aligned}
\varphi_3(\alpha + \beta) &= (a_{n^2} + b_{n^2})_n \\
&= (a_{n^2})_n + (b_{n^2})_n \\
&= \varphi_3(\alpha) + \varphi_3(\beta).
\end{aligned}
$$

Thus, φ_3 respects addition. For scalar multiplication, we take $\alpha = (a_n)_n \in \mathcal{S}$ and $\lambda \in \mathbb{R}$. The nth term in $\lambda\alpha$ is λa_n, so

$$
\begin{aligned}
\varphi_3(\lambda\alpha) &= (\lambda a_{n^2})_n \\
&= \lambda(a_{n^2})_n \\
&= \lambda\varphi_3(\alpha).
\end{aligned}
$$

(d) φ_4 is not a linear transformation. For example, it does not respect scalar multiplication. To see this, let $p = x$, take any $a \in \mathbb{R}$, and let $q = ap = ax$. Then

$$\varphi_4(ap) = \varphi_4(q)$$

$$\begin{aligned} &= q(q(x)) \\ &= q(ax) \\ &= a(ax) \\ &= a^2x, \end{aligned}$$

while

$$\begin{aligned} a\varphi_4(p) &= ap(p(x)) \\ &= ap(x) \\ &= ax. \end{aligned}$$

Therefore, if $a \notin \{0, 1\}$, then $\varphi_4(ap) \neq a\varphi_4(p)$.

54. (a) φ_1 is not a linear transformation. For example, it does not respect addition. To see this, observe that

$$\varphi_1\left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right) = \varphi_1\begin{pmatrix}1\\1\end{pmatrix} = 2,$$

while

$$\varphi_1\begin{pmatrix}1\\0\end{pmatrix} + \varphi_1\begin{pmatrix}0\\1\end{pmatrix} = 0 + 0 = 0.$$

(b) φ_2 is not a linear transformation. For example, it does not respect scalar multiplication. To see this, observe that for $A \in M_3(\mathbb{R})$ and $c \in \mathbb{R}$,

$$\begin{aligned} \varphi_2(cA) &= \operatorname{Tr}(cA)(cA) \\ &= c^2 \operatorname{Tr}(A)A \\ &= c^2\varphi_2(A), \end{aligned}$$

so $\varphi_2(cA) \neq c\varphi_2(A)$ if $\operatorname{Tr}(A) \neq 0$ and $c \neq 0, 1$.

(c) φ_3 is a linear transformation: it respects both addition and scalar multiplication. For brevity, let us write $q(x) = x^2 + 1$. Then for $p_1, p_2 \in \mathcal{P}_3$,

$$\begin{aligned} \varphi_3(p_1 + p_2) &= \frac{d}{dx}(q \cdot (p_1 + p_2)) \\ &= \frac{d}{dx}(qp_1 + qp_2) \\ &= \frac{d}{dx}(qp_1) + \frac{d}{dx}(qp_2) \\ &= \varphi_3(p_1) + \varphi_3(p_2). \end{aligned}$$

Note that the product rule is unnecessary in this calculation.

For scalar multiplication, we take $p \in \mathcal{P}_3$ and $c \in \mathbb{R}$:

$$\varphi_3(cp) = \frac{d}{dx}(qcp)$$

$$
\begin{aligned}
&= c\frac{d}{dx}(qp) \quad \text{because } c \text{ is constant}\\
&= c\varphi_3(p).
\end{aligned}
$$

(d) φ_4 is a linear transformation: it respects both addition and scalar multiplication. Indeed, if $f, g \in C[1,2]$, then

$$
\begin{aligned}
\varphi_4(f+g) &= \int_1^2 \frac{(f+g)(x)}{x}\,dx\\
&= \int_1^2 \left(\frac{f(x)}{x} + \frac{g(x)}{x}\right) dx\\
&= \int_1^2 \frac{f(x)}{x}\,dx + \int_1^2 \frac{g(x)}{x}\,dx\\
&= \varphi_4(f) + \varphi_4(g),
\end{aligned}
$$

and if $c \in \mathbb{R}$, then

$$
\begin{aligned}
\varphi_4(cf) &= \int_1^2 \frac{(cf)(x)}{x}\,dx\\
&= \int_1^2 \frac{cf(x)}{x}\,dx\\
&= c\int_1^2 \frac{f(x)}{x}\,dx\\
&= c\varphi_4(f).
\end{aligned}
$$

55. The linear transformation φ is injective. To see this, suppose that $p = ax^2 + bx + c$ is in $\mathrm{Ker}(\varphi)$. Then

$$
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \varphi(p) = \begin{pmatrix} a+b+c \\ 4a+b \\ 2a \end{pmatrix},
$$

so $a+b+c = 4a+b = 2a = 0$. The only solution to these equations is $a = b = c = 0$, so $p = 0$.

56. The map φ is not injective, because

$$
\varphi\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.
$$

The map is surjective, though, because given any $a \in \mathbb{R}$, we see that

$$
\varphi\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = a.
$$

57. For the first part of this question, we are looking for a non-zero polynomial $p = ax^2 + bx + c$ such that

$$\begin{pmatrix} a+b+c & a+2b+3c \\ -a+b+3c & a-c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

i.e.,

$$\begin{aligned} a + b + c &= 0 \\ a + 2b + 3c &= 0 \\ -a + b + 3c &= 0 \\ a - c &= 0 \end{aligned}$$

This homogeneous system is represented by the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The solutions (a, b, c) therefore take the form $\lambda(1, -2, 1)$ with $\lambda \in \mathbb{R}$. Taking $\lambda = 1$, for example, gives the non-zero polynomial $p = x^2 - 2x + 1 \in \operatorname{Ker}(\varphi)$.

To find a non-zero matrix in the image, let us take $a = 1$ and $b = c = 0$ in the definition of φ:

$$\varphi(x^2) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

This is a non-zero matrix in the image.

58. To decide surjectivity, the question is this: Given any matrix $A = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \in M_2(\mathbb{R})$, does there exist $p = ax^2 + bx + c \in \mathcal{P}_2$ such that $\varphi(p) = A$, i.e., such that

$$\begin{pmatrix} a+b+c & a+2b+3c \\ -a+b+3c & a-c \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}?$$

Deciding this amounts to deciding whether the system

$$\begin{aligned} a + b + c &= y_1 \\ a + 2b + 3c &= y_2 \\ -a + b + 3c &= y_3 \\ a - c &= y_4 \end{aligned} \tag{20}$$

has a solution for any given y_1, y_2, y_3, y_4, which is the same as deciding whether the coefficient matrix has a pivot in every row of a row-echelon form. But, of course, the coefficient matrix has more rows than columns, so a row-echelon form cannot possibly have a pivot in every row, and so φ is not surjective.

Let us find a matrix in $M_2(\mathbb{R})$ that is not in Image(φ). We may do this by finding $y_1, y_2, y_3, y_4 \in \mathbb{R}$ such that the system in (20) has no solution. The reduced row-echelon form of the augmented matrix representing that system is

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & y_4 \\ 0 & 1 & 2 & y_3 + y_4 \\ 0 & 0 & 0 & y_1 - y_3 - 2y_4 \\ 0 & 0 & 0 & y_2 - 2y_3 - 3y_4 \end{array}\right).$$

A matrix $\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$ in Image(φ) satisfies, in particular, $y_1 - y_3 - 2y_4 = 0$. Therefore, the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is not in Image(φ), because here $y_1 - y_3 - 2y_4 = 1 \neq 0$.

59. Suppose that $p \in \text{Ker}(\varphi)$, which is to say that $p(-1) = p(0) = p(1) = 0$. Then $p = 0$, because a polynomial of degree 2 or less that vanishes at three different values must be the zero polynomial. Thus, Ker(φ) consists only of the zero polynomial, so φ is injective.

Alternatively, write $p = ax^2 + bx + c$. The equations $p(-1) = p(0) = p(1) = 0$ say

$$\begin{aligned} a - b + c &= 0 \\ c &= 0 \\ a + b + c &= 0 \end{aligned}$$

The only solution to this system is $a = b = c = 0$, so $p = 0$.

60. We take an arbitrary vector

$$\mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3$$

and decide whether it is possible to solve $\varphi(p) = \mathbf{v}$ for $p = ax^2 + bx + c \in \mathcal{P}_2$. The equation $\varphi(p) = \mathbf{v}$ says

$$\begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix},$$

i.e.,

$$\begin{aligned} a - b + c &= a_1 \\ c &= a_2 \\ a + b + c &= a_3 \end{aligned}$$

The question of surjectivity is therefore whether this system has a solution for every $a_1, a_2, a_3 \in \mathbb{R}$. This amounts to deciding whether the coefficient matrix of the system has a pivot in every row of a row-echelon form. Row reducing, we see that

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which has a pivot in every row, so φ is surjective.

61. For injectivity, suppose that $p \in \operatorname{Ker}(\varphi)$. Then

$$p(0) + p(1) = 0 \tag{21}$$
$$p(0) - p(1) = 0 \tag{22}$$
$$p'(0) + p'(1) = 0 \tag{23}$$
$$p'(0) - p'(1) = 0 \tag{24}$$

Equations (21) and (22) taken together are equivalent to the equations $p(0) = p(1) = 0$. Similarly, (23) and (24) taken together are equivalent to the equations $p'(0) = p'(1) = 0$. Hence, if $p = ax^2 + bx + c$, then

$$\begin{aligned} c &= 0 \\ a + b + c &= 0 \\ b \quad\;\; &= 0 \\ 2a + b \quad\;\; &= 0 \end{aligned}$$

The only solution to this system is $a = b = c = 0$, so $p = 0$. Thus, φ is injective.

To decide on surjectivity, we attempt to solve the equation $\varphi(p) = A$ given an arbitrary $A \in M_2(\mathbb{R})$. That is, we wish to decide whether, given arbitrary $y_1, y_2, y_3, y_4 \in \mathbb{R}$, there is $p = ax^2 + bx + c \in \mathcal{P}_2$ such that

$$\begin{pmatrix} p(0) + p(1) & p'(0) + p'(1) \\ p(0) - p(1) & p'(0) - p'(1) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$$

i.e., such that

$$\begin{pmatrix} a + b + 2c & 2a + 2b \\ -a - b & -2a \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}.$$

Put another, given any $y_1, y_2, y_3, y_4 \in \mathbb{R}$, can we solve

$$\begin{aligned} a + b + 2c &= y_1 && (25) \\ 2a + 2b &= y_2 && (26) \\ -a - b &= y_3 && (27) \\ -2a &= y_4 && (28) \end{aligned}$$

for $a, b, c \in \mathbb{R}$? We can see, without any row reducing, that the answer must be *no*, because the coefficient matrix has more rows than columns, so a row-echelon form of the coefficient matrix must have at least one row of zeroes. Thus, φ is not surjective.

In fact, it is not too hard to spot a matrix that is not in the image of φ, because if $A = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$ *is* in the image of φ, then by (26) and (27), we see that $y_2 = -2y_3$. Therefore, we have only to find a matrix A in which $y_2 \neq -2y_3$, such as

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This matrix is not in $\mathrm{Image}(\varphi)$.

62. (a) Because

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ a+c & b+d \end{pmatrix},$$

we see that $\varphi(A) = \mathrm{Tr}(A)x + \mathrm{Tr}(BA) = (a+d)x + (a+c+b+d)$.

(b) We are to show that, given $p = \lambda x + \mu \in \mathcal{P}_1$, where $\lambda, \mu \in \mathbb{R}$, there are $a, b, c, d \in \mathbb{R}$ such that

$$(a+d)x + (a+c+b+d) = \lambda x + \mu,$$

i.e., $a+d = \lambda$ and $a+c+b+d = \mu$. There are many possibilities for a, b, c, d. For example, we may take $a = \lambda$, $b = \mu - \lambda$, and $c = d = 0$. That is

$$\varphi \begin{pmatrix} \lambda & \mu - \lambda \\ 0 & 0 \end{pmatrix} = \lambda x + \mu.$$

(c) We find a non-zero solution to the equations $a + d = 0$ and $a + c + b + d = 0$. One possibility is $a = 1$, $d = -1$, $b = c = 0$. Thus, the non-zero matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, for example, is in $\mathrm{Ker}(\varphi)$.

63. In our solution to Question 61, we saw that φ is injective. Therefore, its nullity is 0, so by the rank-nullity theorem,

$$\mathrm{rank}(\varphi) = \dim(\mathcal{P}_2) - \mathrm{nullity}(\varphi) = 3 - 0 = 3.$$

This gives us another way to see that φ is not surjective, for Image(φ) is a 3-dimensional subspace of the 4-dimensional space $M_2(\mathbb{R})$.

64. If

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

$$\text{then} \quad XA = \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix}$$

$$\text{and} \quad X^2A = \begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix},$$

$$\text{so} \quad \varphi(A) = \begin{pmatrix} a+e+i \\ g+b+f \\ d+h+c \end{pmatrix}.$$

Hence, given any $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$, we may choose A such that $\varphi(A) = \mathbf{v}$ by choosing the entries of A to satisfy

$$\begin{aligned} a + e + i &= x \\ g + b + f &= y \\ d + h + c &= z \end{aligned}$$

For example, we may take $a = x$, $b = y$, and $c = z$, and take all the other entries to be zero. Thus,

$$\varphi \begin{pmatrix} x & y & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Having shown that φ is surjective, i.e., that Image(φ) $= \mathbb{R}^3$, we may use the rank-nullity theorem to see that

$$\begin{aligned} \text{nullity}(\varphi) &= \dim(M_3(\mathbb{R})) - \text{rank}(\varphi) \\ &= 9 - 3 \\ &= 6. \end{aligned}$$

65. (a) Note that

$$\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}.$$

Now, for a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}),$$

we have $\varphi(A) = 0$ if and only if $a = b = 0$, if and only if

$$A = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} = cE_3 + dE_4.$$

Thus, $\mathrm{Ker}(\varphi) = \mathrm{Span}(E_3, E_4)$. But

$$\begin{aligned} \mathrm{Image}(\varphi) &= \left\{ \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\} \\ &= \{aE_3 + bE_4 \mid a, b \in \mathbb{R}\} \\ &= \mathrm{Span}(E_3, E_4), \end{aligned}$$

so $\mathrm{Ker}(\varphi)$ and $\mathrm{Image}(\varphi)$ are equal.

(b) Suppose that v is in both $\mathrm{Ker}(\pi)$ and $\mathrm{Image}(\pi)$. Choose $u \in V$ such that $v = \pi(u)$. Then

$$\begin{aligned} v &= \pi(u) \\ &= \pi(\pi(u)) \quad \text{by the property given in the question} \\ &= \pi(v) \\ &= \mathbf{0}_V. \end{aligned}$$

66. (a) If $p \in \mathrm{Ker}(\varphi)$, then $p(1) = p(2) = p(3) = p(4)$. Therefore, if $q(x) = p(x) - p(1)$,

$$q(1) = q(2) = q(3) = q(4) = 0,$$

i.e., $1, 2, 3, 4$ are all roots of q. Since q has degree at most 4, the only possibility is $q(x) = a(x-1)(x-2)(x-3)(x-4)$ for some $a \in \mathbb{R}$. Hence,

$$\begin{aligned} p(x) &= q(x) + p(1) \\ &= a(x-1)(x-2)(x-3)(x-4) + b \end{aligned}$$

where $b = p(1)$, so $p \in \mathrm{Span}(t, 1)$ where $t(x) = (x-1)(x-2)(x-3)(x-4)$. Conversely, t and 1 are both in $\mathrm{Ker}(\varphi)$, so $\mathrm{Ker}(\varphi) = \mathrm{Span}(t, 1)$. Because $t, 1$ are linearly independent, $\{t, 1\}$ is a basis for $\mathrm{Ker}(\varphi)$.

(b) By part (a), $\mathrm{nullity}(\varphi) = 2$. Therefore,

$$\mathrm{rank}(\varphi) = \dim(\mathcal{P}_4) - \mathrm{nullity}(\varphi) = 5 - 2 = 3.$$

(c) Let U be the subspace of $M_2(\mathbb{R})$ consisting of the matrices $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ with $a_1 + a_2 + a_3 + a_4 = 0$. We observe from the equation

$$\begin{pmatrix} -a_2 - a_3 - a_4 & a_2 \\ a_3 & a_4 \end{pmatrix} = a_2 \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + a_4 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

that U is 3-dimensional with basis $\{A_1, A_2, A_3\}$, where

$$A_1 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

But Image(φ) is contained in U, and we saw in part (b) that Image(φ) has dimension 3, so Image(φ) $= U$. Therefore, $\{A_1, A_2, A_3\}$ is a basis for Image(φ).

Another basis is

$$\left\{ \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \right\}.$$

66. *Alternative solution:* We solve the problem this time by computing $[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}}$, where $\mathcal{B} = \{x^4, x^3, x^2, x, 1\}$ and

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

We find

$$[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} -15 & -7 & -3 & -1 & 0 \\ -65 & -19 & -5 & -1 & 0 \\ -175 & -37 & -7 & -1 & 0 \\ 255 & 63 & 15 & 3 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 1/50 & 0 \\ 0 & 1 & 0 & -1/5 & 0 \\ 0 & 0 & 1 & 7/10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) A basis for the null space of $[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}}$ is

$$\left\{ \begin{pmatrix} -1/50 \\ 1/5 \\ -7/10 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

so a basis for Ker(φ) is

$$\{-\tfrac{1}{50}x^4 + \tfrac{1}{5}x^3 - \tfrac{7}{10}x^2 + x, 1\}.$$

Or we could scale the first basis vector by -50 to obtain the basis

$$\{x^4 - 10x^3 + 35x^2 - 50x, 1\}.$$

(b) The nullity is 2, because a basis for $\mathrm{Ker}(\varphi)$ consists of two vectors. Because we have computed $[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}}$ and its reduced row-echelon form, we can see by the fact that there are three pivots that $\mathrm{rank}(\varphi) = 3$. However, we are asked to use the rank-nullity theorem: $\mathrm{rank}(\varphi) = \dim(\mathcal{P}_4) - \mathrm{nullity}(\varphi) = 5 - 2 = 3$.

