Calculus I – v 1.2

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About these notes

These notes provide the core material for a first course in calculus taught at the University of Alberta. The structure follows that created by Vincent Bouchard, and I am grateful to Saeed Rastgoo for providing applications to physical problems as well as a few nice examples of the theory. Vincent Bouchard's course is available at

https://sites.ualberta.ca/~vbouchar/MATH144/notes.html.

Contents

1 Review

1.1 Review of functions

A *function* $f : \mathbb{R} \to \mathbb{R}$ is a rule that assigns to each $x \in \mathbb{R}$ a real number $f(x)$. For example, one function is $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - x + 1$. Then

$$\begin{aligned} f(1) &= 1^2 - 1 + 1 = 1, \\ f(2) &= 2^2 - 2 + 1 = 3, \\ f(-1/2) &= \frac{1}{4} + \frac{1}{2} + 1 = \frac{7}{4}, \end{aligned}$$

and so on. Other examples are $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \cos(x)$ and $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = e^x$.

1.2 Domain of a function

Example 1.1 Consider a function f satisfying

$$f(x) = \frac{x^2}{x^2 - 1}.$$

Observe that the formula on the right is not defined when $x = \pm 1$, because $x^2 - 1 = 0$ in that case, so we take the "domain" of f to exclude ± 1. Let

$$D = \mathbb{R} \smallsetminus \{1, -1\}$$

(notation for the set obtained by removing 1 and -1 from the set of real numbers). Then we may define $f : D \to \mathbb{R}$ by $f(x) = x^2/(x^2 - 1)$. The set D is called the *domain* of f.

Example 1.2 Consider $\ln(x)$, the natural logarithm of x. This is defined for $x > 0$ (but not otherwise in this course), so if

$$D = (0, \infty),$$

we may define a function $f : D \to \mathbb{R}$ by $f(x) = \ln(x)$. The domain of f is D.

Some other common types of domain are

$$\begin{aligned} D &= \{x \in \mathbb{R} \mid 3 \leq x < 4\} = [3, 4), \\ D &= \{x \in \mathbb{R} \mid x \leq -2\} = (-\infty, -2], \\ D &= \{x \in \mathbb{R} \mid x < -4 \text{ or } x > 9\} = (-\infty, -4) \cup (9, \infty) = \mathbb{R} \smallsetminus [-4, 9]. \end{aligned}$$

1.3 Range of a function

The *range* of a function $f : D \to \mathbb{R}$ is the set of values $f(x)$ as x runs through D, i.e.,

$$\{f(x) \mid x \in D\}.$$

Example 1.3 Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2+2$. Here, every number $y = f(x)$ is greater than or equal to 2, but conversely every number $y \geq 2$ is of the form $y = x^2 + 2 = f(x)$, because we may take $x = \sqrt{y-2}$. Thus, the range of f is

$$\{f(x) \mid x \in \mathbb{R}\} = \{y \in \mathbb{R} \mid y \geq 2\} = [2, \infty).$$

Example 1.4 The range of the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin(x)$ is $[-1, 1]$: Every number $y = f(x)$ lies in the interval $[-1, 1]$, and conversely every real number y in this interval is equal to $\sin(x) = f(x)$ for some $x \in \mathbb{R}$.

Example 1.5 The range of the function $f : \mathbb{R}_{>0} \to \mathbb{R}$ given by $f(x) = \ln(x)$ is all of $\mathbb{R}$: Every real number y is the logarithm of some positive real number x.

1.4 Lines

A line in the (x, y)-plane has equation

$$ax + by = d$$

where a, b, d are constants such that a and b are not both zero.

- If $b = 0$, then $x = d/a$, constant, so the line is vertical.
- If $a = 0$, then $y = d/b$, constant, so the line is horizontal.

As long as the line is not vertical, it has an equation of the form

$$y = mx + c$$

where

- m is called the *slope*,
- c is called the *y-intercept*.

If (x_1, y_1) and (x_2, y_2) are two points in the (x, y)-plane with $x_1 \neq x_2$, then the line through these points has equation

$$y - y_1 = m(x - x_1)$$

where

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

the slope.

Example 1.6 Find the equation of the line through the points $(3, 7)$ and $(-2, 4)$.

Solution: Let us take $(x_1, y_1) = (3, 7)$ and $(x_2, y_2) = (-2, 4)$. Then the equation is

$$y - y_1 = m(x - x_1) \quad \text{where } m = \tfrac{y_2-y_1}{x_2-x_1} = \tfrac{4-7}{-2-3} = \tfrac{3}{5}$$
$$\text{i.e.,} \quad y - 7 = \frac{3}{5}(x-3) = \frac{3}{5}x - \frac{9}{5},$$
$$\text{i.e.,} \quad y = \frac{3}{5}x - \frac{9}{5} + 7 = \frac{3}{5}x + \frac{26}{5}.$$

1.5 Essential trigonometric identities

The following three trigonometric identities should be memorized:

$$\sin^2(x) + \cos^2(x) = 1, \tag{1.1}$$
$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y), \tag{1.2}$$
$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y). \tag{1.3}$$

Many other trigononmetric identities are derivable from these three. For example, taking $y = x$ in (1.2) and (1.3) gives

$$\sin(2x) = 2\sin(x)\cos(x),$$
$$\cos(2x) = \cos^2(x) - \sin^2(x).$$

Then, using (1.1) to replace either $\cos^2(x)$ or $\sin^2(x)$ in this last equation, we obtain either

$$\cos(2x) = \cos^2(x) - (1 - \cos^2(x)) = 2\cos^2(x) - 1$$
$$\text{or} \quad \cos(2x) = (1 - \sin^2(x)) - \sin^2(x) = 1 - 2\sin^2(x).$$

2 Preview of calculus

2.1 Studying motion

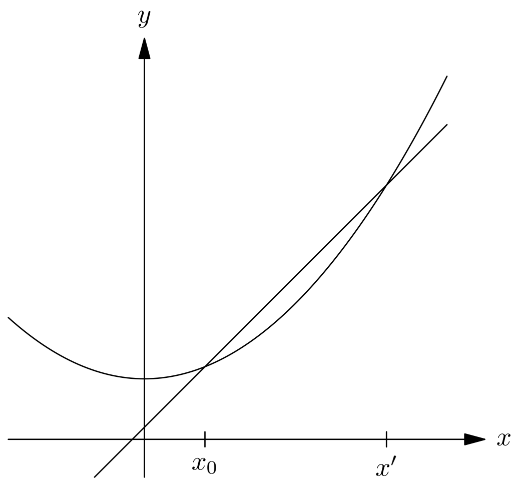
Rates of change

Consider a function $f : D \to \mathbb{R}$ and an interval $[x_1, x_2] \subseteq D$. The *average rate of change* of f over $[x_1, x_2]$ is

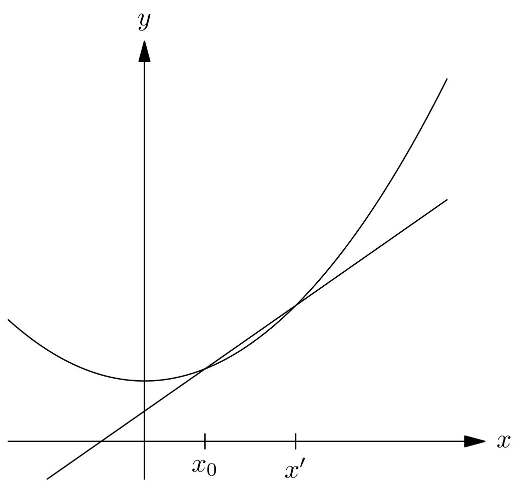
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

i.e., the vertical change divided by the horizontal change.

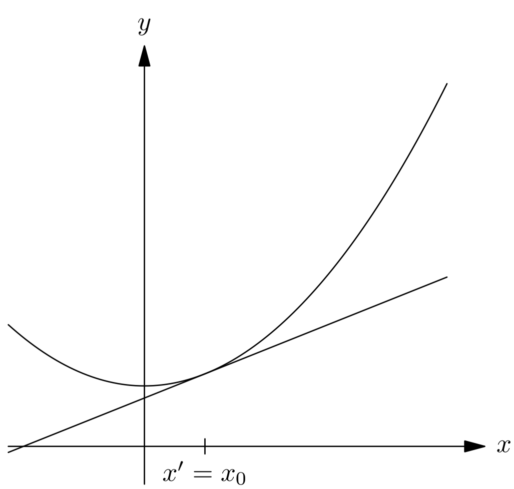
Often, the *instantaneous rate of change* at a particular $x = x_0$ is more important to know. For this, we consider the average rate of change over intervals $[x_0, x']$ as x' gets closer and closer to x_0. When x' is very close to x_0, the average rate of change is very close to the instantaneous rate. In the *limit*, as x' approaches x_0, we obtain the actual instantaneous rate. Pictorially, the instantaneous rate of change is the slope of the *tangent line* to the graph of f where $x' = x_0$. (A tangent line is one that just touches the curve at some point.) This idea is illustrated in the following three diagrams, the first one showing a secant line for some choice of x', the second showing a secant line with x' taken closer to x_0, and the third showing the tangent line at x_0:



y
x
x_0
x'



y
x
x_0
x'



y
x
$x' = x_0$

Formally, the instantaneous rate of change of f at x_0 is the limit of

$$\frac{f(x) - f(x_0)}{x - x_0}$$

as x tends to x_0. We will make this notion precise in a later lecture.

Example 2.1 Suppose that a vehicle is moving along the real line and that its position at time t is $f(t) = 3 + t^2$. The instantaneous rate of change in this setting is simply the velocity of the vehicle. Graphing this function, we see that the slope of the tangent line at time t increases as $t \geq 0$ increases. Thus, the vehicle is speeding up over time.

Example 2.2 Suppose, now, that the vehicle's position at time t is $f(t) = 5 + 4t - t^2 = 9 - (t - 2)^2$. This time, when we graph the function, we see that the tangent line is horizontal when $t = 2$. Thus, at this point in time, the vehicle's velocity is zero, i.e., the vehicle is stationary. Note also that for $t > 2$, the slope is negative, so the vehicle's velocity is negative, i.e., the vehicle is moving backwards.

We will see soon how to calculate instantaneous rates of change precisely, rather than simply referring to sketches of graphs.

Distance travelled

Recall that if speed is constant, then

$$\text{distance} = \text{speed} \times \text{change in time}.$$

Technically, we should consider velocity instead of speed:

Positive velocity v:	Object moves rightwards along the real line with speed $\lvert v\rvert$.
Negative velocity v:	Object moves leftwards along the real line with speed $\lvert v\rvert$.
Zero velocity v:	Object is stationary.

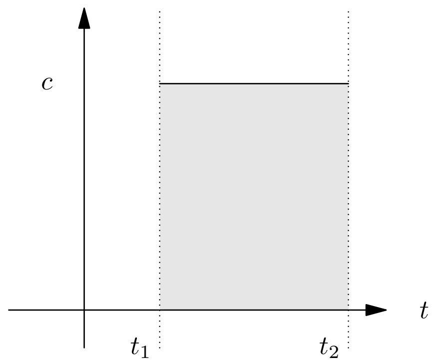
Correspondingly, we consider *displacement* instead of distance.

Positive displacement d:	Object's new position is a distance $\lvert d\rvert$ to the right of its previous position.
Negative displacement d:	Object's new position is a distance $\lvert d\rvert$ to the left of its previous position.
Zero displacement d:	Object is in the same position as before.

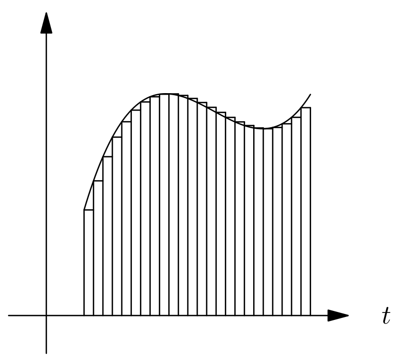
Then

$$\text{displacement} = \text{velocity} \times \text{change in time}.$$

If an object has constant velocity $v(t) = c$ and travels during the time interval $[t_1, t_2]$, then the displacement is $c(t_2 - t_1)$. We can think of this as the area under the horizontal line at height c from $t = t_1$ to $t = t_2$:



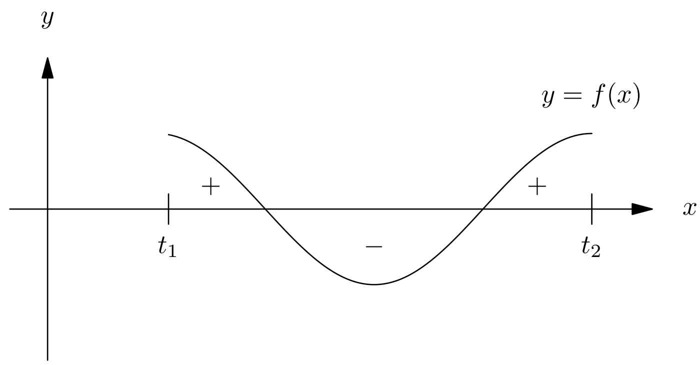
If the velocity is not constant, we approximate the object's motion by imagining that it moves at a constant speed for a tiny duration, then at another constant speed for another tiny duration, and so on. Then the graph of this approximate motion is represented by steps, and the approximate displacement, according to the above rule, is the sum of the areas of the rectangles under the steps.



This suggests that the overall displacement from the initial time to the final time is the area under the velocity curve, and this is indeed the case.

Key points:

- The displacement of an object between time t_1 and time t_2 is the area under the velocity curve between those points in time.
- The portions of the graph where the velocity is negative (i.e., the curve is below the horizontal axis) count negatively, as in the illustration below.



2.2 Tangent lines

A precise calculation of an instantaneous rate of change

Key point: The slope of the line through points (x_1, y_1) and (x_2, y_2), if $x_1 \neq x_2$, is

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Example 2.3 Consider again a vehicle moving along the real line in such a way that its position at time t is

$$f(t) = 5 + 4t - t^2.$$

Let us find the instantaneous rate of change (velocity) at time $t = 3$. For this, we consider the *secant line* through the points

$$(3, f(3)) = (3, 8) \quad \text{and} \quad (3 + h, f(3 + h))$$

where h is a very small change in time. This slope, by the formula above, is

$$\begin{aligned}\frac{f(3+h) - f(3)}{(3+h) - 3} &= \frac{(5 + 4(3+h) - (3+h)^2) - 8}{h} \\ &= \frac{5 + 12 + 4h - 9 - 6h - h^2 - 8}{h} = \frac{-2h - h^2}{h} = -2 - h.\end{aligned}$$

As h approaches zero, the slope approaches -2. Thus, the velocity at time $t = 3$ is -2. (The object is moving left along the real line with speed 2 at this point in time.)

Example 2.4 Consider the same object as before (same position function f). We now find the time (or times) t at which the object is stationary, i.e., when the velocity is zero. To achieve this, leave t unknown for now and consider the secant line through points

$$(t, f(t)) \quad \text{and} \quad (t + h, f(t + h))$$

where h is again a very small change in time. The slope of the secant line is

$$\begin{aligned}\frac{f(t+h)-f(t)}{(t+h)-t} &= \frac{(5+4(t+h)-(t+h)^2)-(5+4t-t^2)}{(t+h)-t} \\ &= \frac{5+4t+4h-t^2-2th-h^2-5-4t+t^2}{h} \\ &= \frac{(4-2t)h-h^2}{h} = 4-2t-h.\end{aligned}$$

As h approaches zero, the slope approaches $4-2t$, so the velocity at time t is $4-2t$. Therefore, the vehicle is stationary if and only if $4-2t=0$, i.e., $t=2$.

Equations of tangent lines

Let $f : D \to \mathbb{R}$ be a function, let $x_1 \in D$, and suppose that the slope

$$\frac{f(x_1+h)-f(x_1)}{(x_1+h)-x_1} = \frac{f(x_1+h)-f(x_1)}{h}$$

of the secant line through $(x_1, f(x_1))$ and $(x_1+h, f(x_1+h))$ approaches some value m as h approaches zero. Then the tangent line to the graph of f at $(x_1, f(x_1))$ is the line through that point with slope m, so the equation is

$$y - f(x_1) = m(x - x_1).$$

(Slope m is equal to change in y over change in x.)

Example 2.5 Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x^2 - 3x + 5$. Find the tangent line to the graph of f at $(1, f(1))$.

Solution: First, we find the slope m at this point, beginning with the slope of a secant line:

$$\begin{aligned}\frac{f(1+h)-f(1)}{h} &= \frac{(2(1+h)^2-3(1+h)+5)-4}{h} \\ &= \frac{h+2h^2}{h} = 1+2h,\end{aligned}$$

so $m = 1$ (let h approach 0). Hence, the desired tangent line has equation

$$\begin{aligned} y-4 &= m(x-1) = x-1, \\ \text{i.e.,} \quad y &= x+3.\end{aligned}$$

Example 2.6 Repeat for the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 4x + 2$, taking the point this time to be $(-3, f(-3))$.

Solution: The secant line between $t = -3$ and $t = -3 + h$ has slope

$$\frac{f(-3+h)-f(-3)}{h} = \frac{((-3+h)^2+4(-3+h)+2)-(-1)}{h}$$

$$= \frac{9 - 6h + h^2 - 12 + 4h + 2 + 1}{h}$$
$$= \frac{-2h + h^2}{h} = -2 + h.$$

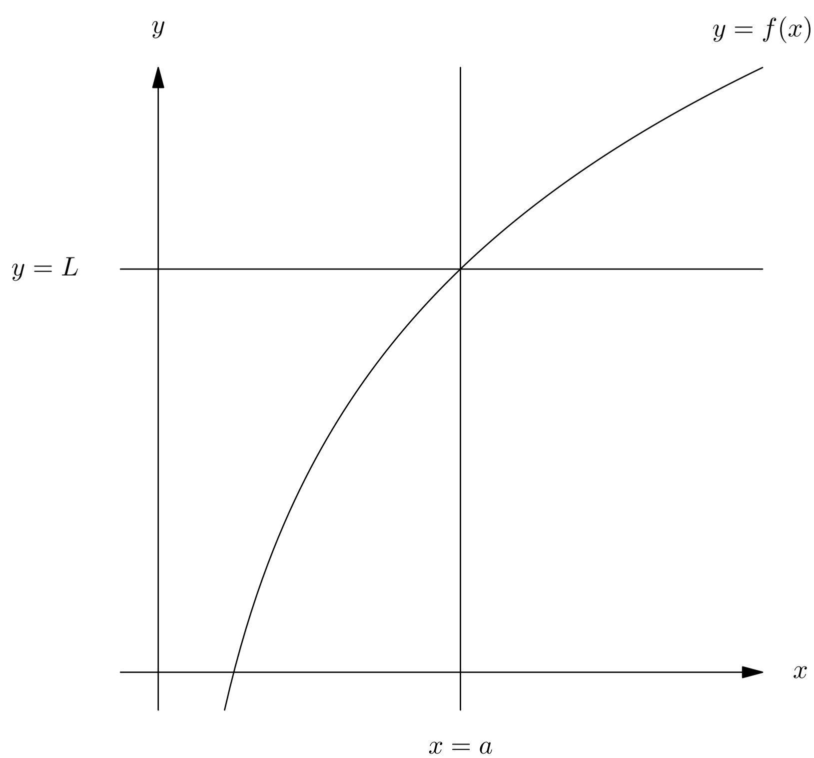
The slope of the tangent line, obtained by letting h approach 0, is therefore $m = -2$. Hence, the equation of the tangent line is

$$y - f(-3) = m(x - (-3)),$$
$$\text{i.e.,} \quad y + 1 = -2(x + 3),$$
$$\text{i.e.,} \quad y = -2x - 7.$$

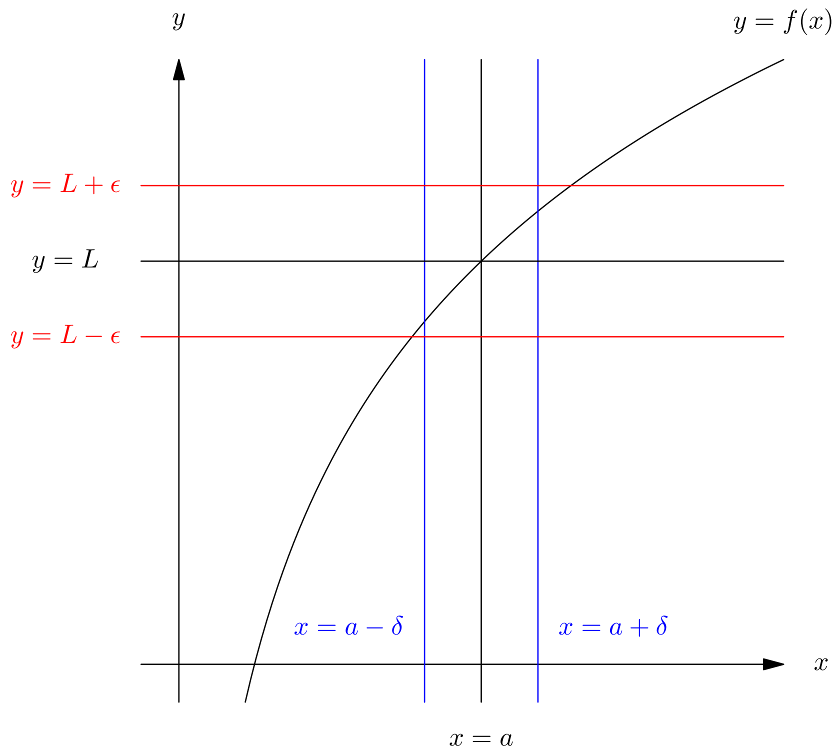
3 Limits

3.1 Introduction to limits

Consider some function f with a graph as follows:



Imagine bringing x close to a and looking at the behaviour of the function as that happens. More precisely, we consider horizontal "train tracks", in red below, either side of the line $y = L$, say, $y = L + \epsilon$ and $y = L - \epsilon$ where $\epsilon > 0$:



In the above, as long as x is between $a - \delta$ and $a + \delta$, we have

$$L - \epsilon < f(x) < L + \epsilon.$$

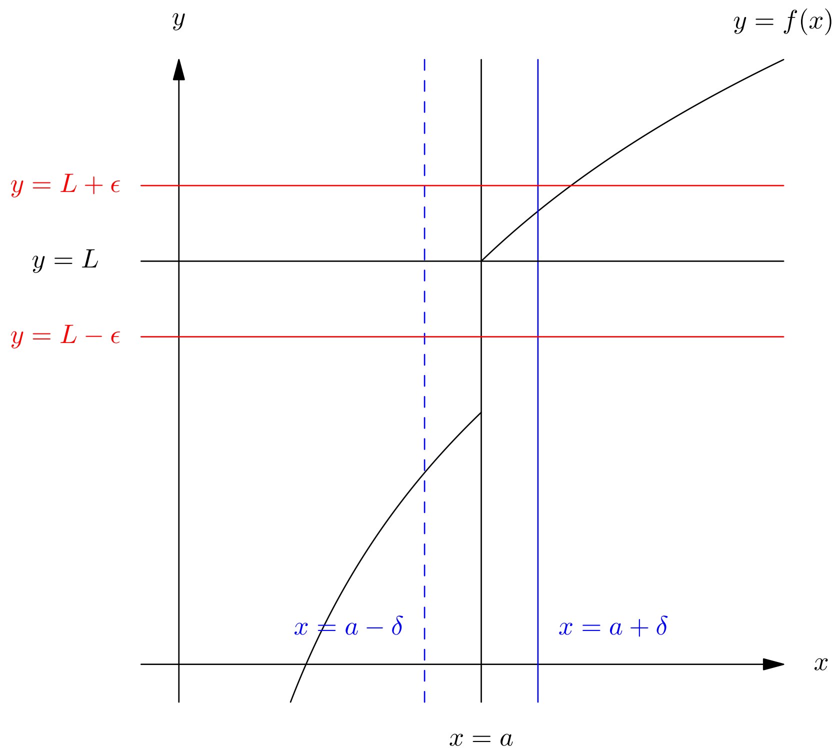
That is, if $|x - a| < \delta$, then $|f(x) - L| < \epsilon$.

What if we made ϵ smaller, i.e., brought the horizontal lines closer together? Well, with a suitable δ (smaller than our previous δ if necessary), corresponding to vertical lines suitably close, we could again ensure that the portion of the graph within the vertical lines was also within the horizontal ones. Since we may do this no matter how small $\epsilon > 0$ is, we say that $f(x)$ *approaches* L as x approaches a, and write

$$\lim_{x \to a} f(x) = L.$$

One-sided limits

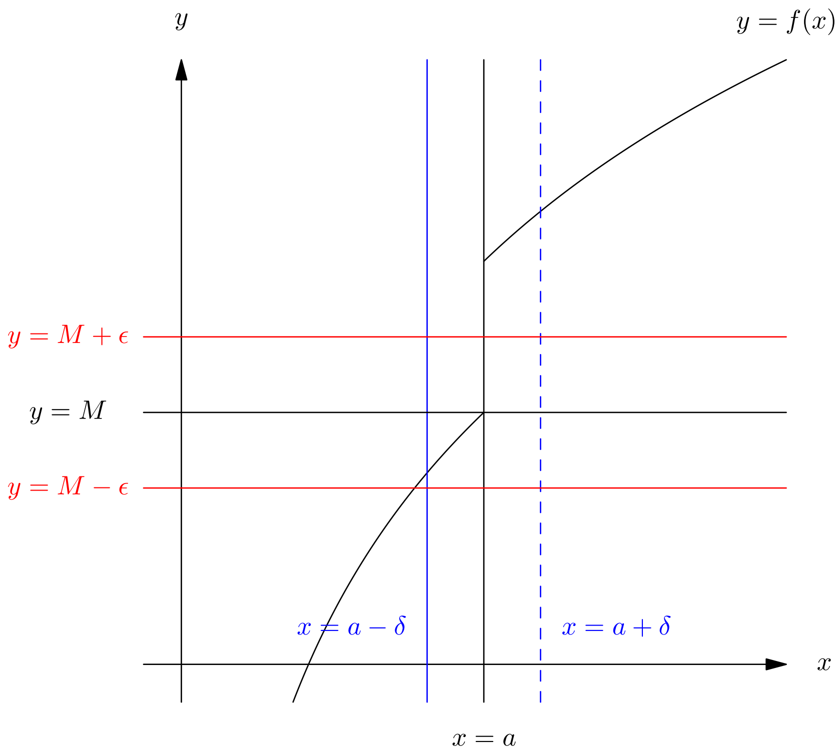
Consider instead a function f with a graph like this:



In this situation, with ϵ as given in the diagram, there is no way to choose $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all x with $|x - a| < \delta$. If x is just less than a, then $f(x) < L - \epsilon$. However, if we restrict our attention only to values of x greater than a, then we can again bring $f(x)$ within the prescribed horizontal train tracks, no matter how close they are to the line $y = L$. We say that $f(x)$ approaches L as x approaches a *from the right* (or *from above*), and write

$$\lim_{x \to a^+} f(x) = L.$$

Similarly, if we restrict x to be less than a instead, then we obtain a limit *from the left* (or *from below*):



Here,

$$\lim_{x \to a^-} f(x) = M.$$

3.2 Formal definition of limits

In a limit, we are actually concerned only with the behaviour of the function just either side of $x = a$ (or on one side, if necessary), not the value at $x = a$ itself. Indeed, the function need not be defined at $x = a$. Therefore, in place of the condition $|x - a| < \delta$, we actually take the condition $0 < |x - a| < \delta$, i.e., x should be considered to be different from a. The formal definitions of limit, right-sided limit, and left-sided limit may now be given as follows.

Formal definition of a limit

Let f be a function, a a value in its domain, and L a real number. We say that $f(x)$ approaches L as x approaches a if for every $\epsilon > 0$, there is $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

In this case, we say that the limit of $f(x)$ as $x \to a$ exists, and write

$$\lim_{x \to a} f(x) = L.$$

Note that the condition $0 < |x - a| < \delta$ can also be written

$$a - \delta < x < a + \delta \quad \text{and} \quad x \neq a.$$

Formal definition of a right-handed limit

Let f be a function, a a value in its domain, and L a real number. We say that $f(x)$ approaches L as x approaches a from the right (or from above) if for every $\epsilon > 0$, there is $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - L| < \epsilon.$$

In this case, we say that the limit of $f(x)$ as $x \to a^+$ exists, and write

$$\lim_{x \to a+} f(x) = L.$$

Formal definition of a left-handed limit

Let f be a function, a a value in its domain, and L a real number. We say that $f(x)$ approaches L as x approaches a from the left (or from below) if for every $\epsilon > 0$, there is $\delta > 0$ such that

$$a - \delta < x < a \implies |f(x) - L| < \epsilon.$$

In this case, we say that the limit of $f(x)$ as $x \to a^-$ exists, and write

$$\lim_{x \to a-} f(x) = L.$$

3.3 Infinite limits and vertical asymptotes

Consider $f : \mathbb{R} \setminus \{3\} \to \mathbb{R}$ given by

$$f(x) = \frac{1}{x^2 - 6x + 9}.$$

What happens as x approaches 3? Observe that

$$f(x) = \frac{1}{(x-3)^2},$$

so if x is a very little more than 3, say $x = 3 + h$ with $h > 0$ and small, then

$$f(x) = \frac{1}{h^2},$$

a positive number of very large magnitude. In fact, we can make $f(x)$ as large as we wish by taking $x = 3 + h$ with $h > 0$ small enough. We say that f *approaches* (or *tends to*) ∞ as x approaches (or tends to) 3 from above (or from the right), and write

$$\lim_{x \to 3^+} f(x) = \infty.$$

The line $x = 3$ is then called a *vertical asymptote* of the graph of the function.

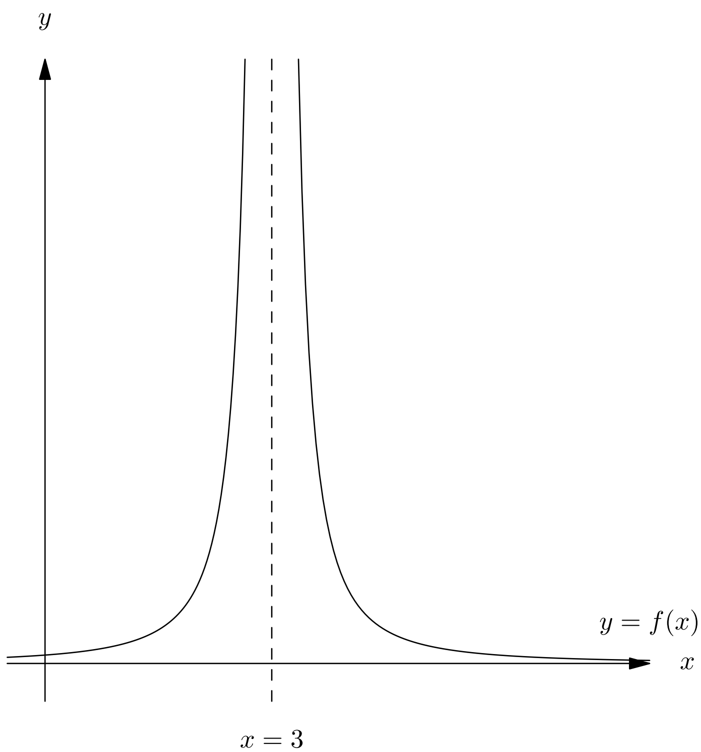
What if $x \to 3^-$, that is, x approaches 3 from below (from the left)? Here, $x = 3 - h$ with $h > 0$ and small, so

$$f(x) = \frac{1}{(-h)^2} = \frac{1}{h^2},$$

so again $f(x)$ approaches ∞ as $x \to 3^-$, and we write

$$\lim_{x \to 3^-} f(x) = \infty.$$

A sketch of the graph of the function illustrates the behaviour from both sides:



Example 3.1 Consider $f : \mathbb{R} \setminus \{-2\} \to \mathbb{R}$ given by

$$f(x) = \frac{x - 1}{x + 2}.$$

We examine the behaviour of f near -2. First, if $x \to -2^+$, i.e., $x = -2 + h$ where $h > 0$ is small, then

$$f(-2 + h) = \frac{(-2 + h) - 1}{(-2 + h) + 2} = \frac{h - 3}{h},$$

a negative number of large magnitude, $h-3$ being negative and h being positive and small. Therefore,

$$\lim_{x\to -2^+} f(x) = -\infty.$$

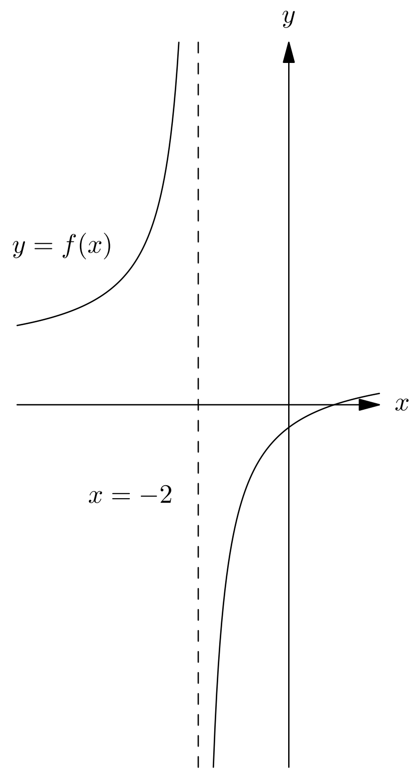
Next, if $x \to -2^-$, i.e., $x = -2 - h$ where $h > 0$ is small, then

$$f(x) = \frac{(-2-h)-1}{(-2-h)+2} = \frac{-3-h}{-h} = \frac{3+h}{h},$$

a positive number of large magnitude, $3+h$ being positive and h being positive and small. Therefore,

$$\lim_{x\to -2^-} f(x) = \infty.$$

The behaviour is illustrated below:



Because the left and right limits are different, we say that $f(x)$ does not have a limit as x approaches (or tends to) -2.

