A Few Calculus Questions

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I hope to add to this small collection of problems in due course. The solutions can be found accompanying this document at https://sites.ualberta.ca/~pbucking/.

1. In each of parts (a), (b), and (c) below, a function f and a real number a are given such that the line $x = a$ is a vertical asymptote of f. In each part, do all of the following, briefly explaining your answers:

   - Decide whether $f(x) \to \infty$ as $x \to a^+$, $f(x) \to -\infty$ as $x \to a^+$, or neither.
   - Decide whether $f(x) \to \infty$ as $x \to a^-$, $f(x) \to -\infty$ as $x \to a^-$, or neither.
   - Decide whether $f(x) \to \infty$ as $x \to a$, $f(x) \to -\infty$ as $x \to a$, or neither.

   (a) $f(x) = -\dfrac{1}{x^2 + 14x + 49}$, $a = -7$

   (b) $f(x) = \dfrac{x-2}{(x^2 - 4x + 4)(32x - 65)}$, $a = 2$

   (c) $f(x) = \dfrac{x-5}{|x-5|^3}$, $a = 5$

2. For each of the following functions g, find all its vertical asymptotes, explaining your answers. Further, for each asymptote $x = a$, decide with reasoning whether $g(x) \to \infty$ as $x \to a$, $g(x) \to -\infty$ as $x \to a$, or neither.

   (a) $g(x) = \dfrac{x^2 + 8x + 12}{x^2 - x - 6}$

   (b) $g(x) = \dfrac{x+9}{x^2 + 2x - 35}$

   (c) $g(x) = 5 + \dfrac{x+10}{x^2 + 6x + 9}$

   (d) $g(x) = 2x - \dfrac{7}{x-8}$

3. Evaluate each of the limits below, showing your working.

   (a) $\displaystyle\lim_{x\to 1} \frac{x - 4 + \frac{3}{x}}{1 - \frac{1}{x}}$

   (b) $\displaystyle\lim_{x\to 11} \frac{\sqrt{x+5} - 4}{x - 11}$

   (c) $\displaystyle\lim_{x\to 6} \frac{2 - \sqrt{x-2}}{x^2 - x - 30}$

(d) $\lim_{x \to -2} \dfrac{\frac{3}{x} + \sqrt{\frac{1}{x} + \frac{11}{x^2}}}{4 + \frac{8}{x}}$

4. Consider the function $f : \mathbb{R} \setminus \{-2, 0\} \to \mathbb{R}$ defined by

$$f(x) = \frac{\frac{3}{x} + \sqrt{\frac{1}{x} + \frac{11}{x^2}}}{4 + \frac{8}{x}}.$$

(a) Find $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$.

(b) Does $\lim_{x \to 0} f(x)$ exist? Briefly explain your answer.

5. (a) If $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ are functions such that $\lim_{x \to 7} f_1(x) = -2$ and $\lim_{x \to 7} f_2(x) = 9$, find

$$\lim_{x \to 7} \frac{f_1(x)^3 f_2(x) + (3x+1) f_2(x)}{f_1(x) + f_2(x)}.$$

(b) Let $L \in \mathbb{R}$. If $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ are functions such that $\lim_{x \to 5} g_1(x) = L$ and $\lim_{x \to 5} g_2(x) = 2L$, find

$$\lim_{x \to 5} \frac{4g_1(x)^2 - 6g_1(x) + 5x}{g_2(x)^2 - 3g_2(x) + x^2},$$

showing all working. *Hint: Use limit laws.*

6. Let $a \in \mathbb{R}$, and suppose that f and g are functions such that

$$\lim_{x \to a} (f(x) + g(x)) = L \quad \text{and} \quad \lim_{x \to a} (f(x) - g(x)) = M.$$

By considering $L + M$ and $L - M$, express $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ in terms of L and M.

7. Find $f'(x)$ for each function f below, simplifying your answers as much as is reasonably possible. Additionally, in parts (a), (b), and (c), find the values of x where the tangent line to the graph of f at x is horizontal.

(a) $f(x) = \dfrac{x}{\ln(x)} \quad (x > 0,\ x \neq 1)$

(b) $f(x) = \dfrac{\sin(x)}{1 + \sin(x)} \quad (x \neq 2n\pi - \frac{\pi}{2} \text{ where } n \in \mathbb{Z})$

(c) $f(x) = \dfrac{\ln(x)}{x} \quad (x > 0)$

(d) $f(x) = \dfrac{1 - \cos(x)}{x^2} \quad (x \neq 0)$

(e) $f(x) = \dfrac{\ln(x)}{\ln(2) + \ln(x)} \quad (x > 0,\ x \neq 1/2)$

8. Find $f'(x)$ for each function f below. You do not need to investigate where f is defined, and you do not need to simplify your answers except where a very obvious simplification is possible.