(c) The pivot columns of $[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}}$ are the first, second, and third, so a basis for the column space is

$$\left\{ \begin{pmatrix} -15 \\ -65 \\ -175 \\ 255 \end{pmatrix}, \begin{pmatrix} -7 \\ -19 \\ -37 \\ 63 \end{pmatrix}, \begin{pmatrix} -3 \\ -5 \\ -7 \\ 15 \end{pmatrix} \right\}.$$

Therefore, a basis for $\mathrm{Image}(\varphi)$ is

$$\left\{ \begin{pmatrix} -15 & -65 \\ -175 & 255 \end{pmatrix}, \begin{pmatrix} -7 & -19 \\ -37 & 63 \end{pmatrix}, \begin{pmatrix} -3 & -5 \\ -7 & 15 \end{pmatrix} \right\}.$$

67. Note that $\mathrm{Image}(\varphi)$ has dimension at most 3, because it is contained in the 3-dimensional space V. Therefore, by the rank-nullity theorem,

$$\mathrm{nullity}(\varphi) = \dim(U) - \mathrm{rank}(\varphi) \geq 5 - 3 = 2. \tag{29}$$

If every $\mathbf{y} \in \mathrm{Ker}(\varphi)$ were a scalar multiple of $\mathbf{x}$, then $\mathrm{Ker}(\varphi)$ would have dimension at most 1, contradicting (29). Thus, there exists some $\mathbf{y} \in \mathrm{Ker}(\varphi)$ that is not a scalar multiple of $\mathbf{x}$.

68. (a)

$$\begin{aligned} && \varphi(f) &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{e}_1 + 2\mathbf{e}_2 \\ \text{and} && \varphi(g) &= \begin{pmatrix} 1/2 \\ 2/3 \end{pmatrix} = \frac{1}{2}\mathbf{e}_1 + \frac{2}{3}\mathbf{e}_2, \\ \text{so} && [\varphi]_{\mathcal{C}\leftarrow\mathcal{B}} &= \begin{pmatrix} 1 & 1/2 \\ 2 & 2/3 \end{pmatrix}. \end{aligned}$$

(b) We compute that $\det([\varphi]_{\mathcal{C}\leftarrow\mathcal{B}}) = -1/3 \neq 0$, so a row-echelon form of $[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}}$ has a pivot in each column and in each row. The fact that there is a pivot in each column shows that the null space of $[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}}$ is zero, so $\mathrm{Ker}(\varphi)$ is zero by Proposition 6.1 in Section II of the course notes, and so φ is injective. The fact that there is a pivot in each row shows that the column space of $[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}}$ is $\mathbb{R}^2$, so $\mathrm{Image}(\varphi) = \mathbb{R}^2$ by Proposition 6.1 in Section II again, which is to say that φ is surjective.

69. Let the matrices in $\mathcal{E}$ be A_1, A_2, A_3, A_4 in that order. Then

$$\varphi(A_1) = 1, \quad \varphi(A_2) = 0, \quad \varphi(A_3) = 0, \quad \varphi(A_4) = 1,$$

so

$$[\varphi]_{\mathcal{C}\leftarrow\mathcal{E}} = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}.$$

70. (a) Let the matrices in $\mathcal{E}$ be A_1, A_2, A_3, A_4 in that order. Then

$$\begin{aligned} \varphi(x^2) &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = A_1 + A_2 - A_3 + A_4, \\ \varphi(x) &= \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = A_1 + 2A_2 + A_3 + 0A_4, \\ \varphi(1) &= \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix} = A_1 + 3A_2 + 3A_3 - A_4, \end{aligned}$$

so

$$[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}.$$

(b) Row reducing, we obtain

$$[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{30}$$

We read off from the reduced row-echelon form that a basis for the null space of $[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}}$ is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\},$$

so a basis for $\mathrm{Ker}(\varphi)$ is $\{x^2 - 2x + 1\}$. We also see from (30) that a basis for the column space of $[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}}$ is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\},$$

so a basis for $\mathrm{Image}(\varphi)$ is

$$\left\{ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \right\}.$$

71. (a) Let the matrices in $\mathcal{E}$ be A_1, A_2, A_3, A_4 in that order. Then

$$\varphi(x^2) = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} = A_1 + 2A_2 - A_3 - 2A_4,$$
$$\varphi(x) = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = A_1 + 2A_2 - A_3 + 0A_4,$$
$$\varphi(1) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 2A_1 + 0A_2 + 0A_3 + 0A_4,$$

so

$$[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 0 \\ -1 & -1 & 0 \\ -2 & 0 & 0 \end{pmatrix}.$$

(b) Row reducing, we obtain

$$[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 0 \\ -1 & -1 & 0 \\ -2 & 0 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{31}$$

The null space of $[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}}$ is zero, so $\mathrm{Ker}(\varphi)$ is the zero space, and so φ is injective. We also see from (31) that the three columns of $[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}}$ form a basis for the column space of $[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}}$, so a basis for $\mathrm{Image}(\varphi)$ is

$$\left\{ \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Note that, although we found the reduced row-echelon form for $[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}}$, any row-echelon form would have been enough. Indeed, all we were asked to do in this particular question was to demonstrate injectivity and to find a basis for the image, both of which require only a row-echelon form.

72. (a) We calculate that

$$\begin{aligned} \varphi(x^3) &= 3x^2 + 6x + 6 \\ \varphi(x^2) &= 2x + 2 \\ \varphi(x) &= 1 \\ \varphi(1) &= -x^2 - x - 1, \end{aligned}$$

so

$$[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 3 & 0 & 0 & -1 \\ 6 & 2 & 0 & -1 \\ 6 & 2 & 1 & -1 \end{pmatrix}.$$

(b) Row reducing the matrix found in part (a), we find that

$$[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{32}$$

so $\mathrm{Nul}([\varphi]_{\mathcal{C}\leftarrow\mathcal{B}})$ is spanned by $(1/3, -1/2, 0, 1)$, or, clearing denominators, we may take instead the vector $(2, -3, 0, 6)$. Therefore, a basis for $\mathrm{Ker}(\varphi)$ is $\{2x^3 - 3x^2 + 6\}$ by part (i) of Proposition 6.1 in Section II of the course notes.

(c) A row-echelon form of $[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}}$ has a pivot in every row, as we see from (32), so $\mathrm{Col}([\varphi]_{\mathcal{C}\leftarrow\mathcal{B}}) = \mathbb{R}^3$, and so $\mathrm{Image}(\varphi) = \mathcal{P}_2$ by part (ii) of Proposition 6.1 in Section II. Thus, φ is surjective. Alternatively, we may just use the fact, given on page 38 of the course notes, that a linear transformation is surjective if and only if there is a pivot in every row of a row-echelon form of an associated matrix.

73. (a) From the reduced row-echelon form given in the question, we see that $\mathrm{Nul}([\varphi]_{\mathcal{C}\leftarrow\mathcal{B}})$ has basis

$$\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\},$$

so $\mathrm{Ker}(\varphi)$ has basis $\{-x^4 - x^3 + x^2, -x^3 + 2x + 1\}$.

(b) Because the pivots of the given row-echelon form are in columns 1, 2, and 4, a basis for $\mathrm{Col}([\varphi]_{\mathcal{C}\leftarrow\mathcal{B}})$ is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Therefore, a basis for $\mathrm{Image}(\varphi)$ is

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

74. (a) We see immediately that

$$\varphi(s) = \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \mathbf{e}_1 + 5\mathbf{e}_2 \quad \text{and} \quad \varphi(t) = \begin{pmatrix} 1 \\ 8 \end{pmatrix} = \mathbf{e}_1 + 8\mathbf{e}_2,$$

so

$$[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ 5 & 8 \end{pmatrix}.$$

(b) The matrix $[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}}$ has determinant $3 \neq 0$, so it is invertible and therefore has zero null space. Hence, by Proposition 6.1 in Section II of the course notes, $\mathrm{Ker}(\varphi) = \{0\}$, so φ is injective. Now, if $(a_n)_n$ and $(b_n)_n$ in V satisfy $a_2 = b_2$ and $a_6 = b_6$, then $\varphi((a_n)_n) = \varphi((b_n)_n)$, so the injectivity of φ implies that $(a_n)_n = (b_n)_n$, which is to say that $a_n = b_n$ for all $n \geq 0$.

(There is nothing special about the indices 2 and 6 in this question. A sequence in V is determined by any two of its terms. Nor is there anything particularly special about V. A similar property holds, in typical cases, for spaces of sequences defined by other homogeneous recurrence relations.)

75. (a)

$$\begin{aligned} \psi\circ\varphi(p) &= \psi(p(0)s + p(1)t) \\ &= \psi\big((p(0), 0, -p(0), -p(0), \ldots) + (0, p(1), p(1), 0, \ldots)\big) \\ &= \psi(p(0), p(1), p(1)-p(0), -p(0), \ldots) \\ &= \begin{pmatrix} p(0) & p(1) \\ p(1)-p(0) & -p(0) \end{pmatrix}. \end{aligned}$$

(The dots in the sequences above merely indicate omitted entries, not the continuation of a pattern.)

For the change-of-basis matrix, we observe that

$$\psi\circ\varphi(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \psi\circ\varphi(1) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

so

$$[\psi\circ\varphi]_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(b) We see that

$$\begin{aligned} \varphi(x) &= 0s + t & \qquad \psi(s) &= \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} = A_1 + 0A_2 - A_3 - A_4 \\ \varphi(1) &= s + t & \qquad \psi(t) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0A_1 + A_2 + A_3 + 0A_4, \end{aligned}$$

so

$$[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad [\psi]_{\mathcal{E}\leftarrow\mathcal{C}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence, we calculate that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix},$$

as required.

76. (a) We find

$$\begin{aligned} \varphi(s) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = A_1 + 0A_2 + 0A_3 - A_4 \\ \varphi(t) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0A_1 + A_2 - A_3 + 0A_4, \end{aligned}$$

so

$$[\varphi]_{\mathcal{E}\leftarrow\mathcal{C}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Next,

$$\begin{aligned} \psi(A_1) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{e}_1 + 0\mathbf{e}_2 \\ \psi(A_2) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0\mathbf{e}_1 + \mathbf{e}_2 \\ \psi(A_3) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0\mathbf{e}_1 + \mathbf{e}_2 \\ \psi(A_4) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{e}_1 + 0\mathbf{e}_2, \end{aligned}$$

so

$$[\psi]_{\mathcal{D}\leftarrow\mathcal{E}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

(b)

$$[\psi]_{\mathcal{D}\leftarrow\mathcal{E}}[\varphi]_{\mathcal{E}\leftarrow\mathcal{C}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so $[\psi\circ\varphi]_{\mathcal{D}\leftarrow\mathcal{C}} = [\psi]_{\mathcal{D}\leftarrow\mathcal{E}}[\varphi]_{\mathcal{E}\leftarrow\mathcal{C}} = 0$. Thus, $\psi\circ\varphi$ is the zero map.

(c) Part (b) shows that $\text{Image}(\varphi) \subseteq \text{Ker}(\psi)$. However, we see from

$$[\varphi]_{\mathcal{E}\leftarrow\mathcal{C}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{33}$$

that $\text{Image}(\varphi)$ has dimension 2, because there are two pivots in (33), and we see immediately from

$$[\psi]_{\mathcal{D}\leftarrow\mathcal{E}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \tag{34}$$

that $\text{Ker}(\psi)$ also has dimension 2, because there are two non-pivot columns in (34). Thus, $\text{Image}(\varphi)$ is a dimension-2 subspace of the dimension-2 space $\text{Ker}(\psi)$, so $\text{Image}(\varphi) = \text{Ker}(\psi)$.

77. Let $y \in \mathbb{R}_{<2}$. Then for $x \in \mathbb{R}_{>0}$,

$$\begin{aligned} f(x) &= y \\ \iff 2 - \frac{1}{x} &= y, \\ \iff 2 - y &= \frac{1}{x}, \\ \iff x &= \frac{1}{2-y}. \end{aligned}$$

Thus, there is a unique $x \in \mathbb{R}_{>0}$ such that $f(x) = y$, namely, $x = \frac{1}{2-y}$. (Note that $1/(2-y)$ is indeed positive when $y < 2$.) Thus, f is invertible and $f^{-1}(y) = \frac{1}{2-y}$.

78. We first find the coordinate vector of $\psi\circ\varphi(p)$ with respect to $\mathcal{D}$:

$$\begin{aligned} [\psi\circ\varphi(p)]_{\mathcal{D}} &= [\psi\circ\varphi]_{\mathcal{D}\leftarrow\mathcal{B}}[p]_{\mathcal{B}} \\ &= [\psi\circ\varphi]_{\mathcal{D}\leftarrow\mathcal{B}} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &= [\psi]_{\mathcal{D}\leftarrow\mathcal{C}}[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ -1 & 3 & 0 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ -a+2b \\ -a+3b \\ a-b \end{pmatrix}.$$

Hence,

$$\begin{aligned} \psi \circ \varphi(p) &= bX_1 + (-a+2b)X_2 + (-a+3b)X_3 + (a-b)X_4 \\ &= b \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (-a+2b) \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\ &\quad + (-a+3b) \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + (a-b) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -a+5b & -a+4b \\ 2b & a-b \end{pmatrix}. \end{aligned}$$

79. Let $y \in \mathbb{R}_{\geq 1}$. Then for $x \in \mathbb{R}_{\geq 2}$,

$$\begin{aligned} f(x) &= y \\ \iff e^{x^2-4x+4} &= y, \\ \iff x^2-4x+4 &= \ln(y), \\ \iff (x-2)^2 &= \ln(y), \\ \iff x-2 &= \pm\sqrt{\ln(y)}, \quad \text{(recall that } y \geq 1, \text{ so } \ln(y) \geq 0) \\ \iff x-2 &= \sqrt{\ln(y)} \quad \text{because } x \geq 2 \text{ by assumption,} \\ \iff x &= 2 + \sqrt{\ln(y)}. \end{aligned}$$

Thus, there is a unique $x \in \mathbb{R}_{\geq 2}$ such that $f(x) = y$, namely, $x = 2 + \sqrt{\ln(y)} \geq 2$, so f is invertible and $f^{-1}(y) = 2 + \sqrt{\ln(y)}$.

80. Let $p = b_2x^2 + b_1x + b_0 \in \mathcal{P}_2$. For $\mathbf{u} = (a_1, a_2, a_3) \in \mathbb{R}$, $\varphi(\mathbf{u}) = p$ if and only if

$$(a_1 - a_2)x^2 + (a_2 - a_3)x + a_1 + a_3 = b_2x^2 + b_1x + b_0,$$

if and only if

$$\begin{aligned} a_1 - a_2 \qquad &= b_2 \\ a_2 - a_3 &= b_1 \\ a_1 \qquad + a_3 &= b_0 \end{aligned}$$

We are to decide whether this system has a unique solution for a_1, a_2, a_3 in terms of b_2, b_1, b_0, and to find the solution if so. We can solve this problem by row reducing:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & b_2 \\ 0 & 1 & -1 & b_1 \\ 1 & 0 & 1 & b_0 \end{array}\right) \leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}b_2 + \frac{1}{2}b_1 + \frac{1}{2}b_0 \\ 0 & 1 & 0 & -\frac{1}{2}b_2 + \frac{1}{2}b_1 + \frac{1}{2}b_0 \\ 0 & 0 & 1 & -\frac{1}{2}b_2 - \frac{1}{2}b_1 + \frac{1}{2}b_0 \end{array}\right).$$

Thus, there is a unique solution to the system, so there is a unique $\mathbf{u} \in \mathbb{R}^3$ such that $\varphi(\mathbf{u}) = p$, namely,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{2}b_2 + \frac{1}{2}b_1 + \frac{1}{2}b_0 \\ -\frac{1}{2}b_2 + \frac{1}{2}b_1 + \frac{1}{2}b_0 \\ -\frac{1}{2}b_2 - \frac{1}{2}b_1 + \frac{1}{2}b_0 \end{pmatrix}. \tag{35}$$

The linear transformation φ is therefore invertible, and its inverse sends $p = b_2x^2 + b_1x + b_0 \in \mathcal{P}_2$ to the vector $\mathbf{u}$ in (35).

81.

$$\begin{aligned} \varphi(A_1) &= \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix} \\ \varphi(A_2) &= \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix} \\ \varphi(A_3) &= \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \\ \varphi(A_4) &= \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}, \end{aligned}$$

so

$$\begin{aligned} [\varphi]_{\mathcal{C}} &= \begin{pmatrix} 3 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & -1 & 3 \end{pmatrix} \\ &\leftrightarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 13 & 7 \\ 0 & 0 & -5 & 1 \end{pmatrix}, \end{aligned}$$

which we see, without any further row reducing, has determinant $48 \neq 0$. Thus, $[\varphi]_{\mathcal{C}}$ is invertible, so φ is an isomorphism.

82. (a) By row reducing, we can simultaneously show that the matrix $[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}}$ is invertible and find its inverse:

$$\left(\begin{array}{ccc|ccc} 3 & -2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \leftrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1 & 1/2 \\ 0 & 1 & 0 & 1/2 & -2 & 3/2 \\ 0 & 0 & 1 & 1/2 & -1 & 3/2 \end{array}\right).$$

The matrix $[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}}$ is invertible because the above 3×6 row-echelon form has the 3×3 identity matrix to the left of the vertical line, so the linear transformation φ is invertible. Further, the inverse matrix is the matrix to the right of the vertical line, and so

$$[\varphi^{-1}]_{\mathcal{B}\leftarrow\mathcal{E}} = [\varphi]^{-1}_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} 1/2 & -1 & 1/2 \\ 1/2 & -2 & 3/2 \\ 1/2 & -1 & 3/2 \end{pmatrix}.$$

Hence,

$$\begin{aligned}[\varphi^{-1}(a_1, a_2, a_3)]_{\mathcal{B}} &= [\varphi^{-1}]_{\mathcal{B}\leftarrow\mathcal{E}}[(a_1, a_2, a_3)]_{\mathcal{E}} \\ &= \begin{pmatrix} 1/2 & -1 & 1/2 \\ 1/2 & -2 & 3/2 \\ 1/2 & -1 & 3/2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}a_1 - a_2 + \frac{1}{2}a_3 \\ \frac{1}{2}a_1 - 2a_2 + \frac{3}{2}a_3 \\ \frac{1}{2}a_1 - a_2 + \frac{3}{2}a_3 \end{pmatrix}.\end{aligned}$$

Therefore,

$$\begin{aligned}\varphi^{-1}(a_1, a_2, a_3) = (\tfrac{1}{2}a_1 - a_2 + \tfrac{1}{2}a_3)x^2 + (\tfrac{1}{2}a_1 - 2a_2 + \tfrac{3}{2}a_3)x \\ + \tfrac{1}{2}a_1 - a_2 + \tfrac{3}{2}a_3.\end{aligned}$$

(b) We are looking for the unique $p \in \mathcal{P}_2$ such $\varphi(p) = (0, -1, 0)$, i.e.,

$$p = \varphi^{-1}(0, -1, 0) = x^2 + 2x + 1$$

according to the expression found in part (a).