Formal definition

- We say that $f(x)$ *approaches* (or *tends to*) ∞ as $x \to a$ if for every $y \in \mathbb{R}$, there is $\delta > 0$ such that $f(x) > y$ for all $x \neq a$ with $|x - a| < \delta$. In this case, we write $\lim_{x\to a} f(x) = \infty$.

- We say that $f(x)$ *approaches* (or *tends to*) $-\infty$ as $x \to a$ if for every $y \in \mathbb{R}$, there is $\delta > 0$ such that $f(x) < y$ for all $x \neq a$ with $|x - a| < \delta$. In this case, we write $\lim_{x\to a} f(x) = -\infty$.
- For limits from the right, replace "$x \neq a$" by "$x > a$".
- For limits from the left, replace "$x \neq a$" by "$x < a$".

In any of these situations (including when the limit is one-sided only), we say that the line $x = a$ is a *vertical asymptote* of the graph of the function.

Existence of limits

In this course, we will say that $\lim_{x\to a} f(x)$ exists if $f(x)$ converges to a finite limit as $x \to a$ (from both sides). Otherwise, we will say that the limit does not exist. Thus, not existing means any of the following:

(i) The left limit is $\pm\infty$ or does not exist.

(ii) The right limit is $\pm\infty$ or does not exist.

(iii) The left and right limits exist and are finite, but they are not equal.

Caution: In this course, if $f(x) \to \infty$ as $x \to a$ (from both sides), then we still write $\lim_{x\to a} f(x) = \infty$ even though we have agreed to say that the limit does not exist (see above). The same goes for the situation where $f(x) \to -\infty$, where we would write $\lim_{x\to a} f(x) = -\infty$.

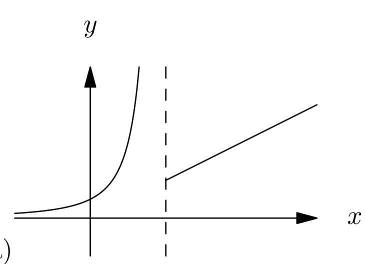
In each of the sketches below, the limit does not exist as x approaches the value indicated by the dashed line. The reasons, in order, are as follows:

(a) The left limit is ∞.

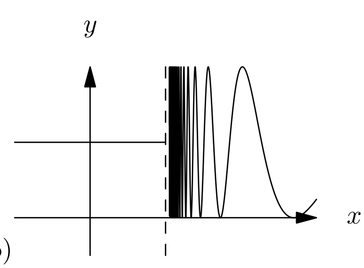
(b) The right limit does not exist.

(c) The left and right limits are finite but different from each other.

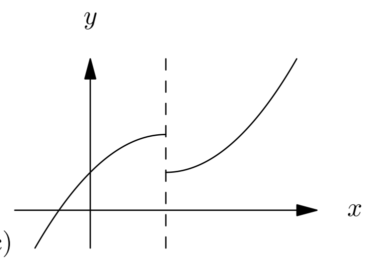
(d) The value of the function approaches ∞.



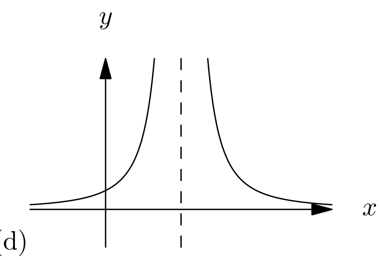
(a)



(b)



(c)



(d)

Typical examples of infinite limits

Example 3.2 If

$$f(x) = \frac{(x+4)^3(x-9)}{(x^2-18x+81)(x-10)},$$

determine the asymptotic behaviour of f at $x = 9$ (i.e., whether $f(x) \to \infty$ from the left, right, both sides, or neither, and so on).

Solution:

$$f(x) = \frac{(x+4)^3(x-9)}{(x-9)^2(x-10)} = \frac{(x+4)^3}{(x-9)(x-10)},$$

so if $h \neq 0$, then

$$f(9+h) = \frac{(13+h)^3}{h(-1+h)},$$

which approaches $-\infty$ as $h \to 0^+$ and approaches ∞ as $h \to 0^-$, so $f(x) \to -\infty$ as $x \to 9^+$, and $f(x) \to \infty$ as $x \to 9^-$. Note, then, that $f(x)$ approaches neither ∞ nor $-\infty$ as $x \to 9$ (in the two-sided sense).

Example 3.3 Find the vertical asymptotes of

$$g(x) = \frac{2x-12}{(x^2-8x+16)(x+12)(x-6)}$$

and the nature of those asymptotes.

Solution: If $x \notin \{-12, 4, 6\}$, then

$$f(x) = \frac{2(x-6)}{(x-4)^2(x+12)(x-6)} = \frac{2}{(x-4)^2(x+12)}.$$

The only values of x where the denominator $(x-4)^2(x+12)$ is zero are -12 and 4, so the lines $x = -12$ and $x = 4$ are the only possibilities for vertical asymptotes. In particular, the line $x = 6$ is not a vertical asymptote.

Let us consider the behaviour near $x = -12$. If $h \neq 0$, then

$$f(-12+h) = \frac{2}{(-16+h)^2 h},$$

which approaches ∞ as $h \to 0^+$ and approaches $-\infty$ as $h \to 0^-$, so $f(x) \to \infty$ as $x \to -12^+$, and $f(x) \to -\infty$ as $x \to 0^-$. Thus, the line $x = -12$ is a vertical asymptote, but $f(x)$ approaches neither ∞ nor $-\infty$ as $x \to -12$.

Now we turn to the behaviour near $x = 4$. Again, if $h \neq 0$, then

$$f(4+h) = \frac{2}{h^2(16+h)},$$

which approaches ∞ as $h \to 0$ (from both sides). The line $x = 4$ is a vertical asymptote.

3.4 Limit laws

Assume that the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then we have the following:

(i) $\lim\limits_{x\to a}(f(x)+g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$.

(ii) $\lim\limits_{x\to a}(f(x)-g(x)) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$.

(iii) $\lim\limits_{x\to a} f(x)g(x) = \left(\lim\limits_{x\to a} f(x)\right)\left(\lim\limits_{x\to a} g(x)\right)$.

(iv) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ as long as $\lim_{x\to a} g(x) \neq 0$.

(v) $\lim\limits_{x\to a}(f(x)^{1/n}) = \left(\lim\limits_{x\to a} f(x)\right)^{1/n}$ if n is a positive integer, provided that $\lim_{x\to a} f(x) > 0$ in the case where n is even.

(vi) If c is constant, then $\lim_{x\to a} c = c$. This combines with (iii) above to give $\lim\limits_{x\to a}(cf(x)) = c\lim\limits_{x\to a} f(x)$ if c is constant.

(vii) If k is a positive integer, then $\lim_{x\to a} x^k = a^k$.

Example 3.4 Evaluate

$$\lim_{x\to 4} \frac{x^2-x-12}{x-4}.$$

Solution:

$$\begin{aligned}
\lim_{x\to 4} \frac{x^2-x-12}{x-4} &= \lim_{x\to 4} \frac{(x+3)(x-4)}{x-4} \\
&= \lim_{x\to 4}(x+3) \quad \text{by cancellation} \\
&= \lim_{x\to 4} x + \lim_{x\to 4} 3 \quad \text{by (i)} \\
&= 4+3 \quad \text{by (vii) and (vi)} \\
&= 7.
\end{aligned}$$

Example 3.5 Evaluate

$$\lim_{x \to -2} \frac{2x^2 + 7x + 6}{x^2 - 3x - 10}.$$

Solution:

$$\begin{aligned}
\lim_{x \to -2} \frac{2x^2 + 7x + 6}{x^2 - 3x - 10} &= \lim_{x \to -2} \frac{(2x+3)(x+2)}{(x-5)(x+2)} \\
&= \lim_{x \to -2} \frac{2x+3}{x-5} \quad \text{by cancellation} \\
&= \frac{\lim_{x \to -2}(2x+3)}{\lim_{x \to -2}(x-5)} \quad \text{by (iv)} \\
&= \frac{2 \lim_{x \to -2} x + \lim_{x \to -2} 3}{\lim_{x \to -2} x - \lim_{x \to -2} 5} \quad \text{by (i), (ii), and (vi)} \\
&= \frac{2(-2)+3}{-2-5} \quad \text{by (vii) and (vi)} \\
&= \frac{1}{7}.
\end{aligned}$$

Observe that limit laws (i), (ii), (vi), and (vii) combine to show that if f is a polynomial function, then

$$\lim_{x \to a} f(x) = f(a).$$

For example, if $f(x) = 3x^2 - 4x + 1$, then

$$\lim_{x \to -1} f(x) = 3(-1)^2 - 4(-1) + 1 = 8.$$

Example 3.6 Use the limit laws to evaluate

$$\lim_{x \to 1} \frac{\sqrt{15+x} - 4}{x - 1}.$$

Solution:

$$\begin{aligned}
\lim_{x \to 1} \frac{\sqrt{15+x} - 4}{x - 1} &= \lim_{x \to 1} \frac{(\sqrt{15+x} - 4)(\sqrt{15+x} + 4)}{(x-1)(\sqrt{15+x} + 4)} \\
&= \lim_{x \to 1} \frac{15 + x - 16}{(x-1)(\sqrt{15+x} + 4)} \\
&= \lim_{x \to 1} \frac{x-1}{(x-1)(\sqrt{15+x} + 4)} \\
&= \lim_{x \to 1} \frac{1}{\sqrt{15+x} + 4} \quad \text{by cancellation} \\
&= \frac{1}{\sqrt{\lim_{x \to 1}(15+x)} + 4} \quad \text{(several laws, in partic. (iv), (v))} \\
&= \frac{1}{8}.
\end{aligned}$$

Example 3.7 Evaluate

$$\lim_{x\to -2} \frac{\frac{1}{x^2}+\frac{1}{x}+\frac{1}{4}}{\frac{1}{x}+\frac{1}{2}}.$$

Solution:

$$\begin{aligned}
\lim_{x\to -2} \frac{\frac{1}{x^2}+\frac{1}{x}+\frac{1}{4}}{\frac{1}{x}+\frac{1}{2}} &= \lim_{x\to -2} \frac{4x^2}{4x^2}\,\frac{\frac{1}{x^2}+\frac{1}{x}+\frac{1}{4}}{\frac{1}{x}+\frac{1}{2}} \\
&= \lim_{x\to -2} \frac{x^2+4x+4}{2x^2+4x} \\
&= \lim_{x\to -2} \frac{(x+2)^2}{2x(x+2)} \\
&= \lim_{x\to -2} \frac{x+2}{2x} \\
&= \frac{0}{-4} \\
&= 0.
\end{aligned}$$

Example 3.8 Evaluate

$$\lim_{x\to -3} \frac{|x-1|-4}{2x+6}.$$

Solution: As $x \to -3$, $x-1 \to -4 < 0$, so in the limit expression, $|x-1| = -(x-1) = 1-x$. Hence,

$$\begin{aligned}
\lim_{x\to -3} \frac{|x-1|-4}{2x+6} &= \lim_{x\to -3} \frac{(1-x)-4}{2x+6} \\
&= \lim_{x\to -3} \frac{-3-x}{2x+6} \\
&= \lim_{x\to -3} \left(-\frac{x+3}{2(x+3)}\right) \\
&= -\frac{1}{2}.
\end{aligned}$$

Example 3.9 Decide whether $\lim_{x\to 5} \frac{\sqrt{(x-5)^2}}{x^2+x-30}$ exists.

Solution:

$$\sqrt{(x-5)^2} = |x-5| = \begin{cases} x-5 & \text{if } x-5 \geq 0, \text{ i.e., } x \geq 5, \\ -(x-5) & \text{if } x-5 < 0, \text{ i.e., } x < 5. \end{cases}$$

Also, $x^2+x-30 = (x-5)(x+6)$, so

$$\frac{\sqrt{(x-5)^2}}{x^2+x-30} = \begin{cases} \frac{x-5}{(x-5)(x+6)} = \frac{1}{x+6} & \text{if } x > 5, \\ -\frac{x-5}{(x-5)(x+6)} = -\frac{1}{x+6} & \text{if } x < 5. \end{cases}$$

Hence,

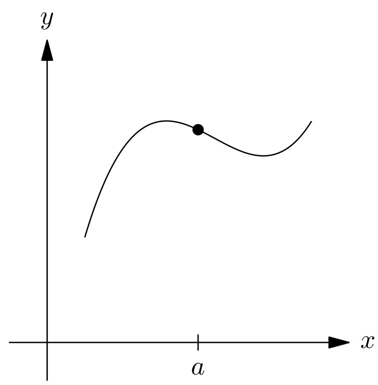
$$\lim_{x\to 5^+} \frac{\sqrt{(x-5)^2}}{x^2+x-30} = \lim_{x\to 5^+} \frac{1}{x+6} = \frac{1}{11},$$
$$\text{while} \quad \lim_{x\to 5^-} \frac{\sqrt{(x-5)^2}}{x^2+x-30} = \lim_{x\to 5^-} \left(-\frac{1}{x+6}\right) = -\frac{1}{11}.$$

The left and right limits are different, so $\frac{\sqrt{(x-5)^2}}{x^2+x-30}$ has no limit as $x \to 5$.

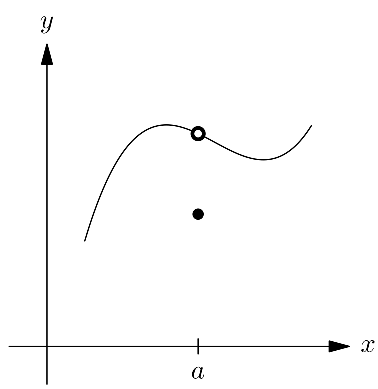
3.5 Continuity

Let $f : D \to \mathbb{R}$ be a function and $a \in D$. We say that f is *continuous at a* if $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$. The illustrations below will help, in which a filled circle emphasizes that the function takes that value there and an unfilled one indicates that the function does not:

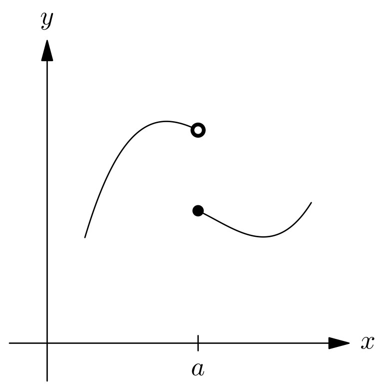
(a)



(b)



(c)



(a) The function is continuous at a: the limit as $x \to a$ is equal to $f(a)$.

(b) The function is not continuous at a: $f(a)$ is different from $\lim_{x\to a} f(x)$.

(c) The function is not continuous at a: the left and right limits as $x \to a$ are not equal.

If $f, g : D \to \mathbb{R}$ are continuous at $a \in D$, then the following functions are also continuous at a:

- $f + g$ defined by $(f + g)(x) = f(x) + g(x)$
- $f - g$ defined by $(f - g)(x) = f(x) - g(x)$
- fg defined by $(fg)(x) = f(x)g(x)$
- a constant function $f(x) = c$
- cf defined by $(cf)(x) = cf(x)$ where c is constant (because of the two previous points)
- f/g provided that $g(a) \neq 0$, where f/g is the function defined everywhere that $g(x) \neq 0$ by $(f/g)(x) = f(x)/g(x)$

A function $f : D \to \mathbb{R}$ is said to be *continuous* if it is continuous at every point of D.

Example 3.10 The following functions are all continuous where they are defined:

$$\begin{aligned} f(x) &= x \\ f(x) &= \cos(x) \\ f(x) &= \sin(x) \\ f(x) &= b^x \quad \text{where } b > 0 \\ f(x) &= \ln(x) \end{aligned}$$

Consequently, by the rules above, the following are continuous where they are defined:

$$\begin{aligned}
&f(x) = e^x - \sin(x)\\
&f(x) = x\cos(x)\\
&f(x) = x\cos(x) + \sin(x)\\
&f(x) = \tan(x) \quad \text{because } \tan(x) = \sin(x)/\cos(x).
\end{aligned}$$

Remark. We have just stated that $f + g$ is continuous if f and g are, but this implies also that $f + g + h$ is continuous if f, g, h are, and so on for any finite number of continuous functions. The same applies to products of continuous functions.

Remark. Because the function f defined by $f(x) = x$ is continuous, the rules above show that any polynomial function $f(x) = a_n x^n + \cdots + a_1 x + a_0$ is continuous.

Composition of continuous functions

If $f : D_1 \to D_2$ and $g : D_2 \to \mathbb{R}$ are continuous, then so is the composition $g \circ f : D_1 \to \mathbb{R}$ defined by $(g \circ f)(x) = g(f(x))$.

Example 3.11 Consider f, g, h defined by

$$f(x) = \sin(x), \quad g(x) = e^x, \quad h(x) = \cos(x),$$

all continuous. Then $g \circ f$ is also continuous, where

$$(g \circ f)(x) = g(f(x)) = e^{\sin(x)}.$$

Hence, $h \circ g \circ f$ is continuous as well, where

$$(h \circ g \circ f)(x) = h(g(f(x))) = \cos(e^{\sin(x)}).$$

Limits and compositions 1

Loosely speaking, if f and g are functions and g is continuous, then

$$\lim_{x \to x_0} g(f(x)) = g\left(\lim_{x \to x_0} f(x)\right).$$

A precise statement is given in Proposition 3.13 below. You do not need to know that precise statement or its proof, but you are expected to be able to apply the idea when presented with a problem of that kind. The following example illustrates the point.

Example 3.12 It is a fact, which we will prove later, that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Therefore, because ln is continuous at 1, we have

$$\begin{aligned} \lim_{x\to 0} \ln\left(\frac{\sin(x)}{x}\right) &= \ln\left(\lim_{x\to 0} \frac{\sin(x)}{x}\right) \quad \text{by Proposition 3.13 below} \\ &= \ln(1) = 0. \end{aligned}$$

The precise statement is as follows.

Proposition 3.13 *Assume the following three hypotheses:*

- $f : D_1 \to D_2$ *is a function (not necessarily continuous).*
- x_0 *is a limit point of* D_1*, and the real number* $y_0 = \lim_{x\to x_0} f(x)$ *exists and belongs to* D_2*.*
- $g : D_2 \to \mathbb{R}$ *is a function that is continuous at* y_0 *(but not necessarily anywhere else).*

Then

$$\lim_{x\to x_0} g(f(x)) = g\left(\lim_{x\to x_0} f(x)\right).$$

Proof. Let $\epsilon > 0$, and choose $\delta' > 0$ such that $|g(y) - g(y_0)| < \epsilon$ for all $y \in D_2$ satisfying $|y - y_0| < \delta'$, possible because g is continuous at y_0. Then choose $\delta > 0$ such that $|f(x) - y_0| < \delta'$ for all $x \in D_1$ satisfying $0 < |x - x_0| < \delta$, possible because $\lim_{x\to x_0} f(x) = y_0$. For such x, $|g(f(x)) - g(y_0)| < \epsilon$. Thus,

$$\lim_{x\to x_0} g(f(x)) = g(y_0) = g\left(\lim_{x\to x_0} f(x)\right).$$

■

Limits and compositions 2

Under some technical assumptions, which you do not need to worry about in this course,

$$\lim_{x\to x_0} g(f(x)) = \lim_{y\to y_0} g(y)$$

where $y_0 = \lim_{x\to x_0} f(x)$. Please see Proposition 3.15 for a precise formulation, but note that details of the statement and the proof are not required understanding in this course, only an ability to solve related problems, as in the following example.

Example 3.14 We will see later that

$$\lim_{x\to 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2}.$$

Use this fact to find $\lim_{x\to 0} \frac{1-\cos(5x)}{x^2}$.

Solution:

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos(5x)}{x^2} &= 5^2 \lim_{x\to 0} \frac{1-\cos(5x)}{5^2x^2} \\ &= 25 \lim_{x\to 0} \frac{1-\cos(5x)}{(5x)^2} \\ &= 25 \lim_{y\to 0} \frac{1-\cos(y)}{y^2} \quad \text{by Proposition 3.15 below} \\ &= \frac{25}{2}.\end{aligned}$$

The precise statement is as follows.

Proposition 3.15 *Assume the following three hypotheses:*

- $f : D_1 \to D_2$ *is a function (not necessarily continuous).*
- x_0 *is a limit point of* D_1*, the real number* $y_0 = \lim_{x\to x_0} f(x)$ *exists, and there is* $\delta_0 > 0$ *such that* $f(x) \neq y_0$ *for all* $x \in D_1$ *satisfying* $0 < |x - x_0| < \delta_0$.
- $g : D_2 \to \mathbb{R}$ *is a function (not necessarily continuous) such that* $\lim_{y\to y_0} g(y)$ *exists.*

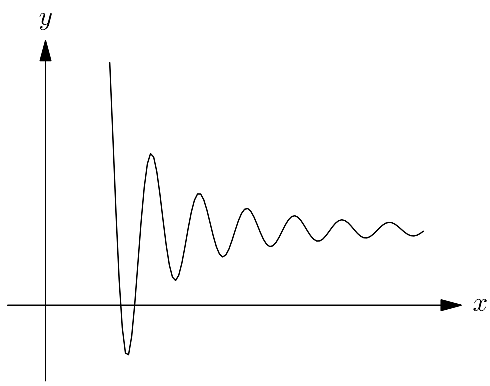
Then

$$\lim_{x\to x_0} g(f(x)) = \lim_{y\to y_0} g(y).$$

Proof. Let $z = \lim_{y\to y_0} g(y)$. Let $\epsilon > 0$, and choose $\delta' > 0$ such that $|g(y) - z| < \epsilon$ for all $y \in D_2$ with $0 < |y - y_0| < \delta'$. Then choose $\delta > 0$ with $\delta \leq \delta_0$ such that $|f(x) - y_0| < \delta'$ for all $x \in D_1$ satisfying $0 < |x - x_0| < \delta$. For such x, we in fact have $0 < |f(x) - y_0| < \delta'$ by the second hypothesis in the proposition. Thus, $\lim_{x\to x_0} g(f(x)) = z = \lim_{y\to y_0} g(y)$. ■

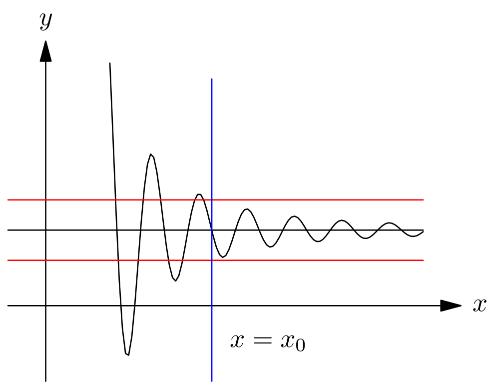
3.6 Limits at infinity

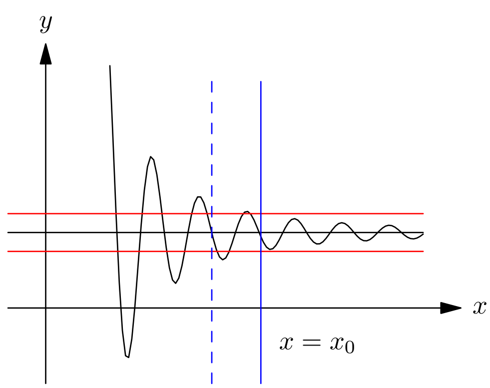
Consider a function f with the following graph:



Imagine moving x further and further along the horizontal axis, in the positive direction, and looking at the behaviour of the function as that happens.

More precisely, we imagine placing "train tracks" either side of some line $y = L$ and see whether it is possible to bring the values $f(x)$ within the train tracks by taking x large enough.





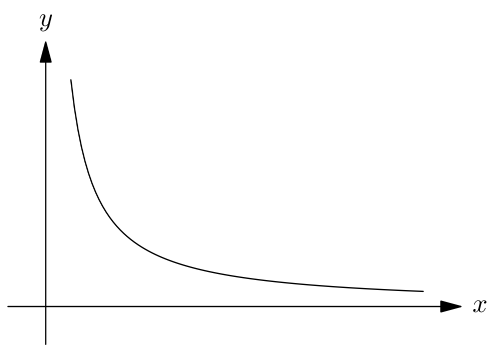
In the first of these two diagrams, all of the graph to the right of the vertical blue line is within the red horizontal train tracks. In the second diagram, the horizontal train tracks have been brought closer together. To achieve the same this time, we needed to move the blue line further to the right.

Formally, we make these definitions:

- Suppose $f : (a, \infty) \to \mathbb{R}$, where $a \in \mathbb{R}$. We say that $f(x) \to L$ as $x \to \infty$ if for every $\epsilon > 0$, there is $x_0 > a$ such that $|f(x) - L| < \epsilon$ for all $x > x_0$. In this case, we write $\lim_{x\to\infty} f(x) = L$.
- Suppose $f : (-\infty, a) \to \mathbb{R}$, where $a \in \mathbb{R}$. We say that $f(x) \to L$ as $x \to -\infty$ if for every $\epsilon > 0$, there is $x_0 < a$ such that $|f(x) - L| < \epsilon$ for all $x < x_0$. In this case, we write $\lim_{x\to-\infty} f(x) = L$.

In either situation, we say that the line $y = L$ is a *horizontal asymptote* of the graph of the function.

Example 3.16 Consider $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = 1/x$. Here, $\lim_{x\to\infty} f(x) = 0$.



Let us show this using the formal definition. Let $\epsilon > 0$, and take $x_0 = 1/\epsilon$. If $x > x_0 = 1/\epsilon$, then $1/x < \epsilon$, so

$$|f(x) - 0| = \frac{1}{x} < \epsilon.$$

This justifies the claim.

Example 3.17 If $r > 0$, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0,$$

for a similar reason to the previous example. Also, if r is a positive integer, then $1/x^r$ is defined for $x < 0$ as well, and then $\lim_{x\to-\infty} \frac{1}{x^r} = 0$.

Rational functions

If

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions, then to analyze $\lim_{x\to\pm\infty} f(x)$, one extracts the highest power of x from each of $p(x)$ and $q(x)$.

Example 3.18

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3+x}{2x^3-x^2-1} &= \lim_{x\to\infty} \frac{x^3(1+\frac{1}{x^2})}{x^3(2-\frac{1}{x}-\frac{1}{x^3})} \\
&= \lim_{x\to\infty} \frac{1+\frac{1}{x^2}}{2-\frac{1}{x}-\frac{1}{x^3}} \\
&= \frac{\lim_{x\to\infty}(1+\frac{1}{x^2})}{\lim_{x\to\infty}(2-\frac{1}{x}-\frac{1}{x^3})} \\
&= \frac{1+0}{2-0-0} \\
&= \frac{1}{2}.
\end{aligned}$$

Thus, the line $y = 1/2$ is a horizontal asymptote of the graph of f.

A method for a type of problem involving square roots

Example 3.19 Find

$$\lim_{x\to\infty}(\sqrt{4x^2-8x+3}-2x).$$

Solution: For $x \geq 2$ (say),

$$\begin{aligned}
\sqrt{4x^2-8x+3}-2x &= \sqrt{x^2(4-\tfrac{8}{x}+\tfrac{3}{x^2})}-2x \\
&= |x|\sqrt{4-\tfrac{8}{x}+\tfrac{3}{x^2}}-2x \\
&= x\left(\sqrt{4-\tfrac{8}{x}+\tfrac{3}{x^2}}-2\right) \qquad \text{because } x \geq 0 \\
&= x\,\frac{\left(\sqrt{4-\frac{8}{x}+\frac{3}{x^2}}-2\right)\left(\sqrt{4-\frac{8}{x}+\frac{3}{x^2}}+2\right)}{\sqrt{4-\frac{8}{x}+\frac{3}{x^2}}+2} \\
&= x\,\frac{\left(4-\frac{8}{x}+\frac{3}{x^2}\right)-4}{\sqrt{4-\frac{8}{x}+\frac{3}{x^2}}+2} \\
&= \frac{-8+\frac{3}{x}}{\sqrt{4-\frac{8}{x}+\frac{3}{x^2}}+2}.
\end{aligned}$$

Hence,

$$\lim_{x\to\infty}(\sqrt{4x^2-8x+3}-2x) = \lim_{x\to\infty}\frac{-8+\frac{3}{x}}{\sqrt{4-\frac{8}{x}+\frac{3}{x^2}}+2} = \frac{-8}{4} = -2.$$

Remark. If we wished instead to find $\lim_{x\to-\infty}(\sqrt{4x^2-8x+3}+2x)$, with x approaching $-\infty$ and $-2x$ replaced by $+2x$, then we would have to remember that $|x| = -x$ when $x < 0$:

$$\sqrt{x^2(4-\tfrac{8}{x}+\tfrac{3}{x^2})}+2x = |x|\sqrt{4-\tfrac{8}{x}+\tfrac{3}{x^2}}+2x = -x\left(\sqrt{4-\tfrac{8}{x}+\tfrac{3}{x^2}}-2\right).$$

Infinite limits at infinity

A function may also approach ∞ or $-\infty$ as $x \to \infty$ or as $x \to -\infty$. Here are the formal definitions:

- $f(x) \to \infty$ as $x \to \infty$ if for every $y \in \mathbb{R}$, there is x_0 such that $f(x) > y$ for all $x > x_0$.
- $f(x) \to -\infty$ as $x \to \infty$ if for every $y \in \mathbb{R}$, there is x_0 such that $f(x) < y$ for all $x > x_0$.
- $f(x) \to \infty$ as $x \to -\infty$ if for every $y \in \mathbb{R}$, there is x_0 such that $f(x) > y$ for all $x < x_0$.
- $f(x) \to -\infty$ as $x \to -\infty$ if for every $y \in \mathbb{R}$, there is x_0 such that $f(x) < y$ for all $x < x_0$.

Example 3.20 Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ where

$$f(x) = \frac{3x^5+x^2-1}{2x^4+x}.$$

Solution: The idea is again to extract the highest power of x from each of the numerator and the denominator:

$$\frac{3x^5+x^2-1}{2x^4+x} = \frac{x^5(3+\frac{1}{x^3}-\frac{1}{x^5})}{x^4(2+\frac{1}{x^3})} = x\,\frac{3+\frac{1}{x^3}-\frac{1}{x^5}}{2+\frac{1}{x^3}}.$$

Now, both of the limits

$$\lim_{x\to\infty}\frac{3+\frac{1}{x^3}-\frac{1}{x^5}}{2+\frac{1}{x^3}} \quad \text{and} \quad \lim_{x\to-\infty}\frac{3+\frac{1}{x^3}-\frac{1}{x^5}}{2+\frac{1}{x^3}}$$

are equal to $3/2$, a positive real number, so

$$\lim_{x\to\infty} f(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty.$$

4 Differentiation

4.1 Differentiability and the derivative of a function

Let $f : D \to \mathbb{R}$ be a function and $x \in D$. We say that f is *differentiable* at x if the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists. In this case, we let

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

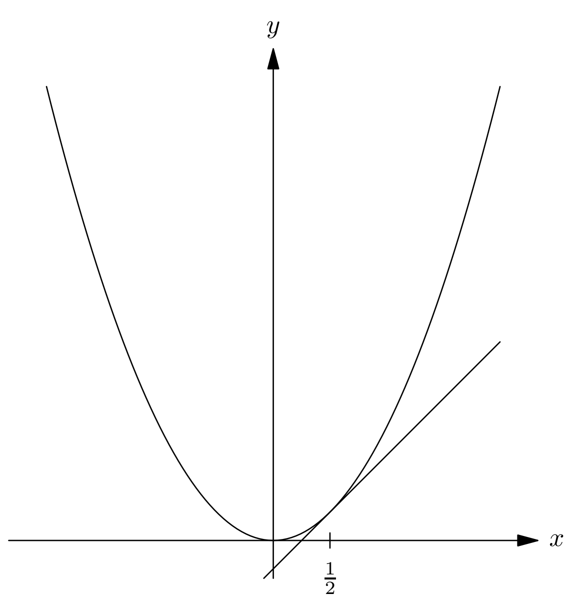
and call this number the *derivative* of f at x.

