Linear Algebra II (MATH 225): Practice Problems – v 1.12

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Searching for questions related to a given section

Each question in this set of practice problems is tagged with sections from the course notes that the question is related to. For example, Question 28 is tagged as follows: ⟨**I−3**, **I−4**, **I−5**, **I−6**⟩. These tags are searchable (except for the four in the previous sentence!). For example, to search for questions related to Section II−1, type `(II-1)` or `(ii-1)` into your PDF viewer's search bar. Note that the separator in the search term should be a hyphen with no surrounding spaces. Also, the brackets in the search should be (round) parentheses and not angle brackets. The parentheses should be included to prevent your search returning III−1 or II−10 when you mean II−1.

1. ⟨**I−1**⟩ Let V be a vector space.

   (a) Show that $0v = \mathbf{0}$ for all $v \in V$, where $\mathbf{0}$ is the zero vector in V. *Hint: Consider* $(0+0)v$.

   (b) Show that $(-a)v = -(av)$ for all $a \in \mathbb{R}$ and all $v \in V$, that is, $(-a)v$ is the additive inverse of av. *Hint: Use part (a).*

2. ⟨**I−1**, **I−2**⟩ In the vector space $\mathcal{P}_2$, show that we may express the polynomial $p = 2x^2 + 7x + 10$ as a linear combination of the polynomials

$$p_1 = (x+1)^2, \quad p_2 = x+1, \quad p_3 = 1.$$

   Find explicit $a_1, a_2, a_3 \in \mathbb{R}$ such that $p = a_1p_1 + a_2p_2 + a_3p_3$.

3. ⟨**I−1**, **I−2**⟩ Define functions $f, g, h \in \mathcal{F}$ by

$$f(x) = x^3, \quad g(x) = x+1, \quad h(x) = \ln(x^2+1).$$

   Decide whether f is a linear combination of g and h. If it is, find an explicit linear combination. Otherwise, prove that no such linear combination exists.

4. ⟨**I−1**, **I−2**⟩ Define sequences $s, t, u \in \mathcal{S}$ by

$$s = ((n+1)^3)_n, \quad t = (n^3+1)_n, \quad u = (n(n+1))_n.$$

   Express s as a linear combination of t and u, that is, find $a, b \in \mathbb{R}$ such that $s = at + bu$.

5. ⟨**I−1**, **I−2**⟩ Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

an element of the vector space $M_2(\mathbb{R})$. Show that A^2 is a linear combination of A and I, where I is the 2×2 identity matrix. Find explicit $c, d \in \mathbb{R}$ such that $A^2 = cA + dI$.

6. $\langle \mathbf{I-1}, \mathbf{I-2} \rangle$ Let $A \in M_2(\mathbb{R})$, and suppose further that A has integer entries, has trace zero, and is invertible. Show that there is a positive integer n such that $A^4 = nI$, where I is the 2×2 identity matrix. *Hint: Compute A^2 first.*

7. $\langle \mathbf{I-1}, \mathbf{I-2} \rangle$ Define functions $f, g, h \in \mathcal{F}$ by

$$f(x) = \cos(2x), \quad g(x) = \cos^2(x), \quad h(x) = 1.$$

Show that f is a linear combination of g and h. Find explicit $a, b \in \mathbb{R}$ such that $f = ag + bh$. (Note that the equality $f = ag + bh$ means that $f(x) = ag(x) + bh(x)$ for all $x \in \mathbb{R}$.)

8. $\langle \mathbf{I-1}, \mathbf{I-2} \rangle$ Define functions $f, g, h \in \mathcal{F}$ by

$$f(x) = \sin(2x), \quad g(x) = \cos(x), \quad h(x) = \sin(x).$$

Is f a linear combination of g and h? Justify your answer either way.

9. $\langle \mathbf{I-1} \rangle$ Suppose we give $\mathbb{R}_{\geq 0}$ the addition operation $u \oplus v = |u - v|$ and the scalar multiplication operation $a \odot u = |a|u$. Decide which vector space axioms are satisfied by $\mathbb{R}_{\geq 0}$ with these operations. Justify your answers.

10. $\langle \mathbf{I-3} \rangle$ Decide whether each of the following sets is a subspace of $\mathcal{P}$. If it is, justify your answer by showing that the set is non-empty and is closed under both addition and scalar multiplication. Otherwise, explain why the set is not a subspace of $\mathcal{P}$.

(a) $B_1 = \{p \in \mathcal{P} \mid p'(a) = 0\}$, where a is some fixed real number and p' denotes the derivative of p.

(b) $B_2 = \{p \in \mathcal{P} \mid p'(a) = 1\}$, where a is some fixed real number.

(c) $B_3 = \{p \in \mathcal{P} \mid p' \text{ is the zero polynomial}\}$.

(d) $B_4 = \{p \in \mathcal{P} \mid p(0) > \deg(p)\}$, where $\deg(p)$ denotes the degree of p.

11. $\langle \mathbf{I-3} \rangle$ Decide whether each of the following sets is a subspace of $\mathcal{S}$, the space of sequences. If it is, justify your answer by showing that the set is non-empty and is closed under both addition and scalar multiplication. Otherwise, explain why the set is not a subspace of $\mathcal{S}$.

(a) $B_1 = \{(a_n)_n \in \mathcal{S} \mid a_n = 2a_{n-1} - 3a_{n-2} \text{ for all } n \geq 2\}$.

(b) $B_2 = \{(a_n)_n \in \mathcal{S} \mid a_n = 0 \text{ whenever 3 divides } n\}$.

(c) $B_3 = \{(a_n)_n \in \mathcal{S} \mid a_n = a_{n-1} + 1 \text{ for all } n \geq 1\}$.

(d) $B_4 = \{(a_n)_n \in \mathcal{S} \mid |a_n| \geq n \text{ for all } n\}$.

12. $\langle \mathbf{I-3} \rangle$ For each of the following sets B_i, decide whether it is a subspace of the given vector space V_i. If it is, show that it is non-empty and is closed under both addition and scalar multiplication. Otherwise, explain why the set is not a subspace of V_i.

(a) $V_1 = \mathcal{P}_2$, $B_1 = \{p \in \mathcal{P}_2 \mid p + xp' \text{ has degree exactly } 2\}$.

(b) $V_2 = M_2(\mathbb{R})$, $B_2 = \{X \in M_2(\mathbb{R}) \mid XA = 0\}$, where $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.

(c) $V_3 = \mathcal{S}$, $B_3 = \{(a_n)_n \in \mathcal{S} \mid a_n = a_{n-1}^2 \text{ for all } n \geq 1\}$.

(d) $V_4 = \mathcal{F}$, $B_4 = \{f \in \mathcal{F} \mid f(0) = f(1)\}$.

13. $\langle \mathbf{I-3} \rangle$ For each of the subsets of $\mathbb{R}^3$ below, decide whether it is closed under addition, justifying your answer either way.

(a) $A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1x_2x_3 > 1 \text{ and } x_1, x_2, x_3 > 0\}$.

(b) $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1x_2x_3 > 1\}$.

14. $\langle \mathbf{I-4}, \mathbf{I-5} \rangle$ For each of the following sets of polynomials in $\mathcal{P}_2$, decide whether it is linearly independent, a spanning set, both, or neither.

(a) $B_1 = \{2x + 1, 3x + 2, 4x + 3\}$

(b) $B_2 = \{x, x + 2, -x^2\}$

(c) $B_3 = \{2, x^2\}$

(d) $B_4 = \{x + 2, x - 1, x^2, x^2 - 3\}$

15. $\langle \mathbf{I-3} \rangle$ Let $\mathcal{F}$ be the space of functions $f : \mathbb{R} \to \mathbb{R}$.

(a) Find a non-empty subset of $\mathcal{F}$ that is closed under addition but not scalar multiplication. Justify your claim.

(b) Find a non-empty subset of $\mathcal{F}$ that is closed under scalar multiplication but not addition. Justify your claim.

16. $\langle \mathbf{I-4} \rangle$ Let $A = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$. Decide whether the matrices I, A, A^2 are linearly independent in $M_2(\mathbb{R})$. Justify your answer either way.

17. $\langle \mathbf{I-4} \rangle$ Decide whether the sequences

$$s = (n+1)_n, \quad t = (2n+1)_n, \quad u = (n^2)_n$$

in $\mathcal{S}$ are linearly independent. Justify your answer either way.

18. $\langle \mathbf{I-4} \rangle$

(a) Show that if a is a non-zero real number and $f \in \mathcal{F}$ is the function $x \mapsto e^{ax^2}$, then f, f', and f'' are linearly independent in $\mathcal{F}$. (In fact, the whole family $\{f^{(n)}\}_{n\geq 0}$ is linearly independent, but you do not need to prove this more general assertion.)

(b) Show that if $g \in \mathcal{F}$ is the function $x \mapsto x\sin(x)$, then $g, g', g'', g''', g^{(4)}$ are linearly dependent.

19. $\langle \mathbf{I-4}, \mathbf{I-5} \rangle$ Let $u = (4^n)_n$, $v = (2^n)_n$, and $w = (n^2)_n$ be sequences in $\mathcal{S}$.

(a) Show that u, v, w are linearly independent in $\mathcal{S}$.

(b) Using part (a), show that the sequences $\alpha = (4^n - 2^{n+2})_n$ and $\beta = (4^n + 2^{n+2} - 2n^2)_n$ are linearly independent. *Hint: First express each of α and β as a linear combination of u, v, w.*

(c) Show that the sequences

$$\begin{aligned} \gamma_1 &= ((2^n - n)(2^n + n))_n \\ \gamma_2 &= ((2^{\frac{n}{2}} - \tfrac{n}{2})(2^{\frac{n}{2}} + \tfrac{n}{2}))_n \end{aligned}$$

are both in $\mathrm{Span}(\alpha, \beta)$, and express each as a linear combination of α and β.

20. $\langle \mathbf{I-5} \rangle$ Let V be a vector space, and suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span V. If

$$\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{u}_2 = \mathbf{v}_1 - \mathbf{v}_2, \quad \mathbf{u}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3,$$

show that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ span V.

21. $\langle \mathbf{I-5} \rangle$ For an integer $k \geq 0$, let $s_k \in \mathcal{S}$ be the sequence

$$s_k = (0, \ldots, 0, 1, 1, 1, 1, \ldots),$$

where the number of zeroes occurring is k. Thus,

$$\begin{aligned} s_0 &= (1,1,1,1,\ldots) \\ s_1 &= (0,1,1,1,1,\ldots) \\ s_2 &= (0,0,1,1,1,1,\ldots), \end{aligned}$$

and so on. Let $X = \{s_0, s_1, s_2, s_3, \ldots\}$.

(a) Is the sequence $t = (1, 2, 3, 4, \ldots)$ in $\mathrm{Span}(X)$?

(b) Can you characterize the sequences in $\mathrm{Span}(X)$?

22. $\langle \mathbf{I-6} \rangle$ Let $V = \{p \in \mathcal{P}_3 \mid p(1) = p(-1)\}$.

(a) If $p = a_3x^3 + a_2x^2 + a_1x + a_0 \in \mathcal{P}_3$, find a condition on the coefficients of p for p to be in V.

(b) Find a basis for V, and write down $\dim(V)$.

23. $\langle \mathbf{I-6}, \mathbf{I-7} \rangle$ Let $U = \{A \in M_2(\mathbb{R}) \mid A^{\mathrm{T}} = A\}$, and let

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(a) Show that $\mathcal{B} = \{A_1, A_2, A_3\}$ is a basis for U.

(b) Any matrix of the form $X^{\mathrm{T}}X$ is symmetric. If $X \in M_2(\mathbb{R})$ has columns $\mathbf{u}, \mathbf{v}$, i.e., $X = \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}$, find $[X^{\mathrm{T}}X]_{\mathcal{B}}$ in terms of $\mathbf{u} \cdot \mathbf{u}$, $\mathbf{v} \cdot \mathbf{v}$, and $\mathbf{u} \cdot \mathbf{v}$.