83. Let $t = (c_n)_n \in \mathcal{S}$. For $s = (a_n)_n \in \mathcal{S}$,

$$\begin{aligned} \varphi(s) &= t \\ \iff \left(\sum_{k=0}^{n} a_k\right)_n &= (c_n)_n, \\ \iff \sum_{k=0}^{n} a_k &= c_n \quad \text{for all } n \geq 0, \end{aligned}$$

if and only if

$$\begin{aligned} a_0 &= c_0 \\ a_1 + a_0 &= c_1 \\ a_2 + a_1 + a_0 &= c_2 \\ a_3 + a_2 + a_1 + a_0 &= c_3 \\ &\vdots \end{aligned}$$

if and only if

$$a_0 = c_0$$

$$a_1 = c_1 - c_0$$
$$a_2 = c_2 - c_1$$
$$a_3 = c_3 - c_2$$
$$\vdots$$

Thus, there is a unique $(a_n)_n \in \mathcal{S}$ such that $\varphi((a_n)_n) = (c_n)_n$, its terms being given by $a_0 = c_0$ and $a_n = c_n - c_{n-1}$ for $n \geq 1$. Therefore, φ is invertible, and

$$\varphi^{-1}((c_n)_n) = (c_n - c_{n-1})_n,$$

where we have set $c_{-1} = 0$.

84. (a) Assume $\mathrm{Tr}(C) \neq s$, and let $A \in \mathrm{Ker}(\varphi)$, i.e., $\mathrm{Tr}(A)C - sA = 0$. Taking the trace of both sides, we find $\mathrm{Tr}(A)\,\mathrm{Tr}(C) - s\,\mathrm{Tr}(A) = 0$, i.e.,

$$\mathrm{Tr}(A)(\mathrm{Tr}(C) - s) = 0.$$

Because $\mathrm{Tr}(C) \neq s$ by assumption, we have $\mathrm{Tr}(A) = 0$. Hence,

$$A = \frac{1}{s}\,\mathrm{Tr}(A)C = 0.$$

Thus, φ is injective, so it is in fact an isomorphism because the domain and codomain have the same dimension.

(b) (*i*) We first observe that $C \in \mathrm{Ker}(\varphi)$. Indeed, because $s = \mathrm{Tr}(C)$, we have

$$\varphi(C) = \mathrm{Tr}(C)C - \mathrm{Tr}(C)C = 0.$$

Conversely, suppose $A \in \mathrm{Ker}(\varphi)$, i.e., $\mathrm{Tr}(A)C - sA = 0$. Then

$$A = \frac{1}{s}\,\mathrm{Tr}(A)C \in \mathrm{Span}(C).$$

(*ii*) Suppose $C = \varphi(A)$ for some $A \in M_n(\mathbb{R})$, i.e., $C = \mathrm{Tr}(A)C - sA$. Then taking the trace of both sides and remembering that $\mathrm{Tr}(C) = s$, we obtain

$$s = \mathrm{Tr}(A)s - s\,\mathrm{Tr}(A) = 0,$$

contradicting our assumption that $s \neq 0$.

85. (a) Suppose $\lambda, \mu \in \mathbb{R}$ satisfy

$$\lambda e^{cx}\sin(sx) + \mu e^{cx}\cos(sx) = 0$$

for all $x \in \mathbb{R}$. Since $e^{cx} > 0$, we in fact have

$$\lambda \sin(sx) + \mu \cos(sx) = 0$$

for all $x \in \mathbb{R}$. Taking $x = 0$ gives $\mu = 0$, and then taking $x = \frac{\pi}{2s}$ gives $\lambda = 0$.

(b)

$$\begin{aligned} f'(x) &= ce^{cx}\sin(sx) + se^{cx}\cos(sx) = cf(x) + sg(x) \\ \text{and} \quad g'(x) &= ce^{cx}\cos(sx) - se^{cx}\sin(sx) = -sf(x) + cg(x), \end{aligned}$$

so

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}.$$

(c) $\det([\varphi]_{\mathcal{B}}) = c^2 + s^2 > 0$, since $s \neq 0$ by assumption.

(d) One way is to observe that the linear transformation φ is invertible, because the matrix $[\varphi]_{\mathcal{B}}$ is. Invertible transformations are surjective, so every $h \in V$ is in the image of φ, i.e., is the derivative of some function H in V. The function H is unique because φ is injective.

In short, $H = \varphi^{-1}(h)$.

(e) We are trying to find the function $\int h = \varphi^{-1}(\lambda f + \mu g)$. The coordinate vector of $\int h$ is

$$\begin{aligned} [\varphi^{-1}]_{\mathcal{B}} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} &= [\varphi]_{\mathcal{B}}^{-1} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \\ &= \begin{pmatrix} c & -s \\ s & c \end{pmatrix}^{-1} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \\ &= \frac{1}{c^2+s^2} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \\ &= \frac{1}{c^2+s^2} \begin{pmatrix} c\lambda + s\mu \\ -s\lambda + c\mu \end{pmatrix}. \end{aligned}$$

Thus,

$$\int h = \tfrac{1}{c^2+s^2}\big((c\lambda + s\mu)f + (-s\lambda + c\mu)g\big).$$

86. (a) We have

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} = \begin{pmatrix} \cos(2\pi/9) & -\sin(2\pi/9) \\ \sin(2\pi/9) & \cos(2\pi/9) \end{pmatrix},$$

and this is the form of a rotation matrix, the angle in this case being $2\pi/9$.

(b) Because the 9th power of the rotation matrix

$$\begin{pmatrix} \cos(2\pi/9) & -\sin(2\pi/9) \\ \sin(2\pi/9) & \cos(2\pi/9) \end{pmatrix}$$

is the 2×2 identity matrix I, we have

$$\begin{aligned}[\varphi^9]_{\mathcal{B}} &= [\varphi]_{\mathcal{B}}^9 \\ &= I \\ &= [\mathbf{1}_V]_{\mathcal{B}},\end{aligned}$$

so $\varphi^9 = \mathbf{1}_V$. In other words, the 9th derivative of any function $h \in V$ is just h.

(c) By definition, $\int h$ is the unique function in V whose derivative is h. But $(h^{(8)})' = h^{(9)} = h$ by part (b), so $\int h = h^{(8)}$.

87. The map φ is invertible. We are told that it is surjective, so by Proposition 9.1 in Section II of the course notes, it remains only to show that it is injective. But we are also told that a row-echelon form of $[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}}$ has a pivot in every column, so $\mathrm{Nul}([\varphi]_{\mathcal{C}\leftarrow\mathcal{B}})$ is the zero space. Thus, $\mathrm{Ker}(\varphi) = \{\mathbf{0}_U\}$ by Proposition 6.1 in Section II, and so φ is injective. (Alternatively, we may use the paragraph in the middle of page 38 of the course notes.)

88. We group together vector spaces of the same dimension: $\{4,6\}$ (dimension 2), $\{1,3\}$ (dimension 3), and $\{2,5\}$ (dimension 8).

89. We group together vector spaces of the same dimension: $\{3,5\}$ (dimension 2), $\{2,4\}$ (dimension 4), and $\{1,6\}$ (dimension 6).

90. The spaces $\mathbb{R}^3$ and V are isomorphic to each other. To show this, we first establish that f_1, f_2, f_3 are linearly independent. Suppose that $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$, where $c_1, c_2, c_3 \in \mathbb{R}$. That is,

$$c_1 + c_2 \cos(x) + c_3 \cos(x + \tfrac{\pi}{4}) = 0 \quad \text{for all } x \in \mathbb{R}.$$

Taking $x = 0, \pi/4, -\pi/4$ gives, respectively, the equations

$$\begin{aligned} c_1 + c_2 + \frac{\sqrt{2}}{2}c_3 &= 0 \\ c_1 + \frac{\sqrt{2}}{2}c_2 \qquad &= 0 \\ c_1 + \frac{\sqrt{2}}{2}c_2 + c_3 &= 0 \end{aligned}$$

While we could solve this system by Gaussian elimination, there is a quicker way. The second and third equations imply immediately that $c_3 = 0$, and then the difference of the first and second equations shows that $c_2 = 0$, from which we finally deduce (via the first or the second equation) that $c_1 = 0$. Thus, f_1, f_2, f_3

are linearly independent. Hence, because they span V by definition, they form a basis for V.

By Section II$-$10 in the course notes, we may therefore define an isomorphism

$$\begin{aligned} \varphi : \mathbb{R}^3 &\to V \\ a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 &\mapsto a_1 f_1 + a_2 f_2 + a_3 f_3, \end{aligned}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$. Note that

$$\varphi(a_1, a_2, a_3) = a_1 f_1 + a_2 f_2 + a_3 f_3.$$

91. (a)

$$\begin{aligned} p_A(x) &= \det(xI - A) \\ &= \det \begin{pmatrix} x-8 & -6 \\ 3 & x+1 \end{pmatrix} \\ &= (x-8)(x+1) + 18 \\ &= x^2 - 7x + 10 \\ &= (x-2)(x-5). \end{aligned}$$

The eigenvalues are therefore 2 and 5.

The eigenspace for 2 is the null space of

$$2I - A = \begin{pmatrix} -6 & -6 \\ 3 & 3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

which is spanned by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The eigenspace for 5 is the null space of

$$5I - A = \begin{pmatrix} -3 & -6 \\ 3 & 6 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix},$$

which is spanned by $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

(b)

$$\begin{aligned} p_B(x) &= \det(xI - B) \\ &= (x-7)((x+5)(x-4)+18) \\ &\quad - 12(-3(x-4)-9) + 6(18 - 3(x+5)) \\ &= x^3 - 6x^2 + 9x - 4 \end{aligned}$$

$$= \quad (x-1)^2(x-4).$$

The eigenvalues are therefore 1 and 4.

The eigenspace for 1 is the null space of

$$I - B = \begin{pmatrix} -6 & 12 & 6 \\ -3 & 6 & 3 \\ 3 & -6 & -3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which has basis

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The eigenspace for 4 is the null space of

$$4I - B = \begin{pmatrix} -3 & 12 & 6 \\ -3 & 9 & 3 \\ 3 & -6 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which has basis

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

92. Let us first find the eigenspace associated to -1:

$$-I - C = \begin{pmatrix} -6 & 3 & 9 \\ 12 & -6 & -18 \\ -6 & 3 & 9 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 & -1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so this eigenspace has basis $\{(1,2,0),(3,0,2)\}$. Note that the geometric multiplicity is $d_{-1} = 2$.

Next, we turn to the eigenspace associated to 2:

$$2I - C = \begin{pmatrix} -3 & 3 & 9 \\ 12 & -3 & -18 \\ -6 & 3 & 12 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

so this eigenspace has basis $\{(1,-2,1)\}$. The geometric multiplicity is $d_2 = 1$.

The sum of the geometric multiplicities is $d_{-1} + d_2 = 2 + 1 = 3$, so C is diagonalizable. Specifically, $P^{-1}CP = D$ where

$$P = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -2 \\ 0 & 2 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

93. Let us first find the eigenspace associated to -2:

$$-2I - C = \begin{pmatrix} -6 & 1 & -1 \\ -2 & -3 & -1 \\ 10 & -5 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 5 & 0 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

so this eigenspace has basis $\{(1, 1, -5)\}$. The geometric multiplicity is $d_{-2} = 1$.

Next, we turn to the eigenspace associated to 2:

$$2I - C = \begin{pmatrix} -2 & 1 & -1 \\ -2 & 1 & -1 \\ 10 & -5 & 5 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so this eigenspace has basis $\{(1, 2, 0), (-1, 0, 2)\}$. Note that the geometric multiplicity is $d_2 = 2$.

The sum of the geometric multiplicities is $d_{-2} + d_2 = 1 + 2 = 3$, so C is diagonalizable. Specifically, $P^{-1}CP = D$ where

$$P = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -5 & 0 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

94. (a) We consider the eigenspace associated to 3:

$$3I - A = \begin{pmatrix} 0 & -1 & 3 \\ 42 & 6 & 3 \\ 14 & 2 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 14 & 2 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

We do not need the reduced row-echelon form, because we can see already from this row-echelon form that the eigenspace is one-dimensional, that is, the geometric multiplicity of 3 is $d_3 = 1$. This is less than the algebraic multiplicity m_3 (which is 2), so A is not diagonalizable.

(b) Because $p_B(x) = (x - 2)(x^2 + 1)$ and $x^2 + 1$ has no real roots, we see that the roots of $p_B(x)$ are not all real, so B is not diagonalizable over $\mathbb{R}$. (It is, however, diagonalizable over $\mathbb{C}$.)

Alternatively, we may observe that the only real eigenvalue is 2, and it has algebraic multiplicity $m_2 = 1$. Hence, its geometric multiplicity d_2 satisfies $1 \leq d_2 \leq m_2 = 1$, so $d_2 = 1$. The sum of the (real) geometric multiplicities is therefore $1 < 3$, so B is not diagonalizable over $\mathbb{R}$.

95. (a) The characteristic polynomial of A is

$$p_A(x) \quad = \quad \det(xI - A)$$

$$
\begin{aligned}
&= \det\begin{pmatrix} x+1 & 5 & -5 \\ -5 & x+1 & -5 \\ -5 & 5 & x-9 \end{pmatrix} \\
&= (x+1)(x-4)^2,
\end{aligned}
$$

so the eigenvalues of A are 4 and -1.

The eigenvalue 4 has algebraic multiplicity 2, but the corresponding eigenspace has dimension $1 < 2$:

$$4I - A = \begin{pmatrix} 5 & 5 & -5 \\ -5 & 5 & -5 \\ -5 & 5 & -5 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

There is therefore no basis of $\mathbb{R}^3$ consisting of eigenvalues of A, so A is not diagonalizable.

Just for interest: The matrix A has Jordan normal form

$$\begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Specifically,

$$Q^{-1}AQ = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

where

$$Q = \begin{pmatrix} 0 & 1 & -1 \\ 10 & -1 & 1 \\ 10 & 0 & 1 \end{pmatrix}.$$

(b) The characteristic polynomial of B is

$$
\begin{aligned}
p_B(x) &= \det(xI - B) \\
&= \det\begin{pmatrix} x+1 & -5 & 5 \\ -5 & x+1 & -5 \\ -5 & 5 & x-9 \end{pmatrix} \\
&= (x+1)(x-4)^2,
\end{aligned}
$$

so the eigenvalues of B are again 4 and -1.

This time, the eigenspace corresponding to 4 has dimension 2:

$$4I - B = \begin{pmatrix} 5 & -5 & 5 \\ -5 & 5 & -5 \\ -5 & 5 & -5 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis for this eigenspace is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Further,

$$-I - B = \begin{pmatrix} 0 & -5 & 5 \\ -5 & 0 & -5 \\ -5 & 5 & -10 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

so the eigenspace corresponding to -1 has basis

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

The set

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

is therefore a basis of $\mathbb{R}^3$ consisting of eigenvectors of B, so if

$$P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

then P is invertible and

$$P^{-1}BP = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

96. The matrix C has characteristic polynomial $\det(xI - C) = x^5$, so 0 is the only eigenvalue. The corresponding eigenspace is the null space of

$$0I - C = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which has dimension 4. Therefore, any linearly independent set of eigenvectors has size 4 or less. Consequently, there is no basis of $\mathbb{R}^5$ (or even $\mathbb{C}^5$) consisting of eigenvectors of C, so C is not diagonalizable.

97. (a) A is diagonalizable, because it is a 5×5 matrix with 5 distinct eigenvalues.

(b) A is not invertible. Indeed, because 0 is an eigenvalue, there is $v \in \mathbb{R}^5 \setminus \{\mathbf{0}\}$ such that $Av = 0v = \mathbf{0}$, so A has a non-zero null space.

(c) The eigenvalues of A^2 are the squares of the eigenvalues of A. To see this, we reason as follows. Using the fact that A is diagonalizable, we choose an invertible 5×5 matrix P such that $P^{-1}AP = D$, where

$$D = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Then

$$P^{-1}A^2P = (P^{-1}AP)^2 = D^2 = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Hence, $p_{A^2}(x) = p_{D^2}(x) = x(x-1)^2(x-4)^2$, so A^2 has eigenvalues 0, 1, and 4.

98. (a) Because A is upper triangular, and therefore $xI - A$ as well, we may compute $p_A(x)$ easily:

$$\begin{aligned} p_A(x) = \det(xI - A) &= \det \begin{pmatrix} x-1 & b & -b^2 & b^3 \\ 0 & x-b & b^2 & -b^3 \\ 0 & 0 & x-b^2 & b^3 \\ 0 & 0 & 0 & x-b^3 \end{pmatrix} \\ &= (x-1)(x-b)(x-b^2)(x-b^3). \end{aligned}$$

(b) The numbers $1, b, b^2, b^3$ are distinct. Indeed, if $i < j$ are integers, then $j - i > 0$ and so $b^{j-i} > 1$ because $b > 1$ (by assumption). Hence, $b^i < b^j$. Therefore, by part (a), the eigenvalues of A are the four distinct real numbers $1, b, b^2, b^3$, and each has algebraic multiplicity 1. Because $1 \leq d_\lambda \leq m_\lambda$ for each eigenvalue λ, every eigenvalue has geometric multiplicity 1.

(c) The sum of the geometric multiplicities is $d_1 + d_b + d_{b^2} + d_{b^3} = 1+1+1+1 = 4$, so A is diagonalizable by Theorem 1.1 in Section III of the course notes.

99. We first diagonalize the matrix

$$A = \begin{pmatrix} 8 & 2 \\ -15 & -3 \end{pmatrix}.$$

The characteristic polynomial is $\det(xI - A) = x^2 - 5x + 6 = (x-2)(x-3)$, so the eigenvalues of A are 2 and 3. For the eigenspace associated to 2, we row reduce as follows:

$$\begin{aligned} 2I - A &= \begin{pmatrix} -6 & -2 \\ 15 & 5 \end{pmatrix} \\ &\leftrightarrow \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

A basis for the eigenspace is therefore

$$\left\{ \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\}.$$

Similarly, from the row reduction

$$\begin{aligned} 3I - A &= \begin{pmatrix} -5 & -2 \\ 15 & 6 \end{pmatrix} \\ &\leftrightarrow \begin{pmatrix} 5 & 2 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

we see that a basis for the eigenspace associated to 3 is

$$\left\{ \begin{pmatrix} -2 \\ 5 \end{pmatrix} \right\}.$$

Therefore, if

$$P = \begin{pmatrix} -1 & -2 \\ 3 & 5 \end{pmatrix}, \ D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

then P is invertible and $P^{-1}AP = D$.