Example 4.1 If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$, show that f is differentiable at every x and find $f'(x)$.

Solution: If $h \neq 0$, then

$$\frac{f(x+h) - f(x)}{h} = \frac{(x^2 + 2xh + h^2) - x^2}{h} = 2x + h.$$

The limit $\lim_{h\to 0}(2x + h)$ exists and is equal to $2x$, so f is differentiable at x, and $f'(x) = 2x$. For example, at $x = 1/2$, the tangent line has slope $f'(1/2) = 1$ and therefore has equation $y - f(1/2) = 1(x - \frac{1}{2})$, i.e., $y = x - \frac{1}{4}$:



The notation $\frac{df}{dx}$ is often used in place of $f'(x)$, but it has its drawbacks, so we will use it only when it is convenient.

Derivative of $f(x) = x^n$

If n is a positive integer, then

$$y^n - x^n = (y - x) \sum_{m=0}^{n-1} x^{n-1-m} y^m. \tag{4.1}$$

The proof is short:

$$\begin{aligned}
(y - x) \sum_{m=0}^{n-1} x^{n-1-m} y^m &= \sum_{m=0}^{n-1} x^{n-1-m} y^{m+1} - \sum_{m=0}^{n-1} x^{n-m} y^m \\
&= \sum_{m=1}^{n} x^{n-m} y^m - \sum_{m=0}^{n-1} x^{n-m} y^m \\
&= y^n - x^m.
\end{aligned}$$

We use this observation to prove the following.

Proposition 4.2 *If n is a positive integer, then the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^n$ is differentiable everywhere and satisfies $f'(x) = nx^{n-1}$.*

Proof. If $h \neq 0$, then

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^n - x^n}{h} &= \frac{y^n - x^n}{y - x} \quad \text{where } y = x + h \\
&= \sum_{m=0}^{n-1} x^{n-1-m} y^m \quad \text{by (4.1)} \\
&= \sum_{m=0}^{n-1} x^{n-1-m} (x+h)^m. \qquad (4.2)
\end{aligned}$$

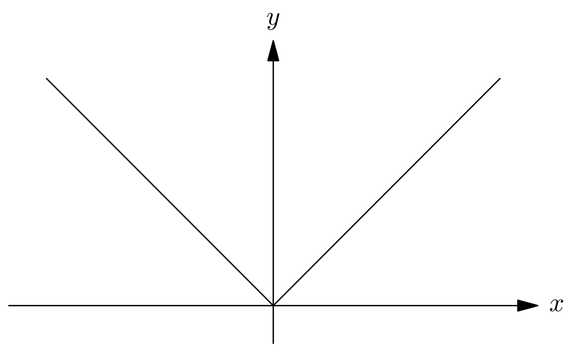
We see that the limit as $h \to 0$ exists in (4.2), and that

$$\begin{aligned}
\lim_{h \to 0} \left(\sum_{m=0}^{n-1} x^{n-1-m} (x+h)^m \right) &= \sum_{m=0}^{n-1} x^{n-1-m} x^m \\
&= \sum_{m=0}^{n-1} x^{n-1} \\
&= nx^{n-1}.
\end{aligned}$$

■

A function whose derivative is undefined at some point

Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$.



The derivative at $x = 0$ does not exist, because the left and right limits are different:

$$\lim_{h\to 0^+} \frac{|0+h|-0}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1,$$
$$\lim_{h\to 0^-} \frac{|0+h|-0}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1.$$

However, f is differentiable at all other points.

Repeated differentiation

If $f : D \to \mathbb{R}$ is differentiable, then because f' is again a function, one can consider whether it, too, is differentiable.

Example 4.3 The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$, is differentiable with derivative $f'(x) = 4x^3$. Being a polynomial function again, f' is differentiable as well.

If a differentiable function f is such that its derivative f' is differentiable at a point x, then the derivative of f' at x is denoted $f''(x)$. That is,

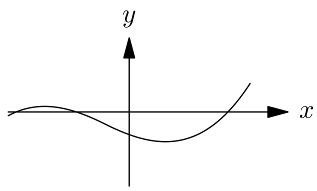
$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}.$$

Thus, $f''(x)$ describes the rate of change of $f'(x)$, i.e., the rate of change of the slope of the graph of f at x. The second derivative of f is also sometimes denoted $\frac{d^2 f}{dx^2}$.

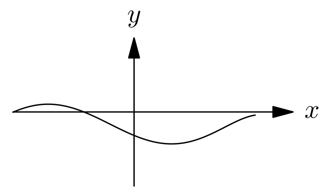
Example 4.4 If $f(x) = x^4$, then $f'(x) = 4x^3$ and $f''(x) = 4 \cdot 3x^2 = 12x^2$.

The third derivative, i.e., the derivative of f'', if it exists, is denoted f''', and higher derivatives, if they exist, are typically denoted by $f^{(4)}, f^{(5)}, f^{(6)}$, and so on.

It is not always easy to tell from the graph of a differentiable function whether its derivative is again differentiable. For example, only one of the two differentiable functions f_1 and f_2 below has a differentiable derivative.

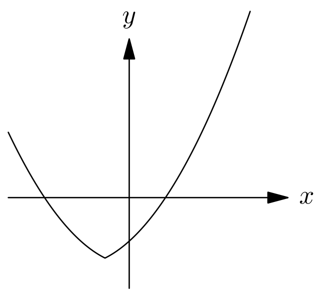


Graph of f_1

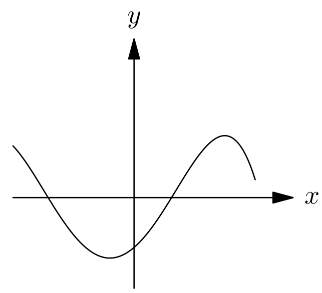


Graph of f_2

Here are the graphs of their derivatives:



Graph of f_1'



Graph of f_2'

4.2 More on differentiability

Differentiability implies continuity

Proposition 4.5 *If f is a function that is differentiable at some point a in its domain, then it is continuous at a as well.*

Proof. We are to show that $f(x) \to f(a)$ as $x \to a$. If $x \neq a$, then

$$f(x) = f(x) - f(a) + f(a) = (x-a)\frac{f(x)-f(a)}{x-a} + f(a),$$

so because f is differentiable at a, meaning that $(f(x) - f(a))/(x - a)$ has a finite limit as $x \to a$, we have

$$\lim_{x\to a} f(x) = \left(\lim_{x\to a}(x-a)\right)\left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right) + f(a) = 0(f'(a)) + f(a) = f(a).$$

■

Equivalently, if a function is not continuous at a, then it is not differentiable there either.

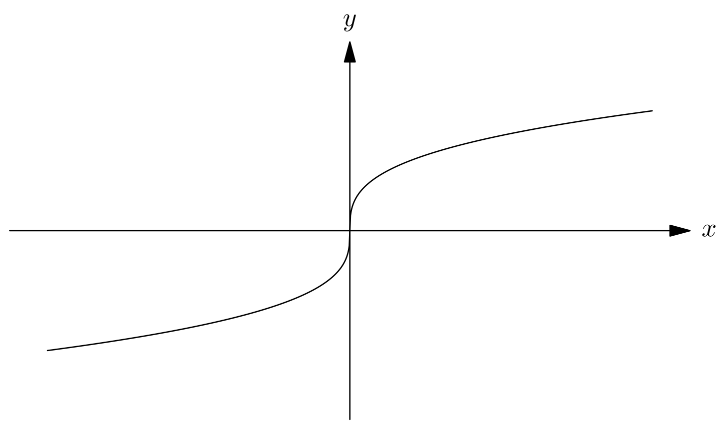
Example 4.6 The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

is not continuous at 0, so it is not differentiable there either.

Another example of failure of differentiability

Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^{1/3}$, the cube root of x:



Although f is continuous everywhere, it is not differentiable at 0. Indeed, if $h \neq 0$, then

$$\frac{f(0+h) - f(0)}{h} = \frac{h^{1/3}}{h} = h^{-2/3} = \frac{1}{h^{2/3}},$$

and this does not have a finite limit as $h \to 0$. The slopes of the secant lines approach ∞.

A differentiable piecewise-defined function

Consider $f : \mathbb{R}_{\geq -4} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x(1 + \ln(1+x)) & \text{if } x \geq 0, \\ 4\sqrt{x+4} - 8 & \text{if } -4 \leq x < 0. \end{cases}$$

We show that f is differentiable at 0. This entails showing that

$$(f(h) - f(0))/h$$

has finite left and right limits that are equal to each other as h approaches 0. Because the expressions for $f(h)$ for $h > 0$ and $h < 0$ are different, we will need to consider the two cases separately. Note that $f(0) = 0(1 + \ln(1+0)) = 0$.

If $h < 0$, then

$$\frac{f(h) - f(0)}{h} = \frac{4\sqrt{h+4} - 8}{h} = \frac{16(h+4) - 64}{h(4\sqrt{h+4} + 8)}$$
$$= \frac{4}{\sqrt{h+4} + 2} \to 1 \quad \text{as } h \to 0^-,$$

and if $h > 0$, then

$$\frac{f(h) - f(0)}{h} = \frac{h(1 + \ln(1+h))}{h} = 1 + \ln(1+h) \to 1 \quad \text{as } h \to 0^+.$$

The left and right limits exist and are the same, equal to 1, so f is differentiable at 0, and $f'(0) = 1$.

Differentiation rules

(i) For all positive integers n, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^n$ is differentiable and has derivative $f'(x) = nx^{n-1}$ (proven earlier). If $x \neq 0$, this holds also when $n < 0$. We will see later that if $a \in \mathbb{R}$ (i.e., a is any real number) and $f : \mathbb{R}_{>0} \to \mathbb{R}$ is defined by $f(x) = x^a$, then

$$f'(x) = ax^{a-1} \quad \text{for all } x > 0.$$

(ii) A constant function $f(x) = c$ has zero derivative, because

$$\frac{f(x+h) - f(x)}{h} = \frac{c - c}{h} = 0 \quad \text{for all } h \neq 0.$$

(iii) If $c \in \mathbb{R}$ (constant) and f is differentiable, then cf is differentiable, and $(cf)' = cf'$. For example, if $f(x) = 7x^3$, then

$$f'(x) = 7 \cdot 3x^2 = 21x^2.$$

(iv) If f and g are differentiable, then so is $f + g$, and

$$(f + g)' = f' + g'.$$

For example, if $f(x) = 6x^4 + 5x^8$, then differentiating term by term gives

$$f'(x) = 24x^3 + 40x^7.$$

Similarly, $f - g$ is differentiable, and $(f - g)' = f' - g'$.

(v) The product rule: If f and g are differentiable, then so is fg, and

$$(fg)' = f'g + fg'.$$

For example, if $f(x) = (2x + 3)(x - 5)$, then we can think of this as a product of two functions and apply the product rule to obtain

$$f'(x) = 2(x - 5) + (2x + 3) \cdot 1 = 4x - 7.$$

The product rule may be proven as follows. If $h \neq 0$, then

$$\begin{aligned}\frac{(fg)(x+h)-(fg)(x)}{h} &= \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\ &= \frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} \\ &= \frac{f(x+h)-f(x)}{h}\,g(x+h)+f(x)\,\frac{g(x+h)-g(x)}{h},\end{aligned}$$

and this approaches $f'(x)g(x)+f(x)g'(x)$ as $h \to 0$.

4.3 The quotient rule, the Squeeze Theorem, and some trigonometry

The quotient rule

Proposition 4.7 *If f and g are differentiable functions, then f/g is differentiable everywhere that g is non-zero, and*

$$\left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}.$$

Proof. We provide a proof along the lines of that of the product rule. The equality

$$\frac{(f/g)(x+h)-(f/g)(x)}{h} = \frac{\frac{f(x+h)-f(x)}{h}\,g(x)-f(x)\,\frac{g(x+h)-g(x)}{h}}{g(x)g(x+h)}$$

is readily verified, and the limit laws, along with the assumptions in the proposition, show that the limit of this expression as $h \to 0$ is

$$\frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}.$$

■

Example 4.8 If

$$j(x) = \frac{x^2-2x+2}{3x-5},$$

find $j'(x)$ for $x \neq 5/3$.

Solution: By the quotient rule,

$$\begin{aligned}j'(x) &= \frac{(2x-2)(3x-5)-(x^2-2x+2)\cdot 3}{(3x-5)^2} \\ &= \frac{6x^2-16x+10-3x^2+6x-6}{(3x-5)^2} \\ &= \frac{3x^2-10x+4}{(3x-5)^2}.\end{aligned}$$

Example 4.9 If

$$f(x) = \frac{x + \sqrt{x}}{x - \sqrt{x}} \quad (x > 0 \text{ and } x \neq 1),$$

find $f'(x)$.

Solution: It simplifies matters marginally to divide top and bottom by $\sqrt{x}$:

$$f(x) = \frac{\sqrt{x} + 1}{\sqrt{x} - 1}.$$

Hence, by the quotient rule,

$$\begin{aligned} f'(x) = \frac{\frac{1}{2}x^{-1/2}(\sqrt{x} - 1) - (\sqrt{x} + 1) \cdot \frac{1}{2}x^{-1/2}}{(\sqrt{x} - 1)^2} &= -\frac{x^{-1/2}}{(\sqrt{x} - 1)^2} \\ &= -\frac{1}{\sqrt{x}(\sqrt{x} - 1)^2}. \end{aligned}$$

The Squeeze Theorem

If f, g, j are functions such that $f(x) \leq g(x) \leq j(x)$ for all x near some given point a, except possibly at $x = a$ itself, and if the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} j(x)$ both exist and take the same value L, then

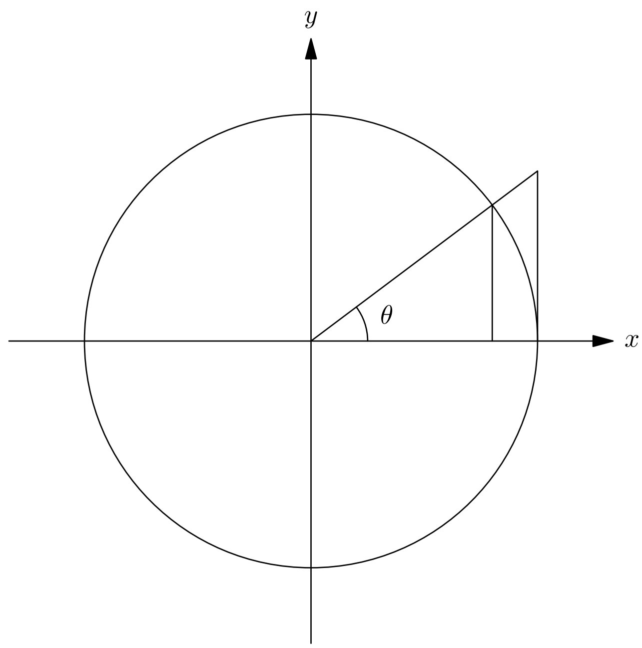
$$\lim_{x \to a} g(x) = L$$

as well. We will see a nice application of this fact to the calculation of the limit $\lim_{x \to 0} \frac{\sin(x)}{x}$ below.

Some trigonometry and the derivative of sine

We use some trigonometry to show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x)$ is differentiable and to find its derivative. Having done that, we will consequently be able to find derivatives of other trigonometric functions.

The diagram below features a circle of radius 1 together with a large triangle and a small one.



The dimensions of the triangles are as follows:

Triangle	Base	Height
Large	1	$\tan(\theta)$
Small	$\cos(\theta)$	$\sin(\theta)$

We consider three areas:

$A(\theta)$:	area of the small triangle
$B(\theta)$:	area of the circular sector between the horizontal and the line defined by θ
$C(\theta)$:	area of the large triangle

From the dimensions above, we see that

$$A(\theta) = \frac{1}{2}\sin(\theta)\cos(\theta) \quad \text{and} \quad C(\theta) = \frac{1}{2}\tan(\theta) = \frac{1}{2}\,\frac{\sin(\theta)}{\cos(\theta)},$$

and the fact that the area of the full circle is π shows that

$$B(\theta) = \frac{\theta}{2\pi}\cdot\pi = \frac{\theta}{2}.$$

Now, the diagram illustrates that

$$A(\theta) < B(\theta) < C(\theta),$$

so

$$\frac{1}{2}\sin(\theta)\cos(\theta) < \frac{\theta}{2} < \frac{1}{2}\,\frac{\sin(\theta)}{\cos(\theta)},$$

i.e.,

$$\cos(\theta) < \frac{\theta}{\sin(\theta)} < \frac{1}{\cos(\theta)}.$$

Therefore, by the Squeeze Theorem,

$$\lim_{\theta\to 0} \frac{\theta}{\sin(\theta)} = 1,$$

so

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1 \tag{4.3}$$

as well.

We now turn to the derivative of sin. First, for any $h \neq 0$,

$$\frac{\cos(h)-1}{h} = \frac{\cos^2(h)-1}{h(\cos(h)+1)} = -\frac{\sin^2(h)}{h(\cos(h)+1)} = -\frac{\sin(h)}{h}\,\frac{\sin(h)}{\cos(h)+1},$$

and this approaches $(-1)\cdot 0 = 0$ as $h \to 0$ by (4.3). That is,

$$\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0. \tag{4.4}$$

Hence, if $x \in \mathbb{R}$,

$$\begin{aligned}\frac{\sin(x+h)-\sin(x)}{h} &= \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\\ &= \cos(x)\,\frac{\sin(h)}{h} + \sin(x)\,\frac{\cos(h)-1}{h},\end{aligned}$$

and in light of (4.3) and (4.4), we see that this approaches

$$\cos(x)\cdot 1 + \sin(x)\cdot 0 = \cos(x)$$

as $h \to 0$. In summary, if $f(x) = \sin(x)$, then f is differentiable, and

$$f'(x) = \cos(x). \tag{4.5}$$

4.4 Derivatives and limits of trigonometric functions

Derivatives of trigonometric functions

Recall the following definitions:

$$\begin{aligned}\tan(x) &= \frac{\sin(x)}{\cos(x)}\\ \cot(x) &= \frac{\cos(x)}{\sin(x)}\\ \sec(x) &= \frac{1}{\cos(x)}\end{aligned}$$

$$\csc(x) = \frac{1}{\sin(x)}$$

These functions, along with sin and cos, are differentiable where they are defined, and one may find their derivatives, as well as that of cos, using the derivative of sin found just above, in combination with differentiation rules and trigonometric identities. We summarize:

$f(x)$	$f'(x)$
$\sin(x)$	$\cos(x)$
$\tan(x)$	$\sec^2(x) = 1 + \tan^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\cos(x)$	$-\sin(x)$
$\cot(x)$	$-\csc^2(x) = -(1 + \cot^2(x))$
$\csc(x)$	$-\csc(x)\cot(x)$

Let us prove three of these, cos, tan, and sec, by way of illustration.

First, for $\frac{d}{dx}(\cos(x))$, we use

$$\sin(x + \tfrac{\pi}{2}) = \sin(x)\cos(\tfrac{\pi}{2}) + \cos(x)\sin(\tfrac{\pi}{2}) = \cos(x).$$

Hence, if $f(x) = \sin(x)$, then

$$\begin{aligned}
\lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} &= \lim_{h\to 0} \frac{\sin(x+h+\frac{\pi}{2}) - \sin(x+\frac{\pi}{2})}{h} \\
&= \lim_{h\to 0} \frac{f(x+\frac{\pi}{2}+h) - f(x+\frac{\pi}{2})}{h} \\
&= f'(x+\tfrac{\pi}{2}) \\
&= \cos(x+\tfrac{\pi}{2}) \quad \text{by (4.5)} \\
&= \cos(x)\cos(\tfrac{\pi}{2}) - \sin(x)\sin(\tfrac{\pi}{2}) \\
&= -\sin(x).
\end{aligned}$$

Next, we turn to $\frac{d}{dx}(\tan(x))$. This uses the quotient rule and the derivatives of sin and cos:

$$\begin{aligned}
\frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) &= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} = \sec^2(x).
\end{aligned}$$

The derivative of sec, too, may be found via the quotient rule:

$$\begin{aligned}
\frac{d}{dx}(\sec(x)) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) &= \frac{0\cos(x) - 1(-\sin(x))}{\cos^2(x)} \\
&= \frac{\sin(x)}{\cos^2(x)} = \sec(x)\tan(x).
\end{aligned}$$

Limits of trigonometric functions

In Example 3.14, we used the fact that

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}. \tag{4.6}$$

Let us now prove this equality. If $x \neq 0$,

$$\begin{aligned}\frac{1-\cos(x)}{x^2} = \frac{1-\cos(x)}{x^2}\,\frac{1+\cos(x)}{1+\cos(x)} &= \frac{1-\cos^2(x)}{x^2(1+\cos(x))} \\ &= \frac{\sin^2(x)}{x^2(1+\cos(x))} \\ &= \left(\frac{\sin(x)}{x}\right)^2 \frac{1}{1+\cos(x)},\end{aligned}$$

so

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{\left(\lim_{x\to 0} \frac{\sin(x)}{x}\right)^2}{\lim_{x\to 0}(1+\cos(x))} = \frac{1^2}{2} = \frac{1}{2}.$$

Example 4.10 If $a, b \neq 0$, find

$$\lim_{x\to 0} \frac{1-\cos(ax)}{1-\cos(bx)}.$$

Solution: In fact, all we need for this problem is Proposition 3.15 and the observation that $(1-\cos(x))/x^2$ has a finite non-zero limit as $x \to 0$. Indeed,

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos(ax)}{1-\cos(bx)} &= \lim_{x\to 0} \frac{1-\cos(ax)}{(ax)^2}\,\frac{(bx)^2}{1-\cos(bx)}\,\frac{a^2}{b^2} \\ &= \left(\lim_{x\to 0} \frac{1-\cos(ax)}{(ax)^2}\right)\left(\lim_{x\to 0} \frac{(bx)^2}{1-\cos(bx)}\right)\frac{a^2}{b^2} \\ &= \left(\lim_{y\to 0} \frac{1-\cos(y)}{y^2}\right)\left(\lim_{y\to 0} \frac{y^2}{1-\cos(y)}\right)\frac{a^2}{b^2} \quad \text{by Prop. 3.15} \\ &= \left(\lim_{x\to 0} \frac{1-\cos(y)}{y^2}\,\frac{y^2}{1-\cos(y)}\right)\frac{a^2}{b^2} \\ &= \left(\lim_{x\to 0} 1\right)\frac{a^2}{b^2} \\ &= \frac{a^2}{b^2}.\end{aligned}$$

Example 4.11 Another problem in the same vein as the previous one is to find the limit

$$\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)},$$

where again $a, b \neq 0$. This time, we use the fact that $\sin(x)/x$ has a finite non-zero limit as $x \to 0$:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)} &= \lim_{x\to 0} \frac{\sin(ax)}{ax} \frac{bx}{\sin(bx)} \frac{a}{b} \\
&= \left(\lim_{x\to 0} \frac{\sin(ax)}{ax}\right) \left(\lim_{x\to 0} \frac{bx}{\sin(bx)}\right) \frac{a}{b} \\
&= \left(\lim_{y\to 0} \frac{\sin(y)}{y}\right) \left(\lim_{y\to 0} \frac{y}{\sin(y)}\right) \frac{a}{b} \quad \text{by Prop. 3.15} \\
&= \left(\lim_{y\to 0} \frac{\sin(y)}{y} \frac{y}{\sin(y)}\right) \frac{a}{b} \\
&= \left(\lim_{x\to 0} 1\right) \frac{a}{b} \\
&= \frac{a}{b}.
\end{aligned}$$

4.5 The chain rule

Theorem 4.12 *If $f : D_1 \to D_2$ is differentiable at $x \in D_1$ and $g : D_2 \to \mathbb{R}$ is differentiable at $f(x)$, then $g \circ f : D_1 \to \mathbb{R}$ is differentiable at x, and*

$$(g \circ f)'(x) = g'(f(x))f'(x).$$

We prove the chain rule in Section 1 of the Appendix.

Example 4.13 Find $j'(x)$ where $j(x) = (\sin(x) + \cos(x))^3$.

Solution: Here, $j = g \circ f$ where

$$\begin{aligned}
f(x) &= \sin(x) + \cos(x), \\
g(y) &= y^3,
\end{aligned}$$

so

$$\begin{aligned}
j'(x) &= g'(f(x))f'(x) \\
&= 3f(x)^2(\cos(x) - \sin(x)) \\
&= 3(\sin(x) + \cos(x))^2(\cos(x) - \sin(x)).
\end{aligned}$$

Example 4.14 Find $j'(x)$ where $j(x) = \tan(x \sin(x))$.

Solution: This time, $j = g \circ f$ where

$$\begin{aligned}
f(x) &= x\sin(x), \\
g(y) &= \tan(y),
\end{aligned}$$

so

$$\begin{aligned}
j'(x) &= g'(f(x))f'(x) \\
&= \sec^2(x\sin(x))(\sin(x) + x\cos(x)) \quad \text{(chain rule and product rule).}
\end{aligned}$$

When applying the chain rule to a composition of several functions, not just two, it can be useful to use the $\frac{d}{dx}$ notation, as illustrated in the following examples.

Example 4.15 Find
$$\frac{d}{dx}\left(\sec((x^3+1)^{10})\right).$$

Solution: Recall that $\frac{d}{dy}(\sec(y)) = \sec(y)\tan(y)$, so

$$\begin{aligned}\frac{d}{dx}\left(\sec((x^3+1)^{10})\right) &= \sec((x^3+1)^{10})\tan((x^3+1)^{10})\frac{d}{dx}((x^3+1)^{10}) \\ &= \sec((x^3+1)^{10})\tan((x^3+1)^{10})\cdot 10(x^3+1)^9\cdot 3x^2 \\ &= 30x^2(x^3+1)^9\sec((x^3+1)^{10})\tan((x^3+1)^{10}).\end{aligned}$$

Example 4.16 Find
$$\frac{d}{dx}\left(\cot\left(\left(\frac{x+1}{x-1}\right)^4\right)\right).$$

Solution: This time, we use the fact that $\frac{d}{dy}(\cot(y)) = -\csc^2(y)$. Hence,

$$\begin{aligned}\frac{d}{dx}\left(\cot\left(\left(\frac{x+1}{x-1}\right)^4\right)\right) &= -\csc^2\left(\left(\frac{x+1}{x-1}\right)^4\right)\frac{d}{dx}\left(\left(\frac{x+1}{x-1}\right)^4\right) \\ &= -\csc^2\left(\left(\frac{x+1}{x-1}\right)^4\right)\cdot 4\left(\frac{x+1}{x-1}\right)^3\frac{d}{dx}\left(\frac{x+1}{x-1}\right) \\ &= -\csc^2\left(\left(\frac{x+1}{x-1}\right)^4\right)\cdot 4\left(\frac{x+1}{x-1}\right)^3\frac{(x-1)-(x+1)}{(x-1)^2} \\ &= 8\csc^2\left(\left(\frac{x+1}{x-1}\right)^4\right)\frac{(x+1)^3}{(x-1)^5}.\end{aligned}$$

Example 4.17 Find $f'(x)$ where $f(x) = \sin^3\Big(\cos((x^2+1)^4)\Big)$.

Solution:

$$\begin{aligned}f'(x) &= 3\sin^2\Big(\cos((x^2+1)^4)\Big)\frac{d}{dx}\left(\sin\Big(\cos((x^2+1)^4)\Big)\right) \\ &= 3\sin^2\Big(\cos((x^2+1)^4)\Big)\cos\Big(\cos((x^2+1)^4)\Big)\frac{d}{dx}\Big(\cos((x^2+1)^4)\Big) \\ &= 3\sin^2\Big(\cos((x^2+1)^4)\Big)\cos\Big(\cos((x^2+1)^4)\Big) \\ &\quad\cdot\Big(-\sin((x^2+1)^4)\Big)\frac{d}{dx}((x^2+1)^4) \\ &= 3\sin^2\Big(\cos((x^2+1)^4)\Big)\cos\Big(\cos((x^2+1)^4)\Big) \\ &\quad\cdot\Big(-\sin((x^2+1)^4)\Big)\cdot 4(x^2+1)^3\cdot 2x\end{aligned}$$

$$= -24x(x^2+1)^3 \sin^2\Big(\cos((x^2+1)^4)\Big) \cos\Big(\cos((x^2+1)^4)\Big) \cdot \sin((x^2+1)^4).$$

Example 4.18 Find $f'(x)$ where

$$f(x) = \sqrt{x + \sqrt{x + \sqrt{x+1}}} = \Big(x + \big(x + (x+1)^{\frac{1}{2}}\big)^{\frac{1}{2}}\Big)^{\frac{1}{2}}.$$

Solution:

$$\begin{aligned} f'(x) &= \frac{1}{2}\Big(x + \big(x + (x+1)^{\frac{1}{2}}\big)\Big)^{-\frac{1}{2}} \frac{d}{dx}\Big(x + \big(x + (x+1)^{\frac{1}{2}}\big)^{\frac{1}{2}}\Big) \\ &= \frac{1}{2}\Big(x + \big(x + (x+1)^{\frac{1}{2}}\big)\Big)^{-\frac{1}{2}} \left(1 + \frac{1}{2}\big(x + (x+1)^{\frac{1}{2}}\big)^{-\frac{1}{2}} \frac{d}{dx}\big(x + (x+1)^{\frac{1}{2}}\big)\right) \end{aligned}$$

and

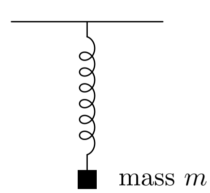
$$\frac{d}{dx}\big(x + (x+1)^{\frac{1}{2}}\big) = 1 + \frac{1}{2}(x+1)^{-\frac{1}{2}},$$

so

$$\begin{aligned} f'(x) &= \frac{1}{2}\Big(x + \big(x + (x+1)^{\frac{1}{2}}\big)\Big)^{-\frac{1}{2}} \\ &\quad \cdot \left(1 + \tfrac{1}{2}\big(x + (x+1)^{\frac{1}{2}}\big)^{-\frac{1}{2}}\big(1 + \tfrac{1}{2}(x+1)^{-\frac{1}{2}}\big)\right) \\ &= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x+1}}}} \left(1 + \frac{1 + \frac{1}{2\sqrt{x+1}}}{2\sqrt{x + \sqrt{x+1}}}\right). \end{aligned}$$

Simple harmonic oscillation

Consider an object of mass m suspended on a spring:



Let its displacement from the equilibrium position be $y(t)$ at time t. If the object is moved by hand to some position y_0 and then released there from rest, then under idealized conditions (no air friction or spring damping), the position $y(t)$ is given by

$$y(t) = y_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right),$$

where k is the *spring constant*. Observe that upon differentiating y twice using the chain rule, we obtain

$$\begin{aligned} y'(t) &= -y_0 \sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{k}{m}}\, t\right), \\ y''(t) &= -y_0 \,\frac{k}{m} \cos\left(\sqrt{\frac{k}{m}}\, t\right) \\ &= -\frac{k}{m}\, y(t), \end{aligned}$$

so

$$my''(t) + ky(t) = 0.$$

This last equation is known as a *differential equation*. In fact, in the situation of this spring-mass system, and in many other models besides, the order of investigation is often instead to

- first find a differential equation satisfied by the quantity to be modelled, and
- then solve the differential equation to obtain a function describing the system at hand.

4.6 Implicit differentiation

There is nothing very special about implicit differentiation. It is just differentiation but in a context that looks a bit different.

Example 4.19 Suppose that f is a differentiable function such that

$$f(x)^3 + xf(x) = 2$$

for all x. Differentiate both sides to obtain an equation involving $f'(x)$. Use the product rule and chain rule as desired or as necessary.

Solution: The right-hand side is constant and therefore has a zero derivative. For the left-hand side, we use the product rule on the second term,

$$\frac{d}{dx}(xf(x)) = f(x) + xf'(x)$$

and the chain rule on the first,

$$\frac{d}{dx}(f(x)^3) = 3f(x)^2 f'(x).$$

In summary,

$$\begin{aligned} \text{derivative of the left-hand side} &= 3f(x)^2 f'(x) + f(x) + xf'(x), \\ \text{derivative of the right-hand side} &= 0, \end{aligned}$$

so

$$3f(x)^2 f'(x) + f(x) + xf'(x) = 0.$$

Example 4.20 Suppose that f is a differentiable function such that

$$f(x)^4 - (x^2+1)f(x) = x^3$$

for all x. Differentiate both sides to obtain an equation involving $f'(x)$, and hence express $f'(x)$ in terms of $f(x)$.