24. $\langle \mathbf{I-4}, \mathbf{I-5}, \mathbf{I-6}, \mathbf{I-7} \rangle$ Define functions $f_1, f_2, f_3 \in \mathcal{F}$ by

$$f_1(x) = \cos(2\pi x), \quad f_2(x) = \sin(2\pi x), \quad f_3(x) = x.$$

(a) Show that the set $\mathcal{B} = \{f_1, f_2, f_3\}$ is linearly independent and is therefore a basis for the space $V = \mathrm{Span}(f_1, f_2, f_3)$.

(b) Let $g \in V$ be the function with coordinate vector $[g]_{\mathcal{B}} = (4, -1, 2)$ with respect to $\mathcal{B}$. Find $g(3/8)$.

25. $\langle \mathbf{I-3}, \mathbf{I-4} \rangle$ Let $U = \{(x_n)_n \in \mathcal{S} \mid x_n = x_{n-1} + x_{n-2} \text{ for all } n \geq 2\}$.

(a) Show that U is a subspace of $\mathcal{S}$.

(b) Let $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $\beta = \frac{1}{2}(1-\sqrt{5})$, and let $s, t \in \mathcal{S}$ be the sequences defined by

$$s = (\alpha^n)_n = (1, \alpha, \alpha^2, \alpha^3, \ldots)$$
$$t = (\beta^n)_n = (1, \beta, \beta^2, \beta^3, \ldots)$$

Show that $s, t \in U$. *Hint: Note that* $\alpha^2 = \alpha + 1$ *and* $\beta^2 = \beta + 1$.

(c) Show that s, t are linearly independent.

26. $\langle \mathbf{I-4}, \mathbf{I-5}, \mathbf{I-6}, \mathbf{I-7}, \mathbf{I-8} \rangle$ Let $f, g \in \mathcal{F}$ be the functions defined by $f(x) = e^{2x} + x$ and $g(x) = e^x + 2x$, and let $V = \text{Span}(f, g)$.

(a) Show that f and g are linearly independent and therefore form a basis $\mathcal{E} = \{f, g\}$ for V.

(b) Write down the coordinate vectors $[h_1]_{\mathcal{E}}$ and $[h_2]_{\mathcal{E}}$ where $h_1(x) = -2e^{2x} + e^x$ and $h_2(x) = 3e^{2x} - e^x + x$.

(c) Is $\{h_1, h_2\}$ a linearly independent set? Does it span V? Explain your answers by using part (b).

27. $\langle \mathbf{I-6}, \mathbf{I-7}, \mathbf{I-8} \rangle$

(a) Define $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^4$ by $\mathbf{v}_1 = (2, 1, 1, 0)$, $\mathbf{v}_2 = (1, 2, 1, 1)$. Extend the linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis of $\mathbb{R}^4$.

(b) Extend the following linearly independent set to a basis of $M_2(\mathbb{R})$:

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

28. $\langle \mathbf{I-3}, \mathbf{I-4}, \mathbf{I-5}, \mathbf{I-6} \rangle$ Let $W = \{A \in M_2(\mathbb{R}) \mid A^{\mathrm{T}} = A \text{ and } \mathrm{Tr}(A) = 0\}$.

(a) Show that W is a subspace of $M_2(\mathbb{R})$.

(b) Find a basis for W, and state the dimension of W.

29. $\langle \mathbf{I-4}, \mathbf{I-8} \rangle$ Consider the polynomials

$$\begin{aligned} p_1 &= x^3 + 2x^2 - x + 3 \\ p_2 &= x^3 + x^2 - x + 1 \\ p_3 &= x^3 + 2x^2 + x - 1 \\ p_4 &= 2x^3 + 6x^2 + 4x - 2 \end{aligned}$$

Decide whether the polynomials p_1, p_2, p_3, p_4 are linearly independent. If they are, explain why. If they are not, explain why not and express one of them as a linear combination of the others, showing your working.

30. $\langle \mathbf{I-6}, \mathbf{I-8} \rangle$

(a) Let U be the subspace of $M_2(\mathbb{R})$ spanned by

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \right\}.$$

Find a basis for U contained in S.

(b) Let W be the subspace of $\mathcal{P}$ spanned by

$$T = \{x^3 + x^2 - 1, -x^3 + 2x + 1, x^3 + 2x^2 + 2x - 1, \\ 2x^3 + x^2 + x - 2, 4x^3 + 2x^2 - x - 4\}.$$

Find a basis for W contained in T.

31. $\langle \mathbf{I-8} \rangle$ In Question 24, we saw that the functions $f_1, f_2, f_3 \in \mathcal{F}$ defined by $f_1(x) = \cos(2\pi x)$, $f_2(x) = \sin(2\pi x)$, and $f_3(x) = x$ are linearly independent. Using this fact, decide in each of the following cases whether the given set of functions is a linearly independent set.

(a) $\{g_1, g_2, g_3\}$, where

$$\begin{aligned} g_1(x) &= \cos(2\pi x) + 2\sin(2\pi x) + 3x \\ g_2(x) &= \cos(2\pi x) - \sin(2\pi x) + x \\ g_3(x) &= -3\cos(2\pi x) + 6\sin(2\pi x) - x \end{aligned}$$

(b) $\{h_1, h_2, h_3\}$, where

$$\begin{aligned} h_1(x) &= \cos(2\pi x) + 3\sin(2\pi x) + -2x \\ h_2(x) &= 2\cos(2\pi x) + \sin(2\pi x) + x \\ h_3(x) &= \cos(2\pi x) + 4\sin(2\pi x) + 3x \end{aligned}$$

32. $\langle \mathbf{I-4}, \mathbf{I-5}, \mathbf{I-6} \rangle$ Find a basis for the space $V = \{p \in \mathcal{P}_3 \mid p(2) = p'(2) = 0\}$.

33. $\langle \mathbf{I-7}, \mathbf{I-8} \rangle$ We saw in Section I−6 of the course notes a basis $\{A_1, A_2, A_3\}$ for $V = \{A \in M_2(\mathbb{R}) \mid \operatorname{Tr}(A) = 0\}$. Denote that basis by $\mathcal{C}$.

(a) Find the coordinate vectors, with respect to the basis $\mathcal{C}$, of each of the following matrices in V:

$$B_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$$
$$B_3 = \begin{pmatrix} -1 & 3 \\ -3 & 1 \end{pmatrix} \qquad B_4 = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}.$$

(b) Decide whether B_1, B_2, B_3, B_4 span V.

34. $\langle \mathbf{I-4}, \mathbf{I-5}, \mathbf{I-6}, \mathbf{I-8} \rangle$ For each of the following subsets of $M_2(\mathbb{R})$, decide whether it is linearly independent, a spanning set, both (i.e., a basis), or neither. You may wish to use Proposition 8.1 in Section I of the course notes.

(a)

$$S_1 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \right\}.$$

(b)

$$S_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \right\}.$$

35. $\langle \mathbf{I-10} \rangle$ Let $\mathcal{B} = \{v_1, v_2, v_3\}$ and $\mathcal{C} = \{w_1, w_2, w_3\}$, where

$$v_1 = \begin{pmatrix} 11 \\ -1 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$$

and

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

Given that $\mathcal{B}$ and $\mathcal{C}$ are bases for $\mathbb{R}^3$, find the change-of-basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

36. $\langle \mathbf{I-10} \rangle$ Consider the following bases $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{E}$ for $\mathcal{P}_1$:

$$\begin{aligned} \mathcal{B} &= \{3x+1, 8x+2\} \\ \mathcal{C} &= \{x+1, x-1\} \\ \mathcal{E} &= \{x, 1\}. \end{aligned}$$

(a) Find each of $P_{\mathcal{B}\leftarrow\mathcal{E}}$, $P_{\mathcal{C}\leftarrow\mathcal{B}}$, and $P_{\mathcal{C}\leftarrow\mathcal{E}}$.

(b) Verify that $P_{\mathcal{C}\leftarrow\mathcal{E}} = P_{\mathcal{C}\leftarrow\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{E}}$.

37. $\langle \mathbf{I-8} \rangle$ Let $p = ax^2 + bx + c$ be a polynomial of degree 2 (so $a \neq 0$).

(a) Show that $\mathcal{B} = \{p, p', p''\}$ is a basis for $\mathcal{P}_2$.

(b) Write down the coordinate vectors of the polynomials

$$p,\ p + p',\ p + p' + p''$$

with respect to the basis $\mathcal{B}$.

(c) Deduce that $\{p, p + p', p + p' + p''\}$ is also a basis of $\mathcal{P}_2$.

38. $\langle \mathbf{I-8} \rangle$ Let V be an n-dimensional vector space with basis $\mathcal{B} = \{u_1, \dots, u_n\}$, let $v_1, \dots, v_n$ be any vectors in V, and let

$$A = \begin{pmatrix} [v_1]_\mathcal{B} & [v_2]_\mathcal{B} & \cdots & [v_n]_\mathcal{B} \end{pmatrix}.$$

Show that $\{v_1, \dots, v_n\}$ is a basis for V if and only if $\det(A) \neq 0$.

39. $\langle \mathbf{I-5}, \mathbf{I-7} \rangle$ In some 4-dimensional vector space V with basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, there are vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ whose coordinate vectors with respect to $\mathcal{B}$ are

$$[\mathbf{u}_1]_\mathcal{B} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 3 \end{pmatrix}, \quad [\mathbf{u}_2]_\mathcal{B} = \begin{pmatrix} -2 \\ 3 \\ -1 \\ -5 \end{pmatrix}, \quad [\mathbf{u}_3]_\mathcal{B} = \begin{pmatrix} 2 \\ -1 \\ 4 \\ 9 \end{pmatrix}.$$

Show that $\mathbf{v}_1 \in \operatorname{Span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, and express $\mathbf{v}_1$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. *Hint: Begin with the equation* $\mathbf{v}_1 = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$, *take coordinate vectors of both sides, and show that the resulting equation has a solution in* c_1, c_2, c_3.

40. $\langle \mathbf{I-10} \rangle$ Consider the following bases for $\mathbb{R}^2$:

$$\begin{aligned} \mathcal{B} &= \left\{ \begin{pmatrix} 11 \\ 2 \end{pmatrix}, \begin{pmatrix} 30 \\ 5 \end{pmatrix} \right\} \\ \mathcal{C} &= \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \end{pmatrix} \right\} \\ \mathcal{E} &= \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}. \end{aligned}$$

(a) Find $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and $P_{\mathcal{E}\leftarrow\mathcal{C}}$.

(b) Use your answers to part (a) to find $P_{\mathcal{E}\leftarrow\mathcal{B}}$ via a single matrix multiplication.

(c) Use your answer to part (b) to find $P_{\mathcal{B}\leftarrow\mathcal{E}}$.

(d) Use your answer to part (c) to write $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 11 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 30 \\ 5 \end{pmatrix}$.

41. $\langle \mathbf{I-10} \rangle$ Let

$$\begin{array}{lll} p_1(x) = 2x^2 + 2x + 2 & p_2(x) = 3x^2 + 5x + 6 & p_3(x) = 2x^2 + 5x + 6 \\ q_1(x) = x^2 + x + 2 & q_2(x) = x^2 + 2x + 2 & q_3(x) = x^2 + 2x + 3. \end{array}$$

Given that the sets $\mathcal{B} = \{p_1, p_2, p_3\}$ and $\mathcal{C} = \{q_1, q_2, q_3\}$ are both bases for $\mathcal{P}_2$, find the change-of-basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$. Show your steps.

42. $\langle \mathbf{I-10} \rangle$ Consider the invertible 3×3 matrix A below and its inverse:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 2 & -3 \\ 1 & 1 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 5 & 3 & -1 \\ -2 & -1 & 1 \\ -3 & -2 & 1 \end{pmatrix}.$$

Suppose that $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are bases for a 3-dimensional vector space V, and suppose that $P_{\mathcal{C}\leftarrow\mathcal{B}} = A$.

(a) Express $2\mathbf{u}_1 + 3\mathbf{u}_2 + 5\mathbf{u}_3$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(b) Express $\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

43. $\langle \mathbf{I-10} \rangle$ Suppose $f_1, f_2, f_3 \in \mathcal{F}$ are linearly independent, and let $V = \mathrm{Span}(f_1, f_2, f_3)$.