Now define the functions g_1, g_2 by

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = P^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} g_1' \\ g_2' \end{pmatrix} = P^{-1} \begin{pmatrix} f_1' \\ f_2' \end{pmatrix} = P^{-1}A \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = D \begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

that is, $g_1' = 2g_1$ and $g_2' = 3g_2$. Thus, there are real constants a_1 and a_2 such that $g_1(x) = a_1 e^{2x}$ and $g_2(x) = a_2 e^{3x}$. To find a_1 and a_2, note that

$$\begin{aligned} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= \begin{pmatrix} g_1(0) \\ g_2(0) \end{pmatrix} \\ &= P^{-1} \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 5 & 2 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

Thus, $g_1(x) = 3e^{2x}$ and $g_2(x) = -2e^{3x}$. Finally,

$$\begin{aligned} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= P \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \\ &= \begin{pmatrix} -g_1 - 2g_2 \\ 3g_1 + 5g_2 \end{pmatrix}, \end{aligned}$$

so

$$\begin{aligned} f_1(x) &= -3e^{2x} + 4e^{3x} \\ f_2(x) &= 9e^{2x} - 10e^{3x}. \end{aligned}$$

100. The system may be written as

$$\begin{pmatrix} f_1' \\ f_2' \end{pmatrix} = A \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

where

$$A = \begin{pmatrix} 4 & 1 \\ -6 & -3 \end{pmatrix}.$$

First, we show that A is diagonalizable and diagonalize it. The characteristic polynomial is

$$\begin{aligned} p_A(x) = \det(xI - A) = (x-4)(x+3) + 6 &= x^2 - x - 6 \\ &= (x+2)(x-3), \end{aligned}$$

so the eigenvalues of A are -2 and 3. The eigenspace associated to -2 is the null space of

$$-2I - A = \begin{pmatrix} -6 & -1 \\ 6 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 6 & 1 \\ 0 & 0 \end{pmatrix},$$

so a basis is

$$\left\{ \begin{pmatrix} -1 \\ 6 \end{pmatrix} \right\}.$$

The eigenspace associated to 3 is the null space of

$$3I - A = \begin{pmatrix} -1 & -1 \\ 6 & 6 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

so a basis is

$$\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

The sum of the geometric multiplicities is $1 + 1 = 2$, so A is diagonalizable. Specifically, if

$$P = \begin{pmatrix} -1 & -1 \\ 6 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix},$$

then P is invertible and $P^{-1}AP = D$.

Now, define functions g_1, g_2 by

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = P^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Then

$$\begin{aligned} \begin{pmatrix} g_1' \\ g_2' \end{pmatrix} = P^{-1} \begin{pmatrix} f_1' \\ f_2' \end{pmatrix} &= P^{-1}A \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ &= P^{-1}AP \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = D \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} -2g_1 \\ 3g_2 \end{pmatrix}. \end{aligned}$$

Hence, there are constants $a_1, a_2 \in \mathbb{R}$ such that

$$g_1(x) = a_1 e^{-2x}, \quad g_2(x) = a_2 e^{3x}.$$

Therefore, because

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = P \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} -g_1 - g_2 \\ 6g_1 + g_2 \end{pmatrix},$$

we have $f_1(x) = -a_1 e^{-2x} - a_2 e^{3x}$ and $f_2(x) = 6a_1 e^{-2x} + a_2 e^{3x}$.

Finally, we use the given constraints. Because $f_1'(x) = 2a_1 e^{-2x} - 3a_2 e^{3x}$, the constaints $f_1(0) = 16$ and $f_1'(0) = -3$ yield the linear system

$$\begin{aligned} -a_1 - a_2 &= -3 \\ 2a_1 - 3a_2 &= 16, \end{aligned}$$

which has solution $a_1 = 5$, $a_2 = -2$. Thus,

$$f_1(x) = -5e^{-2x} + 2e^{3x}, \quad f_2(x) = 30e^{-2x} - 2e^{3x}.$$

101. Define functions g_1, g_2, g_3 by

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = P^{-1} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}.$$

Then

$$\begin{aligned}
\begin{pmatrix} g_1' \\ g_2' \\ g_3' \end{pmatrix} &= P^{-1} \begin{pmatrix} f_1' \\ f_2' \\ f_3' \end{pmatrix} \\
&= P^{-1} A \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \\
&= D \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \\
&= \begin{pmatrix} -g_1 \\ -g_2 \\ g_3 \end{pmatrix}.
\end{aligned}$$

Therefore, there are real constants a_1, a_2, a_3 such that

$$\begin{aligned}
g_1(x) &= a_1 e^{-x} \\
g_2(x) &= a_2 e^{-x} \\
g_3(x) &= a_3 e^{x}.
\end{aligned}$$

Now,

$$\begin{aligned}
\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} &= \begin{pmatrix} g_1(0) \\ g_2(0) \\ g_3(0) \end{pmatrix} \\
&= P^{-1} \begin{pmatrix} f_1(0) \\ f_2(0) \\ f_3(0) \end{pmatrix} \\
&= P^{-1} \begin{pmatrix} 6 \\ -3 \\ 8 \end{pmatrix}.
\end{aligned}$$

To find a_1, a_2, a_3, we solve the system

$$P \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 8 \end{pmatrix}:$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & 6 \\ 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 8 \end{array}\right) \leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right).$$

Thus, $g_1(x) = 2e^{-x}$, $g_2(x) = 3e^{-x}$, and $g_3(x) = 5e^x$. Finally,

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = P \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} -g_1 + g_2 + g_3 \\ g_1 - g_3 \\ g_2 + g_3 \end{pmatrix},$$

so

$$\begin{aligned} f_1(x) &= -2e^{-x} + 3^{-x} + 5e^x \\ &= e^{-x} + 5e^x \\ f_2(x) &= 2e^{-x} - 5e^x \\ f_3(x) &= 3e^{-x} + 5e^x. \end{aligned}$$

102. Note that the matrix describing this system of differential equations is

$$A = \begin{pmatrix} 1 & 1 & 4 \\ 2 & 2 & -4 \\ -2 & 1 & 7 \end{pmatrix}.$$

That is,

$$\begin{pmatrix} f_1' \\ f_2' \\ f_3' \end{pmatrix} = A \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}.$$

We saw in Section III−1 of the course notes that $P^{-1}AP = D$ where

$$P = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Therefore, if we define functions g_1, g_2, g_3 by

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = P^{-1} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix},$$

we have

$$\begin{aligned} \begin{pmatrix} g_1' \\ g_2' \\ g_3' \end{pmatrix} &= P^{-1} \begin{pmatrix} f_1' \\ f_2' \\ f_3' \end{pmatrix} \\ &= P^{-1}A \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \\ &= P^{-1}AP \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \end{aligned}$$

$$= D\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

$$= \begin{pmatrix} 3g_1 \\ 3g_2 \\ 4g_3 \end{pmatrix}.$$

Hence, there exist constants $a_1, a_2, a_3 \in \mathbb{R}$ such that $g_1(x) = a_1e^{3x}$, $g_2(x) = a_2e^{3x}$, and $g_3(x) = a_3e^{4x}$.

Now,

$$\begin{aligned}
\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} &= \begin{pmatrix} f_1(0) \\ f_2(0) \\ f_3(0) \end{pmatrix} \\
&= P\begin{pmatrix} g_1(0) \\ g_2(0) \\ g_3(0) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.
\end{aligned}$$

Solving the system

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 0 & -1 & 3 \\ 0 & 1 & 1 & 1 \end{array}\right) \leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right),$$

we see that $a_1 = 1$, $a_2 = 2$, and $a_3 = -1$, so $g_1(x) = e^{3x}$, $g_2(x) = 2e^{3x}$, and $g_3(x) = -e^{4x}$. Hence, because

$$\begin{aligned}
\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} &= P\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \\
&= \begin{pmatrix} g_1 + 2g_2 + g_3 \\ 2g_1 - g_3 \\ g_2 + g_3 \end{pmatrix},
\end{aligned}$$

we have

$$\begin{aligned}
f_1(x) &= e^{3x} + 4e^{3x} - e^{4x} \\
&= 5e^{3x} - e^{4x}
\end{aligned}$$

$$\begin{aligned} f_2(x) &= 2e^{3x} + e^{4x} \\ f_3(x) &= 2e^{3x} - e^{4x}. \end{aligned}$$

103. (a) The system may be written as

$$\begin{pmatrix} f_1' \\ f_2' \end{pmatrix} = A \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

where

$$A = \begin{pmatrix} -3 & 2 \\ -1 & -6 \end{pmatrix}.$$

First, we show that A is diagonalizable and diagonalize it. The characteristic polynomial is

$$\begin{aligned} p_A(x) = \det(xI - A) = (x+3)(x+6) + 2 &= x^2 + 9x + 20 \\ &= (x+4)(x+5), \end{aligned}$$

so the eigenvalues of A are -4 and -5. The eigenspace associated to -4 is the null space of

$$-4I - A = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix},$$

so a basis is

$$\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}.$$

The eigenspace associated to -5 is the null space of

$$-5I - A = \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

so a basis is

$$\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

The sum of the geometric multiplicities is $1 + 1 = 2$, so A is diagonalizable. Specifically, if

$$P = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -4 & 0 \\ 0 & -5 \end{pmatrix},$$

then P is invertible and $P^{-1}AP = D$.

Now, define functions g_1, g_2 by

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = P^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} g_1' \\ g_2' \end{pmatrix} = P^{-1} \begin{pmatrix} f_1' \\ f_2' \end{pmatrix} = P^{-1}A \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$
$$= P^{-1}AP \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = D \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} -4g_1 \\ -5g_2 \end{pmatrix}.$$

Hence, there are constants $a_1, a_2 \in \mathbb{R}$ such that

$$g_1(x) = a_1 e^{-4x}, \quad g_2(x) = a_2 e^{-5x}.$$

Therefore, because

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = P \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} -2g_1 - g_2 \\ g_1 + g_2 \end{pmatrix},$$

we have

$$\begin{aligned} f_1(x) &= -2a_1 e^{-4x} - a_2 e^{-5x} \\ f_2(x) &= a_1 e^{-4x} + a_2 e^{-5x}. \end{aligned} \tag{36}$$

(b) First, $f_1(0) = f_2(0)$ if and only if $-2a_1 - a_2 = a_1 + a_2$, if and only if

$$3a_1 + 2a_2 = 0. \tag{37}$$

Further, $f_1'(x) = 8a_1 e^{-4x} + 5a_2 e^{-5x}$ and $f_2'(x) = -4a_1 e^{-4x} - 5a_2 e^{-5x}$, so $7f_1'(0) = f_2'(0)$ if and only if $56a_1 + 35a_2 = -4a_1 - 5a_2$, if and only if $60a_1 + 40a_2 = 0$, if and only if $3a_1 + 2a_2 = 0$. This is the same condition as (37), so the desired functions are those in (36) where $a_1 = -\frac{2}{3}a_2$. Letting $c = \frac{1}{3}a_2$ (so $a_1 = -2c$ and $a_2 = 3c$), we obtain the solutions

$$\begin{aligned} f_1(x) &= c(4e^{-4x} - 3e^{-5x}) \\ f_2(x) &= c(-2e^{-4x} + 3e^{-5x}) \end{aligned}$$

with $c \in \mathbb{R}$. (Note that c, although arbitrary, is the same for both functions.)

104. The given system of differential equations can be expressed as

$$\begin{pmatrix} f_1' \\ f_2' \end{pmatrix} = A \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

where the matrix A is as in the question. Hence,

$$\begin{pmatrix} g_1' \\ g_2' \end{pmatrix} = P^{-1} \begin{pmatrix} f_1' \\ f_2' \end{pmatrix} = P^{-1}A \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$
$$= P^{-1}AP \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = B \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} -2g_1 + g_2 \\ -2g_2 \end{pmatrix}.$$

105. (a) The equation $g_2' = -2g_2$ says that there is a constant $b \in \mathbb{R}$ such that $g_2(x) = be^{-2x}$. Hence, the equation $g_1' = -2g_1 + g_2$ says that $g_1'(x) = -2g_1(x) + be^{-2x}$. Therefore, by the fact given in the question, there is a constant $a \in \mathbb{R}$ such that $g_1(x) = (a + bx)e^{-2x}$. In summary,

$$g_1(x) = (a + bx)e^{-2x} \tag{38}$$

$$g_2(x) = be^{-2x} \tag{39}$$

for some constants a, b. Conversely, differentiation shows that functions g_1 and g_2 given by (38) and (39) satisfy the specified differential equations.

(b) We find that

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = P \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 3g_1 + g_2 \\ -3g_1 \end{pmatrix},$$

so

$$\begin{aligned} f_1(x) &= 3(a + bx)e^{-2x} + be^{-2x} \\ &= (3a + b + 3bx)e^{-2x} \\ \text{and} \quad f_2(x) &= -3(a + bx)e^{-2x} \end{aligned}$$

106. The characteristic polynomial of B is

$$p_B(x) = \det(xI - B) = x^2 - 6x + 13,$$

whose complex roots are

$$\frac{1}{2}(6 \pm \sqrt{36 - 52}) = 3 \pm 2i.$$

Note that both eigenvalues have algebraic multiplicity 1, so each geometric multiplicity is necessarily equal to the corresponding algebraic multiplicity because of the inequalities $1 \leq d_\lambda \leq m_\lambda$. Therefore, Theorem 3.2 in Section III tells us already that B is diagonalizable over $\mathbb{C}$.

To diagonalize B, let us find the eigenspace associated to the eigenvalue $3 + 2i$:

$$(3 + 2i)I - B = \begin{pmatrix} 2 + 2i & -2 \\ 4 & -2 + 2i \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 + i & -1 \\ 0 & 0 \end{pmatrix},$$

so the eigenspace is spanned by $\mathbf{w} = \begin{pmatrix} 1 \\ 1 + i \end{pmatrix}$.

By Proposition 3.1 in Section III, the eigenspace associated to the eigenvalue $3-2i$ is spanned by the complex conjugate of $\mathbf{w}$, i.e., $\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$.

Hence, if

$$P = \begin{pmatrix} 1 & 1 \\ 1+i & 1-i \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 3+2i & 0 \\ 0 & 3-2i \end{pmatrix},$$

then P is invertible and $P^{-1}BP = D$.

107. The complex eigenvalues of A are the roots of $p_A(x) = (x+1)(x^2-4x+5)$, which are -1, $2+i$, and $2-i$. Let us find the eigenspace associate to $\lambda = -1$ first:

$$-I - A = \begin{pmatrix} -4 & -4 & -2 \\ -4 & -4 & -2 \\ 4 & 9 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so a basis for this eigenspace is

$$\left\{ \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right\}.$$

Next, we turn to the eigenspace associated to $\lambda = 2+i$:

$$(2+i)I - A = \begin{pmatrix} -1+i & -4 & -2 \\ -4 & -1+i & -2 \\ 4 & 9 & 5+i \end{pmatrix} \leftrightarrow \begin{pmatrix} 13 & 0 & 5+i \\ 0 & 13 & 5+i \\ 0 & 0 & 0 \end{pmatrix}.$$

(Please see eClass for a video showing the row operations that bring us to this row-echelon form.) We may read off from this row-echelon form the basis

$$\left\{ \begin{pmatrix} 5+i \\ 5+i \\ -13 \end{pmatrix} \right\}$$

for the eigenspace associated to $2+i$.

A basis for the remaining eigenspace, for $2-i$, may be obtained by taking the complex conjugate of the basis vector we found for the previous eigenspace:

$$\left\{ \begin{pmatrix} 5-i \\ 5-i \\ -13 \end{pmatrix} \right\}.$$

We have found a basis for $\mathbb{C}^3$ consisting of eigenvectors of A, namely,

$$\left\{ \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 5+i \\ 5+i \\ -13 \end{pmatrix}, \begin{pmatrix} 5-i \\ 5-i \\ -13 \end{pmatrix} \right\},$$

so if

$$P = \begin{pmatrix} -1 & 5+i & 5-i \\ 0 & 5+i & 5-i \\ 2 & -13 & -13 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{pmatrix},$$

then P is invertible and $P^{-1}AP = D$. We have thus diagonalized A over $\mathbb{C}$.

108. The complex eigenvalues of A are the roots of $p_A(x) = (x-3)(x^2-2x+2)$, which are 3, $1+i$, and $1-i$. Let us find the eigenspace associate to $\lambda = 3$ first:

$$3I - A = \begin{pmatrix} 2 & 3 & 2 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so a basis for this eigenspace is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

Next, we turn to the eigenspace associated to $\lambda = 1+i$:

$$\begin{aligned} (1+i)I - A &= \begin{pmatrix} i & 3 & 2 \\ -1 & -1+i & -1 \\ 1 & -1 & -1+i \end{pmatrix} \\ &\leftrightarrow \begin{pmatrix} 1 & -3i & -2i \\ -1 & -1+i & -1 \\ 1 & -1 & -1+i \end{pmatrix} \\ &\leftrightarrow \begin{pmatrix} 1 & -3i & -2i \\ 0 & -1-2i & -1-2i \\ 0 & -1+3i & -1+3i \end{pmatrix} \\ &\leftrightarrow \begin{pmatrix} 1 & -3i & -2i \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &\leftrightarrow \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

We may read off from this row-echelon form the basis

$$\left\{ \begin{pmatrix} i \\ 1 \\ -1 \end{pmatrix} \right\}$$

for the eigenspace associated to $1+i$.

A basis for the remaining eigenspace, for $1-i$, may be obtained by taking the complex conjugate of the basis vector we found for the previous eigenspace:

$$\left\{ \begin{pmatrix} -i \\ 1 \\ -1 \end{pmatrix} \right\}$$

We have found a basis for $\mathbb{C}^3$ consisting of eigenvectors of A, namely,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} i \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \\ -1 \end{pmatrix} \right\},$$

so if

$$P = \begin{pmatrix} 1 & i & -i \\ 0 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{pmatrix},$$

then P is invertible and $P^{-1}AP = D$. We have thus diagonalized A over $\mathbb{C}$.

109. (a) Multiplying both sides of the equation $z^2 + \sqrt{3}\,z + 3 = 0$ by z, we obtain

$$0 = z^3 + \sqrt{3}\,z^2 + 3z = z^3 + \sqrt{3}(z^2 + \sqrt{3}\,z) = z^3 + \sqrt{3}(-3),$$

so $z^3 = 3\sqrt{3}$.