Solution: We differentiate both sides of the given equation, using the chain rule and any other necessary differentiation rules:

$$\begin{aligned} 4f(x)^3 f'(x) - 2xf(x) - (x^2+1)f'(x) &= 3x^2, \\ \text{i.e.,} \quad (4f(x)^3 - x^2 - 1)f'(x) &= 2xf(x) + 3x^2, \\ \text{i.e.,} \quad f'(x) &= \frac{2xf(x)+3x^2}{4f(x)^3 - x^2 - 1}. \end{aligned}$$

It is common in problems of implicit differentiation to let the function be called y and to omit the dependence of y on x, as in the next example. Take heed, though, that when we differentiate a term involving y, we must remember that y is a function of x. For example, the derivative of $\sin(y)$ (with respect to x) is not $\cos(y)$ but $\cos(y)y'$ by the chain rule. Differentiation is always considered to be with respect to x in this context.

Example 4.21 Suppose that y is a function of x such that $\tan(x+y^2) = 1$. Find y' in terms of x and y.

Solution: We differentiate both sides of the equation $1 = \tan(x+y^2)$ with respect to x, using the chain rule:

$$\begin{aligned} 0 = \sec^2(x+y^2)\frac{d}{dx}(x+y^2) &= \sec^2(x+y^2)\left(1 + \frac{d}{dx}(y^2)\right) \\ &= \sec^2(x+y^2)(1+2yy'). \end{aligned}$$

Hence, because $\sec^2(x+y^2) > 0$, we obtain $1 + 2yy' = 0$, i.e.,

$$y' = -\frac{1}{2y}.$$

Tangent lines via implicit differentiation

Recall that if f is a differentiable function and x_0 a point in its domain, then the tangent line to the graph of f at $(x_0, f(x_0))$ has equation

$$y - f(x_0) = f'(x_0)(x - x_0).$$

This is an application of the point-slope formula for the equation of a line of slope m passing through a point (x_0, y_0), that equation being

$$y - y_0 = m(x - x_0).$$

Example 4.22 For some open interval $I \subseteq \mathbb{R}$ containing $\pi/6$, there is a differentiable function $y : I \to \mathbb{R}$ such that

- $\sin(xy(x)) = 1/2$ for all $x \in I$, and
- $y(\pi/6) = 1$.

Given this information, find the equation of the tangent line to the graph of y where $x = \pi/6$.

Solution: To obtain the tangent line at $x = \pi/6$, we first find $y'(\pi/6)$. For this, we can differentiation both sides of the equation $\sin(xy(x)) = 1/2$, remembering the chain rule:

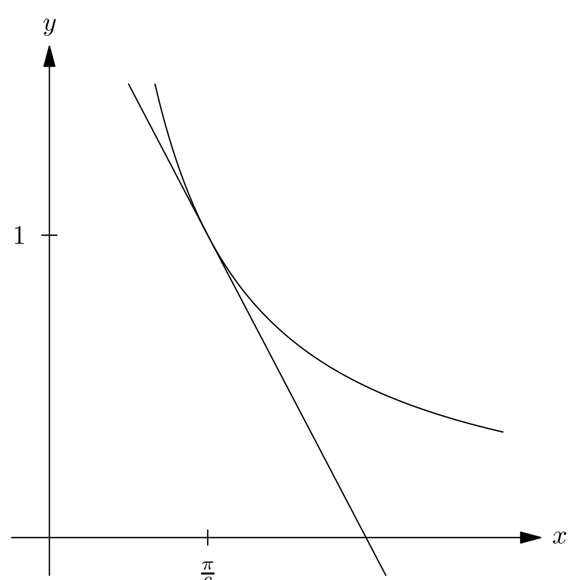
$$\cos(xy(x))(y(x) + xy'(x)) = 0.$$

This holds for all $x \in I$ and, in particular, for $x = \pi/6$, so making this substitution and noting that $y(\pi/6) = 1$, we have

$$\cos(\tfrac{\pi}{6} \cdot 1)(1 + \tfrac{\pi}{6} y'(\tfrac{\pi}{6})) = 0,$$
$$\text{i.e.,} \quad y'(\tfrac{\pi}{6}) = -\frac{6}{\pi} \quad \text{because } \cos(\tfrac{\pi}{6}) \neq 0.$$

Hence, the equation of the tangent line, via the point-slope formula, is

$$y - 1 = -\tfrac{6}{\pi}(x - \tfrac{\pi}{6}),$$
$$\text{i.e.,} \quad y = -\tfrac{6}{\pi}\, x + 2.$$



(Unfortunately, when the symbol y is used for a function of x, it can be confusing when one comes to write the equation of a line, as at the end of the example above, because there y refers not to the function but to the second coordinate of a point (x, y) of the plane.)

Example 4.23 Consider the graph of the function $y : \mathbb{R} \to \mathbb{R}$ defined by $y(x) = 4x^3 + 4x^2 - 17x + 1$. Find all tangent lines to the curve that have slope -2.

Solution: We find all x where $y'(x) = -2$, i.e., $12x^2 + 8x - 17 = -2$. This equation can be rearranged to

$$\begin{aligned} 12x^2 + 8x - 15 &= 0, \\ \text{i.e.,} \quad (6x-5)(2x+3) &= 0, \end{aligned}$$

which has the solutions $x = -3/2$ and $x = 5/6$. Thus, the points where the tangent line has slope -2 are

$$\begin{aligned} (-\tfrac{3}{2}, f(-\tfrac{3}{2})) &= (-\tfrac{3}{2}, 22), \\ (\tfrac{5}{6}, f(\tfrac{5}{6})) &= (\tfrac{5}{6}, -\tfrac{218}{27}). \end{aligned}$$

The equations of the corresponding tangent lines are, in the same order as above,

$$\begin{aligned} y - 22 &= -2(x + \tfrac{3}{2}), \\ \text{i.e.,} \quad y &= -2x + 19, \\[1em] \text{and} \quad y + \tfrac{218}{27} &= -2(x - \tfrac{5}{6}), \\ \text{i.e.,} \quad y &= -2x - \tfrac{173}{27}. \end{aligned}$$

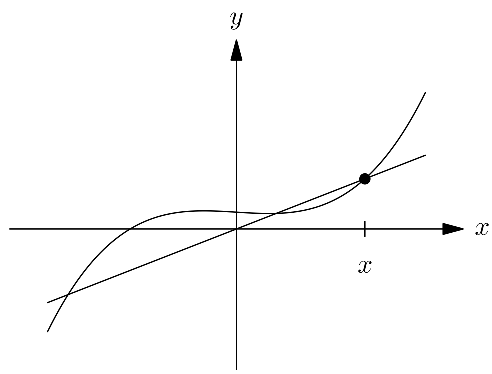
Example 4.24 Consider the curve C with equation $y = \frac{1}{81}(9x^3 - 5x + 18)$. Find all real numbers m such that the line L with equation $y = mx$ is tangent to C.

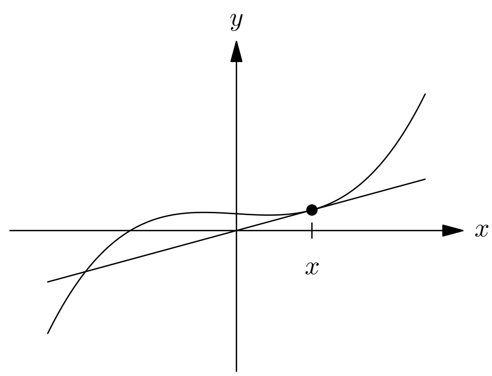
Solution: We present two solutions. For our first, we consider y as a function of x and consider points $(x, y(x))$ on C. Specifically, we wish to find points x such that the slope of the line through $(0, 0)$ and $(x, y(x))$ is equal to the slope of the tangent line to C at $(x, y(x))$, in other words, such that

$$\begin{aligned} \frac{y(x)}{x} &= y'(x), \\ \text{i.e.,} \quad y(x) &= xy'(x), \\ \text{i.e.,} \quad \frac{1}{81}(9x^3 - 5x + 18) &= x \cdot \frac{1}{81}(27x^2 - 5) = \frac{1}{81}(27x^3 - 5x), \\ \text{i.e.,} \quad 18x^3 &= 18. \end{aligned}$$

This has the unique solution $x = 1$, and the corresponding slope in this case is $y'(1) = \frac{1}{81}(27 - 5) = \frac{22}{81}$. Therefore, $m = \frac{22}{81}$ is the answer.

The method is illustrated by the two diagrams below. In the first diagram, the line through $(0, 0)$ and the point $(x, y(x))$, marked with a dot, has a different slope from the tangent line at $(x, y(x))$. In the second diagram, the slopes are equal, showing the situation where a line through $(0, 0)$ is tangent to C.



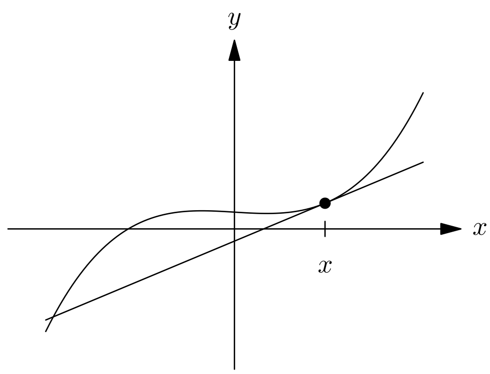


For our second solution, we find the equation of the tangent line to C at a general point $(x_0, y(x_0))$ of C, again considering y as a function of x, and choose x_0 so that the tangent line passes through $(0, 0)$. The tangent line at $(x_0, y(x_0))$ has slope $y'(x_0) = \frac{1}{81}(27x_0^2 - 5)$ and therefore equation

$$\begin{aligned} y - y(x_0) &= \frac{1}{81}(27x_0^2 - 5)(x - x_0), \\ \text{i.e.,} \quad y &= y(x_0) + \frac{1}{81}(27x_0^2 - 5)(x - x_0) \\ &= \frac{1}{81}(9x_0^3 - 5x_0 + 18) + \frac{1}{81}(27x_0^2 - 5)(x - x_0) \\ &= \frac{1}{81}(27x_0^2 - 5)x - \frac{2}{9}x_0^3 + \frac{2}{9}. \end{aligned}$$

This tangent line passes through the point $(0, 0)$ if and only if $-\frac{2}{9}x_0^3 + \frac{2}{9} = 0$, if and only if $x_0 = 1$, and in this case the slope is $y'(1) = \frac{1}{81}(27 - 5) = \frac{22}{81}$.

To illustrate this second approach, we refer to the diagram below, which shows the tangent line to C at an arbitrary point $(x, y(x))$, marked with a dot. Notice how the tangent line does not pass through $(0, 0)$ but will do if we choose x appropriately, resulting in the second diagram of the first solution.



Normal lines

Let P be a point on a curve C. A line passing through P is said to be a *normal line* to C if it is orthogonal (perpendicular) to the tangent line at P (assuming the tangent line exists).

If the tangent line at a point P on C exists and has slope m, then the normal line at P has slope $-1/m$.

Example 4.25 Consider again the function y, defined close to $x = \frac{\pi}{6}$, that satisfies

$$\sin(xy(x)) = \frac{1}{2} \quad \text{and} \quad y(\tfrac{\pi}{6}) = 1,$$

as in Example 4.22. Find the equation of the normal line to the curve $\sin(xy(x)) = \frac{1}{2}$ at $(\frac{\pi}{6}, 1)$.

Solution: The tangent line has slope $y'(\frac{\pi}{6}) = -\frac{6}{\pi}$, as we saw in that earlier example, so the normal line has slope $\frac{\pi}{6}$ and therefore equation

$$y - 1 = \frac{\pi}{6}\left(x - \frac{\pi}{6}\right),$$
$$\text{i.e.,} \quad y = \frac{\pi}{6}\,x + 1 - \frac{\pi^2}{36}.$$

4.7 Invertible functions

Injective functions

A function $f : X \to Y$ is said to be *injective* if for all $x_1, x_2 \in X$,

$$f(x_1) = f(x_2) \implies x_1 = x_2,$$

i.e., $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Equivalently,

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2),$$

i.e., $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

Example 4.26 If S is the set of students in the university, then the function $f : S \to \mathbb{Z}$ mapping a given student to their Student ID is injective, because different students have different Student IDs.

Example 4.27 If S is again the set of students in the university, and $\Omega = \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{Z}\}$, the alphabet, then the function $f : S \to \Omega$ mapping a given student to the first letter of their surname is not injective, because there are at least two different students whose surnames begin with the same letter. (How do we know this?)

Example 4.28 The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^7$ is injective. Let us show this. Suppose that $f(x_1) = f(x_2)$, i.e., $x_1^7 = x_2^7$. If $x_2 = 0$, then the equation $x_1^7 = x_2^7 = 0$ shows that $x_1 = 0$ as well. Otherwise, we may divide both sides by x_2^7 to obtain

$$\left(\frac{x_1}{x_2}\right)^7 = 1,$$

i.e., $a^7 = 1$ where $a = x_1/x_2$, and because 7 is odd, $a > 0$. We can now rule out both $a < 1$ and $a > 1$. If $a < 1$, then $0 < a < 1$, so $a^2 < a$, and so $(a^2)^7 < a^7$, i.e., $(a^7)^2 < a^7$, i.e., $1 < 1$, a contradiction. If, instead, $a > 1$, then $a^2 > a$, so $(a^2)^7 > a^7$, i.e., $(a^7)^2 > a^7$, i.e., $1 > 1$, again a contradiction. Thus, $a = 1$, i.e., $x_1 = x_2$.

Example 4.29 The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not injective, because, for example, $f(-1) = f(1)$.

Surjective functions

A function $f : X \to Y$ is said to be *surjective* if for every $y \in Y$, there is $x \in X$ such that $f(x) = y$. Thus, f is surjective if and only if its range is equal to its codomain, Y.

Example 4.30 The function $f : \mathbb{R} \to [0, \infty)$ defined by $f(x) = x^2$ is surjective, because given any $y \in [0, \infty)$, we can find $x \in \mathbb{R}$ such that $f(x) = y$. Indeed,

$$f(x) = y \iff x^2 = y \iff x = \pm\sqrt{y}.$$

However, the function $g : \mathbb{R} \to \mathbb{R}$ defined again by $g(x) = x^2$ is not surjective. The codomain is now $\mathbb{R}$, which is larger than the range, $[0, \infty)$.

Example 4.31 Consider

$$f : \mathbb{R} \setminus \{-1, 1\} \to \mathbb{R} \setminus (-1, 0]$$

satisfying

$$f(x) = \tfrac{1}{x^2 - 1}.$$

This function is surjective. To show this, take a general $y \in \mathbb{R} \smallsetminus (-1, 0]$ and consider the equation $f(x) = y$, i.e., $\frac{1}{x^2-1} = y$. This has a solution for $x \in \mathbb{R} \smallsetminus \{-1, 1\}$:

$$\begin{aligned} f(x) = y &\iff \frac{1}{x^2 - 1} = y \\ &\iff x^2 - 1 = \frac{1}{y} \\ &\iff x = \pm\sqrt{1 + \frac{1}{y}}. \end{aligned}$$

Thus, f is surjective, and for any $y \in \mathbb{R} \smallsetminus (-1, 0]$,

$$f\left(\sqrt{1 + \frac{1}{y}}\right) = y.$$

Alternatively, $f\left(-\sqrt{1 + \frac{1}{y}}\right) = y$; both possibilities work.

Bijective functions

A function $f : X \to Y$ is said to be *invertible*, or *bijective*, if it is both injective and surjective. This means that for every $y \in Y$, there is a *unique* $x \in X$ such that $f(x) = y$.

Example 4.32 If n is an odd positive integer, then the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^n$ is bijective. The injectivity of f is shown in exactly the same way as the case $n = 7$ in Example 4.28. The surjectivity is harder to show, but it follows from a result we will see later, the Intermediate Value Theorem.

If $f : X \to Y$ is invertible (bijective), it has an inverse function $f^{-1} : Y \to X$. If $y \in Y$, then $f^{-1}(y)$ is equal to the unique $x \in X$ such that $f(x) = y$. That is, to find $f^{-1}(y)$, solve the equation $f(x) = y$ for x in terms of y, and then $f^{-1}(y) = x$. The following relations hold, and in fact characterize f^{-1}:

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for all } x \in X, \\ \text{and} \quad f(f^{-1}(y)) &= y \quad \text{for all } y \in Y. \end{aligned}$$

Example 4.33 Show that the function $f : [0, \infty) \to [1, \infty)$ defined by $f(x) = x^2 + 1$ is invertible, and find its inverse function.

Solution: Let $y \in [1, \infty)$. If $x \in [0, \infty)$, then

$$\begin{aligned} f(x) = y &\iff x^2 + 1 = y \\ &\iff x^2 = y - 1 \\ &\iff x = \sqrt{y - 1} \quad \text{because } x \geq 0. \end{aligned}$$

Therefore, we have a unique solution for any given y, so f is invertible and

$$f^{-1}(y) = \sqrt{y - 1}.$$

Example 4.34 Repeat the previous example with the function $g : (-\infty, 0] \to [1, \infty)$ defined by $g(x) = x^2 + 1$.

Solution: This time, we work with $x \in (-\infty, 0]$, because that is the domain of g. Let us see how this changes the expression for the inverse function:

$$\begin{aligned} f(x) = y &\iff x^2 + 1 = y \\ &\iff x^2 = y - 1 \\ &\iff x = -\sqrt{y-1} \quad \text{because } x \leq 0. \end{aligned}$$

We again have a unique solution for any given y, so f is invertible, but this time,

$$g^{-1}(y) = -\sqrt{y-1}.$$

Example 4.35 Consider $f : (-\infty, -1] \to [0, \infty)$ defined by $f(x) = \sqrt{-1-x}$. Show that f is invertible, and find its inverse.

Solution: We show that for all $y \in [0, \infty)$, there is a unique $x \in (-\infty, 1]$ such that $f(x) = y$, i.e., $\sqrt{-1-x} = y$. Indeed, if $x \in (-\infty, 1]$, then

$$\begin{aligned} \sqrt{-1-x} = y &\iff -1 - x = y^2 \quad \text{because } y \geq 0 \\ &\iff x = -1 - y^2. \end{aligned}$$

Thus, f is invertible, and $f^{-1}(y) = -1 - y^2$.

Remark. Often, we consider f^{-1} as a function of a variable x instead of y, and then we write $f^{-1}(x)$ instead of $f^{-1}(y)$. For instance, in Example 4.35, we may write

$$f^{-1}(x) = -1 - x^2$$

instead of

$$f^{-1}(y) = -1 - y^2.$$

Proposition 4.36 *If $f : X \to Y$ is an invertible function, where X and Y are subsets of $\mathbb{R}$, then the graph of f^{-1} is the reflection in the line $y = x$ of the graph of f.*

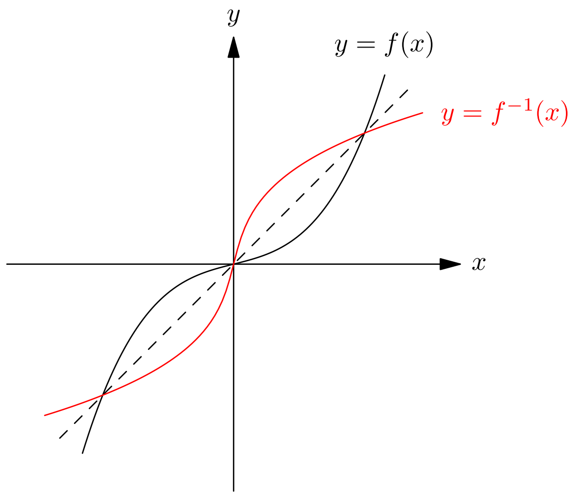
Proof. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = x$, i.e., $T(x, y) = (y, x)$, and let Γ_g denote the graph of a given function g. Then

$$\begin{aligned} \{T(P) \mid P \in \Gamma_f\} &= \{T(x, f(x)) \mid x \in X\} \\ &= \{(f(x), x) \mid x \in X\} \\ &= \{(f(f^{-1}(y)), f^{-1}(y)) \mid y \in Y\} \\ &= \{(y, f^{-1}(y)) \mid y \in Y\} \end{aligned}$$

$$= \Gamma_{f^{-1}}.$$

■

For example, here is an invertible function f and its inverse:



Derivative of an inverse function

Proposition 4.37 *If f is a bijective function such that both f and f^{-1} are differentiable, then*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Proof. By definition, $(f \circ f^{-1})(x) = x$, so

$$1 = (f \circ f^{-1})'(x) = f'(f^{-1}(x))(f^{-1})'(x)$$

by the chain rule. ■

Example 4.38 Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 + x$. You may assume that f is bijective. Find the equation of the tangent line to the curve $y = f^{-1}(x)$ at the point where $x = 30$ and $y = f^{-1}(30)$.

Solution: We first find the slope of the curve at the point in question, which is

$$(f^{-1})'(30) = \frac{1}{f'(f^{-1}(30))}.$$

By inspection, we find that $f(3) = 30$, so $f^{-1}(30) = 3$, and we also have $f'(x) = 3x^2 + 1$, so

$$(f^{-1})'(30) = \frac{1}{3 \cdot 3^2 + 1} = \frac{1}{28}.$$

Hence, the tangent line to the curve at $(30, 3)$ has equation

$$\begin{aligned} y - 3 &= \frac{1}{28}(x - 30), \\ \text{i.e.,} \quad y &= \frac{1}{28}x + 3 - \frac{15}{14} \\ &= \frac{1}{28}x + \frac{27}{14}. \end{aligned}$$

4.8 Exponentials and logarithms

If $a > 0$ and $a \neq 1$, then for all $x > 0$ there is a unique $y \in \mathbb{R}$ such that $a^y = x$. This value y is called the *logarithm* of x to the base a, written $\log_a(x)$. Thus, by definition,

$$a^{\log_a(x)} = x \quad \text{and} \quad \log_a(a^y) = y$$

for all $x > 0$ and all $y \in \mathbb{R}$.

Properties of exponentials and logarithms

If $a, b > 0$ and $x, y \in \mathbb{R}$, then

(i) $a^{x+y} = a^x a^y$,

(ii) $a^{xy} = (a^x)^y$,

(iii) $a^{-x} = \frac{1}{a^x}$ (because $a^0 = 1$),

(iv) $(ab)^x = a^x b^x$.

Now assume that $a > 0$, $a \neq 1$, $x_1, x_2 > 0$, and $r \in \mathbb{R}$. Then

(i) $\log_a(x_1 x_2) = \log_a(x_1) + \log_a(x_2)$,

(ii) $\log_a(x_1/x_2) = \log_a(x_1) - \log_a(x_2)$,

(iii) $\log_a(x^r) = r \log_a(x)$.

Change of base of the logarithm

If $a, b \neq 1$ are positive and x is also positive, then

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

To see this, we apply $\log_a$ to both sides of the equality

$$x = b^{\log_b(x)}$$

to obtain

$$\log_a(x) = \log_a(b^{\log_b(x)}) = \log_b(x) \log_a(b).$$

Example 4.39 If $x > 0$, then

$$\log_{1/3}(x) = \frac{\log_3(x)}{\log_3(1/3)} = \frac{\log_3(x)}{-1} = -\log_3(x).$$

The natural logarithm

For all $a > 0$, the function $\mathbb{R} \to \mathbb{R}$ mapping x to a^x is differentiable. It is a fact that there is a unique positive real number, denoted e, with the property that the function $x \mapsto e^x$ is unchanged upon differentiation, that is

$$\frac{d}{dx}(e^x) = e^x.$$

The value of e is approximately 2.718 28. The logarithm to the base e is denoted in this course by ln and is called the *natural logarithm*. The natural logarithm function is differentiable and satisfies

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}.$$

To show this formula, we may use the chain rule: If $g(y) = e^y$ and $f(x) = \ln(x)$, then $g' = g$ and $x = g(f(x))$, so differentiating gives

$$1 = g'(f(x))f'(x) = g(f(x))f'(x) = xf(x).$$

We also have these formulas, valid for real numbers λ and a with $a > 0$ and $a \neq 1$:

$$\frac{d}{dx}(\log_a(x)) = \frac{1}{\ln(a)x},$$

$$\frac{d}{dx}(e^{\lambda x}) = \lambda e^{\lambda x},$$

$$\frac{d}{dx}(\ln(\lambda x)) = \frac{1}{x} \quad \text{if } x\lambda > 0.$$

For example, the last one may be proven via the chain rule thus:

$$\frac{d}{dx}(\ln(\lambda x)) = \frac{1}{\lambda x} \cdot \lambda = \frac{1}{\lambda}.$$

More general than the formula $\frac{d}{dx}(e^{\lambda x}) = \lambda e^{\lambda x}$ above (valid when λ is constant) is the equality

$$\frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)},$$

valid for any differentiable function f. Of course, this is a particular case of the chain rule.

Example 4.40 If $a > 0$, find $\frac{d}{dx}(a^x)$.

Solution: Observe that

$$a^x = e^{\ln(a^x)} = e^{x\ln(a)},$$

so

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\ln(a)}) = \ln(a)e^{x\ln(a)} = \ln(a)a^x.$$

Example 4.41 Find $f'(x)$ where

$$f(x) = \cos(x)^{x^4}.$$

Solution: First,

$$f(x) = \cos(x)^{x^4} = e^{x^4\ln(\cos(x))},$$

so differentiating gives

$$\begin{aligned}
f'(x) &= e^{x^4\ln(\cos(x))}\frac{d}{dx}(x^4\ln(\cos(x))) \\
&= \cos(x)^{x^4}\Big(4x^3\ln(\cos(x)) + x^4\frac{d}{dx}\big(\ln(\cos(x))\big)\Big) \\
&= \cos(x)^{x^4}\Big(4x^3\ln(\cos(x)) + x^4\frac{-\sin(x)}{\cos(x)}\Big) \quad \text{(chain rule again)} \\
&= \cos(x)^{x^4}\Big(4x^3\ln(\cos(x)) - x^4\tan(x)\Big).
\end{aligned}$$

Example 4.42 Find $f'(x)$ where

$$f(x) = e^{e^{e^x}}.$$

Solution: The chain rule, used twice, gives

$$\begin{aligned}
f'(x) = e^{e^{e^x}}\frac{d}{dx}\left(e^{e^x}\right) &= e^{e^{e^x}}e^{e^x}\frac{d}{dx}(e^x) \\
&= e^{e^{e^x}}e^{e^x}e^x \\
&= e^{e^{e^x}+e^x+x}.
\end{aligned}$$

Logarithmic differentiation

Another approach for handling derivatives of the form $\frac{d}{dx}\Big(g(x)^{j(x)}\Big)$ is *logarithmic differentiation*. While appearing at first sight to be different from the approach of the foregoing examples, it is only cosmetically different, being fundamentally the same idea. It uses this fact, a simple consequence of the chain rule:

$$\frac{d}{dx}\big(\ln(f(x))\big) = \frac{f'(x)}{f(x)}. \tag{4.7}$$

Example 4.43 Let us repeat Example 4.41 using (4.7). Taking $f(x) = \cos(x)^{x^4}$ again, we have

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{d}{dx}\big(\ln(f(x))\big)\\ &= \frac{d}{dx}(x^4 \ln(\cos(x)))\\ &= 4x^3 \ln(\cos(x)) + x^4 \frac{d}{dx}\big(\ln(\cos(x))\big)\\ &= 4x^3 \ln(\cos(x)) + x^4 \frac{-\sin(x)}{\cos(x)}\\ &= 4x^3 \ln(\cos(x)) - x^4 \tan(x),\end{aligned}$$

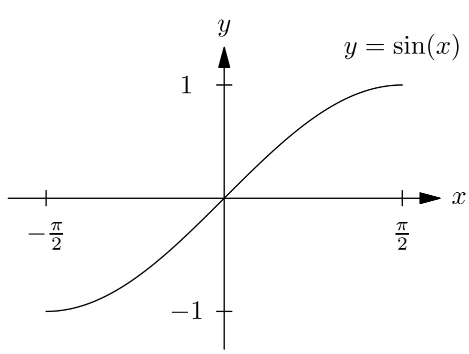
so multiplying by $f(x)$ gives

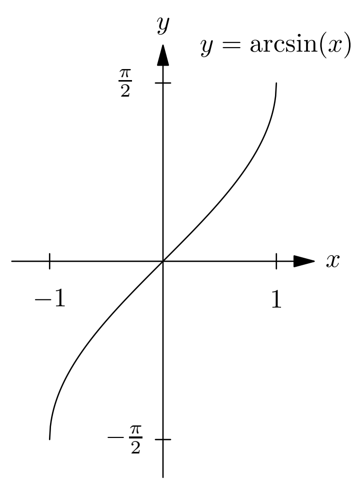
$$f'(x) = \cos(x)^{x^4}\Big(4x^3 \ln(\cos(x)) - x^4 \tan(x)\Big)$$

as before. Observe that the calculations were identical to those in Example 4.41. Only the presentation differed.

4.9 Inverse trigonometric functions

Every trigonometric function is invertible if its domain is restricted suitably. For example, if we restrict sine to a function $[-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$, then it is invertible: For every $y \in [-1, 1]$, there is a unique $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin(x) = y$. The interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is called the *principal domain* of sine, and the inverse function $[-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is denoted arcsin or $\sin^{-1}$. We will use arcsin.





Here are some common values of arcsin:

x	$\arcsin(x)$
-1	$-\frac{\pi}{2}$
$-\frac{\sqrt{3}}{2}$	$-\frac{\pi}{3}$
$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$
$-\frac{1}{2}$	$-\frac{\pi}{6}$
0	0
$\frac{1}{2}$	$\frac{\pi}{6}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$
1	$\frac{\pi}{2}$

Some caution is required when using inverse trigonometric functions, such as arcsin. While the relation

$$\arcsin(\sin(x)) = x$$

holds when $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, it does not hold for all $x \in \mathbb{R}$.

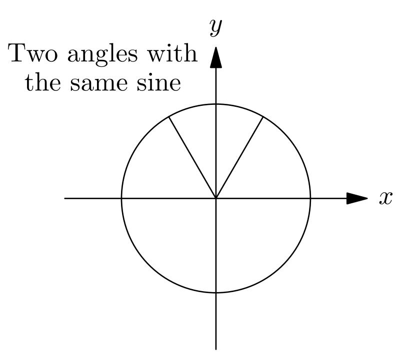
Example 4.44 Find $\arcsin(\sin(\frac{2\pi}{3}))$.

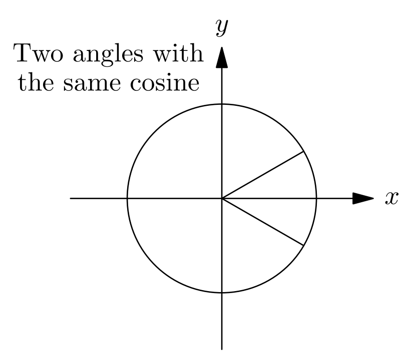
Solution 1:

$$\arcsin(\sin(\tfrac{2\pi}{3})) = \arcsin(\tfrac{\sqrt{3}}{2}) = \tfrac{\pi}{3}.$$

The point is that $\frac{2\pi}{3} \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$. Rather, the unique angle in this interval whose sine is $\frac{\sqrt{3}}{2}$ is $\frac{\pi}{3}$.

Solution 2: We do not need to calculate $\sin(\frac{2\pi}{3})$. It is enough simply to find the unique $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin(\theta) = \sin(\frac{2\pi}{3})$, and this is $\frac{\pi}{3}$. See the left-hand diagram below for an illustration of the relationship between the angles.





Similarly,

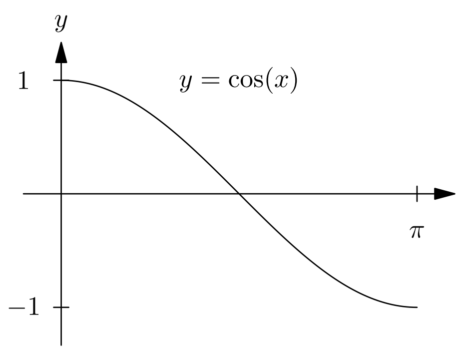
$$\arcsin(\sin(\tfrac{7\pi}{6})) = -\tfrac{\pi}{6}.$$

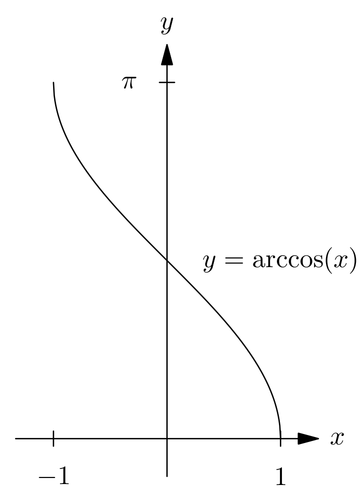
Example 4.45 Find $\arcsin(\sin(\frac{198\pi}{151}))$.

Solution: Let $\theta = \frac{198\pi}{151}$. We are looking for the unique $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin(\phi) = \sin(\theta)$. Because $\sin(\theta) = \sin(\pi - \theta)$ and $\pi - \theta = -\frac{47\pi}{151} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we see that

$$\arcsin(\sin(\tfrac{198\pi}{151})) = -\tfrac{47\pi}{151}.$$

To invert cosine, we take its principal domain to be $[0, \pi]$. The inverse of the invertible function $\cos : [0, \pi] \to [-1, 1]$ is denoted arccos, a function from $[-1, 1]$ to $[0, \pi]$.