(a) Define

$$\begin{aligned} g_1 &= -f_1 - 2f_2 - 3f_3 \\ g_2 &= -2f_1 - f_2 + 2f_3 \\ g_3 &= 2f_1 + 2f_2 \\ h_1 &= f_2 + 2f_3 \\ h_2 &= f_1 + 2f_2 + 3f_3 \\ h_3 &= f_1 + f_2 - f_3. \end{aligned}$$

By row-reducing an appropriate 3×3 matrix, show that $\mathcal{B} = \{g_1, g_2, g_3\}$ is a basis for V.

(b) By row-reducing an appropriate 3×6 matrix, simultaneously show that $\mathcal{C} = \{h_1, h_2, h_3\}$ is a basis for V and find $P_{\mathcal{C} \leftarrow \mathcal{B}}$.

44. $\langle \mathbf{I}-\mathbf{9} \rangle$ Let $f_1, f_2, f_3, f_4 \in \mathcal{F}$, and let $V = \text{Span}(f_1, f_2, f_3, f_4)$. Now suppose that there are $g_1, g_2, g_3, g_4 \in V$ such that the only solution to the equation $c_1 g_1 + c_2 g_2 + c_3 g_3 + c_4 g_4 = 0$ is $c_1 = c_2 = c_3 = c_4 = 0$. Show that both $\{f_1, f_2, f_3, f_4\}$ and $\{g_1, g_2, g_3, g_4\}$ are bases for V.

45. $\langle \mathbf{I}-\mathbf{9} \rangle$ Let V be a 3-dimensional vector space with basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and suppose that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are vectors in V such that every $\mathbf{v}_i \in \mathcal{B}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for V.

46. $\langle \mathbf{I}-\mathbf{6},\ \mathbf{I}-\mathbf{7},\ \mathbf{I}-\mathbf{8},\ \mathbf{I}-\mathbf{10} \rangle$ In this question, you may use the following fact: If $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for $\mathbb{R}^2$, then the area of the parallelogram formed by $\mathbf{u}_1$ and $\mathbf{u}_2$ is $|\det(P_{\mathcal{E} \leftarrow \mathcal{B}})|$, where $\mathcal{E}$ is the standard basis of $\mathbb{R}^2$.

Let $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2\}$ be a basis for $\mathbb{R}^2$, and define

$$\begin{aligned} \mathbf{v}_1 &= \frac{1}{a+b}(\sqrt{2}\,\mathbf{w}_1 + \mathbf{w}_2) \\ \mathbf{v}_2 &= \frac{1}{a-b}(\mathbf{w}_1 + \sqrt{2}\,\mathbf{w}_2) \\ \mathbf{u}_1 &= a\mathbf{v}_1 + b\mathbf{v}_2 \\ \mathbf{u}_2 &= b\mathbf{v}_1 + a\mathbf{v}_2 \end{aligned}$$

where $a, b \in \mathbb{R}$ satisfy $a^2 \neq b^2$ (so all of the above vectors are well defined).

(a) Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for $\mathbb{R}^2$.

(b) Show that the area of the parallelogram formed by $\mathbf{u}_1$ and $\mathbf{u}_2$ remains constant as a and b vary (subject to the constraint that $a^2 \neq b^2$).

47. $\langle \mathbf{I}-\mathbf{6},\ \mathbf{I}-\mathbf{7},\ \mathbf{I}-\mathbf{8},\ \mathbf{I}-\mathbf{10} \rangle$ In this question, you may use the following fact: If $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for $\mathbb{R}^2$, then the area of the parallelogram formed by $\mathbf{u}_1$ and $\mathbf{u}_2$ is $|\det(P_{\mathcal{E} \leftarrow \mathcal{B}})|$, where $\mathcal{E}$ is the standard basis of $\mathbb{R}^2$.

Suppose now that $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$ and that the area of the parallelogram formed by $\mathbf{v}_1$ and $\mathbf{v}_2$ is $1/2$.

(a) Show that, for any given $x \in \mathbb{R}$, the vectors $\mathbf{u}_1 = (x-1)\mathbf{v}_1 - 5\mathbf{v}_2$ and $\mathbf{u}_2 = 5\mathbf{v}_1 + (x-3)\mathbf{v}_2$ form a basis for $\mathbb{R}^2$.

(b) Find the area of the parallelogram formed by $\mathbf{u}_1$ and $\mathbf{u}_2$, expressing your answer in terms of x. *Hint: Use Proposition 10.2 in Section I of the course notes.*

(c) Find x such that the area in part (b) is least. What is the area in that case?

48. $\langle \mathbf{I-2}, \mathbf{I-4} \rangle$ Let $f_1, \dots, f_n \in \mathcal{F}$ be differentiable functions that are linearly independent. Suppose that there are $x_1, \dots, x_m \in \mathbb{R}$, where $m < n$, such that $f_j'(x_i) \in \mathbb{Q}$ for all $i \in \{1, \dots, m\}$ and all $j \in \{1, \dots, n\}$. Show that there are $a_1, \dots, a_n \in \mathbb{Z}$ such that the function $f = a_1 f_1 + \dots + a_n f_n$ satisfies $f'(x_i) = 0$ for all i and is not the zero function.

49. $\langle \mathbf{I-4}, \mathbf{I-5}, \mathbf{I-7} \rangle$ Define functions $f_1, f_2, f_3 \in \mathcal{F}$ by

$$f_1(x) = \frac{1}{x^2+1}, \quad f_2(x) = \frac{1}{x^2+2}, \quad f_3(x) = \frac{1}{x^2+3}.$$

(a) Show that the function $g \in \mathcal{F}$ defined by

$$g(x) = \frac{1}{(x^2+1)(x^2+2)(x^2+3)}$$

is in the space $W = \operatorname{Span}(f_1, f_2, f_3)$, and express g as a linear combination of f_1, f_2, f_3.

(b) Show that $\mathcal{B} = \{f_1, f_2, f_3\}$ is a linearly independent set. You may refer to any relevant calculations you performed in part (a).

(c) In light of part (b), the set $\mathcal{B}$ is a basis for W. Write down $[g]_{\mathcal{B}}$.

50. $\langle \mathbf{I-7}, \mathbf{I-8}, \mathbf{I-9}, \mathbf{I-10} \rangle$ In this question, you may use the fact that the matrix A below is invertible with the given inverse:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 8 & 7 & 5 \\ 15 & 10 & 6 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 4/3 & -2/3 & 1/3 \\ -9/2 & 3/2 & -1/2 \\ 25/6 & -5/6 & 1/6 \end{pmatrix}.$$

Define functions $f_1, f_2, f_3, g_1, g_2, g_3 \in \mathcal{F}$ by

$$\begin{aligned} f_1(x) &= \frac{1}{x^2+2} & f_2(x) &= \frac{1}{x^2+3} & f_3(x) &= \frac{1}{x^2+5} \\ g_1(x) &= \frac{x^4}{p(x)} & g_2(x) &= \frac{x^2}{p(x)} & g_3(x) &= \frac{1}{p(x)} \end{aligned}$$

where $p(x) = (x^2+2)(x^2+3)(x^2+5)$. You may assume that $\mathcal{C} = \{g_1, g_2, g_3\}$ is a linearly independent set and is therefore a basis for the space $V = \operatorname{Span}(g_1, g_2, g_3)$. (Show this yourself for extra practice.)

(a) Using the fact that $f_1(x) = (x^2+3)(x^2+5)/p(x)$, express f_1 as a linear combination of g_1, g_2, g_3 and then write down the coordinate vector $[f_1]_{\mathcal{C}}$. Find $[f_2]_{\mathcal{C}}$ and $[f_3]_{\mathcal{C}}$ in the same way.

(b) Using the coordinate vectors $[f_1]_{\mathcal{C}}, [f_2]_{\mathcal{C}}, [f_3]_{\mathcal{C}}$ found in part (a), show that $\mathcal{B} = \{f_1, f_2, f_3\}$ is a linearly independent set.

(c) Making reference to Proposition 9.2 in Section I of the course notes, explain briefly why $\mathcal{B}$ must be a basis for V.

(d) Express each of g_1, g_2, g_3 as a linear combination of f_1, f_2, f_3.

51. $\langle$**II−1**$\rangle$ Decide whether each of the following maps is a linear transformation. Justify your answer in each case.

(a)

$$\begin{array}{rcl} \varphi_1 : M_n(\mathbb{R}) & \rightarrow & \mathbb{R} \\ A & \mapsto & \mathrm{Tr}(AY) \end{array}$$

where Y is some fixed matrix in $M_n(\mathbb{R})$.

(b)

$$\begin{array}{rcl} \varphi_2 : \mathbb{R}^3 & \rightarrow & \mathcal{S} \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & \mapsto & (x_1^{n+1} + x_2^{n+1} + x_3^{n+1})_n \end{array}$$

52. $\langle$**II−1**$\rangle$ Decide whether each of the following maps is a linear transformation. Justify your answer in each case.

(a)

$$\begin{array}{rcl} \varphi_1 : \mathbb{R}^3 & \rightarrow & \mathcal{P}_2 \\ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} & \mapsto & u_1(x+1)^2 + u_2(x+1) + u_3 \end{array}$$

(b)

$$\begin{array}{rcl} \varphi_2 : \mathcal{F} & \rightarrow & M_2(\mathbb{R}) \\ f & \mapsto & \begin{pmatrix} f(1)^2 & f(1)f(2) \\ f(1)f(2) & f(2)^2 \end{pmatrix} \end{array}$$

53. $\langle$**II−1**$\rangle$ Decide whether each of the following maps is a linear transformation. Justify your answer in each case.

(a)

$$\begin{array}{rcl} \varphi_1 : \mathbb{R}^2 & \to & \mathcal{P}_2 \\ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} & \mapsto & (a_1 - a_2)x^2 + (a_1 + a_2)x + (2a_1 + 3a_2) \end{array}$$

(b)

$$\begin{array}{rcl} \varphi_2 : M_3(\mathbb{R}) & \to & \mathbb{R} \\ (a_{i,j})_{i,j} & \mapsto & \min_{i,j}(a_{i,j}) \end{array}$$

(i.e., $\varphi_2(A)$ is the minimum of all 9 entries of the matrix A)

(c)

$$\begin{array}{rcl} \varphi_3 : \mathcal{S} & \to & \mathcal{S} \\ (a_n)_n & \mapsto & (a_{n^2})_n \end{array}$$

(d)

$$\begin{array}{rcl} \varphi_4 : \mathcal{P}_3 & \to & \mathcal{P}_9 \\ p & \mapsto & p(p(x)) \end{array}$$

54. $\langle \mathbf{II-1} \rangle$ Decide whether each of the following maps is a linear transformation. Justify your answer in each case.

(a)

$$\begin{array}{rcl} \varphi_1 : \mathbb{R}^2 & \to & \mathbb{R} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \mapsto & |x_1 + x_2| - |x_1 - x_2|. \end{array}$$

(b)

$$\begin{array}{rcl} \varphi_2 : M_3(\mathbb{R}) & \to & M_3(\mathbb{R}) \\ A & \mapsto & \mathrm{Tr}(A)A. \end{array}$$

(c)

$$\begin{array}{rcl} \varphi_3 : \mathcal{P}_3 & \to & \mathcal{P}_4 \\ p & \mapsto & \dfrac{d}{dx}((x^2 + 1)p(x)). \end{array}$$

(d)

$$\begin{aligned} \varphi_4 : C[1,2] &\to \mathbb{R} \\ f &\mapsto \int_1^2 \frac{f(x)}{x}\, dx, \end{aligned}$$

where $C[1,2]$ is the space of continuous functions $[1,2] \to \mathbb{R}$.

55. $\langle \mathbf{II-2} \rangle$ Decide whether the linear transformation

$$\begin{aligned} \varphi : \mathcal{P}_2 &\to \mathbb{R}^3 \\ p &\mapsto \begin{pmatrix} p(1) \\ p'(2) \\ p''(3) \end{pmatrix} \end{aligned}$$

is injective. Justify your answer either way.

56. $\langle \mathbf{II-2}, \mathbf{II-3} \rangle$ Is the trace map

$$\begin{aligned} \varphi : M_3(\mathbb{R}) &\to \mathbb{R} \\ A &\mapsto \operatorname{Tr}(A) \end{aligned}$$

injective, surjective, both, or neither?