(b) As noted in the question, A is diagonalizable over $\mathbb{C}$. Let us find the eigenvalues. Let the two roots of the polynomial $x^2 + \sqrt{3}\,x + 3$ be $z, w \in \mathbb{C}$, and note that the roots of $x^2 + 3$ are $\pm\sqrt{3}\,i$. Then the complex eigenvalues of A are $\sqrt{3}\,i, -\sqrt{3}\,i, z, w$, so there is an invertible matrix $P \in M_4(\mathbb{C})$ such that $P^{-1}AP = D$, where D is the diagonal matrix with $\sqrt{3}\,i, -\sqrt{3}\,i, z, w$ on the diagonal. Now, $(\sqrt{3}\,i)^{12} = 3^6\,i^{12} = 9^3 = 729$, and similarly for $(-\sqrt{3}\,i)^{12}$. Further, using part (a), we have $z^{12} = (z^3)^4 = (3\sqrt{3})^4 = 3^6 = 729$ again, and similarly for w^{12}. Hence,

$$A^{12} = (PDP^{-1})^{12} = PD^{12}P^{-1} = P(729I)P^{-1} = 729I,$$

the last equality holding because scalar matrices commute with all matrices.

110. (a) The characteristic polynomial of A is

$$p_A(x) = \det(xI - A) = (x-2)(x-4) + 28 = x^2 - 6x + 36,$$

whose roots are $\frac{1}{2}(6 \pm \sqrt{-3 \cdot 36}) = 3 \pm 3\sqrt{3}i$. We work with the eigenvalue $3 - 3\sqrt{3}i$ (although either eigenvalue is permissible):

$$(3 - 3\sqrt{3}i)I - A = \begin{pmatrix} 1 - 3\sqrt{3}i & 4 \\ -7 & -1 - 3\sqrt{3}i \end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix} 1-3\sqrt{3}i & 4 \\ 7(1-3\sqrt{3}i) & 28 \end{pmatrix} \quad \text{(row 2 times } -1+3\sqrt{3}i\text{)}$$

$$\leftrightarrow \begin{pmatrix} 1-3\sqrt{3}i & 4 \\ 0 & 0 \end{pmatrix} \quad \text{(row 2 minus 7 times row 1),}$$

so an eigenvector with this eigenvalue is

$$\mathbf{w} = \begin{pmatrix} 4 \\ -1+3\sqrt{3}i \end{pmatrix}.$$

Hence, we let

$$Q = \begin{pmatrix} \mathrm{Re}(\mathbf{w}) & \mathrm{Im}(\mathbf{w}) \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ -1 & 3\sqrt{3} \end{pmatrix}.$$

By Proposition 4.1 in Section III, Q is invertible and

$$Q^{-1}AQ = \begin{pmatrix} 3 & -3\sqrt{3} \\ 3\sqrt{3} & 3 \end{pmatrix} = 6\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = 6R,$$

where R is a rotation matrix.

(b)

$$R = \begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix},$$

which is the matrix for rotation anticlockwise by angle $\pi/3$.

If we had chosen to work with the eigenvalue $3+3\sqrt{3}\,i$ instead, then the anticlockwise angle of rotation would have been $5\pi/3$.

111. Note first that $p_A(x) = \det(xI - A) = x^2 - 10x + 169$, which has complex roots

$$\frac{1}{2}(10 \pm \sqrt{100 - 4\cdot 169}) = \frac{1}{2}(10 \pm \sqrt{-576}) = 5 \pm 12i,$$

because $576 = 24^2$. We choose to work with the eigenvalue $\lambda = 5 - 12i$. (Either eigenvalue is acceptable.) For the eigenspace associated to λ, we row reduce as follows:

$$(5-12i)I - A = \begin{pmatrix} 4-12i & -16 \\ 10 & -4-12i \end{pmatrix} \leftrightarrow \begin{pmatrix} 5 & -2-6i \\ 0 & 0 \end{pmatrix}.$$

We see, then, that the eigenspace is spanned by

$$\mathbf{w} = \begin{pmatrix} 2+6i \\ 5 \end{pmatrix}.$$

Hence, we let

$$s = |\lambda| = \sqrt{5^2 + 12^2} = 13,$$

$$Q = \begin{pmatrix} \mathrm{Re}(\mathbf{w}) & \mathrm{Im}(\mathbf{w}) \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 5 & 0 \end{pmatrix},$$

$$\text{and} \quad R = \frac{1}{s}\begin{pmatrix} 5 & -12 \\ 12 & 5 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 5 & -12 \\ 12 & 5 \end{pmatrix}.$$

Then according to Proposition 4.1 in Section III, Q is invertible, and $Q^{-1}AQ = sR$. Further, R is a rotation matrix, because it takes the form

$$\frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

for real numbers a and b (not both zero), specifically, $a = 5$ and $b = -12$.

If we had chosen to work with the eigenvalue $5 + 12i$, then the rotation matrix would have been

$$\frac{1}{13}\begin{pmatrix} 5 & 12 \\ -12 & 5 \end{pmatrix}$$

instead, and the matrix Q would have been different as well, e.g.,

$$Q = \begin{pmatrix} 2 & -6 \\ 5 & 0 \end{pmatrix}.$$

112. (a) The matrix A has characteristic polynomial $p_A(x) = x^2 - 6x + 12$, whose roots are $3 \pm \sqrt{3}i$. We choose to work with the eigenvalue $\lambda = 3 - \sqrt{3}i$. To find the corresponding eigenspace, we row-reduce as follows:

$$\lambda I - A = \begin{pmatrix} 2-\sqrt{3}i & -7 \\ 1 & -2-\sqrt{3}i \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -2-\sqrt{3}i \\ 0 & 0 \end{pmatrix}.$$

A non-zero eigenvector for λ is

$$w = \begin{pmatrix} 2+\sqrt{3}i \\ 1 \end{pmatrix},$$

so letting

$$Q = \begin{pmatrix} \mathrm{Re}(w) & \mathrm{Im}(w) \end{pmatrix} = \begin{pmatrix} 2 & \sqrt{3} \\ 1 & 0 \end{pmatrix},$$

we find that

$$\begin{aligned} Q^{-1}AQ &= \begin{pmatrix} \mathrm{Re}(\lambda) & \mathrm{Im}(\lambda) \\ -\mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} 3 & -\sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix} \\ &= 2\sqrt{3}\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \end{aligned}$$

$$= \quad 2\sqrt{3}R,$$

where

$$R = \begin{pmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{pmatrix},$$

rotation anticlockwise by angle $\pi/6$.

If we had chosen $Q = \begin{pmatrix} \mathrm{Re}(\overline{w}) & \mathrm{Im}(\overline{w}) \end{pmatrix}$ instead, then the anticlockwise angle of rotation would have been $11\pi/6$.

(b) If k is a positive integer, then

$$\begin{aligned} A^k \in \mathrm{Span}(I) &\Leftrightarrow (Q(2\sqrt{3}R)Q^{-1})^k \in \mathrm{Span}(I) \\ &\Leftrightarrow (2\sqrt{3})^k QR^kQ^{-1} \in \mathrm{Span}(I) \\ &\Leftrightarrow QR^kQ^{-1} \in \mathrm{Span}(I) \\ &\Leftrightarrow R^k \in \mathrm{Span}(I). \end{aligned}$$

Because R is a rotation matrix, $R^k \in \mathrm{Span}(I)$ if and only if $R^k = \pm I$, and this happens if and only if 6 divides k, because the angle of rotation of R is $\pi/6$. The smallest such $k > 0$ is 6. Then we compute

$$\begin{aligned} A^6 &= (2\sqrt{3})^6 QR^6Q^{-1} \\ &= 2^6 \cdot 3^3(-I) \\ &= \begin{pmatrix} -1728 & 0 \\ 0 & -1728 \end{pmatrix}. \end{aligned}$$

113. (a) The matrix B has characteristic polynomial $p_B(x) = x^2 - 6x + 25$, whose roots are $3 \pm 4i$. We choose to work with the eigenvalue $\lambda = 3 - 4i$. To find the corresponding eigenspace, we row-reduce as follows:

$$\lambda I - B = \begin{pmatrix} 2-4i & -4 \\ 5 & -2-4i \end{pmatrix} \leftrightarrow \begin{pmatrix} 1-2i & -2 \\ 0 & 0 \end{pmatrix}.$$

A non-zero eigenvector for λ is

$$w = \begin{pmatrix} 2 \\ 1-2i \end{pmatrix},$$

so letting

$$Q = \begin{pmatrix} \mathrm{Re}(w) & \mathrm{Im}(w) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix},$$

we find that

$$Q^{-1}BQ \quad = \quad \begin{pmatrix} \mathrm{Re}(\lambda) & \mathrm{Im}(\lambda) \\ -\mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$$
$$= 5R,$$

where R is the rotation matrix

$$\begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}.$$

If we had chosen $Q = \begin{pmatrix} \mathrm{Re}(\overline{w}) & \mathrm{Im}(\overline{w}) \end{pmatrix}$ instead, then the rotation matrix would have been

$$\begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}.$$

(b) By the same argument as in part (b) of question 112, if k is a positive integer, then $B^k \in \mathrm{Span}(I)$ if and only if $R^k = \pm I$. If this occurred, then R^{2k} would be equal to I. But here, R is a rotation matrix with rational entries and is not equal to any of the four matrices given in the question, so no positive power of it can be the identity.

114. We are told that

$$\frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

so using the fact that $\sqrt{a^2+b^2} = |\lambda|$, we obtain

$$a = \frac{1}{2}|\lambda|, \quad b = -\frac{\sqrt{3}}{2}|\lambda|.$$

It remains to find $|\lambda|$. For this, we use Proposition 4.1 in Section III, which tells us that there is an invertible matrix $Q \in M_2(\mathbb{R})$ such that $Q^{-1}CQ = |\lambda|R$. Note, then, that $C = |\lambda|QRQ^{-1}$. Now, because R represents rotation by angle $\pi/3$, $R^3 = -I$ where I is the 2×2 identity matrix. Therefore,

$$\begin{aligned} C^3 &= |\lambda|^3(QRQ^{-1})^3 \\ &= |\lambda|^3 QR^3Q^{-1} \\ &= |\lambda|^3 Q(-I)Q^{-1} \\ &= -|\lambda|^3 I \quad \text{because } Q^{-1}IQ = Q^{-1}Q = I \\ &= \begin{pmatrix} -|\lambda|^3 & 0 \\ 0 & -|\lambda|^3 \end{pmatrix}. \end{aligned}$$

Hence, because the top-left entry of C^3 is -64, we conclude that $|\lambda|^3 = 64$, i.e., $|\lambda| = 4$. Thus, $a = 2$ and $b = -2\sqrt{3}$.

115. (a) The characteristic polynomial of B is $\det(xI - B) = (x-3)^2(x+2)$, so the eigenvalues of B are 3 and -2. The characteristic polynomial of

$$B^2 = \begin{pmatrix} 9 & 6 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

is $\det(xI - B^2) = (x-9)^2(x-4)$, so the eigenvalues of B^2 are 9 and 4. These are the squares of 3 and -2 respectively.

(b) One could take

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

for example. The matrix

$$C^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

has eigenvalues 1 and -1, so it has a negative eigenvalue.

(c) Let A be any matrix whose square has a negative eigenvalue λ, such as the matrix C above. Since λ is negative, it cannot be the square of any real number, so certainly it is not the square of a real eigenvalue of A.

116. (a) Note that, because $\mathbf{u}^{\mathrm{T}}A\mathbf{v}$ is a 1×1 matrix, we consider it as simply a real number. Note further that any 1×1 matrix is symmetric.

Now, if $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} \rangle &= \mathbf{u}^{\mathrm{T}}A\mathbf{v} \\ &= (\mathbf{u}^{\mathrm{T}}A\mathbf{v})^{\mathrm{T}} \quad \text{because } \mathbf{u}^{\mathrm{T}}A\mathbf{v} \text{ is a } 1 \times 1 \text{ matrix} \\ &= \mathbf{v}^{\mathrm{T}}A^{\mathrm{T}}\mathbf{u} \quad \text{(a standard property of the transpose)} \\ &= \mathbf{v}^{\mathrm{T}}A\mathbf{u} \quad \text{because } A \text{ is symmetric by assumption} \\ &= \langle \mathbf{v}, \mathbf{u} \rangle. \end{aligned}$$

This establishes the first axiom of an inner product.

Next, if $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, then

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle &= \mathbf{u}^{\mathrm{T}}A(\mathbf{v} + \mathbf{w}) \\ &= \mathbf{u}^{\mathrm{T}}A\mathbf{v} + \mathbf{u}^{\mathrm{T}}A\mathbf{w} \\ &= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle, \end{aligned}$$

establishing the second axiom.

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then

$$\begin{aligned} \langle c\mathbf{u}, \mathbf{v} \rangle &= (c\mathbf{u})^{\mathrm{T}} A\mathbf{v} \\ &= c(\mathbf{u}^{\mathrm{T}} A\mathbf{v}) \\ &= c\langle \mathbf{u}, \mathbf{v} \rangle, \end{aligned}$$

so the third axiom holds.

Finally, if $\mathbf{u} \in \mathbb{R}^n$, then $\langle \mathbf{u}, \mathbf{u} \rangle = \mathbf{u}^{\mathrm{T}} A\mathbf{u} \geq 0$ by the assumption given in the question, and $\mathbf{u}^{\mathrm{T}} A\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$, by the same assumption.

(b) The standard inner product corresponds to the case where A is the $n \times n$ identity matrix.

117. (a) Note that $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^{\mathrm{T}} A\mathbf{v}$ where

$$A = \begin{pmatrix} 3 & -5 \\ -5 & -8 \end{pmatrix}.$$

Axioms (i), (ii), and (iii) hold by the arguments given in the solution to Question 116. However, axiom (iv) does not hold, because we can find $\mathbf{u} \in \mathbb{R}^2$ such that $\langle \mathbf{u}, \mathbf{u} \rangle < 0$. An example is $\mathbf{u} = (0, 1)$, which satisfies $\langle \mathbf{u}, \mathbf{u} \rangle = -8$.

(b) The pairing does not define an inner product, because it fails axiom (iv).

118. (a) Note that $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^{\mathrm{T}} A\mathbf{v}$ where

$$A = \begin{pmatrix} 16 & 12 \\ 12 & 9 \end{pmatrix}.$$

Axioms (i), (ii), and (iii) hold by the arguments given in the solution to Question 116. However, axiom (iv) does not hold, because we can find a non-zero vector $\mathbf{u} \in \mathbb{R}^2$ such that $\langle \mathbf{u}, \mathbf{u} \rangle = 0$. An example is $\mathbf{u} = (-3, 4)$.

(b) The pairing does not define an inner product, because it fails axiom (iv).

119. (a) Note that $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^{\mathrm{T}} A\mathbf{v}$ where

$$A = \begin{pmatrix} 13 & -11 \\ -7 & 10 \end{pmatrix}.$$

Axioms (ii) and (iii) hold by the arguments given in the solution to Question 116, but axiom (i) fails because, for example, $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = -11$ while $\langle \mathbf{e}_2, \mathbf{e}_1 \rangle = -7$.

Finally, we show that axiom (iv) holds. Observe that, for a vector $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$,

$$\langle \mathbf{u}, \mathbf{u} \rangle = 13u_1^2 - 18u_1u_2 + 10u_2^2$$

$$= \quad (2u_1 - 3u_2)^2 + (3u_1 - u_2)^2$$

by the identity given in the question. This expression is always non-negative, and further, it is zero if and only if

$$\begin{aligned} 2u_1 - 3u_2 &= 0 \\ 3u_1 - u_2 &= 0, \end{aligned}$$

if and only if $u_1 = u_2 = 0$ (because the determinant of the system is non-zero), if and only if $\mathbf{u} = \mathbf{0}$.

(b) The pairing does not define an inner product, because it fails axiom (i).

120. Let $\mathbf{u} = (x_1, x_2, x_3)$. Then

$$\langle \mathbf{u}, \mathbf{u} \rangle = x_1^2 + 5x_2^2 + 4x_2x_3 + 25x_3^2 = x_1^2 + (2x_2 + 3x_3)^2 + (x_2 - 4x_3)^2$$

by the equation given in the question. Because each term on the right is the square of a real number, $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$. Further, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $x_1^2 = (2x_2 + 3x_3)^2 = (x_2 - 4x_3)^2 = 0$, if and only if

$$x_1 = 2x_2 + 3x_3 = x_2 - 4x_3 = 0.$$

We thus have immediately that $x_1 = 0$, and $x_2 = x_3 = 0$ because the matrix $\begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix}$ is invertible. Therefore, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

121. Let $p = ax^2 + bx + c$. Then

$$\begin{aligned} \langle p, x \rangle &= (a - b + c)(-1) + 0 + (a + b + c)(1) = 2b \\ \langle p, x^2 \rangle &= (a - b + c)(1) + 0 + (a + b + c)(1) = 2a + 2c, \end{aligned}$$

so p is orthogonal to both x and x^2 if and only if $2b = 2a + 2c = 0$, if and only if $b = 0$ and $c = -a$, if and only if $p = a(x^2 - 1)$. Among the polynomials of the form $p = a(x^2 - 1)$, we want those of norm 1, i.e., $\|p\| = 1$. But

$$\|p\| = \|a(x^2 - 1)\| = |a| \, \|x^2 - 1\| = |a|\sqrt{0 + 1 + 0} = |a|,$$

so $\|p\| = 1$ if and only if $|a| = 1$, if and only if $a \in \{-1, 1\}$. Thus, the polynomials of norm 1 that are orthogonal to both x and x^2 are $x^2 - 1$ and $-(x^2 - 1)$.

122. If $p = ax^2 + bx + c$, then

$$\begin{aligned} 11 &= \langle p, 1 \rangle = (a - b + c) + c + (a + b + c) = 2a + 3c, \\ -6 &= \langle p, x \rangle = (a - b + c)(-1) + 0 + (a + b + c) = 2b, \end{aligned}$$

$$10 = \langle p, x^2 \rangle = (a - b + c) + 0 + (a + b + c) = 2a + 2c.$$

The unique solution to the system

$$\begin{aligned} 2a \qquad + 3c &= 11 \\ 2b \qquad &= -6 \\ 2a \qquad + 2c &= 10 \end{aligned}$$

is $a = 4$, $b = -3$, $c = 1$. Thus, $p = 4x^2 - 3x + 1$.