Example 4.46 Find $\arccos(\cos(\frac{\pi}{4}))$ and $\arccos(\cos(\frac{11\pi}{6}))$.

Solution: Because $\pi/4$ is in the principal domain of cos,

$$\arccos(\cos(\tfrac{\pi}{4})) = \tfrac{\pi}{4}.$$

However, $11\pi/6$ is outside the principal domain, so we look for the unique angle in $[0, \pi]$ with the same cosine, which is $\frac{\pi}{6}$. That is,

$$\arccos(\cos(\tfrac{11\pi}{6})) = \tfrac{\pi}{6}.$$

The principal domain of tan for inversion is $(-\frac{\pi}{2}, \frac{\pi}{2})$, and the inverse function, arctan, is then a function from $\mathbb{R}$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\arctan : \mathbb{R} \to (-\tfrac{\pi}{2}, \tfrac{\pi}{2}).$$

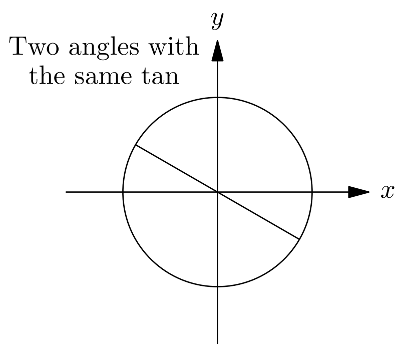
Example 4.47 Find $\arctan(\tan(\frac{\pi}{3}))$ and $\arctan(\tan(\frac{5\pi}{6}))$.

Solution: The angle $\pi/3$ is already in the principal domain of tan, so

$$\arctan(\tan(\tfrac{\pi}{3})) = \tfrac{\pi}{3}.$$

However, $5\pi/6$ is outside the principal domain, so we seek the unique angle in the principal domain with the same tangent, which is $-\pi/6$. Thus,

$$\arctan(\tan(\tfrac{5\pi}{6})) = -\tfrac{\pi}{6}.$$



Finally, the principal domain of cot is $(0, \pi)$, and its inverse function is

$$\operatorname{arccot} : \mathbb{R} \to (0, \pi).$$

For example, $\operatorname{arccot}(\sqrt{3}) = \frac{\pi}{6}$, because $\pi/6$ is the unique angle in $(0, \pi)$ whose cotangent is $\sqrt{3}$.

Derivatives of inverse trigonometric functions

We have the following table of derivatives:

f	$f'(x)$
arcsin	$\frac{1}{\sqrt{1-x^2}}$
arctan	$\frac{1}{1+x^2}$
arccos	$-\frac{1}{\sqrt{1-x^2}}$
arccot	$-\frac{1}{1+x^2}$

Let us prove the formula for the derivative of arcsin by way of example. Let $g = \sin$, so $g^{-1} = \arcsin$. If $x \in (-1, 1)$, then

$$(g^{-1})'(x) = \frac{1}{g'(g^{-1}(x))} = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\cos(\theta)} \tag{4.8}$$

where $\theta = \arcsin(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Now,

$$\begin{aligned} 1 &= \cos^2(\theta) + \sin^2(\theta) \\ &= \cos^2(\theta) + \sin(\arcsin(x))^2 \\ &= \cos^2(\theta) + x^2, \end{aligned}$$

so $\cos^2(\theta) = 1 - x^2$. Hence, because $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, so that $\cos(\theta) > 0$, it follows that

$$\cos(\theta) = \sqrt{1 - x^2}.$$

Combining this with (4.8), we obtain

$$(g^{-1})'(x) = \frac{1}{\sqrt{1 - x^2}}.$$

Example 4.48 Find $f'(x)$ where $f : (e^{-1}, e) \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is defined by

$$f(x) = \arcsin(\ln(x)).$$

Solution: By the chain rule,

$$f'(x) = \frac{1}{\sqrt{1 - \ln(x)^2}} \frac{1}{x} = \frac{1}{x\sqrt{1 - \ln(x)^2}}.$$

Example 4.49 Find $f'(x)$ where $f : (1, \infty) \to (0, \pi)$ is defined by

$$f(x) = 2 \arctan(\sqrt{x - 1}).$$

Solution: The chain rule gives

$$f'(x) = \frac{2}{1 + \sqrt{x - 1}^2} \cdot \frac{1}{2}(x - 1)^{-1/2} = \frac{1}{x\sqrt{x - 1}}.$$

5 Integration

5.1 Antiderivatives and indefinite integrals

If $f : D \to \mathbb{R}$ is a function, then an *antiderivative* of f, if one exists, is a differentiable function $F : D \to \mathbb{R}$ such that $F' = f$.

Example 5.1 If $f(x) = 3x^2 + x^4$, then an antiderivative of f is

$$F(x) = x^3 + \frac{1}{5}x^5,$$

but another is

$$F(x) = x^3 + \frac{1}{5}x^5 + 4.$$

Constants added make no difference: The derivative will still be the same.

Proposition 5.2

(i) If $F : (a, b) \to \mathbb{R}$ is differentiable and $F(x) = 0$ for all $x \in (a, b)$, then F is constant.

(ii) Suppose that $D \subseteq \mathbb{R}$ is an interval, a half-line, or $\mathbb{R}$. (A half-line is a subset of $\mathbb{R}$ of the form $(-\infty, a)$, $(-\infty, a]$, (a, ∞), or $(a, \infty]$.) Any two antiderivatives of f differ by a constant.

Proof. See Corollary A3.4. ■

Example 5.3 If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 3x^2 + x^4$, then every antiderivative of f is of the form

$$F(x) = x^3 + \frac{1}{5}x^5 + C$$

where C is constant.

Remark. A word of caution is appropriate. If D is not one of the types of domain in the proposition above, then the difference of two antiderivatives of a continuous function $f : D \to \mathbb{R}$ may be non-constant. For example, let C_1, C_2 be two constants, and consider $F : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} \ln(-x) + C_1 & \text{if } x < 0, \\ \ln(x) + C_2 & \text{if } x > 0, \end{cases}$$

a continuous function. (Yes, this is continuous! Remember that 0 is not in the domain of F.) Then

$$F'(x) = \ln(x)$$

for all $x \neq 0$, whether $x < 0$ or $x > 0$. So, although some people take $\ln(|x|) + C$ as a general antiderivative of $1/x$, this is not true; the constants C_1 and C_2 above may be different.

Integral notation for antiderivatives

It is common to write

$$\int f(x)\, dx$$

for the general antiderivative of a function f defined on an interval, a half-line, or $\mathbb{R}$ (if an antiderivative of f exists). This notation for an antiderivative is not ideal, but it is almost universally accepted and does have its uses. The expression $\int f(x)\, dx$ is called an *indefinite integral*.

Example 5.4

$$\int (x^7 + 2)\, dx = \frac{1}{8}x^8 + 2x + C$$

$$\int \sec^2(x)\, dx = \tan(x) + C$$

$$\int \frac{1}{\sqrt{1 - x^2}} = \arcsin(x) + C$$

Note that we could also write

$$\int \frac{1}{\sqrt{1 - x^2}} = -\arccos(x) + C,$$

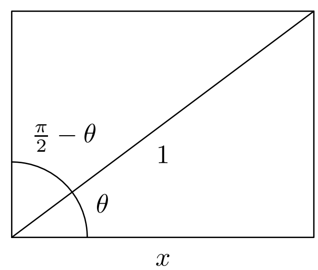
because $\frac{d}{dx}(\arccos(x)) = -\frac{1}{\sqrt{1-x^2}}$.

Example 5.5 Show that

$$\arccos(x) + \arcsin(x) = \frac{\pi}{2}$$

for all $x \in [-1, 1]$.

Solution 1 (direct): First, assume that $x \in [0, 1]$, and consider a rectangle with base of length x and diagonal of length 1:



If $\theta \in [0, \pi/2]$ is the angle from the base to the diagonal line, then

$$\begin{aligned} x &= \cos(\theta), \\ \text{i.e.,} \qquad \arccos(x) &= \theta. \end{aligned} \tag{5.1}$$

Further, the angle from the diagonal line to the vertical is $\frac{\pi}{2} - \theta$, so we also have

$$\begin{aligned} x &= \sin(\tfrac{\pi}{2} - \theta), \\ \text{i.e.,} \qquad \arcsin(x) &= \frac{\pi}{2} - \theta. \end{aligned} \tag{5.2}$$

Adding (5.1) and (5.2) gives the desired equality.

If, instead, $x \in [-1, 0)$, then we use the facts that

$$\begin{aligned} \arccos(x) &= \pi - \arccos(-x), \\ \arcsin(x) &= -\arcsin(-x). \end{aligned}$$

Hence,

$$\arccos(x) + \arcsin(x) = \pi - (\arccos(-x) + \arcsin(-x)) = \pi - \frac{\pi}{2} = \frac{\pi}{2},$$

the second equality being obtained by applying the previous calculation with x replaced by $-x \in (0, 1]$.

Solution 2 (via differentiation): Let

$$f(x) = \arccos(x) + \arcsin(x).$$

Then

$$f'(x) = -\frac{1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - x^2}} = 0,$$

so f is constant, say $f(x) = C$, i.e.,

$$\arccos(x) + \arcsin(x) = C.$$

Taking $x = 0$, for example, gives $C = \frac{\pi}{2} + 0 = \frac{\pi}{2}$.

Although this second solution appears to be shorter, it is in fact less direct because it uses the machinery of differentiation, which takes quite some time to build up. Solution 1 works more directly with the definitions of arccos and arcsin.

Example 5.6 Find

$$\int (\sec(x)\tan(x) + x)\,dx.$$

Solution:

$$\int (\sec(x)\tan(x) + x)\,dx = \sec(x) + \frac{1}{2}x^2 + C.$$

(Remember that $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$.)

Example 5.7 Find

$$\int \frac{x}{x^2+1}\,dx.$$

Solution:

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + C,$$

because

$$\frac{d}{dx}(\ln(x^2+1)) = \frac{2x}{x^2+1}.$$

Example 5.8 Find

$$\int \frac{x^3}{x^8+1}\,dx.$$

Hint: Think about arctan*.*

Solution: Recall that

$$\frac{d}{dy}(\arctan(y)) = \frac{1}{y^2+1},$$

so if we observe that the denominator of the integrand is $(x^4)^2 + 1$ and that the numerator is almost $\frac{d}{dx}(x^4)$, then with minor adjustment we obtain an antiderivative via the chain rule:

$$\int \frac{x^3}{x^8+1}\,dx = \frac{1}{4}\arctan(x^4) + C.$$

We have already seen how to find antiderivatives of $\frac{1}{x^2+1}$ and $\frac{x}{x^2+1}$, so we ought also to be able to find antiderivatives of $\frac{f(x)}{x^2+1}$ for any polynomial function f. Let us illustrate how with a couple of examples.

Example 5.9 Find

$$\int \frac{(x+1)^2}{x^2+1}\,dx.$$

Solution:

$$\begin{aligned}\int \frac{(x+1)^2}{x^2+1}\,dx &= \int \frac{x^2+2x+1}{x^2+1}\,dx \\ &= \int \left(\frac{x^2+1}{x^2+1} + \frac{2x}{x^2+1}\right) dx \\ &= \int \left(1 + \frac{2x}{x^2+1}\right) dx \\ &= x + \ln(x^2+1) + C.\end{aligned}$$

Example 5.10 Find

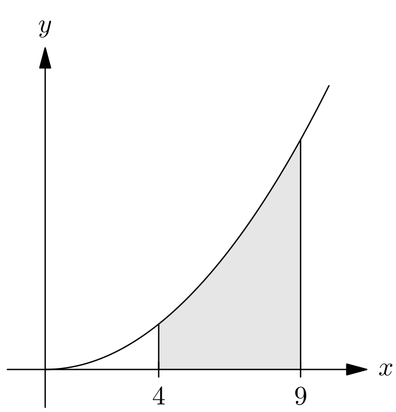
$$\int \frac{(x+1)(x-1)}{x^2+1}\,dx.$$

Solution:

$$\begin{aligned}\int \frac{(x+1)(x-1)}{x^2+1}\,dx &= \int \frac{x^2-1}{x^2+1}\,dx \\ &= \int \frac{(x^2+1)-2}{x^2+1}\,dx \\ &= \int \left(1 - \frac{2}{x^2+1}\right) dx \\ &= x - 2\arctan(x) + C.\end{aligned}$$

5.2 Riemann sums and area

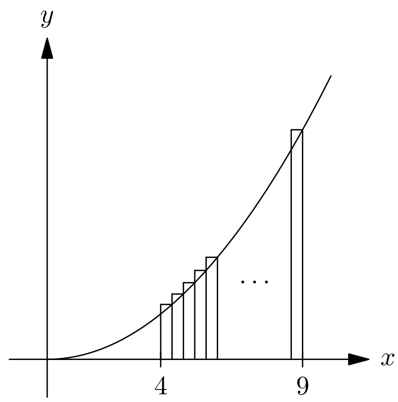
Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{1}{10}x^2$, and consider the area under the graph of f between $x = 4$ and $x = 9$:



For a given integer $n \geq 1$, we approximate this area using n rectangles, each of width $\Delta = (9-4)/n = 5/n$, constructed so that the height of the kth rectangle is the value of f at the right-hand x-value of the rectangle, i.e., the height is

$$f(4 + k\Delta) = f(4 + \tfrac{5k}{n}).$$

This diagram illustrates the situation:



(Note that in the case $k = n$, the height of the rectangle is $f(4+5) = f(9)$, as expected.) Now, the area of the kth rectangle, via the formula

$$\text{area} = \text{width} \times \text{height},$$

is

$$\frac{5}{n} f(4 + \tfrac{5k}{n}),$$

so the total area of the rectangles is

$$R_n = \sum_{k=1}^{n} \frac{5}{n} f(4 + \tfrac{5k}{n}).$$

This number R_n is called a *Riemann sum*. It approximates the area under the curve, with the approximation typically getting closer as n becomes larger. In fact,

$$\text{area under curve} = \lim_{n\to\infty} R_n$$

under reasonable assumptions.

For the function f at hand, given by $f(x) = \frac{1}{10}x^2$, it is not hard to evaluate R_n explicitly and then take the limit to obtain the area under the curve. Let us do this.

Lemma 5.11 *Let n be a positive integer.*

(i) $\displaystyle\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.

(ii) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1).$

Proof. See Section 4 of the Appendix. ∎

Now, the nth Riemann sum is

$$\begin{aligned} R_n &= \frac{5}{n}\sum_{k=1}^{n} f(4+\tfrac{5k}{n}) \\ &= \frac{5}{n}\sum_{k=1}^{n} \frac{1}{10}\left(4+\tfrac{5k}{n}\right)^2 \\ &= \frac{1}{2n}\sum_{k=1}^{n}\left(16+\frac{40k}{n}+\frac{25k^2}{n^2}\right) \\ &= \frac{1}{2n}\left(16n+\frac{40}{n}\frac{1}{2}n(n+1)+\frac{25}{n^2}\frac{1}{6}n(n+1)(2n+1)\right) \\ &= 8+10\left(1+\frac{1}{n}\right)+\frac{25}{12}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right). \end{aligned}$$

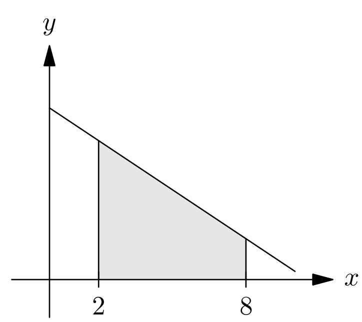
Therefore, as $n \to \infty$,

$$R_n \to 8+10(1+0)+\frac{25}{12}(1+0)(2+0) = \frac{133}{6}.$$

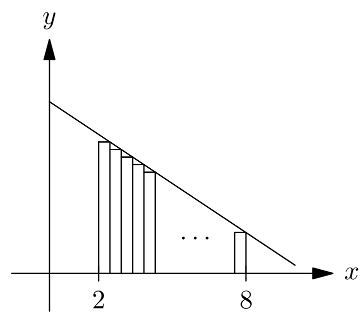
This is the area of under the graph of f between $x = 4$ and $x = 9$.

More practice with Riemann sums

Just for practice, let us use Riemann sums to find an area under the graph of the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = 7 - \frac{2}{3}x$, taking the end-points this time to be $x = 2$ and $x = 8$.



We illustrate the Riemann sums:



If we are considering the nth Riemann sum, in which each rectangle has width $\Delta = (8-2)/n = 6/n$, then the kth rectangle has height $g(2+k\Delta)$ and area

$$\Delta \cdot g(2+k\Delta) = \frac{6}{n} g(2+\tfrac{6k}{n}).$$

Thus, the nth Riemann sum is

$$\begin{aligned} R_n = \sum_{k=1}^{n} \frac{6}{n} g(2+\tfrac{6k}{n}) &= \frac{6}{n} \sum_{k=1}^{n} \left(7 - \frac{2}{3}\Big(2 + \frac{6k}{n}\Big)\right) \\ &= \frac{6}{n} \sum_{k=1}^{n} \left(\frac{17}{3} - \frac{4k}{n}\right) \\ &= \frac{6}{n} \left(\frac{17}{3} n - \frac{4}{n}\frac{1}{2} n(n+1)\right) \\ &= 34 - 12\left(1 + \frac{1}{n}\right). \end{aligned}$$

Hence, $R_n \to 34 - 12(1+0) = 22$ as $n \to \infty$, so the area in question is 22.

5.3 Definite integrals

A rigorous and flexible definition of the notion of definite integral would take us beyond the scope of the course, so we give the following slightly less precise definition, which is nonetheless adequate for illustrating the key idea. Given a function f defined on an interval I, and given $a, b \in I$ with $a < b$, we define

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n$$

where R_n is the nth Riemann sum as defined in Section 5.2, as long as this limit exists. If, instead, $b < a$, we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

In the special case where $a = b$, we assign $\int_a^a f(x)\,dx$ the value 0. The real number $\int_a^b f(x)\,dx$ is called the *integral* of f from a to b.

When $a < b$, $\int_a^b f(x)\,dx$ is the area under the graph of f between the two x-values a and b, the word *under* here meaning "between the x-axis and the curve". Of course, this opens the question of what happens when the graph is below the x-axis, and the answer is that the negative portions of the graph count negatively towards the integral, as in the diagram at the end of Section 2.1.

Example 5.12 If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \frac{1}{10}x^2$, then

$$\int_4^9 f(x)\,dx = \frac{133}{6}$$

by the calculation involving Riemann sums that we carried out in Section 5.2. One may also write

$$\int_4^9 \frac{1}{10}x^2\,dx = \frac{133}{6}.$$

Example 5.13 If $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = 7 - \frac{2}{3}x$, then

$$\int_2^8 g(x)\,dx = 22.$$

Again, see Section 5.2 for the calculation via Riemann sums. We may write, alternatively,

$$\int_2^8 \left(7 - \frac{2}{3}x\right)dx = 22.$$

Two key properties

Let f and g be functions, and assume that the integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ exist. Then

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\text{and} \quad \int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \quad \text{for every constant } c.$$

Example 5.14 Find

$$\int_4^9 \left(x^2 + 2x - 6\right)dx$$

by considering the areas of elementary geometric shapes and using a previous example.

Solution: By the properties above, we know that

$$\int_4^9 \left(x^2 + 2x - 6\right)dx = 10\int_4^9 \tfrac{1}{10}x^2\,dx + 2\int_4^9 x\,dx - \int_4^9 6\,dx$$

$$
\begin{aligned}
&= 10 \cdot \frac{133}{6} + 2\int_4^9 x\,dx - \int_4^9 6\,dx \quad \text{by Example 5.12} \\
&= \frac{665}{3} + 2\int_4^9 x\,dx - \int_4^9 6\,dx.
\end{aligned}
$$

For the remaining two integrals, we use some basic geometry. The integrand in $\int_4^9 6\,dx$ is the constant function 6, so interpreting this integral as the area under the graph of a constant function, we see that it is equal to the area of a rectangle of width $9 - 4 = 5$ and height 6. Thus,

$$
\int_4^9 6\,dx = 5 \cdot 6 = 30.
$$

As for $\int_4^9 x\,dx$, the region under the graph consists of a rectangle and a triangle, both having width $9 - 4 = 5$, and having heights 4 and $9 - 4 = 5$ respectively, so

$$
\begin{aligned}
\int_4^9 x\,dx &= \text{area of rectangle} + \text{area of triangle} \\
&= 5 \cdot 4 + \frac{1}{2} \cdot 5 \cdot 5 = \frac{65}{2}.
\end{aligned}
$$

Hence,

$$
\int_4^9 \left(x^2 + 2x - 6\right) dx = \frac{665}{3} + 2 \cdot \frac{65}{2} - 30 = \frac{770}{3}.
$$

In the next section, we will see a more efficient way to calculate integrals.

5.4 The Fundamental Theorems of Calculus

Theorem 5.15 (The First Fundamental Theorem) *Let $f : [a, b] \to \mathbb{R}$ be a continuous function, where $a < b$, and define $F : [a, b] \to \mathbb{R}$ by*

$$
F(x) = \int_a^x f(t)\,dt.
$$

Then F is continuous on $[a, b]$ and differentiable on (a, b), and

$$
F'(x) = f(x) \quad \text{for all } x \in (a, b).
$$

The First Theorem says, in particular, that every continuous function $f : [a, b] \to \mathbb{R}$ has an antiderivative.

Theorem 5.16 (The Second Fundamental Theorem) *Suppose that $F : [a, b] \to \mathbb{R}$, where $a < b$, is continuous on $[a, b]$ and differentiable on (a, b), and assume that its derivative is Riemann integrable. (The interested reader may investigate the notion of a* Riemann integrable *function.) Then for all $x \in [a, b]$,*

$$
\int_a^x F'(t)\,dt = F(x) - F(a).
$$

Proofs of these theorems are given in Section 5 of the Appendix.

Remark. It is common practice to abbreviate $F(b) - F(a)$ to $[F(x)]_a^b$. For example,

$$\left[x^5 + \tfrac{1}{7}x^7 - e^x\right] = \left(2^5 - \tfrac{1}{7}\cdot 2^7 - e^2\right) - \left(1^5 - \tfrac{1}{7}\cdot 1^7 - e^1\right).$$

The First Theorem may be used to construct differentiable functions with a prescribed derivative, and the Second Theorem is used to compute definite integrals. We first give examples of the Second Theorem.

Example 5.17 Find

$$\int_1^4 x^2\,dx.$$

Solution: An antiderivative of x^2 is $\frac{1}{3}x^3$, so

$$\int_1^4 x^2\,dx = \left[\frac{1}{3}x^3\right]_1^4 = \frac{1}{3}\cdot 4^3 - \frac{1}{3}\cdot 1^3 = \frac{64-1}{3} = 21.$$

Example 5.18 Find

$$\int_{-2}^1 (3 - 8x^3)\,dx.$$

Solution:

$$\int_{-2}^1 (3 - 8x^3)\,dx = [3x - 2x^4]_{-2}^1 = (3-2) - (-6-32) = 39.$$

Example 5.19 Find

$$\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx.$$

Solution:

$$\begin{aligned}\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx &= [\arcsin(x)]_{-1/2}^{1/2} \\ &= \arcsin(1/2) - \arcsin(-1/2) \\ &= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) \\ &= \frac{\pi}{3}.\end{aligned}$$

Example 5.20 Find

$$\int_0^{3^{-1/8}} \frac{x^3}{x^8+1}\,dx.$$

Solution: As we observed when discussing antiderivatives in Section 5.1,

$$\frac{d}{dx}\Big(\arctan(x^4)\Big) = \frac{4x^3}{x^8+1},$$

so

$$\begin{aligned}\int_0^{3^{-1/8}} \frac{x^3}{x^8+1}\,dx &= \left[\frac{1}{4}\arctan(x^4)\right]_0^{3^{-1/8}} \\ &= \frac{1}{4}\arctan(3^{-1/2}) - \frac{1}{4}\arctan(0) \\ &= \frac{1}{4}\arctan(1/\sqrt{3}) - 0 \\ &= \frac{1}{4}\frac{\pi}{6} = \frac{\pi}{24}.\end{aligned}$$

Example 5.21 Find

$$\int_{\pi/6}^{\pi/3} \sec(x)\Big(\tan(x) - 3\sec(x)\Big)\,dx.$$

$$\begin{aligned}\int_{\pi/6}^{\pi/3} \sec(x)\Big(\tan(x) - 3\sec(x)\Big)\,dx &= \int_{\pi/6}^{\pi/3} \Big(\sec(x)\tan(x) - 3\sec^2(x)\Big)\,dx \\ &= [\sec(x) - 3\tan(x)]_{\pi/6}^{\pi/3} \\ &= (2 - 3\sqrt{3}) - (\tfrac{2}{\sqrt{3}} - \sqrt{3}) \\ &= 2 - 3\sqrt{3} - \tfrac{2}{3}\sqrt{3} + \sqrt{3} \\ &= 2 - \frac{8}{3}\sqrt{3}.\end{aligned}$$

Now we turn to examples of the First Theorem.

Example 5.22 Construct, via an integral, a differentiable function $F : \mathbb{R} \to \mathbb{R}$ such that $F'(x) = \cos(x^3)$ for all x.

Solution: If

$$F(x) = \int_3^x \cos(t^3)\,dt,$$

then $F'(x) = \cos(x^3)$ by the First Theorem.

Example 5.23 If

$$F(x) = \int_{-100}^x e^{e^{t+\sin(t)}}\,dt,$$

find $F'(x)$.

Solution: By the First Theorem,

$$F'(x) = e^{e^{x+\sin(x)}}.$$

Example 5.24 If $F(x) = \int_1^{x^2} \sqrt{1+t^2}\,dt$, find $F'(x)$.

Solution: Here, we use the chain rule in conjunction with the First Theorem. If

$$G(y) = \int_1^y \sqrt{1+t^2}\,dt,$$

then

$$F(x) = G(x^2) \quad \text{and} \quad G'(y) = \sqrt{1+y^2},$$

so

$$F'(x) = G'(x^2) \cdot 2x = 2x\sqrt{1+x^4}.$$

Example 5.25 If $F : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ is defined by

$$F(x) = \int_0^{\sin(x)} \arccos(t)\,dt,$$

find $F'(x)$ in terms of x. Present your answer in such a way that it does not include any inverse trigonometric functions.

Solution: If $G : [-1, 1] \to \mathbb{R}$ is defined by

$$G(y) = \int_0^y \arccos(t)\,dt,$$

then

$$F(x) = G(\sin(x)) \quad \text{and} \quad G'(y) = \arccos(y),$$

so

$$\begin{aligned} F'(x) &= G'(\sin(x))\cos(x) \\ &= \arccos(\sin(x))\cos(x) \\ &= \left(\frac{\pi}{2} - \arcsin(\sin(x))\right)\cos(x) \\ &= \left(\frac{\pi}{2} - x\right)\cos(x). \end{aligned}$$

Note that $\arcsin(\sin(x)) = x$ because of the assumption that $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Example 5.26 Find $F'(x)$ where $F : (1, \infty) \to \mathbb{R}$ is defined by

$$F(x) = \int_{\ln(x)}^{x\ln(x)} e^t \ln(t)\,dt.$$

Solution: Observe that both limits in the integral are varying, but we can handle this situation by breaking the integral up into two, choosing some value in the domain of the integrand as the break point. The choice of point does not matter, but let us take $t = 1$ for concreteness:

$$F(x) = \int_{\ln(x)}^{x\ln(x)} e^t \ln(t)\,dt$$

$$
\begin{aligned}
&= \int_{\ln(x)}^{1} e^t \ln(t)\,dt + \int_{1}^{x\ln(x)} e^t \ln(t)\,dt \\
&= \int_{1}^{x\ln(x)} e^t \ln(t)\,dt - \int_{1}^{\ln(x)} e^t \ln(t)\,dt \\
&= F_1(x) - F_2(x)
\end{aligned}
$$

where

$$
F_1(x) = \int_1^{x\ln(x)} e^t \ln(t)\,dt \quad \text{and} \quad F_2(x) = \int_1^{\ln(x)} e^t \ln(t)\,dt.
$$

Let

$$
G(y) = \int_1^y e^t \ln(t)\,dt.
$$

Then

$$
\begin{aligned}
F_1(x) &= G(x\ln(x)), \\
F_2(x) &= G(\ln(x)), \\
G'(y) &= e^y \ln(y),
\end{aligned}
$$

so the chain rule gives

$$
\begin{aligned}
F'(x) &= F_1'(x) - F_2'(x) \\
&= G'(x\ln(x))\frac{d}{dx}(x\ln(x)) - G'(\ln(x))\frac{d}{dx}(\ln(x)) \\
&= e^{x\ln(x)}\ln(x\ln(x))(\ln(x)+1) - e^{\ln(x)}\ln(\ln(x))\frac{1}{x} \\
&= x^x \ln(x\ln(x))(\ln(x)+1) - \ln(\ln(x)).
\end{aligned}
$$

5.5 Change of integration variable

Suppose that T is an invertible function such that T is differentiable with a continuous derivative. Then for an integrable function f,

$$
\int_a^b f(x)\,dx = \int_{T^{-1}(a)}^{T^{-1}(b)} f(T(u))T'(u)\,du.
$$

There are three ingredients of a change of variables in integration:

- the integrand $f(x)$, which is replaced by $f(T(u))$,
- the limits a and b, which are replaced by $T^{-1}(a)$ and $T^{-1}(b)$ respectively, and
- the *differential* (or *differential form*) dx, which is replaced not simply by du but by $T'(u)\,du$.

We illustrate the process with examples.

Example 5.27 Evaluate

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx$$

via the change of variables $x = \sin(u)$.

Solution: Here, the change of variables is $x = T(u)$ where $T(u) = \sin(u)$. The differential dx is replaced by

$$T'(u)\,du = \cos(u)\,du,$$

and the limits are replaced by $\arcsin(-1) = -\pi/2$ and $\arcsin(1) = \pi/2$, so

$$\begin{aligned}
\int_{-1}^{1} \sqrt{1-x^2}\,dx &= \int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2(u)}\cos(u)\,du \\
&= \int_{-\pi/2}^{\pi/2} \sqrt{\cos^2(u)}\cos(u)\,du \\
&= \int_{-\pi/2}^{\pi/2} \cos(u)\cos(u)\,du \quad (\cos(u) \geq 0 \text{ when } u \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]) \\
&= \int_{-\pi/2}^{\pi/2} \cos^2(u)\,du \\
&= \int_{-\pi/2}^{\pi/2} \frac{1}{2}(1+\cos(2u))\,du \\
&= \left[\frac{1}{2}u + \frac{1}{4}\sin(2u)\right]_{-\pi/2}^{\pi/2} \\
&= \frac{1}{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) + \frac{1}{4}(\sin(\pi) - \sin(-\pi)) \\
&= \frac{\pi}{2}.
\end{aligned}$$

Note: Since the function in the integral describes a semicircle of radius 1, our calculation has found that the area of this semicircle is $\pi/2$, as one would expect.

In the example above, we replaced dx by $T'(u)\,du$, but another common strategy is to express u in terms of x, i.e., $u = T^{-1}(x)$, and then replace $(T^{-1})'(x)\,dx$ by du:

$$du = (T^{-1})'(x)\,dx.$$

Let us illustrate.

Example 5.28 Find

$$\int_{1}^{e} \frac{\ln(x)^4}{x}\,dx$$

via a change of variables.

Solution: We make the change of variables $u = \ln(x)$, corresponding to $x = e^u$. Thus, $T(u) = e^u$ in the notation above, but rather than using T itself, we use its inverse, $T^{-1}(x) = \ln(x)$. Because $u = \ln(x)$, and $\ln(x)$ has derivative $1/x$, we obtain

$$du = (T^{-1})'(x)\,dx = \frac{1}{x}\,dx,$$

so

$$\int_1^e \frac{\ln(x)^4}{x}\,dx = \int_0^1 u^4\,du = \left[\frac{1}{5}u^5\right]_0^1 = \frac{1}{5}.$$

Change of variables in indefinite integrals

Sometimes, it is useful to be able to compute an indefinite integral $\int f(x)\,dx$ via a change of variables. To do this,

(i) Make a change of variables $x = T(u)$ or $u = T^{-1}(x)$.

(ii) Find an antiderivative $F(u)$ of $f(T(u))T'(u)$.

(iii) Replace u by $T^{-1}(x)$ in that antiderivative. Thus, $F(T^{-1}(x))$ is an antiderivative of $f(x)$.