57. $\langle \mathbf{II-2} \rangle$ Find a non-zero polynomial in the kernel of the linear transformation

$$\begin{aligned} \varphi : \mathcal{P}_2 &\to M_2(\mathbb{R}) \\ ax^2 + bx + c &\mapsto \begin{pmatrix} a+b+c & a+2b+3c \\ -a+b+3c & a-c \end{pmatrix}. \end{aligned}$$

Also, find a non-zero matrix in the image of this linear transformation.

58. $\langle \mathbf{II-3} \rangle$ Is the linear transformation φ in Question 57 surjective? If not, find a matrix in $M_2(\mathbb{R})$ that is not in $\operatorname{Image}(\varphi)$.

59. $\langle \mathbf{II-2} \rangle$ Show that the following linear transformation is injective:

$$\begin{aligned} \varphi : \mathcal{P}_2 &\to \mathbb{R}^3 \\ p &\mapsto \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix}. \end{aligned}$$

60. $\langle \mathbf{II-3} \rangle$ Is the linear transformation φ in Question 59 surjective?

61. $\langle \mathbf{II-2}, \mathbf{II-3} \rangle$ Is the linear transformation

$$\begin{aligned} \varphi : \mathcal{P}_2 &\to M_2(\mathbb{R}) \\ p &\mapsto \begin{pmatrix} p(0)+p(1) & p'(0)+p'(1) \\ p(0)-p(1) & p'(0)-p'(1) \end{pmatrix} \end{aligned}$$

injective, surjective, both, or neither?

62. $\langle \mathbf{II-2}, \mathbf{II-3} \rangle$ Consider the linear transformation

$$\begin{aligned} \varphi : M_2(\mathbb{R}) &\to \mathcal{P}_1 \\ A &\mapsto \mathrm{Tr}(A)x + \mathrm{Tr}(BA), \end{aligned}$$

where $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

(a) If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

express $\varphi(A)$ as a polynomial whose coefficients are given explicitly in terms of a, b, c, d.

(b) Show that φ is surjective.

(c) Find a non-zero matrix in $\mathrm{Ker}(\varphi)$.

63. $\langle \mathbf{II-4} \rangle$ Use the rank-nullity theorem to find the dimension of $\mathrm{Image}(\varphi)$, where φ is the linear transformation in Question 61.

64. $\langle \mathbf{II-3}, \mathbf{II-4} \rangle$ Let

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and consider the linear transformation

$$\begin{aligned} \varphi : M_3(\mathbb{R}) &\to \mathbb{R}^3 \\ A &\mapsto \begin{pmatrix} \mathrm{Tr}(A) \\ \mathrm{Tr}(XA) \\ \mathrm{Tr}(X^2A) \end{pmatrix}. \end{aligned}$$

Show that φ is surjective, and then use the rank-nullity theorem to find the dimension of $\mathrm{Ker}(\varphi)$.

65. $\langle \mathbf{II-2}, \mathbf{II-3} \rangle$

(a) Let $E_1, \ldots, E_4 \in M_2(\mathbb{R})$ be the matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and let $\varphi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be the linear transformation such that $\varphi(E_1) = E_3$, $\varphi(E_2) = E_4$, $\varphi(E_3) = 0$, and $\varphi(E_4) = 0$. Show that $\mathrm{Ker}(\varphi) = \mathrm{Image}(\varphi)$.

(b) Let V be a vector space and $\pi : V \to V$ a linear transformation such that $\pi(\pi(v)) = \pi(v)$ for all $v \in V$. Show that if $\mathbf{v}$ is in both $\mathrm{Ker}(\pi)$ and $\mathrm{Image}(\pi)$, then $\mathbf{v} = \mathbf{0}_V$.

66. $\langle \mathbf{II}-\mathbf{2}, \mathbf{II}-\mathbf{3}, \mathbf{II}-\mathbf{4} \rangle$ Define

$$\begin{aligned} \varphi : \mathcal{P}_4 &\to M_2(\mathbb{R}) \\ p &\mapsto \begin{pmatrix} p(1)-p(2) & p(2)-p(3) \\ p(3)-p(4) & p(4)-p(1) \end{pmatrix}. \end{aligned}$$

(a) Find a basis for $\mathrm{Ker}(\varphi)$. *Hint: If* $p \in \mathrm{Ker}(\varphi)$, *consider the polynomial* $q(x) = p(x) - p(1)$. *What can you deduce about* q*?*

(b) Write down $\mathrm{nullity}(\varphi)$, and use the rank-nullity theorem to find $\mathrm{rank}(\varphi)$.

(c) Find a basis for $\mathrm{Image}(\varphi)$.

67. $\langle \mathbf{II}-\mathbf{4} \rangle$ Let U and V be vector spaces with $\dim(U) = 5$ and $\dim(V) = 3$, and suppose that $\varphi : U \to V$ is a linear transformation. If $\mathbf{x} \in \mathrm{Ker}(\varphi)$, show that there is $\mathbf{y} \in \mathrm{Ker}(\varphi)$ such that $\mathbf{y}$ is not a scalar multiple of $\mathbf{x}$. *Hint: Suppose that every* $\mathbf{y} \in \mathrm{Ker}(\varphi)$ *were a scalar multiple of* $\mathbf{x}$. *Use the rank-nullity theorem to arrive at a contradiction.*

68. $\langle \mathbf{II}-\mathbf{5}, \mathbf{II}-\mathbf{6} \rangle$ Let $f, g : \mathbb{R} \to \mathbb{R}$ be the functions given by $f(x) = e^x$ and $g(x) = e^{-x}$, and let $V = \mathrm{Span}(f, g) \subseteq \mathcal{F}$. Define

$$\begin{aligned} \varphi : V &\to \mathbb{R}^2 \\ h &\mapsto \begin{pmatrix} \int_0^{\ln(2)} h(x)\,dx \\ \int_0^{\ln(3)} h(x)\,dx \end{pmatrix}. \end{aligned}$$

(a) Find $[\varphi]_{\mathcal{C} \leftarrow \mathcal{B}}$, where $\mathcal{B} = \{f, g\}$ and $\mathcal{C} = \{\mathbf{e}_1, \mathbf{e}_2\}$, the standard basis of $\mathbb{R}^2$.

(b) Using $[\varphi]_{\mathcal{C} \leftarrow \mathcal{B}}$, show that φ is both injective and surjective.

69. $\langle \mathbf{II}-\mathbf{5} \rangle$ For the linear transformation

$$\begin{aligned} \varphi : M_2(\mathbb{R}) &\to \mathbb{R} \\ A &\mapsto \mathrm{Tr}(A), \end{aligned}$$

find the matrix $[\varphi]_{\mathcal{C}\leftarrow\mathcal{E}}$, where

$$\mathcal{E} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and $\mathcal{C} = \{1\}$.

70. $\langle$**II−5, II−6**$\rangle$ Consider the linear transformation

$$\begin{aligned} \varphi : \mathcal{P}_2 &\to M_2(\mathbb{R}) \\ ax^2 + bx + c &\mapsto \begin{pmatrix} a+b+c & a+2b+3c \\ -a+b+3c & a-c \end{pmatrix}. \end{aligned}$$

(a) Find the matrix $[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}}$, where $\mathcal{B} = \{x^2, x, 1\}$ and $\mathcal{E}$ is as in Question 69.

(b) Use $[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}}$ to find a basis for $\mathrm{Ker}(\varphi)$ and a basis for $\mathrm{Image}(\varphi)$.

71. $\langle$**II−5, II−6**$\rangle$ Consider the linear transformation

$$\begin{aligned} \varphi : \mathcal{P}_2 &\to M_2(\mathbb{R}) \\ p &\mapsto \begin{pmatrix} p(0)+p(1) & p'(0)+p'(1) \\ p(0)-p(1) & p'(0)-p'(1) \end{pmatrix}. \end{aligned}$$

(a) Find the matrix $[\varphi]_{\mathcal{E}\leftarrow\mathcal{B}}$, where $\mathcal{B} = \{x^2, x, 1\}$ and $\mathcal{E}$ is as in Question 69.

(b) Use part (a) to show that φ is injective and to find a basis for $\mathrm{Image}(\varphi)$.

72. $\langle$**II−5, II−6**$\rangle$ Consider the linear transformation

$$\begin{aligned} \varphi : \mathcal{P}_3 &\to \mathcal{P}_2 \\ p &\mapsto p' + p'' + p''' - p(0)(x^2 + x + 1). \end{aligned}$$

(a) Find $[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}}$, where $\mathcal{B} = \{x^3, x^2, x, 1\}$ and $\mathcal{C} = \{x^2, x, 1\}$.

(b) Using your answer to part (a), find a basis for $\mathrm{Ker}(\varphi)$.

(c) Is φ surjective? Briefly justify your answer.

73. $\langle$**II−5, II−6**$\rangle$ A linear transformation $\varphi : \mathcal{P}_4 \to M_2(\mathbb{R})$ has matrix

$$[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 1 & 2 & 1 & -1 \\ 1 & 2 & 3 & 0 & 2 \\ 1 & 3 & 4 & 1 & 1 \\ 1 & -1 & 0 & 0 & -1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the basis $\mathcal{B} = \{x^4, x^3, x^2, x, 1\}$ of $\mathcal{P}_4$ and the basis

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

of $M_2(\mathbb{R})$.

(a) Find a basis for $\text{Ker}(\varphi)$.

(b) Find a basis for $\text{Image}(\varphi)$.

74. $\langle \mathbf{II-5}, \mathbf{II-6} \rangle$ Let $V = \{(a_n)_n \in \mathcal{S} \mid a_n = a_{n-1} + a_{n-2} \text{ for all } n \geq 2\}$, and let $s, t \in V$ be the sequences beginning

$$\begin{aligned} s &= (1, 0, 1, 1, 2, 3, 5, \ldots) \\ t &= (0, 1, 1, 2, 3, 5, 8, \ldots) \end{aligned}$$

It is a fact that $\mathcal{B} = \{s, t\}$ is a basis for V, and you may assume this fact. Now consider the linear transformation

$$\begin{aligned} \varphi : V &\to \mathbb{R}^2 \\ (a_n)_n &\mapsto \begin{pmatrix} a_2 \\ a_6 \end{pmatrix}. \end{aligned}$$

(a) Find $[\varphi]_{\mathcal{E} \leftarrow \mathcal{B}}$, where $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$, the standard basis of $\mathbb{R}^2$.

(b) Using part (a), show that if $(a_n)_n$ and $(b_n)_n$ are sequences in V satisfying $a_2 = b_2$ and $a_6 = b_6$, then $a_n = b_n$ for all $n \geq 0$. *Hint: What does your answer to part (a) tell you about whether φ is injective?*

75. $\langle \mathbf{II-5}, \mathbf{II-7} \rangle$ Let $V = \{(a_n)_n \in \mathcal{S} \mid a_n = a_{n-1} - a_{n-2} \text{ for all } n \geq 2\}$. The space V has basis $\mathcal{C} = \{s, t\}$, where

$$\begin{aligned} s &= (1, 0, -1, -1, 0, 1, \ldots), \\ t &= (0, 1, 1, 0, -1, -1, \ldots). \end{aligned}$$

Now consider the linear transformations

$$\begin{aligned} \varphi : \mathcal{P}_1 &\to V \\ p &\mapsto p(0)s + p(1)t \end{aligned}$$

$$\begin{aligned} \psi : V &\to M_2(\mathbb{R}) \\ (a_n)_n &\mapsto \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}. \end{aligned}$$

(a) For $p \in \mathcal{P}_1$, find $\psi \circ \varphi(p)$ as a matrix whose entries are given explicitly in terms of p. Use your description of $\psi \circ \varphi$ to find $[\psi \circ \varphi]_{\mathcal{E} \leftarrow \mathcal{B}}$, where $\mathcal{B} = \{x, 1\}$ and $\mathcal{E}$ is as in Question 69.