123. (a) We use the well-known identity $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$:

$$\begin{aligned} \cos((m+n)x) &= \cos(mx)\cos(nx) - \sin(mx)\sin(nx) \\ \cos((m-n)x) &= \cos(mx)\cos(nx) + \sin(mx)\sin(nx). \end{aligned}$$

Subtracting the first of these two equations from the second and then dividing through by 2 yields

$$\sin(mx)\sin(nx) = \frac{1}{2}\big(\cos((m-n)x) - \cos((m+n)x)\big). \tag{40}$$

(b) The assumption $|m| \neq |n|$ says that both $m + n$ and $m - n$ are non-zero. Hence,

$$\begin{aligned} \langle f_m, f_n \rangle &= \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \big(\cos((m-n)x) - \cos((m+n)x)\big)\,dx \quad \text{by (40)} \\ &= \frac{1}{2}\left[\frac{1}{m-n}\sin((m-n)x) - \frac{1}{m+n}\sin((m+n)x)\right]_{-\pi}^{\pi} \\ &= 0, \end{aligned}$$

because $m - n, m + n \in \mathbb{Z}$.

124. Note first that, for any $i \in \{1, \ldots, k\}$,

$$\begin{aligned} \langle \mathbf{u}_i, \mathbf{w} \rangle &= \langle \mathbf{u}_i, b_1\mathbf{u}_1 + \cdots + b_k\mathbf{u}_k \rangle \\ &= b_1\langle \mathbf{u}_i, \mathbf{u}_1 \rangle + \cdots + b_k\langle \mathbf{u}_i, \mathbf{u}_k \rangle \quad \text{by linearity} \\ &= b_i\langle \mathbf{u}_i, \mathbf{u}_i \rangle \quad \text{by orthogonality} \\ &= b_i\|\mathbf{u}_i\|^2. \end{aligned}$$

Hence,

$$\begin{aligned} \langle \mathbf{v}, \mathbf{w} \rangle &= \langle a_1\mathbf{u}_1 + \cdots + a_k\mathbf{u}_k, \mathbf{w} \rangle \\ &= a_1\langle \mathbf{u}_1, \mathbf{w} \rangle + \cdots + a_k\langle \mathbf{u}_k, \mathbf{w} \rangle \quad \text{by linearity} \end{aligned}$$

$$= a_1b_1\|\mathbf{u}_1\|^2 + \cdots + a_kb_k\|\mathbf{u}_k\|^2.$$

125. (a)

$$\begin{aligned}
\|f_1\|^2 = \langle f_1, f_1\rangle &= \int_{-\pi}^{\pi} dx = 2\pi,\\
\|f_2\|^2 = \langle f_2, f_2\rangle &= \int_{-\pi}^{\pi} \sin^2(x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}(1-\cos(2x))\,dx = \pi,\\
\|f_3\|^2 = \langle f_3, f_3\rangle &= \int_{-\pi}^{\pi} \sin^2(2x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}(1-\cos(4x))\,dx = \pi.
\end{aligned}$$

(b) Let $h = a_1f_1 + a_2f_2 + a_3f_3$, where $a_1, a_2, a_3 \in \mathbb{R}$. Then

$$\begin{aligned}
\langle h, g_1\rangle &= a_2\|f_2\|^2 - a_3\|f_3\|^2 \quad \text{by Question 124}\\
&= \pi(a_2 - a_3),\\
\langle h, g_2\rangle &= a_1\|f_1\|^2 + 2a_2\|f_2\|^2 + 4a_3\|f_3\|^2 \quad \text{by Question 124 again}\\
&= \pi(2a_1 + 2a_2 + 4a_3).
\end{aligned}$$

Therefore, the equations $\langle h, g_1\rangle = \langle h, g_2\rangle = 0$ are equivalent to the system

$$\begin{aligned}
a_2 - a_3 &= 0\\
2a_1 + 2a_2 + 4a_3 &= 0
\end{aligned}$$

This system has general solution $a_1 = 3c$, $a_2 = -c$, $a_3 = -c$, where $c \in \mathbb{R}$. Hence, the desired functions are the functions $h = c(3f_1 - f_2 - f_3)$ with $c \in \mathbb{R}$.

(c) For functions h as above,

$$\begin{aligned}
\|h\|^2 &= c^2\|3f_1 - f_2 - f_3\|^2\\
&= c^2(9\|f_1\|^2 + \|f_2\|^2 + \|f_3\|^2) \quad \text{by Question 124}\\
&= c^2(20\pi),
\end{aligned}$$

so $\|h\| = 1$ if and only if $|c|\sqrt{20\pi} = 1$, if and only if

$$c = \pm\frac{1}{2\sqrt{5\pi}}.$$

Thus, there are only two functions meeting all the criteria:

$$\frac{1}{2\sqrt{5\pi}}(3f_1 - f_2 - f_3), \quad -\frac{1}{2\sqrt{5\pi}}(3f_1 - f_2 - f_3).$$

126. (a) We let

$$p_1 = q_1$$

$$\begin{aligned}
&= 1, \\
p_2 &= q_2 - \frac{\langle p_1, q_2\rangle}{\langle p_1, p_1\rangle} p_1 \\
&= x - 0p_1 \\
&= x, \\
p_3 &= q_3 - \frac{\langle p_1, q_3\rangle}{\langle p_1, p_1\rangle} p_1 - \frac{\langle p_2, q_3\rangle}{\langle p_2, p_2\rangle} p_2 \\
&= x^2 - \frac{8}{3} - 0p_2 \\
&= x^2 - \frac{8}{3}.
\end{aligned}$$

Then $\{p_1, p_2, p_3\}$ is an orthogonal basis. The norms of p_1, p_2, and p_3 are $\sqrt{3}$, $\sqrt{8}$, and $\sqrt{32/3}$ respectively, so

$$\left\{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{8}} x, \sqrt{\frac{3}{32}} \left(x^2 - \tfrac{8}{3}\right) \right\}$$

is an orthonormal basis for $\mathcal{P}_2$ with respect to the given inner product.

(b) We let

$$\begin{aligned}
p_1 &= q_1 \\
&= 1, \\
p_2 &= q_2 - \frac{\langle p_1, q_2\rangle}{\langle p_1, p_1\rangle} p_1 \\
&= x - 0p_1 \\
&= x, \\
p_3 &= q_3 - \frac{\langle p_1, q_3\rangle}{\langle p_1, p_1\rangle} p_1 - \frac{\langle p_2, q_3\rangle}{\langle p_2, p_2\rangle} p_2 \\
&= x^2 - \frac{16/3}{4} - 0p_2 \\
&= x^2 - \frac{4}{3}.
\end{aligned}$$

Then $\{p_1, p_2, p_3\}$ is an orthogonal basis. The norms of p_1, p_2, and p_3 are 2, $4/\sqrt{3}$, and $16/\sqrt{45}$ respectively, so

$$\left\{ \frac{1}{2}, \frac{\sqrt{3}}{4} x, \frac{\sqrt{45}}{16} \left(x^2 - \tfrac{4}{3}\right) \right\}$$

is an orthonormal basis for $\mathcal{P}_2$ with respect to the given inner product.

127. We first let $\mathbf{u}_1 = \mathbf{v}_1$. Next,

$$\mathbf{v}_2 - \frac{\langle \mathbf{u}_1, \mathbf{v}_2\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle} \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \quad \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix},$$

so scaling, as we may, we let

$$\mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}.$$

Finally,

$$\begin{aligned} \mathbf{v}_3 - \frac{\langle \mathbf{u}_1, \mathbf{v}_3 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \frac{\langle \mathbf{u}_2, \mathbf{v}_3 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle}\mathbf{u}_2 \quad &= \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} \\ &= \quad \frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 3 \end{pmatrix}. \end{aligned}$$

Again, scaling this vector, we let

$$\mathbf{u}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 3 \end{pmatrix}.$$

If we normalize the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, we find the orthonormal basis

$$\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{12}}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 3 \end{pmatrix} \right\}$$

of U.

128. We use the formula for orthogonal projection:

$$\text{proj}_{\mathcal{P}_2}(q) = \frac{\langle p_1, q \rangle}{\langle p_1, p_1 \rangle}\, p_1 + \frac{\langle p_2, q \rangle}{\langle p_2, p_2 \rangle}\, p_2 + \frac{\langle p_3, q \rangle}{\langle p_3, p_3 \rangle}\, p_3.$$

Here,

$$\langle p_1, q \rangle = \langle 1, x^3 \rangle = 0 + 1 + 8 + 27 = 36$$

$$\begin{aligned}\langle p_2, q\rangle &= \langle x - \tfrac{3}{2}, x^3\rangle = 0 - \tfrac{1}{2} + 4 + \tfrac{3}{2}\cdot 27 = 44\\ \langle p_3, q\rangle &= \langle x^2 - 3x + 1, x^3\rangle = 0 - 1 - 8 + 27 = 18.\end{aligned}$$

Thus, using the given equalities $\langle p_1, p_1\rangle = 4$, $\langle p_2, p_2\rangle = 5$, $\langle p_3, p_3\rangle = 4$, we obtain

$$\operatorname{proj}_{\mathcal{P}_2}(q) = \tfrac{36}{4} + \tfrac{44}{5}(x - \tfrac{3}{2}) + \tfrac{18}{4}(x^2 - 3x + 1) = \tfrac{9}{2}x^2 - \tfrac{47}{10}x + \tfrac{3}{10}.$$

129. (a) Following the Gram–Schmidt process, we let

$$\begin{aligned}p_1 &= q_1 = 1,\\ p_2 &= q_2 - \frac{\langle p_1, q_2\rangle}{\langle p_1, p_1\rangle}\, p_1 = x^2 - \frac{6}{4} = x^2 - \frac{3}{2},\\ p_3 &= q_3 - \frac{\langle p_1, q_3\rangle}{\langle p_1, p_1\rangle}\, p_1 - \frac{\langle p_2, q_3\rangle}{\langle p_2, p_2\rangle}\, p_2\\ &= x^3 - \frac{8}{4} - \frac{20}{9}\left(x^2 - \frac{3}{2}\right) = x^3 - \frac{20}{9}x^2 + \frac{4}{3}.\end{aligned}$$

Then $\{p_1, p_2, p_3\}$ is an orthogonal basis for U, and the p_i are all monic.

(b)

$$\begin{aligned}\operatorname{proj}_U(r) &= \frac{\langle p_1, r\rangle}{\langle p_1, p_1\rangle}\, p_1 + \frac{\langle p_2, r\rangle}{\langle p_2, p_2\rangle}\, p_2 + \frac{\langle p_3, r\rangle}{\langle p_3, p_3\rangle}\, p_3\\ &= \frac{10}{4} + \frac{25}{9}\left(x^2 - \frac{3}{2}\right) + \frac{76}{50}\left(x^3 - \frac{20}{9}x^2 + \frac{4}{3}\right)\\ &= \frac{38}{25}x^3 - \frac{3}{5}x^2 + \frac{9}{25}.\end{aligned}$$

130. (a) Let

$$\begin{aligned}\mathbf{u}_1 &= \mathbf{v}_1 = (1, 2, 1, -2) &(41)\\ \mathbf{u}_2 &= \mathbf{v}_2 - \frac{\mathbf{u}_1\cdot\mathbf{v}_2}{\mathbf{u}_1\cdot\mathbf{u}_1}\,\mathbf{u}_1\\ &= (0, 1, 0, -1) - \frac{4}{10}(1, 2, 1, -2) = \frac{1}{5}(-2, 1, -2, -1). &(42)\end{aligned}$$

To avoid working with denominators, we choose to scale $\mathbf{u}_2$ by 5 and instead take $\mathbf{u}_2 = (-2, 1, -2, -1)$. Note that we have scaled by a positive scalar, which will be important when we come to find the totally positive QR-factorization of A in part (b). Next, we let

$$\begin{aligned}\mathbf{u}_3 &= \mathbf{v}_3 - \frac{\mathbf{u}_1\cdot\mathbf{v}_3}{\mathbf{u}_1\cdot\mathbf{u}_1}\,\mathbf{u}_1 - \frac{\mathbf{u}_2\cdot\mathbf{v}_3}{\mathbf{u}_2\cdot\mathbf{u}_2}\,\mathbf{u}_2\\ &= (2, 1, 1, -1) - \frac{7}{10}(1, 2, 1, -2) - \frac{-4}{10}(-2, 1, -2, -1)\\ &= \frac{1}{2}(1, 0, -1, 0). &(43)\end{aligned}$$

Finally, we let $\mathbf{w}_i = \frac{1}{\|\mathbf{u}_i\|}\mathbf{u}_i$ for $i = 1, 2, 3$:

$$\mathbf{w}_1 = \frac{1}{\sqrt{10}}(1, 2, 1, -2)$$
$$\mathbf{w}_2 = \frac{1}{\sqrt{10}}(-2, 1, -2, -1)$$
$$\mathbf{w}_3 = \frac{1}{\sqrt{2}}(1, 0, -1, 0)$$

Then $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthonormal basis for $\mathrm{Col}(A)$.

(b) From (41), (42), and (43), along with the definitions of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$, we see that

$$\begin{aligned}
\mathbf{v}_1 &= (1, 2, 1, -2) = \sqrt{10}\,\mathbf{w}_1 \\
\mathbf{v}_2 &= \frac{2}{5}(1, 2, 1, -2) + \frac{1}{5}(-2, 1, -2, -1) \\
&= \frac{2\sqrt{10}}{5}\mathbf{w}_1 + \frac{\sqrt{10}}{5}\mathbf{w}_2 \\
\mathbf{v}_3 &= \frac{7}{10}(1, 2, 1, -2) - \frac{2}{5}(-2, 1, -2, -1) + \frac{1}{2}(1, 0, -1, 0) \\
&= \frac{7\sqrt{10}}{10}\mathbf{w}_1 - \frac{2\sqrt{10}}{5}\mathbf{w}_2 + \frac{\sqrt{2}}{2}\mathbf{w}_3
\end{aligned}$$

Thus, the totally positive QR-factorization of A is

$$A = \begin{pmatrix} \frac{1}{\sqrt{10}} & -\frac{2}{\sqrt{10}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \\ \frac{1}{\sqrt{10}} & -\frac{2}{\sqrt{10}} & -\frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{10} & \frac{2\sqrt{10}}{5} & \frac{7\sqrt{10}}{10} \\ 0 & \frac{\sqrt{10}}{5} & -\frac{2\sqrt{10}}{5} \\ 0 & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

131. Let

$$\mathbf{w}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{w}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{w}_3 = \frac{1}{\sqrt{12}}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 3 \end{pmatrix}.$$

From the calculations in Question 127, we see that

$$\begin{aligned}
\mathbf{v}_1 &= \sqrt{2}\,\mathbf{w}_1 \\
\mathbf{v}_2 &= \frac{\sqrt{2}}{2}\,\mathbf{w}_1 + \frac{\sqrt{6}}{2}\,\mathbf{w}_2 \\
\mathbf{v}_3 &= \frac{\sqrt{2}}{2}\,\mathbf{w}_1 + \frac{\sqrt{6}}{6}\,\mathbf{w}_2 + \frac{\sqrt{12}}{3}\,\mathbf{w}_2
\end{aligned}$$

Hence, $A = QR$ where

$$Q = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{12} \\ 0 & 2/\sqrt{6} & -1/\sqrt{12} \\ 0 & 0 & 3/\sqrt{12} \end{pmatrix}$$

$$\text{and} \quad R = \begin{pmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{6} \\ 0 & 0 & \frac{\sqrt{12}}{3} \end{pmatrix}$$

132. Let the columns of A be $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in that order. Then we let

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{v}_1 = (1,1,1,1) \\ \mathbf{u}_2 &= \mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 \\ &= (0,1,1,1) - \frac{3}{4}(1,1,1,1) = \frac{1}{4}(-3,1,1,1). \end{aligned}$$

At this point, we will replace $\mathbf{u}_2$ by $(-3,1,1,1)$ to avoid working with denominators. Note that, to obtain the totally positive QR-factorization, we should scale the vectors $\mathbf{u}_i$ only by positive scalars (in this case, the positive scalar 4). Hence, we let

$$\begin{aligned} \mathbf{u}_3 &= \mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 \\ &= (0,0,1,1) - \frac{2}{4}(1,1,1,1) - \frac{2}{12}(-3,1,1,1) = \frac{1}{3}(0,-2,1,1). \end{aligned}$$

Now we normalize by letting $\mathbf{w}_i = \frac{1}{\|\mathbf{u}_i\|}\mathbf{u}_i$ for $i = 1,2,3$:

$$\mathbf{w}_1 = \frac{1}{2}(1,1,1,1), \quad \mathbf{w}_2 = \frac{1}{\sqrt{12}}(-3,1,1,1), \quad \mathbf{w}_3 = \frac{1}{\sqrt{6}}(0,-2,1,1).$$

Next, we express the $\mathbf{v}_j$ in terms of the $\mathbf{w}_i$:

$$\begin{aligned} \mathbf{v}_1 &= (1,1,1,1) = 2\mathbf{w}_1 \\ \mathbf{v}_2 &= \frac{3}{4}(1,1,1,1) + \frac{1}{4}(-3,1,1,1) = \frac{3}{2}\mathbf{w}_1 + \frac{\sqrt{3}}{2}\mathbf{w}_2 \\ \mathbf{v}_3 &= \frac{1}{2}(1,1,1,1) + \frac{1}{6}(-3,1,1,1) + \frac{1}{3}(0,-2,1,1) \\ &= \mathbf{w}_1 + \frac{\sqrt{3}}{3}\mathbf{w}_2 + \frac{\sqrt{6}}{3}\mathbf{w}_3. \end{aligned}$$

Therefore, the QR-factorization of A is

$$A = \begin{pmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 2 & 3/2 & 1 \\ 0 & \sqrt{3}/2 & \sqrt{3}/3 \\ 0 & 0 & \sqrt{6}/3 \end{pmatrix}.$$

133.

$$\begin{aligned}
\operatorname{dist}(u,v) &= \|u-v\| \\
&= \sqrt{\langle u-v, u-v\rangle} \\
&= \sqrt{\langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle} \\
&= \sqrt{\langle u,u\rangle + \langle v,v\rangle} \quad \text{because } \langle u,v\rangle = 0 \text{ by assumption} \\
&= \sqrt{\|u\|^2 + \|v\|^2}.
\end{aligned}$$

134. Following the Gram–Schmidt process, we define vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as follows:

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{e}_1 = (1,0,0) \\
\mathbf{v}_2 &= \mathbf{e}_2 - \frac{\langle \mathbf{v}_1, \mathbf{e}_2\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 \\
&= (0,1,0) - \frac{1}{2}(1,0,0) \\
&= (-1/2, 1, 0) \\
\mathbf{v}_3 &= \mathbf{e}_3 - \frac{\langle \mathbf{v}_1, \mathbf{e}_3\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 - \frac{\langle \mathbf{v}_2, \mathbf{e}_3\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\mathbf{v}_2 \\
&= (0,0,1) - 0\mathbf{v}_1 - 0\mathbf{v}_2 \\
&= (0,0,1)
\end{aligned}$$