Example 5.29 Use the change of variables $x = u^2$ to find the general antiderivative of the function $f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{\sqrt{x - x^2}}.$$

Solution: There is no harm in assuming that $u > 0$, so that $\sqrt{u^2} = u$ and $u = \sqrt{x}$. Now, noting that the relationship $x = u^2$ gives $dx = 2u\,du$, we have

$$\begin{aligned}
\int f(x)\,dx &= \int \frac{1}{\sqrt{x - x^2}}\,dx \\
&= \int \frac{1}{\sqrt{u^2 - u^4}} \cdot 2u\,du \\
&= \int \frac{2u}{u\sqrt{1 - u^2}}\,du \\
&= \int \frac{2}{\sqrt{1 - u^2}}\,du \\
&= 2\arcsin(u) + C \\
&= 2\arcsin(\sqrt{x}) + C.
\end{aligned}$$

Example 5.30 Find the general antiderivative of the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \frac{1}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}.$$

Solution: We make the change of variables $u = e^{\frac{x}{2}}$. Note that this gives

$$du = \frac{1}{2}e^{\frac{x}{2}}\,dx = \frac{1}{2}u\,dx,$$

so $dx = \frac{2}{u}\,du$. Hence,

$$\begin{aligned}\int g(x)\,dx &= \int \frac{1}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}\,dx \\ &= \int \frac{1}{u + u^{-1}}\,\frac{2}{u}\,du \\ &= 2\int \frac{1}{u^2+1}\,du \\ &= 2\arctan(u) + C \\ &= 2\arctan\left(e^{\frac{x}{2}}\right) + C.\end{aligned}$$

Example 5.31 Consider $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{\sin(\ln(x))}{x}.$$

Find

$$\int_{e^{\pi/3}}^{e^{\pi/2}} f(x)\,dx$$

in two ways:

(i) by making a change of variables in the definite integral and using the Second Theorem, and

(ii) by finding the indefinite integral $\int f(x)\,dx$ via a change of variables and only then using the Second Theorem.

Solution: We make the change of variables $t = \ln(x)$, observing that $dt = \frac{1}{x}\,dx$. (There is nothing special about the symbol u for changes of variables, so let us use a different symbol in this example.)

(i)

$$\begin{aligned}\int_{e^{\pi/3}}^{e^{\pi/2}} f(x)\,dx &= \int_{e^{\pi/3}}^{e^{\pi/2}} \frac{\sin(\ln(x))}{x}\,dx \\ &= \int_{\pi/3}^{\pi/2} \sin(t)\,dt \\ &= \Big[-\cos(t)\Big]_{\pi/3}^{\pi/2} \\ &= \cos(\pi/3) - \cos(\pi/2)\end{aligned}$$

$$= \frac{1}{2}.$$

(ii)

$$\begin{aligned}\int f(x)\,dx &= \int \frac{\sin(\ln(x))}{x}\,dx \\ &= \int \sin(t)\,dt \\ &= -\cos(t) + C \\ &= -\cos(\ln(x)) + C,\end{aligned}$$

so

$$\int_{e^{\pi/3}}^{e^{\pi/2}} f(x)\,dx = \Big[-\cos(\ln(x))\Big]_{e^{\pi/3}}^{e^{\pi/2}} = \frac{1}{2}$$

again.

5.6 Integration by parts

Suppose that f and g are differentiable functions with the same domain. The product rule says that

$$(fg)' = f'g + fg',$$

so fg is an antiderivative of $f'g + fg'$. In the notation of indefinite integrals, we would therefore write

$$\int (f'(x)g(x) + f(x)g'(x))\,dx = f(x)g(x).$$

Rearranging this, we obtain

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx. \tag{5.3}$$

The formula in (5.3) is known as *integration by parts*. It can be used in many situations to integrate a product uv of functions u and v.

If we write simply $\int(j)$ for some chosen antiderivative of a function j, then letting $u = f'$ and $v = g$, we may express (5.3) as

$$\int(uv) = \int(u)v - \int(\int(u)v'), \tag{5.4}$$

as long as $\int(u)$ is the same function in both terms on the right-hand side. The viewpoint of (5.4) is that, when applying integration by parts to integrate a product of two functions, we choose one function to integrate (u in our notation) and the other to differentiate (v).

Example 5.32 Use integration by parts to find $\int x\cos(x)\,dx$.

Solution: We choose to integrate $\cos(x)$ and different x. (What would happen if you made the opposite choice?) Thus, in our notation above, $u(x) = \cos(x)$ and $v(x) = x$:

$$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx = x\sin(x) + \cos(x) + C.$$

Example 5.33 Use integration by parts to find $\int x^2\sin(x)\,dx$.

Solution: Again, we choose the trigonometric function to be the one to integrate and the polynomial function the one to differentiate. Thus,

$$\begin{aligned}\int x^2\sin(x)\,dx &= -x^2\cos(x) + \int 2x\cos(x)\,dx\\ &= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C\end{aligned}$$

by the previous example.

Example 5.34 Find $\int x^2 e^{3x}\,dx$.

Solution: Here, we will need to apply integration by parts twice, each time choosing the polynomial function to be the one to differentiate:

$$\begin{aligned}\int x^2e^{3x}\,dx &= x^2\cdot\frac{1}{3}e^{3x} - \int 2x\cdot\frac{1}{3}e^{3x}\,dx\\ &= \frac{1}{3}x^2e^{3x} - 2x\cdot\frac{1}{9}e^{3x} + \int 2\cdot\frac{1}{9}e^{3x}\,dx\\ &= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C\\ &= \left(\frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}\right)e^{3x} + C.\end{aligned}$$

Example 5.35 Find $\int e^{4x}\cos(x)\,dx$.

Solution: Let us apply integration by parts, choosing to integrate the trigonometric function:

$$\begin{aligned}\int e^{4x}\cos(x)\,dx &= e^{4x}\sin(x) - \int 4e^{4x}\sin(x)\,dx\\ &= e^{4x}\sin(x) + 4e^{4x}\cos(x) - \int 16e^{4x}\cos(x)\,dx.\end{aligned}$$

Notice that we have applied integration by parts a second time, integrating the trigonometric function again, and that in so doing we have arrived at an equation that can be rearranged so as to express the desired integral in terms of other functions:

$$\int e^{4x}\cos(x)\,dx = \frac{1}{17}e^{4x}(\sin(x) + 4\cos(x)).$$

In fact, we could have achieved the same outcome by integrating the exponential function both times instead. (What happens if you integrate the trigonometric function in the first instance of integration by parts and then the exponential function in the second, or vice versa?)

Example 5.36 Use integration by parts to find $\int \ln(x)\,dx$.

Solution: Viewing $\ln(x)$ as $1 \cdot \ln(x)$, we integrate the first factor and differentiate the second. Thus,

$$\int \ln(x)\,dx = x\ln(x) - \int x\,\frac{1}{x}\,dx = x\ln(x) - \int 1\,dx = x\ln(x) - x + C.$$

Definite integrals via integration by parts

Integration by parts when calculating definite integrals is similar to the indefinite situation:

$$\int_a^b f'(x)g(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)\,dx.$$

Example 5.37 Find

$$\int_{-\pi/6}^{\pi/3} x\sec^2(x)\,dx$$

via integration by parts.

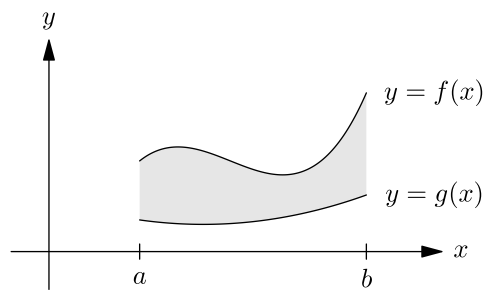
Solution: Note that $\cos(x) > 0$ in the interval of integration, so an antiderivative of $\tan(x)$ is $-\ln(\cos(x))$ for our purposes. Now, choosing $\sec^2(x)$ as the factor to integrate and x as the one to differentiate, we see that

$$\begin{aligned}\int_{-\pi/6}^{\pi/3} x\sec^2(x)\,dx &= \Big[x\tan(x)\Big]_{-\pi/6}^{\pi/3} - \int_{-\pi/6}^{\pi/3} \tan(x)\,dx \\ &= \frac{\pi}{3} + \frac{\pi}{6}\left(-\frac{\sqrt{3}}{3}\right) + \Big[\ln(\cos(x))\Big]_{-\pi/6}^{\pi/3} \\ &= \frac{5\sqrt{3}}{18}\pi + \ln(\tfrac{1}{2}) - \ln(\tfrac{\sqrt{3}}{2}) \\ &= \frac{5\sqrt{3}}{18}\pi - \ln(\sqrt{3}).\end{aligned}$$

5.7 Area between curves

If $f, g : [a,b] \to \mathbb{R}$ are integrable and $f(x) \geq g(x)$ for all $x \in [a,b]$, then the area between the curves $y = g(x)$ and $y = f(x)$ is equal to

$$\int_a^b (f(x) - g(x))\,dx.$$



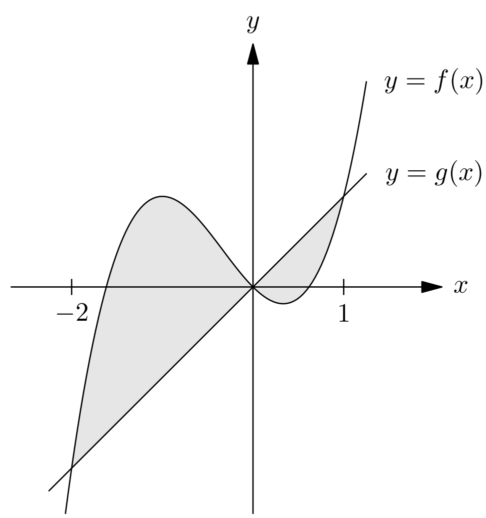
It is crucial to know which function is the greater and which the lesser throughout the region or regions in question.

To find the area enclosed by given curves:

- Sketch the curves. (This may reduce the chance of making errors.)
- Find where the curves intersect, remembering that there may be several points of intersection.
- Find the area of each region via integration and then add all the areas together.

Example 5.38 Find the combined area of the regions bounded by the curves $y = x^3 + x^2 - x$ and $y = x$.

Solution: Let $f(x) = x^3 + x^2 - x$ and $g(x) = x$. The curves in question are as follows, with the shaded regions being the ones whose areas we are to find:



The points of intersection are found by solving $f(x) = g(x)$ for x:

$$f(x) = g(x) \iff x^3 + x^2 - x = x$$

$$\iff 0 = x^3 + x^2 - 2x = x(x-1)(x+2)$$

if and only if $x \in \{-2, 0, 1\}$. Thus, the points of intersection $(x, f(x)) = (x, g(x))$ are

$$(-2, -2), \quad (0, 0), \quad (1, 1).$$

In the left-hand region, $f(x) \geq g(x)$, so this region has area

$$\int_{-2}^{0} (f(x) - g(x))\, dx = \int_{-2}^{0} (x^3 + x^2 - 2x)\, dx = \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2\right]_{-2}^{0} = \frac{8}{3}.$$

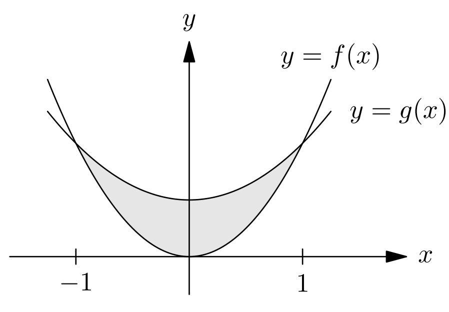
In the right-hand region, the inequality is opposite, i.e., $f(x) \leq g(x)$, so the area this time is

$$\int_{0}^{1} (g(x) - f(x))\, dx = \int_{0}^{1} (-x^3 - x^2 + 2x)\, dx = \left[-\frac{1}{4}x^4 - \frac{1}{3}x^3 + x^2\right]_{0}^{1} = \frac{5}{12}.$$

Thus, the total area is $\frac{8}{3} + \frac{5}{12} = \frac{37}{12}$.

Example 5.39 Find the area of the region enclosed by the curves $y = x^2$ and $y = \frac{1}{2}(x^2 + 1)$.

Solution: Let $f(x) = x^2$ and $g(x) = \frac{1}{2}(x^2 + 1)$. The curves in question are as follows:



For the points of intersection, we again solve $f(x) = g(x)$:

$$f(x) = g(x) \iff x^2 = \tfrac{1}{2}(x^2 + 1)$$
$$\iff x^2 - 1 = 0$$

if and only if $x \in \{-1, 1\}$. The points of intersection are thus

$$(-1, 1) \quad \text{and} \quad (1, 1).$$

Hence, observing that $g(x) \geq f(x)$ in the region in question, we see that the area is

$$\int_{-1}^{1} (g(x) - f(x))\, dx = \frac{1}{2}\int_{-1}^{1} (1 - x^2)\, dx = \frac{1}{2}\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{2}{3}.$$

6 Functions and curves

6.1 The Intermediate Value Theorem

Theorem 6.1 *Let $f : X \to Y$ be a continuous function, where X and Y are subsets of $\mathbb{R}$. Suppose that a and b are elements of X such that $a < b$ and $[a, b] \subseteq X$. Suppose also that y is an element of Y such that either $f(a) < y < f(b)$ or $f(b) < y < f(a)$. Then there is $x \in (a, b)$ such that $f(x) = y$.*

A proof of this theorem is given in Section 2 of the Appendix.

Example 6.2 Show that the equation $x^8 + 3x^2 - 2 = 0$ has at least two solutions $x \in \mathbb{R}$.

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^8 + 3x^2 - 2$, continuous because it is a polynomial function. Now,

$$f(-1) = 2 > 0 \quad \text{and} \quad f(0) = -2 < 0,$$

so by the Intermediate Value Theorem, there is $x_1 \in (-1, 0)$ such that $f(x_1) = 0$. (In the theorem, take $a = -1$, $b = 0$, and $y = 0$.)

Next, $f(1) = 2 > 0$, so the theorem applied again, with $a = 0$, $b = 1$, and $y = 0$, shows that there is $x_2 \in (0, 1)$ such that $f(x_2) = 0$.

Example 6.3 Show that the equation $2^x = \frac{1}{x^2+2}$ has a solution $x \in \mathbb{R}$.

Solution: Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2^x - \frac{1}{x^2+2}$. Then

$$f(0) = \frac{1}{2} > 0 \quad \text{and} \quad f(-5) = \frac{1}{32} - \frac{1}{27} < 0,$$

so by the Intermediate Value Theorem, there is $x \in (-5, 0)$ such that $f(x) = 0$.

Example 6.4 Show that the equation $\ln(x) = x^3 - x - \frac{1}{4}$ has at least two solutions $x \in (\frac{1}{2}, \frac{3}{2})$. You may use the well-known approximations $\sqrt{2} \approx 1.414$ and $e \approx 2.718$.

Solution: Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = \ln(x) - x^3 + x + \frac{1}{4}$. Note to begin with that

$$f(1) = 0 - 1 + 1 + \frac{1}{4} = \frac{1}{4} > 0.$$

Now,

$$\begin{aligned} f\left(\frac{1}{2}\right) &= \ln\left(\frac{1}{2}\right) - \frac{1}{8} + \frac{1}{2} + \frac{1}{4} \\ &= \frac{5}{8} - \ln(2). \end{aligned}$$

This number is negative. Indeed,

$$\frac{5}{8} - \ln(2) < 0 \iff \frac{5}{8} < \ln(2)$$

$$\iff e^{5/8} < 2$$
$$\iff e < 2^{8/5} = 2 \cdot 2^{3/5},$$

so it is sufficient to show that $2 \cdot 2^{3/5} > e$. But

$$2 \cdot 2^{3/5} > 2 \cdot 2^{1/2} = 2\sqrt{2} \approx 2.828,$$

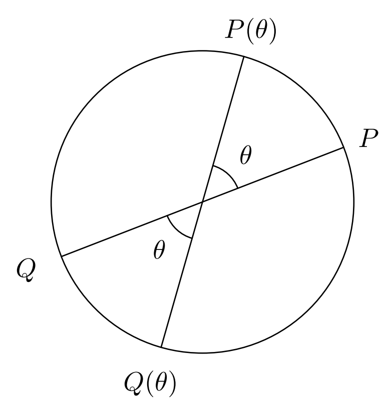
while $e \approx 2.718$. Therefore, by the Intermediate Value Theorem, there is $x_1 \in (\frac{1}{2}, 1)$ such that $f(x_1) = 0$. Also,

$$\begin{aligned} f\left(\frac{3}{2}\right) &= \ln\left(\frac{3}{2}\right) - \frac{27}{8} + \frac{3}{2} + \frac{1}{4} \\ &= \ln\left(\frac{3}{2}\right) - \frac{13}{8} \\ &< 1 - \frac{13}{8} \quad \text{because } \tfrac{3}{2} < e \\ &= -\frac{5}{8} \\ &< 0, \end{aligned}$$

so this time there is $x_2 \in (1, \frac{3}{2})$ such that $f(x_2) = 0$.

Example 6.5 Show that at any time of any day, there are two antipodal points of the Earth's surface at the same temperature.

Solution: We show more. Specifically, given any geodesic, there are antipodal points on that geodesic at the same temperature. To see this, begin with any two antipodal points on the geodesic in question, say P and Q. Move in some chosen direction around the geodesic by angle θ to obtain antipodal points $P(\theta)$ and $Q(\theta)$:



Note that

$$P(0) = P,$$

$$Q(0) = Q,$$
$$P(\pi) = Q,$$
$$Q(\pi) = P.$$

At any point x on the Earth's surface, let $T(x)$ be the temperature at x, and define $f : [0, \pi] \to \mathbb{R}$ by

$$f(\theta) = T(P(\theta)) - T(Q(\theta)).$$

It is reasonable to assume that temperature varies continuously over the Earth's surface, so f is continuous. Now, if $T(P) = T(Q)$, we are done. Otherwise, we may assume that one is greater, say $T(P) > T(Q)$. Let $\Delta = T(P) - T(Q) > 0$. Then

$$f(0) = T(P) - T(Q) = \Delta > 0,$$
$$\text{and} \quad f(\pi) = T(Q) - T(P) = -\Delta < 0.$$

Therefore, by the Intermediate Value Theorem, there is $\theta \in (0, \pi)$ such that $f(\theta) = 0$, i.e., $T(P(\theta)) = T(Q(\theta))$.

Example 6.6 Consider a rotationally symmetric four-legged table, all its legs being, in particular, the same length. We claim that no matter how uneven the ground, it is possible to rotate the table so that all four legs touch the ground simultaneously.

Assume that the four legs do not already all touch the ground. Without yet rotating the table, it is possible to rock the table on two opposite legs, say A and C, in such a way that the other legs, B and D, are an equal vertical distance $d > 0$ from the ground. Call this the *starting point*, and define the *end point* as follows:

- Leg A is at the position leg B was in at the starting point.
- Leg B is at the position leg C was in at the starting point.
- Leg C is at the position leg D was in at the starting point.
- Leg D is at the position leg A was in at the starting point.

The end point is possible because of the assumption of the table's symmetry. Now, to get from the starting point to the end point, rotate the table in such a way that, at all times t,

- the vertical distance $d_A(t)$ from A to the ground is equal to the vertical distance $d_C(t)$ from C to the ground, and
- the vertical distance $d_B(t)$ from B to the ground is equal to the vertical distance $d_D(t)$ from D to the ground.

You may need to tilt and lift the table slightly to achieve this, but it is possible. Consequently, at a given time t, we may let $d_{A,C}(t) = d_A(t) = d_C(t)$, the shared vertical distance of A and C from the ground, and may define $d_{B,D}(t)$ in similar fashion. Hence, we set

$$f(t) = d_{A,C}(t) - d_{B,D}(t).$$

If the ground varies continuously (a reasonable assumption) and we move the table in a continuous way (as we may do, and in fact inevitably will), then f is a continuous function, so we can look to apply the Intermediate Value Theorem. Specifically, if we are at the starting point at time t_0 and are at the end point at time t_1, then

$$\begin{aligned} f(t_0) &= d_{A,C}(t_0) - d_{B,C}(t_0) = 0 - d = -d < 0, \\ f(t_1) &= d_{A,C}(t_1) - d_{B,C}(t_1) = d - 0 = d > 0, \end{aligned}$$

so the theorem guarantees the existence of a time $t \in (t_0, t_1)$ such that $f(t) = 0$, i.e.,

$$d_{A,C}(t) = d_{B,D}(t).$$

Therefore, if we lower the table vertically by this common distance, all four legs will touch the ground simultaneously.

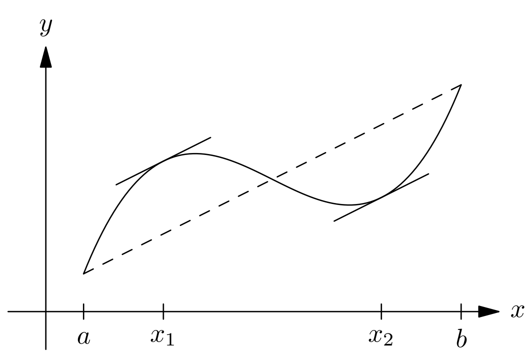
6.2 The Mean Value Theorem

The Mean Value Theorem says the following. Let $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function that is differentiable on (a, b). Then there is $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}. \tag{6.1}$$

A proof is given in Section 3 of the Appendix.

The diagram below illustrates the idea of the Mean Value Theorem. It shows points $x = x_1$ and $x = x_2$ for which (6.1) holds. The dashed line has slope equal to $(f(b) - f(a))/(b - a)$, and the tangent lines have slopes $f'(x_1)$ and $f'(x_2)$. All three of these slopes are equal according to the Mean Value Theorem.



Remark. The special case of the Mean Value Theorem where $f(a) = f(b)$ is called Rolle's Theorem.

Example 6.7 Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4 + 3x^2 + x - 1$. Show that there is $x \in (-1, 1)$ such that $f'(x) = 1$.

Solution: The given function is differentiable everywhere, so we may apply the theorem in the case $a = -1$ and $b = 1$ to see that there is $x \in (-1, 1)$ such that

$$f'(x) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{4 - 2}{2} = 1.$$

Example 6.8 Show that there is exactly one $x \in \mathbb{R}$ such that $\cos(x) = x$.

Solution: Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x - \cos(x)$. Because f is continuous and $f(0) = -1 < 0$ while $f(\frac{\pi}{2}) = \frac{\pi}{2} > 0$, the Intermediate Value Theorem guarantees the existence of some $x_0 \in (0, \frac{\pi}{2})$ such that $f(x_0) = 0$, i.e., $\cos(x_0) = x_0$. This shows existence.

For uniqueness, suppose that there are $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ such that

$$\cos(x_1) = x_1 \quad \text{and} \quad \cos(x_2) = x_2,$$

i.e., $f(x_1) = f(x_2) = 0$. Note that $x_1 = \cos(x_1) \in [-1, 1]$ and similarly for x_2, so

$$-1 \leq x_1 < x_2 \leq 1.$$

Now, f is differentiable, so we may use the Mean Value Theorem on the interval $[x_1, x_2]$ to determine that there is $x \in (x_1, x_2)$ such that

$$\begin{aligned} f'(x) &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0, \\ \text{i.e.,} \quad 1 + \sin(x) &= 0, \\ \text{i.e.,} \quad \sin(x) &= -1, \end{aligned}$$

which is to say that

$$x = \frac{3\pi}{2} + 2\pi k \quad \text{for some } k \in \mathbb{Z}.$$

If $k \geq 0$, then

$$1 \geq x_2 > x \geq \frac{3\pi}{2},$$

a contradiction, and if $k < 0$, then

$$-1 \leq x_1 < x \leq -\frac{\pi}{2},$$

again a contradiction. Therefore, solutions $x_1 < x_2$ do not exist, so x_0 above is the only solution.

Example 6.9 Suppose that $f : [0,1] \to [0,1]$ is continuous, is differentiable on $(0,1)$, and satisfies $f'(x) \geq 1$ for all $x \in (0,1)$. Use the Mean Value Theorem to find f.

Solution: We claim that $f(x) = x$ for all $x \in [0,1]$. First, take any $x \in (0,1)$. By the Mean Value Theorem, there is $x_1 \in (x,1)$ such that

$$f'(x_1) = \frac{f(1) - f(x)}{1 - x} \leq \frac{1 - f(x)}{1 - x} \quad \text{because } f(1) \leq 1,$$

so

$$\frac{1 - f(x)}{1 - x} \geq f'(x_1) \geq 1,$$

and rearranging this gives

$$\begin{aligned} 1 - f(x) &\geq 1 - x, \\ \text{i.e.,} \quad f(x) &\leq x. \end{aligned} \tag{6.2}$$

Further, the Mean Value Theorem applied again shows that there is $x_2 \in (0,x)$ such that

$$f'(x_2) = \frac{f(x) - f(0)}{x - 0} \leq \frac{f(x)}{x} \quad \text{because } f(0) \geq 0,$$

so

$$\frac{f(x)}{x} \geq f'(x_2) \geq 1,$$

i.e., $f(x) \geq x$. Combining this with (6.2) gives $f(x) = x$.

It remains to show that $f(0) = 0$ and $f(1) = 1$. We present two ways to do this. First, we may use the continuity of f, along with the already-proven fact that $f(x) = x$ when $0 < x < 1$, to observe that

$$\begin{aligned} f(0) &= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0, \\ f(1) &= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1. \end{aligned}$$

But the Mean Value Theorem gives us a second approach. Indeed, by that theorem, there is $x_3 \in (0,1)$ such that

$$f'(x_3) = \frac{f(1) - f(0)}{1 - 0} = f(1) - f(0),$$

so

$$1 \geq f(1) = f(0) + f'(x_3) \geq f(0) + 1 \geq 0 + 1 = 1. \tag{6.3}$$

This forces all instances of $\geq$ in (6.3) to be $=$, so $f(1) = 1$ and $f(0) = 0$.

Using the Mean Value Theorem to prove injectivity

Proposition 6.10 *Suppose that $f : [a,b] \to \mathbb{R}$ is continuous, where $a < b$, and assume further that f is differentiable on (a,b).*

(i) If $f'(x) \neq 0$ for all $x \in (a, b)$, then f is injective.

(ii) If $f'(x) > 0$ for all $x \in (a, b)$, then f is monotone increasing.

(iii) If $f'(x) < 0$ for all $x \in (a, b)$, then f is monotone decreasing.

Proof. We may prove this using the Mean Value Theorem. Assume first that $f'(x) \neq 0$ for all $x \in (a, b)$, and suppose that $x_1, x_2 \in [a, b]$ are distinct. By the theorem, there is x strictly between x_1 and x_2 such that

$$f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Because $f'(x) \neq 0$, it follows that $f(x_2) - f(x_1) \neq 0$, i.e., $f(x_2) \neq f(x_1)$.

Now suppose that $f'(x) > 0$ for all $x \in (a, b)$. If $a \leq x_1 < x_2 \leq b$, then we can use the theorem to choose $x \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(x)(x_2 - x_1).$$

Both factors on the right-hand side are positive, so $f(x_2) - f(x_1) > 0$, i.e., $f(x_2) > f(x_1)$. The case where $f'(x) < 0$ for all $x \in (a, b)$ is handled in the same way. ■

Example 6.11 Show that the equation

$$\frac{1}{3^x} = \arctan(x)$$

has a unique solution $x \in \mathbb{R}$.

Solution: First, we will use the Intermediate Value Theorem to show that there is at least one solution. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \arctan(x) - \frac{1}{3^x},$$

a differentiable function (so continuous in particular). We have

$$\begin{aligned} f(0) &= -1 < 0 \\ \text{and} \quad f(1) &= \frac{\pi}{4} - \frac{1}{3} > 0, \end{aligned}$$

so the Intermediate Value Theorem guarantees the existence of an $x_0 \in (0, 1)$ such that $f(x_0) = 0$.

To show that x_0 is the only solution, we consider the derivative of f. Noting that $\frac{1}{3^x} = (\frac{1}{3})^x$, we obtain

$$f'(x) = \frac{1}{x^2 + 1} - \ln\left(\frac{1}{3}\right) \cdot \left(\frac{1}{3}\right)^x = \frac{1}{x^2 + 1} + \frac{1}{3^x} \ln(3),$$

which is positive for all $x \in \mathbb{R}$. Therefore, by Proposition 6.10, f is injective, so the equation $f(x) = 0$ has only the solution $x = x_0$.

6.3 Maxima and minima

Let $f : D \to \mathbb{R}$ be a function. We have the following terms:

- **Point of global maximum:** A point $c \in D$ such that $f(c) \geq f(x)$ for all $x \in D$.
- **Value of global maximum:** The value of f at a point of global maximum.
- **Point of global minimum:** A point $c \in D$ such that $f(c) \leq f(x)$ for all $x \in D$.
- **Value of global minimum:** The value of f at a point of global minimum.
- **Point of local maximum:** A point $c \in D$ such that $f(c) \geq f(x)$ for all x in some open interval containing c. We exclude the end-points of D if $D = [a, b]$. That is, end-points are not considered local maxima.
- **Point of local minimum:** Same definition as for local maximum except with the inequality $f(c) \leq f(x)$ instead.
- **Critical number:** Either of the following:
  - (i) a point $c \in D$ such that $f'(c)$ is zero or undefined, or
  - (ii) an end-point of D.
- **Extremum point:** A point that is either a minimum or a maximum. One has the obvious notions of global extremum point and local extremum point.

In the diagram below, which shows a function defined on an interval $[a, b]$,

(1) is neither a global nor a local extremum. (It is an end-point so is not considered a local extremum.)

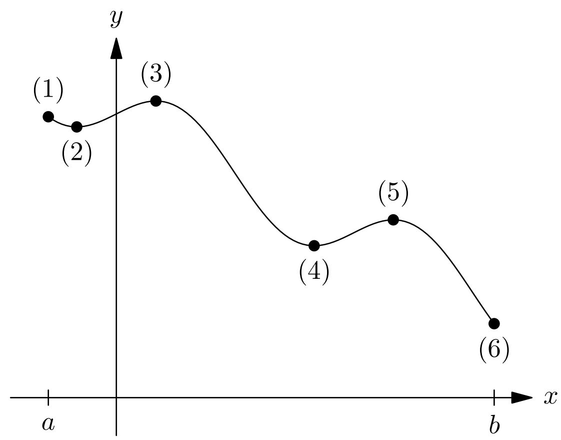
(2) is a local minimum but not a global minimum.

(3) is both a global and a local maximum.

(4) is a local minimum but not a global minimum.

(5) is a local maximum but not a global maximum.

(6) if a global minimum but, being an end-point, is not considered to be a local minimum.



When attempting to find local extrema, a useful way to narrow our search is by using the following fact:

> If c is a point of local extremum, then c is a critical number, i.e., either f is not differentiable at c, or it is and $f'(c) = 0$. (Recall that we do not consider end-points to be local extrema.)

The essence of this assertion is Fermat's Theorem, which is stated and proven in Section 3 of the Appendix.

Remark. Be forewarned that $f'(c) = 0$ does not imply that c is a point of local extremum. For example, if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$, but 0 is not a point of local extremum of f. Indeed, in any open interval I containing 0, there are values $x \in I$ such that $f(x) > f(0)$ (when $x > 0$) and also values $x \in I$ such that $f(x) < f(0)$ (when $x < 0$).

Remark. If c is a point of global maximum, then it is necessarily either a point of local maximum or an end-point. Similarly, a point of global minimum is necessarily either a point of local minimum or an end-point.

Note that some functions have no extrema of any kind, as in the next two examples.

Example 6.12 The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ has no extrema: It has no critical points, so it has no local extrema, so it has no global extrema either.

Example 6.13 The function $g : (0, 1) \to (0, 1)$ given by $g(x) = x$ has no extrema either, because, again, it has no critical points.

Example 6.14 The function $j : \mathbb{R} \to \mathbb{R}$ given by $j(x) = \arctan(x)$ has no extrema, having, indeed, no critical points.

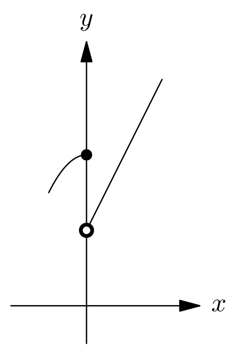
Now let us consider an example where extrema do exist.

Example 6.15 Consider $f : [-\frac{1}{2}, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2 - 2x^2 & \text{if } -\frac{1}{2} \le x \le 0, \\ 1 + 2x & \text{if } 0 < x \le 1. \end{cases}$$

Sketch the graph of f and find its local and global extrema.

Solution:



We contend that f has no global minimum. For this, observe that $f(x) > 1$ for all $x \in [-\frac{1}{2}, 1]$, but also that if $y > 1$, then there exists $x \in [-\frac{1}{2}, 1]$ such that $f(x) < y$, so no value in the range of f is less than or equal to all values in the range. (The range, indeed, is the half-open interval $(1, 3]$.)

There is a global maximum at $x = 1$, where the value of the function is 3, but it is not a local maximum, because it is an end-point of the domain.

There is a local maximum at $x = 0$, where f takes the value 2. Indeed, $f(x) \le 2$ for all $x \in (-\frac{1}{2}, \frac{1}{2})$.

The function f has no local minima. The only critical point is $x = 0$, and every open interval containing 0 possesses points at which the value of the function is less than $f(0)$.