(b) Find $[\varphi]_{\mathcal{C} \leftarrow \mathcal{B}}$ and $[\psi]_{\mathcal{E} \leftarrow \mathcal{C}}$, and verify by direct calculation that $[\psi \circ \varphi]_{\mathcal{E} \leftarrow \mathcal{B}} = [\psi]_{\mathcal{E} \leftarrow \mathcal{C}}[\varphi]_{\mathcal{C} \leftarrow \mathcal{B}}$.

76. $\langle \mathbf{II-2}, \mathbf{II-3}, \mathbf{II-5}, \mathbf{II-7} \rangle$ Consider again the space

$$V = \{(a_n)_n \in \mathcal{S} \mid a_n = a_{n-1} - a_{n-2} \text{ for all } n \geq 2\}.$$

Recall from Question 75 the basis $\mathcal{C} = \{s, t\}$ of V, where

$$\begin{aligned} s &= (1, 0, -1, -1, 0, 1, \ldots), \\ t &= (0, 1, 1, 0, -1, -1, \ldots). \end{aligned}$$

This time, we define linear transformations

$$\begin{aligned} \varphi : V &\to M_2(\mathbb{R}) \\ (a_n)_n &\mapsto \begin{pmatrix} a_0 & a_1 \\ a_4 & a_3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \psi : M_2(\mathbb{R}) &\to \mathbb{R}^2 \\ A &\mapsto \begin{pmatrix} \mathrm{Tr}(A) \\ \mathrm{Tr}(XA) \end{pmatrix}, \end{aligned}$$

where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) Compute $[\varphi]_{\mathcal{E} \leftarrow \mathcal{C}}$ and $[\psi]_{\mathcal{D} \leftarrow \mathcal{E}}$, where $\mathcal{E}$ is as in Question 69 and $\mathcal{D}$ is the standard basis for $\mathbb{R}^2$.

(b) Show by direct calculation that $[\psi]_{\mathcal{D} \leftarrow \mathcal{E}}[\varphi]_{\mathcal{E} \leftarrow \mathcal{C}} = 0$, and deduce that $\psi \circ \varphi : V \to \mathbb{R}^2$ is the zero map.

(c) Show that $\mathrm{Image}(\varphi) = \mathrm{Ker}(\psi)$. *Hint: Use part (b) and compare dimensions.*

77. $\langle \mathbf{II-8} \rangle$ Show that the function

$$\begin{aligned} f : \mathbb{R}_{>0} &\to \mathbb{R}_{<2} \\ x &\mapsto 2 - \frac{1}{x} \end{aligned}$$

is invertible, and find its inverse $f^{-1} : \mathbb{R}_{<2} \to \mathbb{R}_{>0}$. Here, $\mathbb{R}_{<2}$ denotes the set of real numbers less than 2, and $\mathbb{R}_{>0}$ denotes the set of positive real numbers.

78. $\langle \mathbf{II-5}, \mathbf{II-7} \rangle$ In this question, we will work with the basis $\mathcal{B} = \{x^2, x, 1\}$ of $\mathcal{P}_2$ and the basis $\mathcal{D} = \{X_1, X_2, X_3, X_4\}$ of $M_2(\mathbb{R})$, where

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We also let $V = \{(a_n)_n \in \mathcal{S} \mid a_n = 2a_{n-1} + a_{n-2} - a_{n-3} \text{ for all } n \geq 3\}$, a 3-dimensional space.

Suppose that $\varphi : \mathcal{P}_2 \to V$ and $\psi : V \to M_2(\mathbb{R})$ are linear transformations such that

$$[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \qquad [\psi]_{\mathcal{D}\leftarrow\mathcal{C}} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{pmatrix}$$

for some basis $\mathcal{C}$ of V. If $p = ax^2 + bx + c \in \mathcal{P}_2$, find $\psi \circ \varphi(p)$ as a matrix whose entries are given explicitly in terms of the real numbers a, b, c. Show your work.

79. $\langle \mathbf{II-8} \rangle$ Show that the function

$$\begin{aligned} f : \mathbb{R}_{\geq 2} &\to \mathbb{R}_{\geq 1} \\ x &\mapsto e^{x^2-4x+4} \end{aligned}$$

is invertible, and find its inverse $f^{-1} : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 2}$. Here, $\mathbb{R}_{\geq a}$ denotes the set of real numbers greater than or equal to a.

80. $\langle \mathbf{II-8} \rangle$ Without using Proposition 8.1 or Proposition 9.1 in Section II of the course notes, decide whether the linear transformation

$$\begin{aligned} \varphi : \mathbb{R}^3 &\to \mathcal{P}_2 \\ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} &\mapsto (a_1 - a_2)x^2 + (a_2 - a_3)x + a_1 + a_3 \end{aligned}$$

is invertible, and find its inverse $\varphi^{-1} : \mathcal{P}_2 \to \mathbb{R}^3$ if so.

81. $\langle \mathbf{II-5}, \mathbf{II-8} \rangle$ Define

$$\begin{aligned} \varphi : M_2(\mathbb{R}) &\to M_2(\mathbb{R}) \\ A &\mapsto \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} A + \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} A^{\mathrm{T}}, \end{aligned}$$

and let $\mathcal{C}$ be the basis

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

of $M_2(\mathbb{R})$. Calculate $[\varphi]_{\mathcal{C}}$, and use it to decide whether φ is an isomorphism.

82. $\langle$**II − 5**, **II − 8**$\rangle$ Consider the linear transformation

$$\begin{aligned} \varphi : \mathcal{P}_2 &\to \mathbb{R}^3 \\ p &\mapsto \begin{pmatrix} p(-1) - p'(-1) \\ p(0) - p'(0) \\ p(1) - p'(1) \end{pmatrix}. \end{aligned}$$

You may use the fact that

$$[\varphi]_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{pmatrix} 3 & -2 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix},$$

where $\mathcal{B} = \{x^2, x, 1\}$ and $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the standard basis of $\mathbb{R}^3$.

(a) Show that φ is invertible, and find the polynomial $\varphi^{-1}(a_1, a_2, a_3)$ for a given vector $(a_1, a_2, a_3) \in \mathbb{R}^3$.

(b) Find the polynomial $p \in \mathcal{P}_2$ such that $p(-1) = p'(-1)$, $p(1) = p'(1)$, and $p(0) = p'(0) - 1$.

83. $\langle$**II − 8**$\rangle$ Show that the linear transformation

$$\begin{aligned} \varphi : \mathcal{S} &\to \mathcal{S} \\ (a_n)_n &\mapsto \left(\sum_{k=0}^{n} a_k \right)_n \end{aligned}$$

is invertible, and find its inverse $\varphi^{-1} : \mathcal{S} \to \mathcal{S}$.

84. $\langle$**II − 2**, **II − 3**, **II − 8**$\rangle$ Let $C \in M_n(\mathbb{R})$, let s be a non-zero real number, and define

$$\begin{aligned} \varphi : M_n(\mathbb{R}) &\to M_n(\mathbb{R}) \\ A &\mapsto \mathrm{Tr}(A)C - sA. \end{aligned}$$

(a) Show that if $\mathrm{Tr}(C) \neq s$, then φ is an isomorphism. *Hint: First show that φ is injective.*

(b) Now assume $\mathrm{Tr}(C) = s$.

(i) Show that $\mathrm{Ker}(\varphi) = \mathrm{Span}(C)$.

(ii) Show that $C \notin \mathrm{Image}(\varphi)$.

85. $\langle \mathbf{I-4}, \mathbf{II-5}, \mathbf{II-8} \rangle$ Let $c, s \in \mathbb{R}$ with $s \neq 0$, and define differentiable functions f and g by

$$\begin{aligned} f(x) &= e^{cx}\sin(sx) \\ g(x) &= e^{cx}\cos(sx). \end{aligned}$$

(a) Show that f and g are linearly independent.

(b) Let $V = \mathrm{Span}(f, g)$, and define

$$\begin{aligned} \varphi : V &\to V \\ h &\mapsto h'. \end{aligned}$$

Show that

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix},$$

where $\mathcal{B} = \{f, g\}$.

(c) Show that $[\varphi]_{\mathcal{B}}$ is invertible.

(d) Let $h \in V$. Using part (c), show that there is a unique function $H \in V$ such that $H' = h$. Denote the function H by $\int h$.

(e) Suppose $h = \lambda f + \mu g$ with $\lambda, \mu \in \mathbb{R}$. Again using part (c), express $\int h$ as a linear combination of f and g.

86. $\langle \mathbf{II-5}, \mathbf{II-7}, \mathbf{II-8} \rangle$ We continue with the notation of Question 85, taking $c = \cos(2\pi/9)$ and $s = \sin(2\pi/9)$.

(a) Show that $[\varphi]_{\mathcal{B}}$ is equal to the rotation matrix for rotation of the plane by angle $2\pi/9$ anticlockwise.

(b) Show that if $h \in V$, then $h^{(9)} = h$, where $h^{(9)}$ denotes the 9th derivative of h.

(c) Deduce that $\int h = h^{(8)}$.

87. $\langle \mathbf{II-5}, \mathbf{II-6}, \mathbf{II-8}, \mathbf{II-9} \rangle$ Let U and V be finite-dimensional vector spaces with bases $\mathcal{B}$ and $\mathcal{C}$ respectively, and suppose that $\varphi : U \to V$ is a surjective linear transformation. Suppose you also know that a row-echelon form of $[\varphi]_{\mathcal{C}\leftarrow\mathcal{B}}$ has a pivot in every column. Decide whether φ is invertible, explaining your answer carefully. You may use any fact given in the course notes.

88. $\langle \textbf{II}-\textbf{10} \rangle$ Decide which of the following vector spaces are isomorphic to each other. If you think, for example, that V_1 and V_4 are isomorphic, V_2 and V_3 are isomorphic, and V_5 and V_6 are isomorphic, then give your answer as

$$\{1,4\},\{2,3\},\{5,6\}.$$

You do not need to show your reasoning.

$$\begin{aligned}
V_1 &= \{(a_n)_n \in \mathcal{S} \mid a_n = 2a_{n-1} - a_{n-2} + 4a_{n-3} \text{ for all } n \geq 3\} \\
V_2 &= \mathbb{R}^8 \\
V_3 &= \{A \in M_2(\mathbb{R}) \mid \operatorname{Tr}(A) = 0\} \\
V_4 &= \operatorname{Span}(\sin, \cos) \\
V_5 &= M_{2,4}(\mathbb{R}) \\
V_6 &= \operatorname{Span}(f,g,h) \quad \text{where } f(x) = \sin^2(x),\ g(x) = \cos^2(x), \\
& \qquad\qquad\qquad\quad\ \text{and } h(x) = \cos(2x)
\end{aligned}$$

Hint: A basis for V_1 *is* $\{(1,0,0,4,\ldots),(0,1,0,-1,\ldots),(0,0,1,2,\ldots)\}$.

89. $\langle \textbf{II}-\textbf{10} \rangle$ Repeat Question 88 with the following vector spaces:

$$\begin{aligned}
V_1 &= \mathbb{R}^6 \\
V_2 &= \mathcal{P}_3 \\
V_3 &= \{(a_n)_n \in \mathcal{S} \mid a_n = a_{n-1} + a_{n-2} \text{ for all } n \geq 2\} \\
V_4 &= M_2(\mathbb{R}) \\
V_5 &= \mathcal{P}_1 \\
V_6 &= M_{2,3}(\mathbb{R})
\end{aligned}$$

90. $\langle \textbf{II}-\textbf{10} \rangle$ Define functions $f_1, f_2, f_3 \in \mathcal{F}$ by

$$f_1(x) = 1, \quad f_2(x) = \cos(x), \quad f_3(x) = \cos(x + \tfrac{\pi}{4}),$$

and let $V = \operatorname{Span}(f_1, f_2, f_3)$. Are the spaces $\mathbb{R}^3$ and V isomorphic to each other? If so, provide an isomorphism $\varphi : \mathbb{R}^3 \to V$. Otherwise, explain why not. In your answer, you may use facts stated in the course notes, but anything else that you assert or deduce must be justified.

91. $\langle \textbf{III}-\textbf{1} \rangle$ Find the eigenvalues and eigenspaces of each of the following matrices.