The Gram–Schmidt process ensures that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are mutually orthogonal, though not necessarily unit vectors. To obtain an orthonormal basis, we therefore scale each $\mathbf{v}_i$ by $\frac{1}{\|\mathbf{v}_i\|}$, where the norm here is, of course, taken with respect to the given inner product:

$$\left\{ (\tfrac{1}{\sqrt{2}}, 0, 0), (\tfrac{-1}{\sqrt{18}}, \tfrac{2}{\sqrt{18}}, 0), (0,0,1) \right\}.$$

135. (a)

$$\begin{aligned}
\|\alpha\|^2 &= \sum_{n=0}^{\infty} \frac{1}{(n+3)^2} \\
&= \sum_{n=3}^{\infty} \frac{1}{n^2} \\
&= \frac{\pi^2}{6} - \frac{1}{1^2} - \frac{1}{2^2} \\
&= \frac{\pi^2}{6} - \frac{5}{4},
\end{aligned}$$

and

$$\|\beta\|^2 = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(b)

$$
\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{n^2-1} &= \sum_{n=2}^{\infty} \frac{1}{(n+1)(n-1)} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+3)(n+1)} \\
&= \langle \alpha, \beta \rangle \\
&\leq \|\alpha\| \, \|\beta\| \quad \text{by Cauchy–Schwarz} \\
&= \sqrt{\frac{\pi^2}{6} - \frac{5}{4}} \sqrt{\frac{\pi^2}{6}} \\
&= \frac{\pi^2}{6} \sqrt{1 - \frac{15}{2\pi^2}}.
\end{aligned}
$$

136. Let $f, g \in B(M)$ and $t \in [0,1]$. Then

$$
\begin{aligned}
&\int_a^b \big(tf(x) + (1-t)g(x)\big)^2 \, dx \\
&= \|tf + (1-t)g\|^2 \\
&\leq \Big(\|tf\| + \|(1-t)g\|\Big)^2 \quad \text{by the triangle inequality} \\
&= \Big(|t| \, \|f\| + |1-t| \, \|g\|\Big)^2 \\
&\leq \Big(|t|\sqrt{M} + |1-t|\sqrt{M}\Big)^2 \quad \text{because } f, g \in B(M) \\
&= M\Big(|t| + |1-t|\Big) \\
&= M(t + 1 - t) \quad \text{because } t \in [0,1] \\
&= M.
\end{aligned}
$$

137. According to the proof of the triangle inequality on page 65 of the course notes, we have equality if and only if

$$
\begin{aligned}
\langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle &= \langle \mathbf{u}, \mathbf{u} \rangle + 2|\langle \mathbf{u}, \mathbf{v} \rangle| + \langle \mathbf{v}, \mathbf{v} \rangle \\
\text{and} \quad \langle \mathbf{u}, \mathbf{u} \rangle + 2|\langle \mathbf{u}, \mathbf{v} \rangle| + \langle \mathbf{v}, \mathbf{v} \rangle &= \langle \mathbf{u}, \mathbf{u} \rangle + 2\|\mathbf{u}\| \, \|\mathbf{v}\| + \langle \mathbf{v}, \mathbf{v} \rangle
\end{aligned}
$$

The first equality holds if and only if $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, and the second holds if and only if $\mathbf{v} = c\mathbf{u}$ for some $c \in \mathbb{R}$ by Theorem 5.1 in Section IV of the course notes. But if $\mathbf{v} = c\mathbf{u}$, then $\langle \mathbf{u}, \mathbf{v} \rangle \geq 0$ if and only if $c\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, if and only if $c \geq 0$ (because $\langle \mathbf{u}, \mathbf{u} \rangle > 0$).

138. We verify each of the axioms of a metric in turn:

(i) If $\mathbf{u}, \mathbf{v} \in V$, then

$$\begin{aligned} d(\mathbf{u},\mathbf{v}) = 0 \quad &\iff \quad \|\mathbf{u}-\mathbf{v}\| = 0 \\ &\iff \quad \mathbf{u}-\mathbf{v} = \mathbf{0} \quad \text{by axiom (iv) of an inner product} \\ &\iff \quad \mathbf{u} = \mathbf{v}. \end{aligned}$$

(ii) If $\mathbf{u}, \mathbf{v} \in V$, then

$$\begin{aligned} d(\mathbf{u},\mathbf{v}) &= \|\mathbf{u}-\mathbf{v}\| \\ &= \|(-1)(\mathbf{v}-\mathbf{u})\| \\ &= |-1|\,\|\mathbf{v}-\mathbf{u}\| \\ &= \|\mathbf{v}-\mathbf{u}\| \\ &= d(\mathbf{v},\mathbf{u}). \end{aligned}$$

(iii) If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then

$$\begin{aligned} d(\mathbf{u},\mathbf{w}) &= \|\mathbf{u}-\mathbf{w}\| \\ &= \|(\mathbf{u}-\mathbf{v}) + (\mathbf{v}-\mathbf{w})\| \\ &\le \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}-\mathbf{w}\| \quad \text{by the triangle inequality for inner product spaces} \\ &= d(\mathbf{u},\mathbf{v}) + d(\mathbf{v},\mathbf{w}). \end{aligned}$$

139. (a) Let

$$v = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \; x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

with x assumed to satisfy $x_1^2 + x_2^2 + x_3^2 = 1$, i.e., $\|x\|^2 = 1$. Then

$$\begin{aligned} |3x_1 + 4x_2 + 5x_3| &= |v \cdot x| \\ &\le \|v\|\,\|x\| \quad \text{by Cauchy–Schwarz} \\ &= \|v\| \\ &= 5\sqrt{2}, \end{aligned}$$

with equality holding if and only if x is a scalar times v. Only when x is a positive scalar times v can $v \cdot x$ be the maximum, for otherwise $v \cdot x < 0$. Therefore, to find the x where the maximum is attained, we solve $\|cv\| = 1$ for $c > 0$, i.e., $c = 1/\|v\| = 1/5\sqrt{2}$. So

$$x = cv = \frac{1}{5\sqrt{2}} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix},$$

and the maximum is attained when

$$x_1 = \frac{3}{5\sqrt{2}}, \quad x_2 = \frac{4}{5\sqrt{2}}, \quad x_3 = \frac{1}{\sqrt{2}}.$$

(b) Let

$$v = \begin{pmatrix} 5 \\ 12 \\ 13 \end{pmatrix}, \ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

with x assumed to satisfy $5x_1 + 12x_2 + 13x_3 = 26$, i.e., $v \cdot x = 26$. Then

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= \|x\|^2 \\ &\geq \frac{(v \cdot x)^2}{\|v\|^2} \quad \text{by Cauchy–Schwarz} \\ &= \frac{26^2}{2 \cdot 13^2} \\ &= \frac{2^2 \cdot 13^2}{2 \cdot 13^2} \\ &= 2, \end{aligned}$$

with equality holding if and only if $x \in \mathrm{Span}(v)$. Therefore, the minimum is 2, attained at $x \in \mathrm{Span}(v)$ satisfying $v \cdot x = 26$. We can find such x by solving $v \cdot (cv) = 26$ for $c \in \mathbb{R}$, i.e.,

$$c = \frac{26}{\|v\|^2} = \frac{2 \cdot 13}{2 \cdot 13^2} = \frac{1}{13}.$$

Then

$$x = cv = \frac{1}{13} \begin{pmatrix} 5 \\ 12 \\ 13 \end{pmatrix} = \begin{pmatrix} 5/13 \\ 12/13 \\ 1 \end{pmatrix},$$

and the minimum of 2 is attained when

$$x_1 = 5/13, \quad x_2 = 12/13, \quad x_3 = 1.$$

140. (a) Let

$$v = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

with x assumed to satisfy $x_1^2 + x_2^2 + x_3^2 = 1$, i.e., $\|x\|^2 = 1$. Then

$$\begin{aligned} -2x_1 + x_2 - x_3 &= v \cdot x \\ &\leq \|v\| \, \|x\| \quad \text{(Cauchy–Schwarz)} \\ &= \|v\| \end{aligned}$$

$$= \sqrt{6},$$

with equality holding if and only if x is a positive scalar times v. (If x were a negative scalar times v, then $v \cdot x$ would be equal to $-\|v\| \, \|x\|$.) The maximum is therefore $\sqrt{6}$, and to find the x where it is attained, we solve $\|cv\| = 1$ for $c > 0$, i.e., $c = 1/\|v\| = 1/\sqrt{6}$. So

$$x = cv = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix},$$

and the maximum is attained when

$$(x_1, x_2, x_3) = (-2/\sqrt{6}, 1/\sqrt{6}, -1/\sqrt{6}).$$

(b) Let

$$v = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

with x assumed to satisfy $4x_1 + x_2 + 5x_3 = 2$, i.e., $v \cdot x = 2$. Then

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= \|x\|^2 \\ &\geq \frac{(v \cdot x)^2}{\|v\|^2} \quad \text{(Cauchy–Schwarz)} \\ &= \frac{4}{42} \\ &= \frac{2}{21}, \end{aligned}$$

with equality holding if and only if $x \in \operatorname{Span}(v)$. Therefore, the minimum is $2/21$, as long as there is $x \in \operatorname{Span}(v)$ with $v \cdot x = 2$. We can find such x by solving $v \cdot (cv) = 2$ for $c \in \mathbb{R}$, i.e.,

$$c = \frac{2}{\|v\|^2} = \frac{2}{42} = \frac{1}{21}.$$

Then

$$x = cv = \frac{1}{21} \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix},$$

and the minimum of $2/21$ is attained when

$$(x_1, x_2, x_3) = (4/21, 1/21, 5/21).$$

141. (a) Let

$$\mathbf{v} = \begin{pmatrix} 1/3 \\ 1/2 \\ 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

with $\mathbf{x}$ assumed to satisfy $x_1^2 + x_2^2 + x_3^2 = 1$, i.e., $\|\mathbf{x}\|^2 = 1$. Then

$$\begin{aligned}\tfrac{1}{3}x_1 + \tfrac{1}{2}x_2 + x_3 = \mathbf{v}\cdot\mathbf{x} &\le |\mathbf{v}\cdot\mathbf{x}| \\ &\le \|\mathbf{v}\|\,\|\mathbf{x}\| \quad \text{by Cauchy–Schwarz} \\ &= \|\mathbf{v}\| = \tfrac{7}{6},\end{aligned}$$

with equality holding if and only if $\mathbf{x}$ is a positive scalar times $\mathbf{v}$. Therefore, to find the $\mathbf{x}$ where the maximum is attained, we solve $\|c\mathbf{v}\| = 1$ for $c > 0$, i.e., $c = 1/\|\mathbf{v}\| = 6/7$. Thus, the maximum is $7/6$, and it is attained when $\mathbf{x} = \frac{6}{7}\mathbf{v} = (2/7, 3/7, 6/7)$.

(b) We find the minimum of $x_1^2 + x_2^2 + x_3^2$ subject to $2x_1 + 3x_2 + 6x_3 = a$, where a is unknown for the time being. Let $\mathbf{v} = (2, 3, 6)$ and $\mathbf{x} = (x_1, x_2, x_3)$, with $\mathbf{x}$ assumed to satisfy $2x_1 + 3x_2 + 6x_3 = a$, i.e., $\mathbf{v}\cdot\mathbf{x} = a$. Then

$$\begin{aligned}x_1^2 + x_2^2 + x_3^2 = \|\mathbf{x}\|^2 &\ge \frac{(\mathbf{v}\cdot\mathbf{x})^2}{\|\mathbf{v}\|^2} \quad \text{by Cauchy–Schwarz} \\ &= \frac{a^2}{49},\end{aligned}$$

with equality holding if and only if $\mathbf{x} \in \mathrm{Span}(\mathbf{v})$. Therefore, the minimum is $a^2/49$, attained at $\mathbf{x} \in \mathrm{Span}(\mathbf{v})$ satisfying $\mathbf{v}\cdot\mathbf{x} = a$. We can find such $\mathbf{x}$ by solving $\mathbf{v}\cdot(c\mathbf{v}) = a$ for $c \in \mathbb{R}$, i.e.,

$$c = \frac{a}{\|\mathbf{v}\|^2} = \frac{a}{49}.$$

Then

$$\mathbf{x} = c\mathbf{v} = \frac{a}{49}(2, 3, 6).$$

The minimum is 1 if and only if $a^2/49 = 1$, i.e., $a = -7$ (since a was assumed negative). The minimum in this case occurs at $\mathbf{x} = -\frac{1}{7}(2, 3, 6) = (-2/7, -3/7, -6/7)$.

142. Note that the condition $\int_0^{\pi/4} f(x)g(x)\,dx = \frac{1}{2}$ says $\langle f, g\rangle = 1/2$ in the notation of inner products. For such a g,

$$\begin{aligned}\int_0^{\pi/4} g(x)^2\,dx = \|g\|^2 & \\ &\ge \frac{\langle f, g\rangle^2}{\|f\|^2} \quad \text{by the Cauchy–Schwarz inequality} \\ &= \frac{1}{4\|f\|^2},\end{aligned}$$

and equality holds if and only if $g = cf$ for some $c \in \mathbb{R}$. The unique $c \in \mathbb{R}$ for which $\langle f, cf\rangle = 1/2$ is

$$c = \frac{1/2}{\langle f, f\rangle} = \frac{1}{2\|f\|^2}.$$

It remains, then, to find $\|f\|^2$, which we achieve via a couple of trigonometric identities:

$$f(x)^2 = \sin^2(x) - 2\sin(x)\cos(x) + \cos^2(x) = 1 - \sin(2x),$$

so

$$\|f\|^2 = \int_0^{\pi/4} (1 - \sin(2x))\, dx = [x + \tfrac{1}{2}\cos(2x)]_0^{\pi/4} = \frac{1}{4}(\pi - 2).$$

Thus, $c = 2/(\pi - 2)$, so the function g we seek is $g = \frac{2}{\pi-2} f$, i.e.,

$$g(x) = \frac{2}{\pi - 2}(\sin(x) - \cos(x)).$$

In this case,

$$\int_0^{\pi/4} g(x)^2\, dx = \frac{1}{4\|f\|^2} = \frac{1}{\pi - 2}.$$

143. (a)

$$\begin{aligned} p &= \frac{1}{2}(x-2)(x-3) + (x-1)(x-3) + \frac{1}{2}(x-1)(x-2) \\ &= 2x^2 - 8x + 7. \end{aligned}$$

Alternatively, we may solve for p by letting $p = ax^2 + bx + c$ and observing that the given conditions on p translate to the system

$$\begin{aligned} a + b + c &= 1 \\ 4a + 2b + c &= -1 \\ 9a + 3b + c &= 1, \end{aligned}$$

which has solution $a = 2$, $b = -8$, $c = 7$ (steps to this solution should be shown).

(b) Endow $\mathcal{P}_2$ with the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle r, q \rangle = r(1)q(1) + r(2)q(2) + r(3)q(3).$$

The condition $q(1)^2 + q(2)^2 + q(3)^2 = 1$ says $\|q\|^2 = 1$, i.e., $\|q\| = 1$, and by part (a),

$$q(1) - q(2) + q(3) = \langle p, q \rangle$$

where $p = 2x^2 - 8x + 7$, so we are trying to maximize $\langle p, q \rangle$ subject to $\|q\| = 1$. Now,

$$\begin{aligned} \langle p, q \rangle &\le |\langle p, q \rangle| \\ &\le \|p\|\,\|q\| \quad \text{by Cauchy–Schwarz} \\ &= \|p\| \quad \text{by the condition } \|q\| = 1. \end{aligned}$$

Further, equality occurs if and only if q is a positive-scalar multiple of p, i.e., $q = cp$ with $c > 0$. The unique $c > 0$ such that $\|cp\| = 1$ is $c = 1/\|p\|$.

It remains, then, to find $\|p\|$. But we already know that $p(1) = p(3) = 1$ and $p(2) = -1$, so $\|p\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$. Thus, the polynomial we seek is $q = \frac{1}{\|p\|}p = \frac{1}{\sqrt{3}}p = \frac{1}{\sqrt{3}}(2x^2 - 8x + 7)$.

144. We first find orthonormal bases for the eigenspaces, starting with the eigenvalue -1:

$$-I - A = \begin{pmatrix} -1 & -2 & 1 \\ -2 & -4 & 2 \\ 1 & 2 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so a basis is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

We apply Gram–Schmidt to

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix},$$

that is, we let $u_1 = v_1$ and

$$u_2 = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} - \frac{-2}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Scaling, we obtain the orthonormal basis

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Now for the eigenspace for the eigenvalue 5:

$$5I - A = \begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

so a basis is

$$\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\},$$

which we normalize to

$$\left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

Hence, if

$$P = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}, \; D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

then $P^{\mathrm{T}}AP = D$.

145. Let us begin with the eigenspace associated to -9:

$$-9I - A = \begin{pmatrix} -20 & 2 & -6 \\ 2 & -20 & -6 \\ -6 & -6 & -4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

so this eigenspace is 1-dimensional, spanned by $(1, 1, -3)$. A unit spanning vector is $(1/\sqrt{11}, 1/\sqrt{11}, -3/\sqrt{11})$.