In the case of a continuous function on a closed interval, which is common, there is a useful method, based on the following fact:

> If $f : [a, b] \to \mathbb{R}$ is continuous, where $a < b$, then f has a global maximum and a global minimum.

This assertion is often known as the Extreme Value Theorem. To find the global extrema in this situation,

- find the set X of critical points in $[a, b]$ (do not forget the end-points), and
- calculate $f(x)$ for each $x \in X$, and let M be the maximum of these values and m the minimum.

The global maximum occurs at every $x \in X$ such that $f(x) = M$, and the global minimum occurs at every $x \in X$ such that $f(x) = m$.

Example 6.16 If $f : [-3, 3] \to \mathbb{R}$ is defined by

$$f(x) = \ln(x^2 + 5),$$

find the points of global maximum and minimum and the values of f at those points.

Solution: First, we find the critical points. These include the end-points, -3 and 3, but also the points x where $f'(x)$ is non-existent or where it exists and takes the value 0. In fact, f is differentiable everywhere, and

$$f'(x) = \frac{2x}{x^2 + 5},$$

so $f'(x) = 0$ if and only if $x = 0$. Thus, the set of critical points is $\{-3, 0, 3\}$, and the values of f at these points are

$$\begin{aligned} f(-3) &= \ln(14), \\ f(0) &= \ln(5), \\ f(3) &= \ln(14). \end{aligned}$$

Therefore, the points of global maximum are -3 and 3, where f takes the value $\ln(14)$, and the only point of global minimum is 0, where f takes the value $\ln(5)$.

If the interval is not closed, we have to be more careful, as we illustrate in the next example.

Example 6.17 Decide whether the function $f : (-2, 2) \to \mathbb{R}$ defined by

$$f(x) = (9 - x^4)^{3/5}$$

has any global extrema. If so, find those points and the values of the function there.

Solution: The domain of f, the open interval $(-2, 2)$, does not contain the end-points, so the only possible critical points are the numbers x where $f'(x)$ is either undefined or is defined and is equal to 0. In this case, f is not differentiable where $9 - x^4 = 0$, i.e., at $x = \pm\sqrt{3}$, but elsewhere we have

$$f'(x) = \frac{3}{5}(9 - x^4)^{-2/5}(-4x^3) = -\frac{12}{5}\,\frac{x^3}{(9 - x^4)^{2/5}},$$

and this is zero if and only if $x = 0$. Thus, the critical points are $-\sqrt{3}, 0, \sqrt{3}$, with the function taking the values $0, 9^{3/5}, 0$ respectively there.

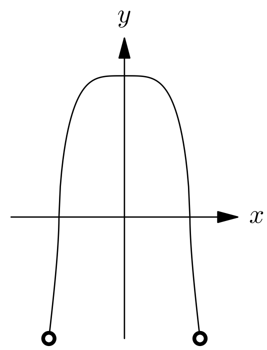
Can we conclude that 0 is the value of global minimum and $9^{3/5}$ the value of global maximum? Let us consider $9^{3/5}$ first. If $x_1, x_2 \in (-2, 2)$, then

$$f(x_1) < f(x_2) \iff (9 - x_1^4)^{3/5} < (9 - x_2^4)^{3/5}$$

$$\begin{aligned} &\iff 9 - x_1^4 < 9 - x_2^4 \\ &\iff x_2^4 < x_1^4 \\ &\iff |x_2| < |x_1|, \end{aligned}$$

so the global maximum occurs where $|x|$ is least, i.e., at $x = 0$, and the value of the function here is $9^{3/5}$.

What about the global minimum? In fact, there is none. The calculation above shows that the greater the absolute value of x, the lesser the value of $f(x)$, so because 2 and -2 are not included in the domain, there is no global minimum. The graph below illustrates the situation.



6.4 Curve sketching

The key features to determine before sketching the graph of a function f are

- the domain of f,
- the axis intercepts,
- the extrema,
- the potential inflection points,
- the regions of upward and downward concavity, and
- the asymptotes.

We will also need the notion of a slant asymptote. If $m \neq 0$, then the line $y = mx + c$ is called a *slant asymptote* of f if either

$$\lim_{x\to\infty} (f(x) - (mx + c)) = 0 \quad \text{or} \quad \lim_{x\to-\infty} (f(x) - (mx + c)) = 0.$$

If f is a rational function, say

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions, and if $\deg(p(x)) = \deg(q(x)) + 1$, then f has a slant asymptote, found by long division of polynomials.

Example 6.18 Show that the graph of the function f given by

$$f(x) = \frac{x^3 - 6x^2 + 12x - 4}{x^2 - 4x + 4}$$

has a slant asymptote, and find it.

Solution: The degree of the numerator is one more than that of the denominator, so there is a slant asymptote. Let us perform long division:

$$\begin{array}{r|l} & x - 2 \\ \hline x^2 - 4x + 4 & x^3 - 6x^2 + 12x - 4 \\ & -x^3 + 4x^2 - 4x \\ \hline & -2x^2 + 8x - 4 \\ & 2x^2 - 8x + 8 \\ \hline & 4 \end{array}$$

We arrive at

$$x^3 - 6x^2 + 12x - 4 = (x-2)(x^2 - 4x + 4) + 4 = (x-2)^3 + 4,$$

so

$$f(x) = \frac{(x-2)^3 + 4}{(x-2)^2} = x - 2 + \frac{4}{(x-2)^2}.$$

Thus,

$$f(x) - (x-2) = \frac{4}{(x-2)^2} \to 0 \quad \text{as } x \to \pm\infty,$$

so the line $y = x - 2$ is a slant asymptote of f in both directions.

Example 6.19 Sketch the graph of the function f given by

$$f(x) = \frac{x^3 + 3x^2}{(x+1)^2}.$$

Solution: Before attending to the key features, we find f' and f'':

$$f'(x) = \frac{(3x^2 + 6x)(x+1)^2 - (x^3 + 3x^2) \cdot 2(x+1)}{(x+1)^4} = \frac{x^3 + 3x^2 + 6x}{(x+1)^3},,$$
$$f''(x) = \frac{(3x^2 + 6x + 6)(x+1)^3 - (x^3 + 3x^2 + 6x) \cdot 3(x+1)^2}{(x+1)^6} = \frac{6 - 6x}{(x+1)^4}.$$

Domain: The denominator is zero when $x = -1$, and the numerator is non-zero at that point, so f is not defined at -1. It is defined everywhere else, so the domain is $\mathbb{R} \setminus \{-1\} = (-\infty, -1) \cup (-1, \infty)$.

Axis intercepts: The y-intercept is $f(0) = 0$, and the x-intercepts occur where $0 = x^3 + 3x^2 = x^2(x+3)$, i.e., at $x = -3, 0$.

Extrema: $f'(x) = 0$ if and only if $x(x^2 + 3x + 6) = 0$, if and only if $x = 0$ because the quadratic factor has negative discriminant. Further, an analysis of the signs of the numerator and denominator shows that the function has the following behaviour:

	$x < -1$	$-1 < x < 0$	$x > 0$
f'	$+$	$-$	$+$
f	$\nearrow$	$\searrow$	$\nearrow$

Thus, $x = 0$ is the only point of local extremum, and there is a local minimum there with value $f(0) = 0$. Another way to see that this is a local minimum rather than a local maximum is to note that $f''(0) = 6 > 0$.

Inflection points: $f''(x) = 0$ if and only if $x = 1$, so this is a potential inflection point. We will confirm that it is when considering the regions of upward and downward concavity below.

Regions of upward and downward concavity: Because the denominator of the quotient in the given expression for $f''(x)$ is always positive, we deduce that $f''(x) > 0$ if $x < 1$ and $f''(x) < 0$ if $x > 1$. In summary:

	$x < -1$	$-1 < x < 1$	$x > 1$
f''	$+$	$+$	$-$
f	$\smile$	$\smile$	$\frown$

Thus, there is indeed an inflection point at $x = 1$, and it has the form $\smile\!\frown$. (For the purposes of sketching the graph, it is also helpful to observe that the y-value of the inflection point is $f(1) = 1$.)

Asymptotes: We begin by investigating possible slant asymptotes. We divide the denominator into the numerator with remainder:

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad x + 1 \\ \hline x^2 + 2x + 1 & x^3 + 3x^2 \\ & -x^3 - 2x^2 - x \\ \hline & \quad\quad x^2 - x \\ & \quad -x^2 - 2x - 1 \\ \hline & \quad\quad\quad -3x - 1 \end{array}$$

Hence,

$$f(x) = \frac{(x+1)(x^2 + 2x + 1) - (3x + 1)}{(x+1)^2} = x + 1 - \frac{3x + 1}{(x+1)^2}.$$

Because

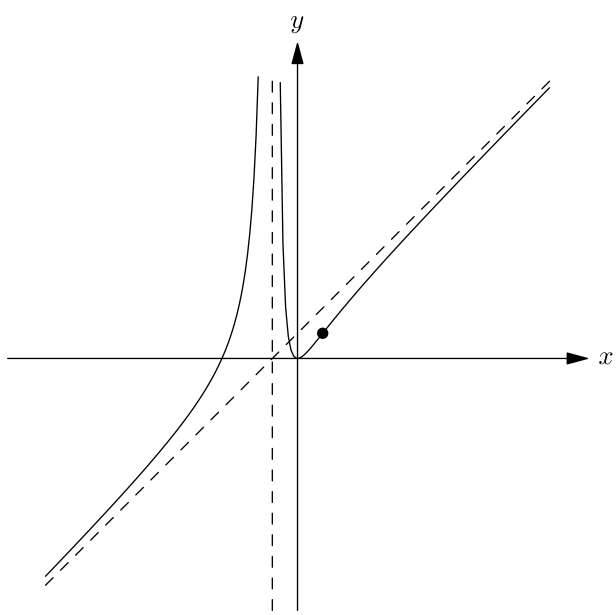
$$\lim_{x \to \infty} \frac{3x + 1}{(x+1)^2} = \lim_{x \to \infty} \frac{1}{x} \frac{3 + \frac{1}{x}}{(1 + \frac{1}{x})^2} = 0,$$

and similarly for the limit as $x \to -\infty$, the line $y = x+1$ is a slant asymptote of the graph in both directions. There are consequently no horizontal asymptotes.

The line $x = -1$ is a vertical asymptote and is the only one in fact. Indeed, the only point where the demoninator is zero is at $x = -1$, and then considering $f(-1+h)$ with $h \neq 0$, we have

$$f(-1+h) = \frac{(-1+h)^2(2+h)}{h^2} \to \infty \quad \text{as } h \to 0,$$

so $f(x) \to \infty$ as $x \to -1$. Thus, the line $x = -1$ is indeed a vertical asymptote, and the graph approaches ∞ on both sides of that line.



The dot indicates the inflection point at $(1, 1)$, and the dashed lines indicate the vertical asymptote $x = -1$ and the slant asymptote $y = x + 1$.

Example 6.20 Sketch the graph of the function f given by

$$f(x) = e^{\frac{1}{x^3}}.$$

Solution: Let us begin by finding f' and f'':

$$\begin{aligned} f'(x) &= -\frac{3}{x^4} e^{\frac{1}{x^3}}, \\ f''(x) &= \left(\frac{12}{x^5} - \frac{3}{x^4} \left(-\frac{3}{x^4} \right) \right) e^{\frac{1}{x^3}} \\ &= \left(\frac{12}{x^5} + \frac{9}{x^8} \right) e^{\frac{1}{x^3}}. \end{aligned}$$

Domain: The function is not defined at 0 but is everywhere else, so the domain is $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$.

Axis intercepts: For all $x \in \mathbb{R} \setminus \{0\}$, $f(x) > 0$, so the graph does not intersect the horizontal axis. It also does not intersect the vertical axis, because f is not defined there.

Extrema: The derivative $f'(x) = -\frac{3}{x^4} e^{\frac{1}{x^3}}$ is never zero. There are no other critical points either, because f is differentiable everywhere it is defined, so f has no extrema.

Inflection points:

$$\begin{aligned} f''(x) = 0 &\iff \frac{12}{x^5} + \frac{9}{x^8} = 0 \\ &\iff 12x^3 + 9 = 0 \\ &\iff x^3 = -\frac{3}{4} \\ &\iff x = -\left(\frac{3}{4}\right)^{1/3}. \end{aligned}$$

This is a candidate for an inflection point, and we will check this when considering regions of upward and downward concavity below.

Regions of upward and downward concavity: The same steps as above show that

$$\begin{aligned} f''(x) > 0 &\iff x > -\left(\frac{3}{4}\right)^{1/3}, \\ f''(x) < 0 &\iff x < -\left(\frac{3}{4}\right)^{1/3}, \end{aligned}$$

so we have the table

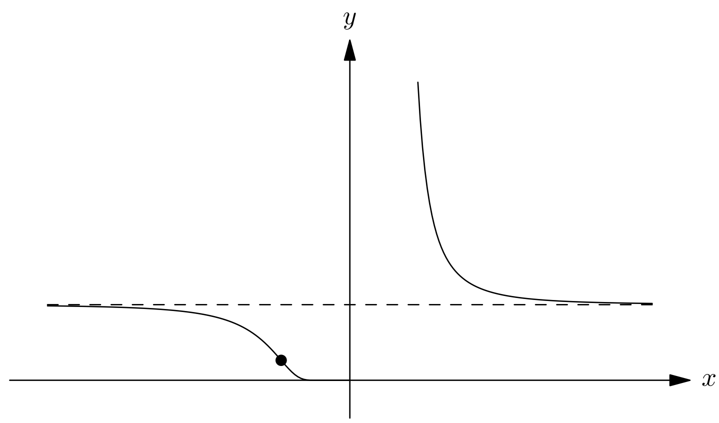
	$x < -\left(\frac{3}{4}\right)^{1/3}$	$-\left(\frac{3}{4}\right)^{1/3} < x < 0$	$x > 0$
f''	$-$	$+$	$+$
f	$\frown$	$\smile$	$\smile$

Thus, there is an inflection point of the form $\frown\!\smile$ at $x = -\left(\frac{3}{4}\right)^{1/3}$. Note that the value of f at this point is $e^{-4/3}$.

Asymptotes: Horizontal: $1/x^3 \to 0$ as $x \to \infty$, so $e^{1/x^3} \to e^0 = 1$. The same reasoning shows that $e^{1/x^3} \to 1$ as $x \to -\infty$.

Vertical: The only possible vertical asymptote is the line $x = 0$, where the function is not defined. As $x \to 0^+$, $1/x^3 \to \infty$, so $e^{1/x^3} \to \infty$. However, $1/x^3 \to -\infty$ as $x \to 0^-$, so from this direction $e^{1/x^3} \to 0$.

Slant: Because $f(x)$ has a finite limit in both directions ($x \to \infty$ and $x \to -\infty$), there are no slant asymptotes.



The dot indicates the inflection point at $\left(-\left(\frac{3}{4}\right)^{1/3}, e^{-4/3}\right)$, and the horizontal dashed line indicates the horizontal asymptote $x = 1$. Note also that the line $x = 0$ is a vertical asymptote.

7 Applications of differentiation

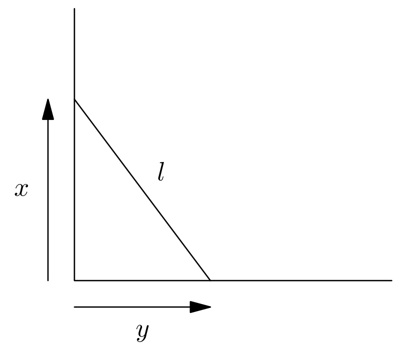
7.1 Related rates

A related-rates problem is a problem where one has to find the rate of change of some quantity x not at a given time but when x takes a given value, or when some quantity related to x takes a given value. Practical steps to solve such a problem are as follows:

- Draw a diagram and label the key quantities.
- Find an equation relating the quantities, eliminating any variables that can be.
- Differentiate both sides of the equation with respect to time.
- Eliminate the time variable if it still appears, substitute known data, and then rearrange the resulting equation to express the unknown rate in terms of everything else.

Example 7.1 A ladder 10 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall horizontally at a rate of $0.5\,\mathrm{m\,s^{-1}}$, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 m from the wall? Assume that the top of the ladder always touches the wall. (Is this a reasonable assumption?)

Solution: We begin with a diagram:



If x is the elevation of the top of the ladder from the ground at time t, and y is the displacement of the bottom of the ladder from the wall at time t, then Pythagoras' Theorem gives us

$$x^2 + y^2 = l^2$$

where $l = 10$ is the length of the ladder. Differentiating both sides of this equation with respect to time—using the chain rule, of course—we obtain

$$2xx' + 2yy' = 0,$$

i.e.,

$$x' = -\frac{yy'}{x} = -\frac{yy'}{\sqrt{l^2 - y^2}}. \tag{7.1}$$

Hence, when $y = 6$,

$$x' = -\frac{6 \cdot 0.5}{\sqrt{10^2 - 6^2}} = -\frac{3}{8}.$$

Thus, the top of the ladder is sliding down the wall at a rate of $\frac{3}{8}\,\mathrm{m\,s^{-1}}$ when the bottom is $6\,\mathrm{m}$ from the wall.

The assumption that the top of the ladder always touch the wall is not reasonable, because it leads to (7.1), which predicts that the top of the ladder would eventually move faster than the speed of light as y approaches l.

Example 7.2 Consider a spherical lollipop. Assuming that when it is in a person's mouth its volume decreases at a constant rate, express the rate of change of the radius in terms of the radius itself.

Solution: Not much of a diagram is needed here, and it is reasonable to omit drawing one. Instead, we simply let r and V be the radius and volume, respectively, of the lollipop at time t. The formula for the volume of a ball in terms of the radius gives

$$V = \frac{4}{3}\pi r^3,$$

and then differentiating with respect to time results in

$$V' = \frac{4}{3}\pi \cdot 3r^2 r' = 4\pi r^2 r'.$$

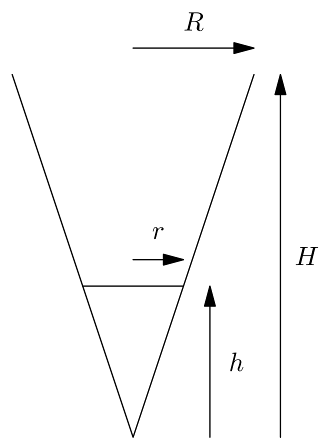
By assumption, V' is constant, say $V' = -C$ where $C > 0$, so

$$r' = -\frac{C}{4\pi r^2}.$$

Observe that the magnitude of the rate of change of the radius increases in inverse proportion to the square of the radius.

Example 7.3 Water is being pumped at a constant rate into a tank in the shape of an inverted cone. The tank is 6 m high, and its radius at the top is 2 m. If the water level is rising at a rate of $0.002\,\mathrm{m\,s^{-1}}$ when the height of the water is 2.5 m, find the rate at which water is entering the tank.

Solution: We begin with a diagram:



Here, H and R are the height and radius of the tank, and h and r are the height and radius of the cone formed by the water in the tank at time t. If V is the volume of water in the tank at time t, then

$$V = \frac{1}{3}\pi r^2 h$$

by the usual formula. But

$$\frac{r}{h} = \frac{R}{H},$$
$$\text{i.e.,} \quad r = \frac{R}{H}h,$$

so

$$V = \frac{1}{3}\pi \left(\frac{R}{H}\right)^2 h^3.$$

Hence, differentiating with respect to time and remembering that the volume is increasing at a constant rate $C > 0$, we arrive at

$$C = V' = \pi \left(\frac{R}{H}\right)^2 h^2 h'.$$

We are told that $R/H = 2/6 = 1/3$, and also that $h' = 0.002$ when $h = 2.5$. Therefore,

$$C = \pi \left(\frac{1}{3}\right)^2 (2.5)^2 \cdot 0.002 = \frac{\pi}{720},$$

so water is entering the tank at a rate of $\frac{\pi}{720}\,\mathrm{m}^3\,\mathrm{s}^{-1}$.

7.2 Optimization

Typically, in an optimization problem, one is given some quantity Q, depending on variables $x_1, \ldots, x_n$, and is required to find values of the variables so that Q is *optimized*, meaning maximized or minimized, if possible. Whether Q is to be maximized or minimized depends on the problem at hand.

The following steps will usually lead to a correct solution in the types of optimization problem you will encounter in this course:

- If it is possible to draw a picture, and if it would be helpful to do so, then draw one, labelling the variables and data.
- Identify a variable x on which Q depends solely, and express Q in terms of that variable alone. (In a general optimization problem, this will not be possible, because Q may depend on several independent variables, but in this course, the variables will be linked in such a way that just one will govern everything.)
- Considering Q now as a function of x on some appropriate domain D, find the global maximum (if maximizing) or the global minimum (if minimizing) as well as the point or points x where the extremum occurs.

Example 7.4 If x and y are non-negative real numbers whose sum is 20, find the least possible value of $x^3 + 4y^3$.

Solution: Let $c = 20$, the problem being solved in the same way for all $c > 0$. Thus, $x + y = c$; this is known as a *constraint*. We use the constraint to express $Q = x^3 + 4y^3$ in terms of x alone: $Q = x^3 + 4(c - x)^3$. The further assumptions that $x, y \geq 0$ imply that $0 \leq x \leq c$, so we have a function $Q : [0, x] \to \mathbb{R}$ given by $Q(x) = x^3 + 4(c - x)^3$.

Because the domain $[0, c]$ of Q is closed, Q has a minimum, which may be found by considering critical points. If $x \in (0, c)$, then

$$Q'(x) = 3x^2 - 12(c - x)^2,$$

so

$$\begin{aligned} Q'(x) = 0 &\iff 3x^2 = 12(c - x)^2 \\ &\iff x^2 = 4(c - x)^2 \\ &\iff x = 2(c - x) \quad \text{because } x, c - x \geq 0 \end{aligned}$$

$$\iff x = \frac{2}{3}c.$$

Thus, the critical points are 0, $\frac{2}{3}c, c$ with corresponding values

$$Q(0) = 4c^3, \quad Q\left(\frac{2}{3}c\right) = \frac{8}{27}c^3 + \frac{4}{27}c^3 = \frac{4}{9}c^3, \quad Q(c) = c^3.$$

The least of these is $\frac{4}{9}c^3$, so this is the least possible value of $x^3 + 4y^3$ subject to $x + y = c$. In the case in question, namely, $c = 20$, the least value is $32\,000/9$. The question did not ask where the minimum occurred, but we note anyway that the location of the minimum is $x = \frac{2}{3}c = \frac{40}{3}$ in this case.

Example 7.5 Let $a, b > 0$, and consider the ellipse defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Find the largest possible area of a rectangle inscribed in the ellipse, the word *inscribed* here meaning that the four vertices of the rectangle are on the ellipse. You may use the fact that if $a \neq b$, any rectangle inscribed in the ellipse must have its sides parallel to the principal axes of the ellipse. (A proof of this supporting fact is given in Section 6 of the Appendix.)

Solution: By the fact given in the question, we may assume that the rectangle is aligned with the coordinate axes. Let (x, y) in the first quadrant be one vertex of the rectangle. The other vertices are therefore $(-x, y)$, $(-x, -y)$, and $(x, -y)$, so the area of the rectangle is

$$2x \cdot 2y = 4xy = 4xb\sqrt{1 - \frac{x^2}{a^2}},$$

where for the second equality we have used the defining equation of the ellipse, along with the fact that $y \geq 0$. Noting that $0 \leq x \leq a$, we define a function $A : [0, a] \to \mathbb{R}$ by

$$A(x) = 4bx\sqrt{1 - \frac{x^2}{a^2}}.$$

Again, the domain is a closed interval, so we need consider only the critical values. Now,

$$\begin{aligned} A'(x) &= 4b\sqrt{1 - \frac{x^2}{a^2}} + 4bx \cdot \frac{1}{2}\left(1 - \frac{x^2}{a^2}\right)^{-1/2}\left(-\frac{2x}{a^2}\right) \\ &= 4b\left(\sqrt{1 - \frac{x^2}{a^2}} - \frac{x^2/a^2}{\sqrt{1 - \frac{x^2}{a^2}}}\right), \end{aligned}$$

so

$$A'(x) = 0 \iff \sqrt{1 - \frac{x^2}{a^2}} = \frac{x^2/a^2}{\sqrt{1 - \frac{x^2}{a^2}}}$$

$$\iff 1 - \frac{x^2}{a^2} = \frac{x^2}{a^2}$$
$$\iff \frac{x^2}{a^2} = \frac{1}{2}$$
$$\iff x = \frac{a}{\sqrt{2}} \quad \text{because } x \geq 0.$$

The critical points are consequently 0, $\frac{a}{\sqrt{2}}$, and a, with corresponding values

$$A(0) = 0, \quad A\left(\frac{a}{\sqrt{2}}\right) = 4b\frac{a}{\sqrt{2}}\sqrt{1 - \frac{1}{2}} = 2ab, \quad A(a) = 0,$$

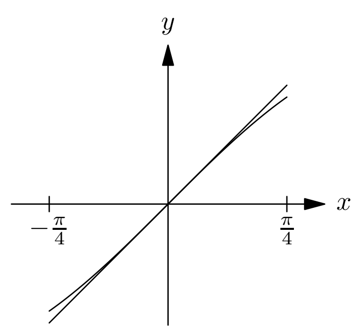
so the maximum value of $A(x)$ is $2ab$ and occurs when $x = \frac{a}{\sqrt{2}}$.

7.3 Linear approximation

Consider a differentiable function $f : D \to \mathbb{R}$. We know that the equation of the tangent line to the graph of f at a is

$$y = f(a) + f'(a)(x - a).$$

If x is close to a, then the tangent line is a good approximation to the curve. Of course, *close* and *good* are not precise terms in this context, but the point holds. For example, here is the graph of sin drawn between $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$, along with its tangent line at 0:

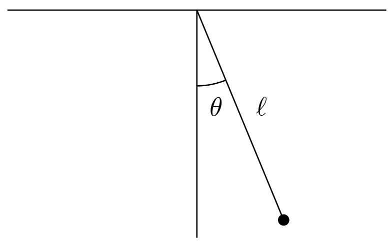


For a given differentiable function $f : D \to \mathbb{R}$ and a given point $a \in D$, the function $L_{f,a} : D \to \mathbb{R}$ defined by

$$L_{f,a}(x) = f(a) + f'(a)(x - a)$$

is called the *linearization* of f at a.

Example 7.6 Consider a pendulum of length ℓ:



By analyzing the forces acting on it, we find that the angle $\theta(t)$ that the pendulum makes with the vertical at time t satisfies the differential equation

$$\theta''(t) + \frac{g}{\ell}\sin(\theta(t)) = 0, \tag{7.2}$$

where g is acceleration due to gravity (approximately $9.81\,\mathrm{m\,s^{-2}}$ at the surface of the Earth). The equation in (7.2) is hard to solve, but a good approximation to the function θ can be obtained by making the simplifying, if not quite accurate, assumption that $\sin(x) = x$ for small angles x. This amounts to replacing sin by its linearization at 0, because if $f(x) = \sin(x)$, then

$$L_{f,0}(x) = f(0) + f'(0)(x-0) = \sin(0) + \cos(0)x = x.$$

A simplified, but still useful, version of (7.2) is thus

$$\theta''(t) + \frac{g}{\ell}\theta(t) = 0, \tag{7.3}$$

which is much easier to solve. In fact, the general solution to (7.3) is

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{\ell}}\,t\right) + B\sin\left(\sqrt{\frac{g}{\ell}}\,t\right),$$

where A and B are constants.

Let us look at some other linearizations.

Example 7.7 If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \cos(x)$, then

$$\begin{aligned} L_{f,0}(x) &= f(0) + f'(0)(x-0) = 1 - \sin(0)x = 1 \quad \text{(constant)}, \\ L_{f,\pi/2}(x) &= f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) \\ &= 0 - \sin\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) \\ &= -\left(x - \frac{\pi}{2}\right). \end{aligned}$$

Example 7.8 If $g : (0,\infty) \to \mathbb{R}$ is given by $g(x) = \ln(x)$, then

$$L_{g,1}(x) = g(1) + g'(1)(x-1) = 0 + \tfrac{1}{1}(x-1) = x - 1.$$

More generally, if $a > 0$, then

$$L_{g,a}(x) = g(a) + g'(a)(x-a) = \ln(a) + \frac{1}{a}(x-a).$$

Example 7.9 If $j : \mathbb{R} \to \mathbb{R}$ is given by $j(x) = e^x$, then

$$L_{j,0}(x) = j(0) + j'(0)(x-0) = e^0 + e^0 x = 1 + x.$$

7.4 Taylor polynomials

Instead of approximating a function f by its linearization at a point a, we approximate it by a function that involves not only $f'(a)$ but also some of the higher derivatives $f^{(k)}(a)$ as well. More precisely, if $f : D \to \mathbb{R}$ is n-times differentiable, where $n \geq 0$, and $a \in D$, the nth *Taylor polynomial* of f at a is

$$\begin{aligned} T_{f,a,n}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \\ + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n, \end{aligned}$$

or, to put it more succinctly,

$$T_{f,a,n}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

By construction,

$$T^{(k)}_{f,a,n}(a) = f^{(k)}(a) \quad \text{for all } k \in \{0, \ldots, n\}.$$

Example 7.10 Find the third Taylor polynomial at 0 of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x)$.

Solution: The derivatives of f of order up to 3 are

$$f'(x) = \cos(x), \quad f''(x) = -\sin(x), \quad f'''(x) = -\cos(x),$$

so

$$\begin{aligned} T_{f,0,3}(x) &= f(0) + f'(0) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ &= 0 + 1 \cdot x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 \\ &= x - \frac{1}{6}x^3. \end{aligned}$$

Example 7.11 Define $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ by $f(x) = \frac{1}{1-x}$. Find the third Taylor polynomial of f at 0.

Solution:

$$T_{f,0,3}(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3.$$

Now,

$$f(x) = \frac{1}{1-x}, \qquad f(0) = 1,$$

$$\begin{aligned} f'(x) &= \frac{1}{(1-x)^2}, & f'(0) &= 1, \\ f''(x) &= \frac{2}{(1-x)^3}, & f''(0) &= 2, \\ f'''(x) &= \frac{6}{(1-x)^4}, & f'''(0) &= 6, \end{aligned}$$

so

$$T_{f,0,3}(x) = 1 + x + x^2 + x^3.$$

More generally, for the function f in this example, if n is a non-negative integer, the nth Taylor polynomial of f at 0 is

$$T_{f,0,n}(x) = \sum_{k=0}^{n} x^k.$$

Compare this with the infinite-series representation of $\frac{1}{1-x}$, valid when $|x| < 1$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots .$$

Example 7.12 Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = \ln(x)$. If n is a non-negative integer, find the nth Taylor polynomial of f at 1.

Solution: Let us try to spot a pattern in the higher derivatives:

$$\begin{aligned} f(x) &= \ln(x), & f(1) &= 0, \\ f'(x) &= \frac{1}{x}, & f'(1) &= 1, \\ f''(x) &= -\frac{1}{x^2}, & f''(1) &= -1, \\ f'''(x) &= \frac{2 \cdot 1}{x^3}, & f'''(1) &= 2!, \\ f^{(4)} &= -\frac{3!}{x^4}, & f^{(4)}(1) &= -3!, \\ f^{(5)} &= \frac{4!}{x^5}, & f^{(5)}(1) &= 4!. \end{aligned}$$

In general, for $k \geq 1$,

$$\begin{aligned} f^{(k)}(x) &= \frac{(-1)^{k-1}(k-1)!}{x^k}, \\ \text{so} \quad f^{(k)}(1) &= (-1)^{k-1}(k-1)!. \end{aligned}$$

(This could be proven using induction.) Therefore,

$$T_{f,1,n}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k$$

$$= f(1) + \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{k!}(x-1)^k$$
$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(x-1)^k.$$

Example 7.13 If $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \int_0^x \cos(z^2)\, dz,$$

find its fifth Taylor polynomial at 0.

Solution: Note to begin with that, by the First Theorem, $f'(x) = \cos(x^2)$. One strategy from here would be to use the infinite series

$$\cos(u) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

to obtain

$$f'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{4k},$$

and hence, because $f(0) = \int_0^0 \cos(z^2)\, dz = 0$,

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!(4k+1)} x^{4k+1} = x - \frac{1}{10}x^5 + \frac{1}{216}x^9 - \cdots$$

via term-by-term integration. However, this approach is beyond the scope of the course, so we instead compute the derivatives $f'(0), \ldots, f^{(5)}(0)$ by hand. As we have remarked, $f(0) = 0$. Now,

$$\begin{aligned}
f'(x) &= \cos(x^2), & f'(0) &= 1,\\
f''(x) &= -2x\sin(x^2), & f''(0) &= 0,\\
f'''(x) &= -2\sin(x^2) - 4x^2\cos(x^2), & f'''(0) &= 0,\\
f^{(4)}(x) &= 8x^3\sin(x^2) - 12x\cos(x^2), & f^{(4)}(0) &= 0,\\
f^{(5)}(x) &= 48x^2\sin(x^2) + (16x^4 - 12)\cos(x^2), & f^{(5)}(0) &= -12.
\end{aligned}$$

Hence,

$$\begin{aligned}
T_{f,0,5}(x) &= f(0) + f'(0) + \frac{f''(0)}{2}x^2\\
&\quad + \frac{f'''(0)}{3!}(0)x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5\\
&= 0 + x + 0 + 0 - \frac{12}{5!}x^5
\end{aligned}$$

$$= x - \frac{1}{10}x^5.$$

For values of x close to 0,

$$\int_0^x \cos(z^2)\,dz \approx x - \frac{1}{10}x^5.$$

7.5 The Newton–Raphson method

Suppose that $f : D \to \mathbb{R}$ is a function. Except in limited situations, there is no simple way to write down solutions to the equation $f(x) = 0$. An example where we can is the equation $ax^2 + bx + c = 0$, but even an equation such as $\ln(x) + 3\sin(x) = 0$ does not have solutions that can be expressed in any useful form.