(a)

$$A = \begin{pmatrix} 8 & 6 \\ -3 & -1 \end{pmatrix}$$

(b)

$$B = \begin{pmatrix} 7 & -12 & -6 \\ 3 & -5 & -3 \\ -3 & 6 & 4 \end{pmatrix}$$

92. $\langle \mathbf{III-1} \rangle$ Decide whether the matrix

$$C = \begin{pmatrix} 5 & -3 & -9 \\ -12 & 5 & 18 \\ 6 & -3 & -10 \end{pmatrix}$$

is diagonalizable over $\mathbb{R}$. If it is, diagonalize it. Otherwise, explain why it is not diagonalizable. You may use the fact that $p_C(x) = (x+1)^2(x-2)$.

93. $\langle \mathbf{III-1} \rangle$ Decide whether the matrix

$$C = \begin{pmatrix} 4 & -1 & 1 \\ 2 & 1 & 1 \\ -10 & 5 & -3 \end{pmatrix}$$

is diagonalizable over $\mathbb{R}$. If it is, diagonalize it. Otherwise, explain why it is not diagonalizable. You may use the fact that $p_C(x) = (x+2)(x-2)^2$.

94. $\langle \mathbf{III-1} \rangle$ Repeat Question 93 for each of the following matrices:

(a)

$$A = \begin{pmatrix} 3 & 1 & -3 \\ -42 & -3 & -3 \\ -14 & -2 & 2 \end{pmatrix}, \qquad p_A(x) = (x-3)^2(x+4).$$

(b)

$$B = \begin{pmatrix} 4 & 5 & 1 \\ -2 & -3 & -1 \\ 3 & 5 & 1 \end{pmatrix}, \qquad p_B(x) = (x-2)(x^2+1).$$

95. $\langle \mathbf{III-1} \rangle$ For each of the matrices below, decide whether it is diagonalizable. If it is, diagonalize it. Otherwise, explain briefly why it is not diagonalizable. Take especial care over signs.

(a)

$$A = \begin{pmatrix} -1 & -5 & 5 \\ 5 & -1 & 5 \\ 5 & -5 & 9 \end{pmatrix}.$$

(b)

$$B = \begin{pmatrix} -1 & 5 & -5 \\ 5 & -1 & 5 \\ 5 & -5 & 9 \end{pmatrix}.$$

96. $\langle \mathbf{III-1} \rangle$ Decide whether the matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is diagonalizable, explaining your answer. Note the value of 1 in the $(1,2)$-entry of C.

97. $\langle \mathbf{III-1} \rangle$ A matrix $A \in M_5(\mathbb{R})$ has eigenvalues $-2, -1, 0, 1, 2$. Answer the following questions about A, explaining your answers.

    (a) Is A diagonalizable?

    (b) Is A invertible?

    (c) What are the eigenvalues of A^2? Take especial care in explaining your answer to this part.

98. $\langle \mathbf{III-1} \rangle$ Let $b > 1$ be a real number, and let

$$A = \begin{pmatrix} 1 & -b & b^2 & -b^3 \\ 0 & b & -b^2 & b^3 \\ 0 & 0 & b^2 & -b^3 \\ 0 & 0 & 0 & b^3 \end{pmatrix}.$$

    (a) Find $p_A(x)$.

    (b) How many eigenvalues does A have, and what is the geometric multiplicity of each one? If you wish, you may use the fact that the geometric multiplicity d_λ and the algebraic multiplicity m_λ always satisfy $1 \leq d_\lambda \leq m_\lambda$. **Caution:** Take care to justify why A has as many eigenvalues as you claim.

    (c) By considering the sum of the geometric multiplicities of the eigenvalues, show that A is diagonalizable.

99. $\langle \mathbf{III-2} \rangle$ Solve the system of first-order differential equations

$$\begin{aligned} f_1' &= 8f_1 + 2f_2 \\ f_2' &= -15f_1 - 3f_2 \end{aligned}$$

subject to the constraints $f_1(0) = 1$ and $f_2(0) = -1$.

100. $\langle \mathbf{III-2} \rangle$ Solve the system

$$\begin{aligned} f_1' &= 4f_1 + f_2 \\ f_2' &= -6f_1 - 3f_2 \end{aligned}$$

of differential equations subject to the constraints $f_1(0) = -3$ and $f_1'(0) = 16$. Note that the second constraint involves the derivative of f_1.

101. $\langle \mathbf{III-2} \rangle$ Solve the system of first-order differential equations

$$\begin{aligned} f_1' &= -3f_1 - 2f_2 + 2f_3 \\ f_2' &= 2f_1 + f_2 - 2f_3 \\ f_3' &= -2f_1 - 2f_2 + f_3 \end{aligned}$$

subject to the constraints $f_1(0) = 6$, $f_2(0) = -3$, $f_3(0) = 8$. You may use the fact that $P^{-1}AP = D$, where

$$\begin{aligned} A &= \begin{pmatrix} -3 & -2 & 2 \\ 2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix} \\ P &= \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \\ D &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

102. $\langle \mathbf{III-2} \rangle$ Solve the system of first-order differential equations

$$\begin{aligned} f_1' &= f_1 + f_2 + 4f_3 \\ f_2' &= 2f_1 + 2f_2 - 4f_3 \\ f_3' &= -2f_1 + f_2 + 7f_3 \end{aligned}$$

subject to the constraints $f_1(0) = 4$, $f_2(0) = 3$, $f_3(0) = 1$.

103. ⟨**III − 2**⟩ Consider the system of differential equations

$$\begin{aligned} f_1' &= -3f_1 + 2f_2 \\ f_2' &= \ \ -f_1 - 6f_2 \end{aligned} \tag{$*$}$$

(a) Find the general solution to the system in $(*)$.

(b) Find all the solutions to $(*)$ that satisfy both of the following conditions simultaneously:

$$\begin{aligned} f_1(0) &= f_2(0) \\ 7f_1'(0) &= f_2'(0) \end{aligned}$$

(If you find only the solution $f_1 = f_2 = 0$, then you have made a mistake somewhere.)

104. ⟨**III − 2**⟩ Consider the following system of differential equations:

$$\begin{aligned} f_1' &= \ \ \ f_1 + 3f_2 \\ f_2' &= -3f_1 - 5f_2 \end{aligned}$$

The matrix

$$A = \begin{pmatrix} 1 & 3 \\ -3 & -5 \end{pmatrix}$$

is not diagonalizable, but it does satisfy $P^{-1}AP = B$ where

$$P = \begin{pmatrix} 3 & 1 \\ -3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}.$$

Using this information, show that the functions g_1, g_2 defined by

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = P^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

satisfy $g_1' = -2g_1 + g_2$ and $g_2' = -2g_2$.

105. ⟨**III − 2**⟩ This question carries on from Question 104.

(a) If $b \in \mathbb{R}$ is constant, the solutions g to the differential equation $g'(x) = -2g(x) + be^{-2x}$ are given by $g(x) = (a + bx)e^{-2x}$ for $a \in \mathbb{R}$. Using this fact, solve the equations $g_1' = -2g_1 + g_2$ and $g_2' = -2g_2$ for g_1, g_2.

(b) Using part (a) and the fact that

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = P \begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

solve the equations $f_1' = f_1 + 3f_2$ and $f_2' = -3f_1 - 5f_2$ introduced in Question 104.

106. $\langle$**III−3**$\rangle$ Decide whether the matrix

$$B = \begin{pmatrix} 1 & 2 \\ -4 & 5 \end{pmatrix}$$

is diagonalizable over $\mathbb{C}$. If it is, diagonalize it. Otherwise, explain why it is not diagonalizable over $\mathbb{C}$.

107. $\langle$**III−3**$\rangle$ Show that the matrix

$$A = \begin{pmatrix} 3 & 4 & 2 \\ 4 & 3 & 2 \\ -4 & -9 & -3 \end{pmatrix}$$

is diagonalizable over $\mathbb{C}$, and diagonalize it. You may use the fact that $p_A(x) = (x+1)(x^2 - 4x + 5)$.

108. $\langle$**III−3**$\rangle$ Repeat Question 107 with the matrix

$$A = \begin{pmatrix} 1 & -3 & -2 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix},$$

which has characteristic polynomial $p_A(x) = (x-3)(x^2 - 2x + 2)$.

109. $\langle$**III−3**$\rangle$

(a) Show that if $z \in \mathbb{C}$ satisfies $z^2 + \sqrt{3}\,z + 3 = 0$, then $z^3 = 3\sqrt{3}$.

(b) Suppose that $A \in M_4(\mathbb{C})$ has characteristic polynomial

$$p_A(x) = (x^2 + 3)(x^2 + \sqrt{3}\,x + 3).$$

Find A^{12}. You may use the fact that A is a 4×4 matrix with 4 distinct complex eigenvalues and is therefore diagonalizable over $\mathbb{C}$.

110. $\langle$**III−4**$\rangle$ Let

$$A = \begin{pmatrix} 2 & -4 \\ 7 & 4 \end{pmatrix}.$$

(a) Find an invertible matrix $Q \in M_2(\mathbb{R})$, a positive real number s, and a rotation matrix R such that $Q^{-1}AQ = sR$.

(b) What is the angle of rotation of R, measured anticlockwise?

111. $\langle$**III**$-$**4**$\rangle$ Let

$$A = \begin{pmatrix} 1 & 16 \\ -10 & 9 \end{pmatrix}.$$

Find an invertible matrix $Q \in M_2(\mathbb{R})$, a positive real number s, and a rotation matrix R such that $Q^{-1}AQ = sR$. You do not need to find the angle of rotation of the rotation matrix R. *Hint: When finding the eigenvalues, you may find it helpful to know that* $576 = 24^2$.

112. $\langle$**III**$-$**4**$\rangle$ Let $A = \begin{pmatrix} 1 & 7 \\ -1 & 5 \end{pmatrix}$.

(a) Find an invertible matrix $Q \in M_2(\mathbb{R})$, a positive real number s, and a rotation matrix R such that $Q^{-1}AQ = sR$. What is the angle of rotation of R, measured anticlockwise?

(b) Use your answer to part (a) to find the smallest positive integer k such that $A^k \in \text{Span}(I)$. What is A^k in this case?

113. $\langle$**III**$-$**4**$\rangle$ Let $B = \begin{pmatrix} 1 & 4 \\ -5 & 5 \end{pmatrix}$.

(a) Find an invertible matrix $Q \in M_2(\mathbb{R})$, a positive real number s, and a rotation matrix R such that $Q^{-1}BQ = sR$.

(b) Show that there is no positive integer k such that $B^k \in \text{Span}(I)$. You may use the following fact: If R is a 2×2 rotation matrix with rational entries, and R is not one of the four matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then there is no positive integer m such that $R^m = I$. (For those interested in number theory, this is equivalent to the fact that the only roots of unity of the form $a + bi$ with $a, b \in \mathbb{Q}$ are $1, i, -1, -i$.)

114. $\langle$**III**$-$**4**$\rangle$ Suppose that $C \in M_2(\mathbb{R})$ has a non-real eigenvalue $\lambda = a + bi$, and suppose that the rotation matrix

$$R = \frac{1}{\sqrt{a^2 + b^2}} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

has anticlockwise angle of rotation $\pi/3$. If the top-left entry of C^3 is -64, find a and b. *Hint: Using Proposition 4.1 from Section III of the course notes, express*

C in terms of $|\lambda|$, R, and an invertible matrix $Q \in M_2(\mathbb{R})$, and then take cubes. Note that it will not be possible to find Q.

115. $\langle \mathbf{III-1} \rangle$

(a) Let

$$B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Find the eigenvalues of B and the eigenvalues of B^2, and show that in this case the eigenvalues of B^2 are exactly the squares of the eigenvalues of B.

(b) Find a matrix $C \in M_3(\mathbb{R})$ such that C^2 has a negative eigenvalue.

(c) Show that the following statement is false: If A is a square matrix with real entries, then the set of real eigenvalues of A^2 is equal to the set of squares of the real eigenvalues of A. *Hint: Use part (b).*

116. $\langle \mathbf{IV-1} \rangle$ Suppose that $A \in M_n(\mathbb{R})$ is a symmetric matrix satisfying $\mathbf{x}^{\mathrm{T}} A\mathbf{x} > 0$ for all non-zero vectors $\mathbf{x} \in \mathbb{R}^n$.