Next, we turn to the eigenspace associated to 13:

$$13I - A = \begin{pmatrix} 2 & 2 & -6 \\ 2 & 2 & -6 \\ -6 & -6 & 18 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so this eigenspace has basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = (-1, 1, 0)$ and $\mathbf{v}_2 = (3, 0, 1)$. To perform Gram–Schmidt on this basis, we let $\mathbf{u}_1 = \mathbf{v}_1$ and

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 = (3, 0, 1) - \tfrac{-3}{2}(-1, 1, 0) = \tfrac{1}{2}(3, 3, 2).$$

Thus, this eigenspace has orthonormal basis

$$\{(-1/\sqrt{2}, 1/\sqrt{2}, 0), (3/\sqrt{22}, 3/\sqrt{22}, 2/\sqrt{22})\}.$$

We have produced an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors, and using this basis we see that $P^{\mathrm{T}}AP = D$, where

$$P = \begin{pmatrix} 1/\sqrt{11} & -1/\sqrt{2} & 3/\sqrt{22} \\ 1/\sqrt{11} & 1/\sqrt{2} & 3/\sqrt{22} \\ -3/\sqrt{11} & 0 & 2/\sqrt{22} \end{pmatrix}, \quad D = \begin{pmatrix} -9 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix}.$$

146.

$$f(x_1, x_2, x_3) = -x_1^2 - x_2^2 - x_3^2 + 4x_1x_2 + 6x_1x_3 + 10x_2x_3.$$

147.

$$A = \begin{pmatrix} 1 & 1/2 & -3/2 \\ 1/2 & -1 & 5/2 \\ 3/2 & 5/2 & 2 \end{pmatrix}.$$

148.

$$A = \frac{1}{2}(B + B^{\mathrm{T}}) = \begin{pmatrix} 2 & 3 & 9 \\ 3 & 4 & 5 \\ 9 & 5 & 6 \end{pmatrix}.$$

149. If $\mathbf{x} = (x_1, x_2)$, then

$$\begin{aligned} f(\mathbf{x}) &= (a^2 - c^2)x_1^2 + 2(ab - cd)x_1x_2 + (b^2 - d^2)x_2^2 \\ &= (a^2x_1^2 + 2abx_1x_2 + b^2x_2^2) - (c^2x_1^2 + 2cdx_1x_2 + d^2x_2^2) \\ &= (ax_1 + bx_2)^2 - (cx_1 + dx_2)^2. \qquad (44) \end{aligned}$$

Now, because $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, there is a solution to the equations

$$\begin{aligned} ax_1 + bx_2 &= 1 \\ cx_1 + dx_2 &= 0, \end{aligned}$$

and for such an $\mathbf{x} = \left(x_1, x_2\right)$, (44) shows that $f(\mathbf{x}) = 1^2 - 0^2 = 1 > 0$. Similarly, there is a solution to

$$\begin{aligned} ax_1 + bx_2 &= 0 \\ cx_1 + dx_2 &= 1, \end{aligned}$$

and then $f(\mathbf{x}) = 0^2 - 1^2 = -1 < 0$.

150. (a) The symmetric matrix associated to f is

$$A = \begin{pmatrix} 1 & 3/2 \\ 3/2 & 3 \end{pmatrix},$$

which has characteristic polynomial $p_A(x) = (x-1)(x-3) - \frac{9}{4} = x^2 - 4x + \frac{3}{4}$. The roots are $\frac{1}{2}(4 \pm \sqrt{13})$, so the maximum and minimum of f on unit vectors are $\frac{1}{2}(4 + \sqrt{13})$ and $\frac{1}{2}(4 - \sqrt{13})$ respectively. Both roots are positive, so f is positive definite.

(b) The symmetric matrix associated to f is

$$A = \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix},$$

which has characteristic polynomial $p_A(x) = (x-1)(x-9) - 9 = x^2 - 10x = x(x-10)$. The roots are 0 and 10, so the maximum and minimum of f on unit vectors are 10 and 0 respectively. Both roots are non-negative, but one is zero, so f is non-negative definite but not positive definite.

(c) The symmetric matrix associated to f is

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -3 & -1 \\ 1 & -1 & -1 \end{pmatrix},$$

which has characteristic polynomial

$$\begin{aligned} p_A(x) &= \det \begin{pmatrix} x-1 & 0 & -1 \\ 0 & x+3 & 1 \\ -1 & 1 & x+1 \end{pmatrix} \\ &= (x-1)(x^2+4x+2) - (x+3) \quad \text{(first row)} \\ &= x^3 + 3x^2 - 3x - 5 = (x+1)(x^2+2x-5). \end{aligned}$$

The roots, in ascending order, are $-1-\sqrt{6}$, -1, and $-1+\sqrt{6}$. Therefore, the maximum and minimum of f on unit vectors are $-1+\sqrt{6}$ and $-1-\sqrt{6}$ respectively. The maximum and minimum have opposite signs, so f is indefinite.

151. We orthogonally diagonalize the real symmetric matrix

$$A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}.$$

The characteristic polynomial is

$$p_A(x) = (x-2)(x+1) - 4 = x^2 - x - 6 = (x+2)(x-3),$$

so the eigenvalues are -2 and 3.

Eigenspace for 3:

$$3I - A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix},$$

so this eigenspace is spanned by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. A unit spanning vector is $\begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$.

Eigenspace for -2:

$$-2I - A = \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix},$$

so this eigenspace is spanned by $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$. A unit spanning vector is $\begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$.

Thus, if

$$P = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix},$$

then $P^{\mathrm{T}}AP = D$. Hence, the quadratic form $g(\mathbf{y}) = f(P\mathbf{y})$ satisfies $g(y_1, y_2) = 3y_1^2 - 2y_2^2$.

The quadratic form f is indefinite, because the corresponding matrix A has both a positive eigenvalue and a negative one (or because the diagonalized quadratic form g is indefinite).

152. We orthogonally diagonalize the real symmetric matrix

$$A = \begin{pmatrix} 3 & 6 \\ 6 & -2 \end{pmatrix}.$$

The characteristic polynomial is

$$p_A(x) = (x-3)(x+2) - 36 = x^2 - x - 42 = (x+6)(x-7),$$

so the eigenvalues are -6 and 7.

Eigenspace for 7:

$$7I - A = \begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix},$$

so this eigenspace is spanned by $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. A unit spanning vector is $\begin{pmatrix} 3/\sqrt{13} \\ 2/\sqrt{13} \end{pmatrix}$.

Eigenspace for -6:

$$-6I - A = \begin{pmatrix} -9 & -6 \\ -6 & -4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix},$$

so this eigenspace is spanned by $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$. A unit spanning vector is $\begin{pmatrix} -2/\sqrt{13} \\ 3/\sqrt{13} \end{pmatrix}$.

Thus, if

$$P = \begin{pmatrix} 3/\sqrt{13} & -2/\sqrt{13} \\ 2/\sqrt{13} & 3/\sqrt{13} \end{pmatrix}, \quad D = \begin{pmatrix} 7 & 0 \\ 0 & -6 \end{pmatrix},$$

then $P^{\mathrm{T}}AP = D$. Hence, the quadratic form $g(\mathbf{y}) = f(P\mathbf{y})$ satisfies $g(y_1, y_2) = 7y_1^2 - 6y_2^2$.

The quadratic form f is indefinite, because the corresponding matrix A has both a positive eigenvalue and a negative one (or because the diagonalized quadratic form g is indefinite).

153. The symmetric matrix corresponding to f is

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{pmatrix}.$$

As we saw in Question 3 in the Week 12 practice problems, $P^{\mathrm{T}}AP = D$ where

$$P = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Therefore, the quadratic form $g(\mathbf{y}) = f(P\mathbf{y})$ satisfies

$$g(y_1, y_2, y_3) = -y_1^2 - y_2^2 + 5y_3^2.$$

The quadratic form f is indefinite, because the corresponding matrix A has both a positive eigenvalue and a negative one (or because the diagonalized quadratic form g is indefinite).

154.

$$\begin{aligned} f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= 5x_1^2 - 2(x_1 + 2x_2)^2 + 4(x_1 + 2x_2 + 3x_3)^2 \\ &= 5y_1^2 - 2y_2^2 + 4y_3^2 \end{aligned}$$

where

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_1 + 2x_2 \\ y_3 &= x_1 + 2x_2 + 3x_3. \end{aligned}$$

Hence,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = P^{\mathrm{T}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

Therefore, if

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

then $f(\mathbf{x}) = \mathbf{y}^{\mathrm{T}} D\mathbf{y} = (P^{\mathrm{T}}\mathbf{x})^{\mathrm{T}} D(P^{\mathrm{T}}\mathbf{x})$.

155. (a) Define $y_1 = x_1 + 3x_2$ and $y_2 = 2x_1 - x_2$. Then $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 4y_1^2 + 3y_2^2$. Now, if $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, then $\mathbf{y} = P^{\mathrm{T}}\mathbf{x}$ where $P = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$, so letting $D = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$, we have

$$f(\mathbf{x}) = 4y_1^2 + 3y_2^2 = \mathbf{y}^{\mathrm{T}} D\mathbf{y} = (P^{\mathrm{T}}\mathbf{x})^{\mathrm{T}} D(P^{\mathrm{T}}\mathbf{x}).$$

(b) By part (a), if $\mathbf{x} \in \mathbb{R}^2$, then

$$\mathbf{x}^{\mathrm{T}} A\mathbf{x} = f(\mathbf{x}) = (P^{\mathrm{T}}\mathbf{x})^{\mathrm{T}} D(P^{\mathrm{T}}\mathbf{x}) = \mathbf{x}^{\mathrm{T}} PDP^{\mathrm{T}}\mathbf{x},$$

so $A = PDP^{\mathrm{T}}$ by the fact given in the question, because both A and PDP^{T} are symmetric. Alternatively, one may compute A and PDP^{T} explicitly and compare them.

156. The quadratic form f is indefinite. To see this, let the eigenvalues of A, counted with multiplicity, be $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, so that $p_A(x) = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)(x-\lambda_4)$. Then $\lambda_1\lambda_2\lambda_3\lambda_4 = p_A(0) = \det(A) < 0$, so at least one of the λ_i is negative and at least one positive (because 4 is even). Now use Proposition 9.2 in Section IV of the course notes.

157. (a) For the quadratic form in Question 151, the maximum and minimum subject to $\|\mathbf{x}\| = 1$ are 3 and -2 respectively by Proposition 10.1 in Section IV of the course notes. For the quadratic from in Question 152, the maximum and minimum are 7 and -6.

(b) For the quadratic form in Question 154, the maximum and minimum subject to $\|\mathbf{x}\| = 1$ are 5 and -1 respectively.

158. (a) Let us first find the eigenspace associated to -6:

$$-6I - A = \begin{pmatrix} -4 & -2 & -6 \\ -2 & -1 & -3 \\ -6 & -3 & -9 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so a basis for this eigenspace is $\{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = (-1, 2, 0)$ and $\mathbf{v}_2 = (-3, 0, 2)$. We apply Gram–Schmidt to these basis vectors, letting

$$\mathbf{u}_1 = \mathbf{v}_1 = (-1, 2, 0)$$

$$\begin{aligned}\mathbf{u}_2 &= \mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 \\ &= (-3,0,2) - \frac{3}{5}(-1,2,0) = \frac{2}{5}(-6,-3,5).\end{aligned}$$

Scaling $\mathbf{u}_1, \mathbf{u}_2$, we obtain the orthonormal basis

$$\left\{ \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{pmatrix}, \begin{pmatrix} -6/\sqrt{70} \\ -3/\sqrt{70} \\ 5/\sqrt{70} \end{pmatrix} \right\}$$

for this eigenspace.

We turn to the eigenspace associated to 8:

$$8I - A = \begin{pmatrix} 10 & -2 & -6 \\ -2 & 13 & -3 \\ -6 & -3 & 5 \end{pmatrix} \leftrightarrow \begin{pmatrix} 3 & 0 & -2 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

so this eigenspace is spanned by $(2,1,3)$. Scaling, we obtain the unit spanning vector

$$\begin{pmatrix} 2/\sqrt{14} \\ 1/\sqrt{14} \\ 3/\sqrt{14} \end{pmatrix}.$$

Hence, if

$$P = \begin{pmatrix} -1/\sqrt{5} & -6/\sqrt{70} & 2/\sqrt{14} \\ 2/\sqrt{5} & -3/\sqrt{70} & 1/\sqrt{14} \\ 0 & 5/\sqrt{70} & 3/\sqrt{14} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 8 \end{pmatrix},$$

then P is orthogonal and $P^{\mathrm{T}}AP = D$.

(b) By Proposition 10.1 in Section IV, the maximum of $\mathbf{x}^{\mathrm{T}}A\mathbf{x}$ subject to $\|\mathbf{x}\| = 1$ occurs at the unit eigenvectors with eigenvalue 8. By our answer to part (a), the corresponding eigenspace is spanned by the unit vector $(2/\sqrt{14}, 1/\sqrt{14}, 3/\sqrt{14})$, so the two vectors of norm 1 in this eigenspace are

$$(2/\sqrt{14}, 1/\sqrt{14}, 3/\sqrt{14}) \quad \text{and} \quad -(2/\sqrt{14}, 1/\sqrt{14}, 3/\sqrt{14}).$$

(c) Let $\mathbf{x} = (x_1, x_2, x_3)$, and define $\mathbf{y} = (y_1, y_2, y_3)$ by

$$\begin{aligned}\mathbf{y} = P^{\mathrm{T}}\mathbf{x} &= \begin{pmatrix} -1/\sqrt{5} & 2/\sqrt{5} & 0 \\ -6/\sqrt{70} & -3/\sqrt{70} & 5/\sqrt{70} \\ 2/\sqrt{14} & 1/\sqrt{14} & 3/\sqrt{14} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{\sqrt{5}}x_1 + \frac{2}{\sqrt{5}}x_2 \\ -\frac{6}{\sqrt{70}}x_1 - \frac{3}{\sqrt{70}}x_2 + \frac{5}{\sqrt{70}}x_3 \\ \frac{2}{\sqrt{14}}x_1 + \frac{1}{\sqrt{14}}x_2 + \frac{3}{\sqrt{14}}x_3 \end{pmatrix}.\end{aligned}$$

Then

$$\begin{aligned}
f(\mathbf{x}) &= \mathbf{x}^{\mathrm{T}} A \mathbf{x} \\
&= \mathbf{y}^{\mathrm{T}} D \mathbf{y} \quad \text{because } \mathbf{x} = P\mathbf{y} \text{ and } P^{\mathrm{T}} A P = D \\
&= -6y_1^2 - 6y_2^2 + 8y_3^2 \\
&= -6\Big(-\tfrac{1}{\sqrt{5}}x_1 + \tfrac{2}{\sqrt{5}}x_2\Big)^2 - 6\Big(-\tfrac{6}{\sqrt{70}}x_1 - \tfrac{3}{\sqrt{70}}x_2 + \tfrac{5}{\sqrt{70}}x_3\Big)^2 \\
&\quad + 8\Big(\tfrac{2}{\sqrt{14}}x_1 + \tfrac{1}{\sqrt{14}}x_2 + \tfrac{3}{\sqrt{14}}x_3\Big)^2 \\
&= -\frac{6}{5}(-x_1 + 2x_2)^2 - \frac{3}{35}(-6x_1 - 3x_2 + 5x_3)^2 \\
&\quad + \frac{4}{7}(2x_1 + x_2 + 3x_3)^2.
\end{aligned}$$

159. Let $y_1 = x_1/x_3$ and $y_2 = x_2/x_3$. Then $x_1x_2/x_3^2 = y_1y_2$, and $x_1^2 + x_2^2 = x_3^2$ if and only if $y_1^2 + y_2^2 = 1$. Thus, we are to find the maximum and minimum of y_1y_2 subject to $y_1^2 + y_2^2 = 1$. The symmetric matrix associated to the quadratic form y_1y_2 is

$$\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix},$$

whose characteristic polynomial is $x^2 - 1/4$. The roots of this polynomial are $\pm 1/2$, so the maximum and minimum in question are $1/2$ and $-1/2$ respectively.

160. The symmetric matrix associated to f is

$$A = \begin{pmatrix} 1 & a/2 & 0 \\ a/2 & 2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix},$$

which has characteristic polynomial

$$\begin{aligned}
p_A(x) &= (x - \tfrac{1}{2})((x-1)(x-2) - (\tfrac{a}{2})^2) \\
&= (x - \tfrac{1}{2})(x^2 - 3x + 2 - (\tfrac{a}{2})^2).
\end{aligned}$$

The roots of the quadratic factor are

$$\frac{1}{2}(3 \pm \sqrt{9 - 8 + 4(\tfrac{a}{2})^2}) = \frac{1}{2}(3 \pm \sqrt{a^2+1}),$$

so A has eigenvalues $\frac{1}{2}$, $\frac{1}{2}(3 + \sqrt{a^2+1})$, and $\frac{1}{2}(3 - \sqrt{a^2+1})$.

(a) By Proposition 10.1 in Section IV of the course notes, we are looking for $a \in \mathbb{R}$ such that the least eigenvalue of A is 0. This amounts to finding a such that $\frac{1}{2}(3 - \sqrt{a^2+1}) = 0$, i.e., $\sqrt{a^2+1} = 3$, i.e., $a^2 + 1 = 9$, i.e., $a = 2\sqrt{2}$ (because $a \geq 0$ by assumption).

(b) Let Δ be the difference between the maximum and minimum of $f(\mathbf{x})$ subject to $\|\mathbf{x}\| = 1$, and for brevity, let $c = \frac{1}{2}\sqrt{a^2+1}$, so that the eigenvalues of A are $\frac{1}{2}$, $\frac{3}{2} - c$, and $\frac{3}{2} + c$. The key observation for this question is that the order of the eigenvalues depends on c, with the crossover between $\frac{1}{2}$ and $\frac{3}{2} - c$ occurring when $c = 1$. Thus,

$$\Delta = \begin{cases} (\frac{3}{2} + c) - \frac{1}{2} = c + 1 & \text{if } c \leq 1 \\ (\frac{3}{2} + c) - (\frac{3}{2} - c) = 2c & \text{if } c \geq 1. \end{cases} \tag{45}$$

(i) Note that if $c < 1$, then according to (45), $\Delta = c + 1 < 2 < 4$. Therefore,

$$\begin{aligned} \Delta = 4 \quad &\iff \quad c \geq 1 \quad \text{and} \quad 2c = 4 \quad \text{by (45) again,} \\ &\iff \quad c \geq 1 \quad \text{and} \quad c = 2, \\ &\iff \quad c = 2, \\ &\iff \quad \tfrac{1}{2}\sqrt{a^2+1} = 2, \\ &\iff \quad a = \sqrt{15} \quad \text{(because } a \geq 0 \text{ by assumption).} \end{aligned}$$

(ii) This time, observe that if $c > 1$, then (45) implies that $\Delta = 2c > 2 > 7/4$. Hence,

$$\begin{aligned} \Delta = \tfrac{7}{4} \quad &\iff \quad c \leq 1 \quad \text{and} \quad c + 1 = \tfrac{7}{4} \quad \text{by (45) again,} \\ &\iff \quad c \leq 1 \quad \text{and} \quad c = \tfrac{3}{4}, \\ &\iff \quad c = \tfrac{3}{4}, \\ &\iff \quad \tfrac{1}{2}\sqrt{a^2+1} = \tfrac{3}{4}, \\ &\iff \quad a = \tfrac{\sqrt{5}}{2} \quad \text{(because } a \geq 0 \text{ by assumption).} \end{aligned}$$