If all we need are numerical approximations to solutions, the Newton–Raphson method can usually provide them, and when it works, the approximations can be made to be very good indeed.

Here is the method for a differentiable function f:

- Choose an initial value x_0.
- For each $n \geq 0$, let
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$
- Repeat as many times as you need.

Example 7.14 Show that the equation $x^5 + x^3 + x = 1$ has a unique solution. Then use the Newton–Raphson method to find a good approximation to the solution.

Solution: Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^5 + x^3 + x - 1$. Because $f(0) = -1 < 0$ and $f(1) = 2 > 0$, the Intermediate Value Theorem tells us that there is a solution between 0 and 1. Further, $f'(x) = 5x^4 + 3x^2 + 1 > 0$ for all $x \in \mathbb{R}$, so f is injective (in fact, monotone increasing), and so the equation $f(x) = 0$ cannot have more than one solution.

Now we apply Newton–Raphson, taking x_0 to be 1 (a fairly arbitrary choice, although it must be close to the solution, which we know to be between 0 and 1). Then

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{2}{9} = \frac{7}{9} \approx 0.777\,778,\\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{7}{9} - \frac{f(\frac{7}{9})}{f'(\frac{7}{9})} \approx 0.663\,039,\\
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.637\,875.
\end{aligned}$$

Continuing, we find that

$$x_4 \approx 0.636\,884, \quad x_5 \approx 0.636\,883, \quad x_6 \approx 0.636\,883.$$

A good approximation to the unique solution to $\ln(x) = \cos(x)$ is $0.636\,883$.

Note how quickly the numbers x_n converged to the solution in the example above. This is quite typical. To take another example, if we apply the Newton–Raphson method to the function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \ln(x) - \cos(x)$ with the starting value $x_0 = 1$, we find that already x_4 agrees with the true solution to 20 significant figures:

$$\begin{aligned}
x_0 &= 1 \\
x_1 &= 1.2934079930260233874 \\
x_2 &= 1.3029554729266951031 \\
x_3 &= 1.3029640012092009523 \\
x_4 &= 1.3029640012160125525 \\
x_5 &= 1.3029640012160125525 \\
x_6 &= 1.3029640012160125525 \\
x_7 &= 1.3029640012160125525
\end{aligned}$$

Although convergence to a solution is common, it is important to note that there is no guarantee, for a given differentiable function f, that the process will converge at all to a solution to $f(x) = 0$.

Appendix

Appendix: 1 Proof of the chain rule

Recall the statement of the chain rule: If $f : D_1 \to D_2$ is differentiable at $x \in D_1$ and $g : D_2 \to \mathbb{R}$ is differentiable at $f(x)$, then $g \circ f : D_1 \to \mathbb{R}$ is differentiable at x, and

$$(g \circ f)'(x) = g'(f(x))f'(x).$$

We prove this here. For any function ϕ differentiable at a point z in its domain, we define functions $(\Delta\phi)_z$ and $(\varepsilon\phi)_z$ of a sufficiently small variable δ by

$$\begin{aligned}(\Delta\phi)_z(\delta) &= \phi(z+\delta) - \phi(z),\\ (\varepsilon\phi)_z(\delta) &= \begin{cases} \frac{(\Delta\phi)_z(\delta)}{\delta} - \phi'(z) & \text{if } \delta \neq 0. \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

By *sufficiently small* in this context, we mean that δ is small enough that $z + \delta$ is in the domain of ϕ. Observe that

$$\lim_{\delta \to 0} (\varepsilon\phi)_z(\delta) = 0,$$

so the function $(\varepsilon\phi)_z$ is continuous at $\delta = 0$. Note also that

$$(\Delta\phi)_z(\delta) = \delta((\varepsilon\phi)_z(\delta) + \phi'(z))$$

for all δ, including $\delta = 0$.

Now, in the case $\phi = g$ and z equal to a point y in the domain of g,

$$(\Delta g)_y(\delta') = \delta'((\varepsilon g)_y(\delta') + g'(y))$$

for all δ', so letting $y = f(x)$ and $\delta' = (\Delta f)_x(\delta)$, we have

$$(\Delta g)_{f(x)}((\Delta f)_x(\delta)) = (\Delta f)_x(\delta)\Big((\varepsilon g)_{f(x)}((\Delta f)_x(\delta)) + g'(f(x))\Big).$$

Hence, if $\delta \neq 0$,

$$\frac{(\Delta g)_{f(x)}((\Delta f)_x(\delta))}{\delta} = \frac{(\Delta f)_x(\delta)}{\delta}\Big((\varepsilon g)_{f(x)}((\Delta f)_x(\delta)) + g'(f(x))\Big). \tag{A1}$$

Because $(\varepsilon g)_{f(x)}$ is continuous at 0,

$$\lim_{\delta \to 0}(\varepsilon g)_{f(x)}((\Delta f)_x(\delta)) = (\varepsilon g)_{f(x)}\Big(\lim_{\delta \to 0}(\Delta f)_x(\delta)\Big) = (\varepsilon g)_{f(x)}(0) = 0.$$

(Note that the fact that $\lim_{\delta \to 0}(\Delta f)_x(\delta) = 0$ follows from the fact that f is continuous—because differentiable—at 0.) Also, the left-hand side of (A1) is equal to

$$\frac{g(f(x) + (\Delta f)_x(\delta)) - g(f(x))}{\delta} = \frac{g(f(x+\delta)) - g(f(x))}{\delta}$$

$$= \frac{(g \circ f)(x+\delta) - (g \circ f)(x)}{\delta},$$

which converges to $(g \circ f)'(x)$ as $\delta \to 0$. Finally,

$$\frac{(\Delta f)_x(\delta)}{\delta} \to f'(x) \quad \text{as } \delta \to 0,$$

so letting δ approach zero in (A1) yields

$$(g \circ f)'(x) = f'(x)g'(f(x)).$$

Appendix: 2 Some topology

We provide some foundational results that will help us to prove some important results, such as the Intermediate Value Theorem.

Open sets

Let $X \subseteq \mathbb{R}$. If $x \in X$ and $r > 0$, we define

$$B_X(x,r) = \{x' \in X \mid |x' - x| < r\}$$

and call this set the *open ball in X of radius r around x*. We then call a subset U of X an *open* subset of X if for every $x \in U$, there is $r > 0$ such that $B_X(x,r) \subseteq U$. Open subsets of X have the following properties, which we leave as exercises:

(i) The empty subset $\emptyset$ of X is open in X, as is the set X itself.

(ii) If $(U_i)_{i\in I}$ is any family of open subsets of X, then the union $U = \bigcup_{i\in I} U_i$ is open in X as well.

(iii) If $(U_i)_{i\in I}$ is any *finite* family of open subsets of X, then the intersection $U = \bigcap_{i\in I} U_i$ is open in X as well.

Proposition A2.1 *Suppose that $U \subseteq X \subseteq Y$. Then U is open in X if and only if $U = V \cap X$ for some open subset V of Y.*

Proof. Suppose that U is open in X. Then for every $x \in U$, there is $r_x > 0$ such that

$$B_X(x, r_x) \subseteq U.$$

Let

$$V = \bigcup_{x\in U} B_Y(x, r_x),$$

an open subset of Y. We claim that $U = V \cap X$. Indeed, it is immediate that $U \subseteq V \cap X$, and if $x \in V \cap X$, then for some $x' \in U$,

$$x \in B_Y(x', r_{x'}) \cap X = B_X(x', r_{x'}) \subseteq U.$$

Conversely, suppose that $U = V \cap X$ for some open subset V of Y. If $x \in U$, then there is $r > 0$ such that $B_Y(x, r) \subseteq V$, and then

$$B_X(x, r) \subseteq V \cap X = U.$$

■

A subset A of X is called *closed* if its complement $X \setminus A$ is open.

Proposition A2.2 *Let A be a closed subset of X, and suppose that $x \in X$ has the property that for all $r > 0$, the ball $B_X(x, r)$ contains a point of A. Then $x \in A$.*

Proof. If x were not in A, i.e., $x \in X \setminus A$, then because $X \setminus A$ is open in X, there would be $r > 0$ such that $B_X(x, r) \subseteq X \setminus A$, a contradiction. ■

Continuous functions

Now suppose that $f : X \to Y$ is a function, where X and Y are subsets of $\mathbb{R}$. We will say here that f is *t-continuous* if for every open subset V of Y, the pre-image $f^{-1}(V)$ is open in X. (In the subject of topology, this would be taken as the definition of continuity, and the t would be dropped from the terminology.)

Proposition A2.3 *A function $f : X \to Y$ is continuous if and only if it is t-continuous.*

Proof. Suppose first that f is continuous, and let V be an open subset of Y. We show that $f^{-1}(V)$ is open in X. Let $x \in f^{-1}(V)$, so that $f(x) \in V$. Because V is open, there is $\epsilon > 0$ such that $B_Y(f(x), \epsilon) \subseteq V$, so because f is continuous, there is $\delta > 0$ such that for all $x' \in X$,

$$|x' - x| < \delta \implies |f(x') - f(x)| < \epsilon.$$

This says that if $x' \in B_X(x, \delta)$, then $f(x') \in B_Y(f(x), \epsilon) \subseteq V$, so $x' \in f^{-1}(V)$. Therefore, $B_X(x, \delta) \subseteq f^{-1}(V)$, so $f^{-1}(V)$ is open. Thus, f is t-continuous.

Conversely, suppose that f is t-continuous. Let $x \in X$, and let $\epsilon > 0$. Because $B_Y(f(x), \epsilon)$ is open in Y, the assumption of t-continuity implies that $f^{-1}(B_Y(f(x), \epsilon))$ is open in X. Therefore, since $x \in f^{-1}(B_Y(f(x), \epsilon))$, there is $\delta > 0$ such that

$$B_X(x, \delta) \subseteq f^{-1}(B_Y(f(x), \epsilon)).$$

Hence, if $x' \in X$ satisfies $|x' - x| < \delta$, i.e., $x' \in B_X(x, \delta)$, then $f(x') \in B_Y(f(x), \epsilon)$, i.e., $|f(x') - f(x)| < \epsilon$. Thus, f is continuous. ■

Lemma A2.4 *Let $f : X \to Y$ be continuous. Then the following hold:*

(i) The function

$$\begin{aligned} \tilde{f} : X &\to f(X) \\ x &\mapsto f(x) \end{aligned}$$

is continuous.

(ii) The restriction map $f|_{X'} : X' \to Y$ *is continuous for any subset* X' *of* X*.*

Proof. (i) In our original definition of continuity, the assertion is immediate, but we may also prove this in the language of t-continuity. Let U be an open subset of $f(X)$. By Proposition A2.1 $U = V \cap f(X)$ for some open subset V of Y, so

$$\begin{aligned} \tilde{f}^{-1}(U) &= \{x \in X \mid \tilde{f}(x) \in V \cap f(X)\} \\ &= \{x \in X \mid f(x) \in V \cap f(X)\} \\ &= \{x \in X \mid f(x) \in V\} \\ &= f^{-1}(V), \end{aligned}$$

which is open in X.

(ii) If V is an open subset of Y, then

$$\begin{aligned} (f|_{X'})^{-1}(V) &= \{x \in X' \mid f(x) \in V\} \\ &= \{x \in X \mid f(x) \in V\} \cap X' \\ &= f^{-1}(V) \cap X', \end{aligned}$$

and this is open in X' by Proposition A2.1. ∎

Connectedness

We say that X is *connected* if whenever U and V are disjoint open subsets of X such that $X = U \cup V$, either $U = \emptyset$ of $V = \emptyset$.

Proposition A2.5 *If* $f : X \to Y$ *is continuous and* X *is connected, then* $f(X)$ *is connected as well.*

Proof. Suppose that U and V are disjoint open subsets of $f(X)$ such that $f(X) = U \cup V$. By Lemma A2.4(i), the function $\tilde{f} : X \to f(X)$ obtained by restricting the codomain is continuous, so the sets $\tilde{f}^{-1}(U) = f^{-1}(U)$ and $\tilde{f}^{-1}(V) = f^{-1}(V)$ are open in X. Further, the assumptions $U \cap V = \emptyset$ and $U \cup V = f(X)$ imply, respectively, that

$$\begin{aligned} &f^{-1}(U) \cap f^{-1}(V) = \emptyset \\ \text{and} \quad &f^{-1}(U) \cup f^{-1}(V) = X, \end{aligned}$$

so because X is connected, either $f^{-1}(U) = \emptyset$ or $f^{-1}(V) = \emptyset$. If $U \neq \emptyset$, say $y \in U$, then because $U \subseteq f(X)$, there is $x \in X$ such that $f(x) = y$, so $x \in f^{-1}(U)$ and $f^{-1}(U) \neq \emptyset$. Similarly, $V \neq \emptyset \implies f^{-1}(V) \neq \emptyset$. Therefore, either U is empty or V is, so $f(X)$ is connected. ∎

Proposition A2.6 *Every interval in* $\mathbb{R}$ *is connected. (By an* interval*, we mean a subset* I *of* $\mathbb{R}$ *such that if* $a, b \in I$*, then all real numbers between* a *and* b *are in* I *as well.)*

Proof. We first prove the assertion in the case where the interval is $[0,1]$ (but see the remark following the proof). Suppose, for a contradiction, that $[0,1]$ is not connected, meaning that there are disjoint non-empty open subsets U and V of $[0,1]$ such that $[0,1] = U \cup V$. Choose $a \in U$ and $b \in V$, and assume, without loss of generality, that $a < b$. (We cannot have $a = b$, because $U \cap V = \emptyset$.) Let

$$T = \{x \in [a,1] \mid [a,x] \subseteq A\},$$

which is non-empty because it contains a, and let s be its *supremum*, meaning the least real number $\geq$ all numbers in T. If s were greater than b, then there would be $x \in (b,s) \cap T$, and then

$$[a,b] \subseteq [a,x] \subseteq A,$$

contradicting the fact that $b \notin A$, so we must have $s \leq b$. Now, by the nature of a supremum, every open ball $B_{[0,1]}(s,r)$, where $r > 0$, contains some $x \in T$, and then $[a,x] \subseteq A$, so $x \in A$. Therefore, because $A = [0,1] \smallsetminus B$, which is closed, Proposition A2.2 implies that $s \in A$, so $s \neq b$, and so $s < b \leq 1$. Hence, because A is open in $[0,1]$, there is $r \in (0, 1-s)$ such that $B_{[0,1]}(s,r) \subseteq A$. But then there is $x \in B_{[0,1]}(s,r)$ with $x > s$, so x is an element of A greater than its supremum s, a contradiction.

Now let I be any interval, and suppose that U and V are non-empty open subsets of I such that $I = U \cup V$. We show that $U \cap V \neq \emptyset$. Choose $a \in U$ and $b \in V$, and choose any continuous function $\gamma : [0,1] \to I$ such that $\gamma(0) = a$ and $\gamma(1) = b$, such as $\gamma(t) = a + t(b-a)$. The fact that $[0,1]$ is connected combines with Proposition A2.5 to show that $\gamma([0,1])$ is connected as well. Now, $\gamma([0,1]) = U' \cup V'$ where

$$U' = U \cap \gamma([0,1]) \quad \text{and} \quad V' = V \cap \gamma([0,1]),$$

and these two sets U' and V' are both non-empty because they contain a and b respectively. They are also both open in $\gamma([0,1])$ because U and V are open in I, so by the connectedness of $\gamma([0,1])$, $U' \cap V'$ is non-empty, i.e., $U \cap V \cap \gamma([0,1])$ is non-empty, so $U \cap V$ is non-empty. Thus, I is connected. ■

Remark. With only minor modification, the proof of the connectedness of $[0,1]$ above works to prove, directly, the connectedness of any interval, without having to use the argument involving the continuous function γ. However, we have presented the proof of Proposition A2.6 this way because the argument in the second stage of the proof is the same as the one that is invoked to show the more general topological fact that any path-connected space is connected.

The Intermediate Value Theorem

Theorem A2.7 *Let $f : X \to Y$ be a continuous function, where X and Y are subsets of $\mathbb{R}$. Suppose that a and b are elements of X such that $a < b$ and $[a, b] \subseteq X$. Suppose also that y is an element of Y such that either $f(a) < y < f(b)$ or $f(b) < y < f(a)$. Then there is $x \in (a, b)$ such that $f(x) = y$.*

Proof. By Lemma A2.4(ii), the restriction map $f|_{[a,b]}$ is continuous. Therefore, because $[a, b]$ is connected by Lemma A2.6, the set $f([a, b])$ is connected as well by Proposition A2.5.

Assume, for a contradiction, that there is no $x \in (a, b)$ such that $f(x) = y$. Because $f(a) \neq y$ and $f(b) \neq y$ either, there is no $x \in [a, b]$ such that $f(x) = y$, so $y \notin f([a, b])$. Consider, then, the sets

$$\begin{aligned} & U = \{y' \in f([a,b]) \mid y' < y\} = (-\infty, y) \cap f([a,b]) \\ \text{and} \quad & V = \{y' \in f([a,b]) \mid y' > y\} = (y, \infty) \cap f([a,b]), \end{aligned}$$

both open in $f([a, b])$. These sets are disjoint, and, by assumption, satisfy $U \cup V = f([a, b])$, so the connectedness of $f([a, b])$ implies that either U is empty or V is. But this is not the case, since if $f(a) < y < f(b)$, then $f(a) \in U$ and $f(b) \in V$, and if $f(b) < y < f(a)$, then $f(b) \in U$ and $f(a) \in V$. This contradiction finishes the proof. ■

Appendix: 3 The Mean Value Theorem

We prove the Mean Value Theorem, starting with two preparatory results.

Theorem A3.1 (Fermat's Theorem) *Let $a < b$, let $f : [a, b] \to \mathbb{R}$ be a continuous function that is differentiable on (a, b), and let its maximum and minimum values on $[a, b]$ be M and m respectively. If there exists $x_0 \in (a, b)$ such that $f(x_0) = M$ or $f(x_0) = m$, then $f'(x_0) = 0$.*

Proof. The case where $f(x_0) = m$ can be proven by applying the maximum case to the function $-f$, so assume that $f(x_0) = M$. Let

$$g(x) = \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0),$$

so that

$$f(x) - f(x_0) = (x - x_0)(f'(x_0) + g(x)).$$

By definition of $f'(x_0)$, $g(x) \to 0$ as $x \to x_0$.

Suppose, for a contradiction, that $f'(x_0) \neq 0$. If $f'(x_0) > 0$, there is $\delta > 0$ such that $0 < |x - x_0| < \delta \implies f'(x_0) + g(x) > 0$. Choose any $x \in (x_0, x_0 + \delta)$. Then

$$f'(x_0) + g(x) > 0$$

$$\text{and} \quad x - x_0 > 0,$$

so

$$f(x) - f(x_0) = (x - x_0)(f'(x_0) + g(x)) > 0,$$

and so

$$f(x) > f(x_0) = M,$$

a contradiction. Similarly, if $f'(x_0) < 0$, there is $\delta > 0$ such that $0 < |x - x_0| < \delta \implies f'(x_0) + g(x) < 0$. This time, choosing any $x \in (x_0 - \delta, x_0)$, we have

$$f'(x_0) + g(x) < 0$$
$$\text{and} \quad x - x_0 < 0,$$

so

$$f(x) - f(x_0) = (x - x_0)(f'(x_0) + g(x)) > 0,$$

a contradiction for the same reason. ∎

Theorem A3.2 (Rolle's Theorem) *Let $a < b$, let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) = f(b)$, and assume that f is differentiable on (a, b). Then there is $x \in (a, b)$ such that $f'(x) = 0$.*

Proof. Let the maximum and minimum of f be M and m respectively. If there is $x \in (a, b)$ such $f(x) = M$, then we are done by Theorem A3.1. Otherwise, $f(x) < M$ for all $x \in (a, b)$, so the maximum must be attained at a and b, i.e., $f(a) = f(b) = M$. But because $f(x) < M$ when $a < x < b$, the minimum m is less than M, so the minimum is attained at neither a nor b and therefore must be attained at some $x \in (a, b)$, and we may again apply Theorem A3.1. ∎

Theorem A3.3 (Mean Value Theorem) *Let $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function that is differentiable on (a, b). Then there is $x \in (a, b)$ such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Define $g : [a, b] \to \mathbb{R}$ by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

This function g satisfies the hypotheses of Theorem A3.2, so there is $x \in (a, b)$ such that $g'(x) = 0$, i.e.,

$$f'(x) - \frac{f(b) - f(a)}{b - a} = 0.$$

∎

Corollary A3.4

(i) If $F : (a, b) \to \mathbb{R}$ is differentiable and $F(x) = 0$ for all $x \in (a, b)$, then F is constant.

(ii) Suppose that $D \subseteq \mathbb{R}$ is an interval, a half-line, or $\mathbb{R}$. (A half-line is a subset of $\mathbb{R}$ of the form $(-\infty, a)$, $(-\infty, a]$, (a, ∞), or $(a, \infty]$.) Any two antiderivatives of a function $f : D \to \mathbb{R}$, if such antiderivatives exist, differ by a constant.

Proof. (i) Take any $c, d \in (a, b)$ with $c \neq d$. By Theorem A3.3, there is x between c and d such that

$$f'(x) = \frac{f(d) - f(c)}{d - c}.$$

But $f'(x) = 0$ by assumption, so $f(d) = f(c)$.

(ii) Let F_1, F_2 be antiderivatives of f, and let $G = F_1 - F_2$. Then

$$G'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0,$$

so G is constant on any open interval inside D by part (i). But then G is constant on all of D. ■

Appendix: 4 Sum of k and sum of k^2

We prove the equalities

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1) \quad \text{and} \quad \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1).$$

There are many proofs of both these facts. We prove them here using the notion of a *telescoping sum*. First, for the sum of k,

$$\begin{aligned} n^2 + 2n &= (n+1)^2 - 1 \\ &= \sum_{k=1}^{n}((k+1)^2 - k^2) \quad \text{(telescoping sum)} \\ &= \sum_{k=1}^{n}(k^2 + 2k + 1 - k^2) \\ &= 2\sum_{k=1}^{n} k + n, \end{aligned}$$

so rearranging gives the desired equality. The approach for the sum of k^2 is similar:

$$n^3 + 3n^2 + 3n = (n+1)^3 - 1$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} ((k+1)^3 - k^3) \quad \text{(telescoping sum)} \\
&= \sum_{k=1}^{n} (k^3 + 3k^2 + 3k + 1 - k^3) \\
&= 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n,
\end{aligned}
$$

so

$$
\begin{aligned}
\sum_{k=1}^{n} k^2 &= \frac{1}{3}\left(n^3 + 3n^2 + 2n - 3\sum_{k=1}^{n} k\right) \\
&= \frac{1}{6}\Big(2n(n+1)(n+2) - 3n(n+1)\Big) \quad \text{by part (i)} \\
&= \frac{1}{6} n(n+1)(2n+1).
\end{aligned}
$$

Appendix: 5 Proofs of the Fundamental Theorems of Calculus

Theorem A5.1 *Let $f : [a,b] \to \mathbb{R}$ be a continuous function, where $a < b$, and define $F : [a,b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is continuous on $[a,b]$ and differentiable on (a,b), and

$$F'(x) = f(x) \quad \textit{for all } x \in (a,b).$$

Proof. Fix $x \in (a,b)$. We are to show that

$$\frac{F(x+h) - F(x)}{h} \to f'(x)$$

as $h \to 0$. To that end, for each $h \neq 0$ (small enough in absolute value that $x + h \in [a,b]$), let $m(h)$ and $M(h)$ be the minimum and maximum of f on the closed interval I_h with end-points x and $x+h$, and choose $c(h), C(h) \in I_h$ such that

$$f(c(h)) = m(h) \quad \text{and} \quad f(C(h)) = M(h).$$

Then $f(c(h)) \leq f(t) \leq f(C(h))$ for all $t \in I_h$, so

$$hf(c(h)) \leq \int_x^{x+h} f(t)\,dt \leq hf(C(h)) \quad \text{(still true if } h < 0\text{)},$$

i.e.,

$$hf(c(h)) \leq F(x+h) - F(x) \leq hf(C(h)),$$

i.e.,

$$f(c(h)) \leq \frac{F(x+h)-F(x)}{h} \leq f(C(h)).$$

Because $c(h)$ and $C(h)$ are both between x and $x+h$, they tend to x as $h \to 0$, so $f(c(h))$ and $f(C(h))$ tend to $f(x)$ as $h \to 0$ by the continuity of f; see Proposition 3.13. Therefore, the Squeeze Theorem gives us the desired result.

Now let us turn to continuity on $[a,b]$. Because F is differentiable on (a,b), it remains to prove continuity at a and b, i.e., that

$$\begin{aligned} & F(x) \to F(a) = 0 \quad \text{as } x \to a \\ \text{and} \quad & F(x) \to F(b) \quad \text{as } x \to b. \end{aligned}$$

Because f is continuous on the closed interval $[a,b]$, it has a maximum M and a minimum m. Therefore, for all $x \in [a,b]$,

$$m(x-a) \leq \int_a^x f(t)\,dt \leq M(x-a),$$

i.e.,

$$m(x-a) \leq F(x) \leq M(x-a),$$

so $F(x) \to 0$ as $x \to a$ by the Squeeze Theorem. Continuity at b is similar: For all $x \in [a,b]$,

$$m(b-x) \leq \int_x^b f(t)\,dt \leq M(b-x),$$

so

$$-M(b-x) \leq -\int_x^b f(t)\,dt \leq -m(b-x),$$

i.e.,

$$F(b) - M(b-x) \leq F(b) - \int_x^b f(t)\,dt \leq F(b) - m(b-x),$$

i.e.,

$$F(b) - M(b-x) \leq F(x) \leq F(b) - m(b-x),$$

so the Squeeze Theorem tells us that $F(x) \to F(b)$ as $x \to a$. ■

Theorem A5.2 *Suppose that $F : [a,b] \to \mathbb{R}$, where $a < b$, is continuous on $[a,b]$ and differentiable on (a,b), and assume that its derivative is Riemann integrable. (The interested reader may investigate the notion of a* Riemann integrable *function.) Then for all $x \in [a,b]$,*

$$\int_a^x F'(t)\,dt = F(x) - F(a).$$

(See also the remark following the proof.)

Proof. Fix $x \in [a, b]$, and for each positive integer n, let $\Delta_n = (x - a)/n$. Further, if $k \in \{1, \ldots, n\}$, choose

$$c_{n,k} \in [a + (k-1)\Delta_n, a + k\Delta_n]$$

such that

$$F'(c_{n,k}) = \frac{F(a + k\Delta_n) - F(a + (k-1)\Delta_n)}{\Delta_n},$$

possible by Theorem A3.3. Then

$$\begin{aligned} F(x) - F(a) &= \sum_{k=1}^{n} \Big(F(a + k\Delta_n) - F(a + (k-1)\Delta_n) \Big) \quad \text{(telescoping sum)} \\ &= \sum_{k=1}^{n} \Delta_n F'(c_{n,k}) \\ &= \frac{x-a}{n} \sum_{k=1}^{n} F'(c_{n,k}). \qquad \text{(A2)} \end{aligned}$$

This last expression is almost the nth Riemann sum of F' on the interval $[a, x]$ as we defined it in Section 5.2, the difference being that the point $c_{n,k}$ in the interval $[a + (k-1)\Delta_n, a + k\Delta_n]$ will typically be somewhere to the left of the right end-point $a + k\Delta_n$ rather than being that end-point itself. The assumption that F' be Riemann integrable ensures that the limit as $n \to \infty$ of the expression in (A2) would remain unchanged if $c_{n,k}$ were replaced by the right end-point, $a + k\Delta_n$, for all n and k. This being the case, the sequence of equalities leading up to (A2) gives

$$F(x) - F(a) = \lim_{n\to\infty} \frac{x-a}{n} \sum_{k=1}^{n} F'(a + k\Delta_n) = \int_a^x F'(t)\,dt.$$

■

Remark. If one does not wish to worry about what the term *Riemann integrable* means, one may state and prove a version of Theorem A5.2 in which this term is replaced by the word *continuous*, as follows. Suppose that F' is continuous, and define $G : [a, b] \to \mathbb{R}$ by

$$G(x) = \int_a^x F'(t)\,dt.$$

Because F' is continuous, Theorem A5.1 says that G is continuous on $[a, b]$ and differentiable on (a, b) and satisfies $G'(x) = F'(x)$ for all $x \in (a, b)$. Therefore, by Corollary A3.4, there is $C \in \mathbb{R}$ such that $G(x) - F(x) = C$ for all $x \in (a, b)$, and hence $G(x) - F(x) = C$ for all $x \in [a, b]$ because $G - F$ is continuous on $[a, b]$. Taking $x = a$, and remembering that $G(a) = 0$, we see that $C = -F(a)$, so

$$G(x) = F(x) - F(a) \quad \text{for all } x \in [a, b].$$

The mean value of a continuous function

Proposition A5.3 *Let $a < b$, and suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function that is differentiable on (a, b). Then there is $x \in (a, b)$ such that*

$$f(x) = \frac{1}{a-b} \int_a^b f(t)\, dt.$$

In other words, f takes its mean value somewhere in (a, b).

Proof. Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\, dt.$$

By Theorem A5.1 (the First Fundamental Theorem), F meets the hypotheses of Theorem A3.3 (the Mean Value Theorem), so there is $x \in (a, b)$ such that

$$F'(x) = \frac{F(b) - F(a)}{b-a}.$$

But $F(b) = \int_a^b f(t)\, dt$, $F(a) = 0$, and, by the First Fundamental Theorem, $F'(x) = f(x)$. ■

Appendix: 6 A fact about rectangles inscribed in ellipses

In Example 7.5, we used the fact that a rectangle inscribed in a non-circular ellipse must be aligned with the principal axes of the ellipse. Here, we justify this using the well-known fact that a parallelogram inscribed in a circle must be a rectangle.

Suppose, then, that we have a rectangle R inscribed in a non-circular ellipse E. Translating and rotating as necessary, we may assume that the ellipse has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

for some positive real numbers a and b with $a \neq b$. We then consider the linear transformation

$$\begin{aligned} T : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (\tfrac{x}{a}, \tfrac{y}{b}), \end{aligned}$$

which maps the ellipse E to the unit circle C centred at the origin. Now, any line in the plane takes the form

$$\{\lambda v + c \mid \lambda \in \mathbb{R}\}$$

where $v, c \in \mathbb{R}^2$ with $v \neq (0,0)$, and T maps this to

$$\{\lambda T(v) + T(c) \mid \lambda \in \mathbb{R}\}. \tag{A3}$$

In this description of a line, two of the parallel sides of R share a direction vector v, and the other two share a direction vector w. If $v = (i, j)$ and $w = (k, l)$, then the fact that the angles in a rectangle are right angles implies that

$$ik + jl = 0. \tag{A4}$$

From the form of the transformed line in (A3), we see that T transforms parallel lines to parallel lines, so T maps R to a parallelogram P. But P is inscribed in a circle (remember that T maps E to the circle C), so P is a rectangle, and so $T(v)$ and $T(w)$ are at right angles to each other. Since

$$T(v) = (\tfrac{i}{a}, \tfrac{j}{b}) \quad \text{and} \quad T(w) = (\tfrac{k}{a}, \tfrac{l}{b}),$$

this means that

$$0 = \frac{i}{a}\frac{k}{a} + \frac{j}{b}\frac{l}{b} = \frac{ik}{a^2} - \frac{ik}{b^2}$$

by (A4), i.e., $ik(\frac{1}{a^2} - \frac{1}{b^2}) = 0$. By assumption, $a^2 \neq b^2$, so $ik = 0$, so we have only two possibilities:

- $i = 0$: In this case, $j \neq 0$, so the equality $ik + jl = 0$ implies that $l = 0$, and then $v = (0, j)$ and $w = (k, 0)$, showing that v and w are aligned with the coordinate axes, which are also the principal axes of E.
- $k = 0$: This case is similar, leading to the conclusion that $v = (i, 0)$ and $w = (0, l)$.