(a) Show that the pairing

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^{\mathrm{T}} A\mathbf{v}$$

defines an inner product on $\mathbb{R}^n$.

(b) What choice of matrix A gives the standard inner product (i.e., the dot product) on $\mathbb{R}^n$?

117. $\langle \mathbf{IV-1} \rangle$ Consider the pairing on $\mathbb{R}^2$ defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 - 5u_1v_2 - 5u_2v_1 - 8u_2v_2$$

for vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in $\mathbb{R}^2$.

(a) Which axioms of an inner product does this pairing satisfy?

(b) Does it define an inner product?

118. $\langle \mathbf{IV-1} \rangle$ Repeat Question 117 for the pairing defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = 16u_1v_1 + 12u_1v_2 + 12u_2v_1 + 9u_2v_2$$

for vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in $\mathbb{R}^2$. You may wish to consider the vector $\mathbf{u} = (-3, 4)$.

119. $\langle \mathbf{IV-1} \rangle$ Repeat Question 117 for the pairing defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = 13u_1v_1 - 11u_1v_2 - 7u_2v_1 + 10u_2v_2$$

for vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in $\mathbb{R}^2$. You may wish to use the fact that

$$13x^2 - 18xy + 10y^2 = (2x - 3y)^2 + (3x - y)^2$$

for all $x, y \in \mathbb{R}$.

120. $\langle \mathbf{IV-1} \rangle$ Consider the pairing on $\mathbb{R}^3$ defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = x_1y_1 + 5x_2y_2 + 2x_2y_3 + 2x_3y_2 + 25x_3y_3,$$

where $\mathbf{u} = (x_1, x_2, x_3)$ and $\mathbf{v} = (y_1, y_2, y_3)$. Decide whether $\langle \cdot, \cdot \rangle$ satisfies axiom (iv) of an inner product, as in Section IV−1 of the course notes. You may use the fact that

$$5a^2 + 4ab + 25b^2 = (2a + 3b)^2 + (a - 4b)^2$$

for all $a, b \in \mathbb{R}$.

121. $\langle \mathbf{IV-2} \rangle$ Endow $\mathcal{P}_2$ with the inner product defined by

$$\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$

Find all polynomials $p \in \mathcal{P}_2$ of norm 1 that are orthogonal to both x and x^2 simultaneously.

122. $\langle \mathbf{IV-2} \rangle$ Endow $\mathcal{P}_2$ with the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$

If $p \in \mathcal{P}_2$ satisfies $\langle p, 1 \rangle = 11$, $\langle p, x \rangle = -6$, and $\langle p, x^2 \rangle = 10$, what is p? Show all steps in your answer.

123. $\langle \mathbf{IV-2} \rangle$ Let $m, n \in \mathbb{Z}$.

(a) Prove the identity

$$\sin(mx)\sin(nx) = \frac{1}{2}\big(\cos((m-n)x) - \cos((m+n)x)\big).$$

(b) Show that if $|m| \neq |n|$, then the functions $f_m, f_n \in C[-\pi, \pi]$ defined by $f_m(x) = \sin(mx)$ and $f_n(x) = \sin(nx)$ are orthogonal in $C[-\pi, \pi]$.

124. $\langle \mathbf{IV-2} \rangle$ Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be pairwise orthogonal vectors in an inner product space $(V, \langle \cdot, \cdot \rangle)$, i.e., $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ for all i, j distinct. If

$$\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_k\mathbf{u}_k$$
$$\mathbf{w} = b_1\mathbf{u}_1 + \cdots + b_k\mathbf{u}_k,$$

prove that $\langle \mathbf{v}, \mathbf{w} \rangle = a_1b_1\|\mathbf{u}_1\|^2 + \cdots + a_kb_k\|\mathbf{u}_k\|^2$. Show your steps carefully.

125. $\langle \mathbf{IV-2} \rangle$ Define the following functions in the inner product space $C[-\pi, \pi]$:

$$f_1(x) = 1, \quad f_2(x) = \sin(x), \quad f_3(x) = \sin(2x),$$
$$g_1(x) = \sin(x) - \sin(2x), \quad g_2(x) = 1 + 2\sin(x) + 4\sin(2x).$$

(a) By evaluating suitable integrals, show that

$$\|f_1\| = \sqrt{2\pi}, \quad \|f_2\| = \sqrt{\pi}, \quad \|f_3\| = \sqrt{\pi}.$$

(b) Using the fact that f_1, f_2, f_3 are pairwise orthogonal, find all functions $h \in \text{Span}(f_1, f_2, f_3)$ such that $\langle h, g_1 \rangle = \langle h, g_2 \rangle = 0$. It will help to use Question 124.

(c) Among the functions h that you found in part (b), find all those that have norm 1.

126. $\langle \mathbf{IV-2}, \mathbf{IV-3} \rangle$

(a) Define an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{P}_2$ by

$$\langle p, q \rangle = p(-2)q(-2) + p(0)q(0) + p(2)q(2),$$

and let $q_1 = 1$, $q_2 = x$, $q_3 = x^2$. By applying the Gram–Schmidt process to the basis $\{q_1, q_2, q_3\}$ of $\mathcal{P}_2$, find an orthonormal basis for $\mathcal{P}_2$ with respect to this inner product.

(b) Repeat part (a) with the inner product

$$\langle p, q \rangle = \int_{-2}^{2} p(x)q(x)\, dx$$

on $\mathcal{P}_2$, using the same polynomials q_1, q_2, and q_3.

127. $\langle \mathbf{IV-2}, \mathbf{IV-3} \rangle$ Endow $\mathbb{R}^4$ with the standard inner product, i.e., the dot product, and let $U = \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \subseteq \mathbb{R}^4$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

By applying the Gram–Schmidt process to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of U, find an orthonormal basis for U.

128. $\langle \mathbf{IV}-\mathbf{3}\rangle$ Endow $\mathcal{P}_3$ with the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle p, q\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2) + p(3)q(3).$$

Find $\mathrm{proj}_{\mathcal{P}_2}(q)$ where $q = x^3$. You may use the fact that the subspace $\mathcal{P}_2$ of $\mathcal{P}_3$ has orthogonal basis $\{p_1, p_2, p_3\}$ where

$$p_1 = 1, \quad p_2 = x - \frac{3}{2}, \quad p_3 = x^2 - 3x + 1,$$

and you may also use the equalities $\|p_1\| = 2$, $\|p_2\| = \sqrt{5}$, and $\|p_3\| = 2$. (The significance of $\mathrm{proj}_{\mathcal{P}_2}(q)$ is that it is the polynomial in $\mathcal{P}_2$ that best approximates the cubic polynomial $q = x^3$ at the four values $x = 0, 1, 2, 3$.)

129. $\langle \mathbf{IV}-\mathbf{3}\rangle$ Endow $\mathcal{P}_3$ with the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1) + p(2)q(2),$$

and let $U = \{p \in \mathcal{P}_3 \mid p'(0) = 0\}$, a subspace of $\mathcal{P}_3$ with basis $\{q_1, q_2, q_3\}$ where $q_1 = 1$, $q_2 = x^2$, and $q_3 = x^3$.

(a) Apply the Gram–Schmidt process to q_1, q_2, q_3 to find an orthogonal basis $\{p_1, p_2, p_3\}$ for U with respect to the given inner product. Choose p_1, p_2, p_3 to all be monic.

(b) You should have found in part (a) that

$$p_1 = 1, \quad p_2 = x^2 - \frac{3}{2}, \quad p_3 = x^3 - \frac{20}{9}x^2 + \frac{4}{3}$$

(but do not use this information when answering part (a)).

Find $\mathrm{proj}_U(r)$ where $r = x^3 + x$. You may use the following information:

$$\langle p_1, r\rangle = 10, \qquad \langle p_2, r\rangle = 25, \qquad \langle p_3, r\rangle = \frac{76}{9},$$
$$\langle p_1, p_1\rangle = 4, \qquad \langle p_2, p_2\rangle = 9, \qquad \langle p_3, p_3\rangle = \frac{50}{9}.$$

(The significance of $\mathrm{proj}_U(r)$ is that it is the unique polynomial $p \in U$ that minimizes $\sum_{k=-1}^{2}(r(k) - p(k))^2$.)

130. $\langle \mathbf{IV}-\mathbf{3}, \mathbf{IV}-\mathbf{4}\rangle$ Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \\ -2 & -1 & -1 \end{pmatrix}.$$

(a) By performing Gram–Schmidt on the columns of A, find an orthonormal basis for $\mathrm{Col}(A)$.

(b) Use your answer to part (a) to find the totally positive QR-factorization of A.

131. $\langle \mathbf{IV-4} \rangle$ Find the totally positive QR-factorization of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hint: Use Question 127.

132. $\langle \mathbf{IV-4} \rangle$ Find the totally positive QR-factorization of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

133. $\langle \mathbf{IV-2} \rangle$ Let u and v be orthogonal vectors in an inner product space $(V, \langle \cdot, \cdot \rangle)$. Show directly from the axioms of an inner product that

$$\mathrm{dist}(u, v) = \sqrt{\|u\|^2 + \|v\|^2}.$$

134. $\langle \mathbf{IV-2}, \mathbf{IV-3} \rangle$ Endow $\mathbb{R}^3$ with the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2x_1y_1 + x_1y_2 + x_2y_1 + 5x_2y_2 + x_3y_3,$$

where $\mathbf{u} = (x_1, x_2, x_3)$ and $\mathbf{v} = (y_1, y_2, y_3)$. (This does indeed define an inner product, and you may assume this fact here.) In this inner product space, apply the Gram–Schmidt process to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$ to find an orthonormal basis for $\mathbb{R}^3$ with respect to this inner product.

135. $\langle \mathbf{IV-5} \rangle$ Let

$$\mathcal{S}' = \{(a_n)_n \in \mathcal{S} \mid \text{there exists } C > 0 \text{ such that, for all } n \geq 1,\ |a_n| \leq C/n\}.$$

This is a subspace of $\mathcal{S}$, and we may endow it with the inner product

$$\langle \alpha, \beta \rangle = \sum_{n=0}^{\infty} a_n b_n,$$

where $\alpha = (a_n)_n$ and $\beta = (b_n)_n$. Now let

$$\alpha = (1/(n+3))_n = \left(\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right) \in \mathcal{S}'$$
$$\beta = (1/(n+1))_n = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) \in \mathcal{S}'.$$

(a) Calculate $\|\alpha\|^2$ and $\|\beta\|^2$. You may use the fact that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(b) Using the Cauchy–Schwarz inequality, deduce from your calculations in part (a) that

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \leq \frac{\pi^2}{6}\sqrt{1-\frac{15}{2\pi^2}}.$$

136. $\langle \mathbf{IV}-\mathbf{5} \rangle$ Fix real numbers $a < b$, let M be a positive real number, and define

$$B(M) = \left\{ f \in C[a,b] \;\middle|\; \int_a^b f(x)^2\,dx \leq M \right\}.$$

Show that if $f, g \in B(M)$, then $tf + (1-t)g \in B(M)$ for all $t \in [0,1]$. *Hint: Apply the triangle inequality to the functions* tf *and* $(1-t)g$.

137. $\langle \mathbf{IV}-\mathbf{5} \rangle$ Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space, and let $\mathbf{u}, \mathbf{v}$ be vectors in V. Recall the triangle inequality, which says that $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$. Under the assumption that $\mathbf{u} \neq \mathbf{0}$, show that $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$ if and only if $\mathbf{v} = c\mathbf{u}$ for some non-negative scalar c. Your proof should be algebraic; a diagram is not sufficient.

138. $\langle \mathbf{IV}-\mathbf{1},\ \mathbf{IV}-\mathbf{5} \rangle$ A *metric space* consists of a non-empty set X together with a function $d : X \times X \to \mathbb{R}$ satisfying the following:

(i) For all $x, y \in X$, $d(x,y) = 0$ if and only if $x = y$.

(ii) For all $x, y \in X$, $d(x,y) = d(y,x)$.

(iii) For all $x, y, z \in X$, $d(x,z) \leq d(x,y) + d(y,z)$.

The function d is called a *metric* (or *distance function*) on X.

Show that if $(V, \langle\cdot,\cdot\rangle)$ is an inner product space and $d : V \times V \to \mathbb{R}$ is defined by $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$, then d is a metric on V, so that V has the structure of a metric space.

139. $\langle \mathbf{IV-5}, \mathbf{IV-6} \rangle$ In this question, do not perform any differentiation; use only the Cauchy–Schwarz inequality.

   (a) Find the maximum value of $3x_1 + 4x_2 + 5x_3$ subject to the constraint $x_1^2 + x_2^2 + x_3^2 = 1$, and find the unique (x_1, x_2, x_3) where the maximum is attained.

   (b) Find the minimum value of $x_1^2 + x_2^2 + x_3^2$ subject to the constraint $5x_1 + 12x_2 + 13x_3 = 26$, and find the unique (x_1, x_2, x_3) where the minimum is attained.

140. $\langle \mathbf{IV-5}, \mathbf{IV-6} \rangle$ In this question, do not perform any differentiation; use only the Cauchy–Schwarz inequality.

   (a) Find the maximum value of $-2x_1 + x_2 - x_3$ subject to the constraint $x_1^2 + x_2^2 + x_3^2 = 1$, and find the unique (x_1, x_2, x_3) where the maximum is attained.

   (b) Find the minimum value of $x_1^2 + x_2^2 + x_3^2$ subject to the constraint $4x_1 + x_2 + 5x_3 = 2$, and find the unique (x_1, x_2, x_3) where the minimum is attained.

141. $\langle \mathbf{IV-5}, \mathbf{IV-6} \rangle$ In this question, do not perform any differentiation; use only the Cauchy–Schwarz inequality.

   (a) Find the maximum value of $\frac{1}{3}x_1 + \frac{1}{2}x_2 + x_3$ subject to the constraint $x_1^2 + x_2^2 + x_3^2 = 1$, and find the unique (x_1, x_2, x_3) where the maximum is attained.

   (b) Find $a < 0$ such that the minimum of $x_1^2 + x_2^2 + x_3^2$ subject to the constraint $2x_1 + 3x_2 + 6x_3 = a$ is 1. Find also the point (x_1, x_2, x_3) where the minimum occurs for that value of a.

142. $\langle \mathbf{IV-5} \rangle$ Define $f \in C[0, \pi/4]$ by $f(x) = \sin(x) - \cos(x)$. Among all functions $g \in C[0, \pi/4]$ such that

$$\int_0^{\pi/4} f(x)g(x)\,dx = \frac{1}{2},$$

find the unique one for which $\int_0^{\pi/4} g(x)^2\,dx$ is least. What is $\int_0^{\pi/4} g(x)^2\,dx$ for that g? Explain your answer carefully.

143. $\langle \mathbf{IV-5} \rangle$

   (a) Find the unique polynomial $p \in \mathcal{P}_2$ such that $p(1) = p(3) = 1$ and $p(2) = -1$.

   (b) Among all polynomials $q \in \mathcal{P}_2$ such that $q(1)^2 + q(2)^2 + q(3)^2 = 1$, find the unique one for which $q(1) - q(2) + q(3)$ is greatest. Do not perform any differentiation.

144. $\langle \mathbf{IV-3}, \mathbf{IV-7} \rangle$ Orthogonally diagonalize the real symmetric matrix

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{pmatrix},$$

whose eigenvalues are -1 and 5.

145. $\langle \mathbf{IV-3}, \mathbf{IV-7} \rangle$ Orthogonally diagonalize the real symmetric matrix

$$A = \begin{pmatrix} 11 & -2 & 6 \\ -2 & 11 & 6 \\ 6 & 6 & -5 \end{pmatrix}.$$

You may use the fact that A has eigenvalues -9 and 13.

146. $\langle \mathbf{IV-8} \rangle$ Find the quadratic form $f : \mathbb{R}^3 \to \mathbb{R}$ associated to the real symmetric matrix

$$\begin{pmatrix} -1 & 2 & 3 \\ 2 & -1 & 5 \\ 3 & 5 & -1 \end{pmatrix}.$$

147. $\langle \mathbf{IV-8} \rangle$ Find the real symmetric 3×3 matrix A corresponding to the quadratic form given by

$$f(x_1, x_2, x_3) = x_1^2 - x_2^2 + 2x_3^2 + x_1x_2 - 3x_1x_3 + 5x_2x_3.$$

148. $\langle \mathbf{IV-8} \rangle$ Let

$$B = \begin{pmatrix} 2 & 5 & 10 \\ 1 & 4 & 3 \\ 8 & 7 & 6 \end{pmatrix}.$$

Find the real symmetric 3×3 matrix A corresponding to the quadratic form

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R} \\ \mathbf{x} &\mapsto \mathbf{x}^{\mathrm{T}} B \mathbf{x}. \end{aligned}$$

149. $\langle \mathbf{IV-8} \rangle$ Suppose that the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$$

is invertible, and let

$$A = \begin{pmatrix} a^2 - c^2 & ab - cd \\ ab - cd & b^2 - d^2 \end{pmatrix}.$$

Show that the quadratic form

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ \mathbf{x} &\mapsto \mathbf{x}^{\mathrm{T}} A\mathbf{x} \end{aligned}$$

is indefinite.

150. $\langle$**IV − 8**, **IV − 9**, **IV − 10**$\rangle$ For each of the following quadratic forms f, find the maximum and minimum of $f(\mathbf{x})$ on unit vectors $\mathbf{x}$, and decide whether f is positive definite, negative definite, non-negative definite (but not positive definite), non-positive definite (but not negative definite), or indefinite.

    (a) $f(x_1, x_2) = x_1^2 + 3x_1x_2 + 3x_2^2$.

    (b) $f(x_1, x_2) = x_1^2 - 6x_1x_2 + 9x_2^2$.

    (c) $f(x_1, x_2, x_3) = x_1^2 - 3x_2^2 - x_3^2 + 2x_1x_3 - 2x_2x_3$. The factorization $x^3 + 3x^2 - 3x - 5 = (x + 1)(x^2 + 2x - 5)$ will help.

151. $\langle$**IV − 7**, **IV − 8**, **IV − 9**$\rangle$ Orthogonally diagonalize the quadratic form

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\mapsto 2x_1^2 + 4x_1x_2 - x_2^2. \end{aligned}$$

Decide whether f is positive definite, negative definite, non-negative definite, non-positive definite, or indefinite.

152. $\langle$**IV − 7**, **IV − 8**, **IV − 9**$\rangle$ Repeat Question 151 for the quadratic form

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\mapsto 3x_1^2 + 12x_1x_2 - 2x_2^2. \end{aligned}$$

153. $\langle$**IV − 7**, **IV − 8**, **IV − 9**$\rangle$ Repeat Question 151 for the quadratic form

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R} \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &\mapsto 3x_2^2 + 4x_1x_2 - 2x_1x_3 - 4x_2x_3. \end{aligned}$$

You may wish to look at Question 144.

154. $\langle\mathbf{IV-8}\rangle$ Consider the quadratic form

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R} \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &\mapsto 5x_1^2 - 2(x_1 + 2x_2)^2 + 4(x_1 + 2x_2 + 3x_3)^2. \end{aligned}$$

Find $P, D \in M_3(\mathbb{R})$, with D diagonal, such that

$$f(\mathbf{x}) = (P^{\mathrm{T}}\mathbf{x})^{\mathrm{T}} D (P^{\mathrm{T}}\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^3$.

155. $\langle\mathbf{IV-8}\rangle$ Consider the quadratic form

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\mapsto 4(x_1 + 3x_2)^2 + 3(2x_1 - x_2)^2. \end{aligned}$$

(a) Find matrices $P, D \in M_2(\mathbb{R})$, with D diagonal, such that

$$f(\mathbf{x}) = (P^{\mathrm{T}}\mathbf{x})^{\mathrm{T}} D (P^{\mathrm{T}}\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^2$. The matrix P need not be orthogonal. *Hint: Begin by writing the expression* $4(x_1 + 3x_2)^2 + 3(2x_1 - x_2)^2$ *in the form* $\mu_1 y_1^2 + \mu_2 y_2^2$ *for some new variables* y_1, y_2. *How are the vectors* (x_1, x_2) *and* (y_1, y_2) *related?*

(b) If $A \in M_2(\mathbb{R})$ is the symmetric matrix associated to f, show that $A = PDP^{\mathrm{T}}$. You may use the fact that if $B, C \in M_n(\mathbb{R})$ are symmetric and $\mathbf{x}^{\mathrm{T}}B\mathbf{x} = \mathbf{x}^{\mathrm{T}}C\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, then $B = C$.

156. $\langle\mathbf{IV-8}, \mathbf{IV-9}\rangle$ Suppose that $A \in M_4(\mathbb{R})$ is symmetric and has negative determinant. Decide whether the quadratic form

$$\begin{aligned} f : \mathbb{R}^4 &\to \mathbb{R} \\ \mathbf{x} &\mapsto \mathbf{x}^{\mathrm{T}}A\mathbf{x} \end{aligned}$$

is positive definite, negative definite, non-negative definite, non-positive definite, or indefinite. Justify your answer.

157. $\langle\mathbf{IV-10}\rangle$

(a) For each of the quadratic forms $f : \mathbb{R}^2 \to \mathbb{R}$ in Questions 151 and 152, find the maximum value and minimum value of $f(\mathbf{x})$ subject to the constraint $\|\mathbf{x}\| = 1$.

(b) For the quadratic form $f : \mathbb{R}^3 \to \mathbb{R}$ in Question 153, find the maximum value and minimum value of $f(\mathbf{x})$ subject to the constraint $\|\mathbf{x}\| = 1$.

158. $\langle \mathbf{IV-7}, \mathbf{IV-8}, \mathbf{IV-9}, \mathbf{IV-10} \rangle$

(a) Orthogonally diagonalize the real symmetric matrix

$$A = \begin{pmatrix} -2 & 2 & 6 \\ 2 & -5 & 3 \\ 6 & 3 & 3 \end{pmatrix}.$$

You may use the fact that the eigenvalues of A are -6 and 8.

(b) Among all $\mathbf{x} \in \mathbb{R}^3$ of norm 1, find the two such that $\mathbf{x}^{\mathrm{T}} A \mathbf{x}$ is greatest. Justify your answer briefly with reference, by number, to a result from the course notes.

(c) A *linear form* is a function $L : \mathbb{R}^n \to \mathbb{R}$ given by $L(\mathbf{x}) = \mathbf{w} \cdot \mathbf{x}$ for some $\mathbf{w} \in \mathbb{R}^n$ (in other words, a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$). Express the quadratic form $f(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} A \mathbf{x}$ in the form

$$f(\mathbf{x}) = \mu_1 L_1(\mathbf{x})^2 + \mu_2 L_2(\mathbf{x})^2 + \mu_3 L_3(\mathbf{x})^2,$$

where L_1, L_2, L_3 are linear forms on $\mathbb{R}^3$ and $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$. *Hint: See the example on p. 72 of the course notes, running into p. 73.*

159. $\langle \mathbf{IV-8}, \mathbf{IV-10} \rangle$ Let C be the cone in $\mathbb{R}^3$ defined by $x_1^2 + x_2^2 = x_3^2$. By applying Proposition 10.1 in Section IV to an appropriate quadratic form, find the maximum and minimum of $x_1 x_2 / x_3^2$ for $(x_1, x_2, x_3) \in C \smallsetminus \{\mathbf{0}\}$. *Hint: Let $y_1 = x_1/x_3$ and $y_2 = x_2/x_3$.*

160. $\langle \mathbf{IV-8}, \mathbf{IV-10} \rangle$ Let a be a non-negative real number, and consider the quadratic form

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R} \\ (x_1, x_2, x_3) &\mapsto x_1^2 + 2x_2^2 + \frac{1}{2} x_3^2 + a x_1 x_2. \end{aligned}$$

(a) Find a such that the minimum of $f(\mathbf{x})$ subject to the constraint $\|\mathbf{x}\| = 1$ is 0.

(b) Find a such that the difference between the maximum and the minimum of $f(\mathbf{x})$ subject to $\|\mathbf{x}\| = 1$ is

(i) 4.

(ii) 7/4.