   (a) $f(x) = \tan(x^2 + \ln(x))$

   (b) $f(x) = \cot\Big(e^{3x} \sin(\ln(x))\Big)$

   (c) $f(x) = \sec\left(\ln\left(x^5 + \dfrac{x^2 - 1}{x^2 + 1}\right)\right)$

   (d) $f(x) = \sqrt{\csc\Big(e^{\tan(5x^6 - 4x)}\Big)}$

9. In this question, give all tangent lines in the form $y = mx + c$.

   (a) If f is the function in part (a) of Question 7, find the equation of the tangent line to the graph of f at $(e^3, f(e^3))$. Check your answer to Question 7 carefully first.

   (b) If f is the function in part (b) of Question 7, find the equation of the tangent line to the graph of f at $(\pi/6, f(\pi/6))$. Check your answer to Question 7 carefully first.

   (c) (*i*) There is a differentiable function $g : (0, \infty) \to \mathbb{R}$ such that

   - $xg(x)^5 - x^5 g(x) = 29x^{10} + 1$,
   - $g(1) = 2$.

   Use this information to find $g'(1)$.

   (*ii*) Let C be the curve defined by the equation $xy^5 - x^5 y = 29x^{10} + 1$. Find the tangent line to C at the point $(1, 2)$. *Hint: Near the point in question, C matches the graph of the function g in part (i).*

10. **Note:** In this question, do not use logarithms or logarithmic differentiation. The purpose of the question is to demonstrate what may be achieved without logarithms.

    If $f : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is a differentiable function, let $T(f) : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$T(f) = \frac{f'}{f}.$$

(a) Use the product rule to show that $T(fg) = T(f) + T(g)$.

(b) It follows from part (a) that if $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ are differentiable, then

$$T(f_1 f_2 \cdots f_n) = T(f_1) + T(f_2) + \cdots + T(f_n).$$

Use this fact to find the derivative of the function $j : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ defined by

$$j(x) = (x^2+1)(x^2+2)^2(x^2+3)^3(x^2+4)^4(x^2+5)^5.$$

You do not need to simplify your answer.

11. For some open interval $I \subseteq \mathbb{R}$ containing $\pi/3$, there is a differentiable function $f : I \to \mathbb{R}$ such that

- $x \sin(f(x)) + f(x)\cos(x) = \dfrac{\pi}{4}$,
- $f(\frac{\pi}{3}) = \frac{\pi}{6}$.

Find the tangent line to the graph of f at $(\pi/3, \pi/6)$, giving your answer in the form $y = mx + c$. Besides giving your answer in this form, you do not need to simplify your expressions.

12. Consider the function $f : (-\infty, -4] \to [7, \infty)$ defined by

$$f(x) = 7 + \sqrt{x^2 + 2x - 8}.$$

(a) Express the polynomial $x^2 + 2x - 8$ in the form $(x+a)^2 + b$ with $a, b \in \mathbb{R}$. (This process is called *completing the square*.)

(b) Show that for any given y in the codomain of f, there is a unique x in the domain such that $f(x) = y$, thus establishing that f is invertible. Give $f^{-1}(y)$ in terms of y.

(c) Find $(f^{-1})'(11)$ in two ways:

(i) by differentiating f^{-1} directly, using the expression you found for $f^{-1}(y)$ in part (b), and

(ii) by calculating f' at an appropriate point and using the formula relating $(f^{-1})'$ and f'.

(d) Finally, consider $f^{-1} : [7, \infty) \to (-\infty, -4]$ now as a function of x instead of y. Write down the slope of the graph of f^{-1} at $x = 11$ (see above!), and hence find the equation of the tangent line to the graph of f^{-1} at $x = 11$, giving your answer in the form $y = mx + c$.

13. Evaluate the following integrals, simplifying your answers as much as possible.

(a) $\displaystyle\int_{1/2}^{1/\sqrt{2}} \left(\frac{1}{\sqrt{1-x^2}} - \frac{2}{x} \right) dx$

(b) $\displaystyle\int_{2}^{6} \frac{3x^2+2x+1}{x^3}\, dx$

(c) $\displaystyle\int_{-1}^{1} (4\sin(x) + x^6)\, dx$

(d) $\displaystyle\int_{-7\pi/6}^{7\pi/6} (5\cos(x) - x^7)\, dx$

(e) $\displaystyle\int_{\pi/6}^{\pi/2} \left(\frac{\cos(x)}{1+\cos(x)} + \frac{\cos(x)}{1-\cos(x)} \right) dx$

Hint: Form a common denominator, use a trigonometric identity, and then consider derivatives of trigonometric functions.

14. (a) Find $F'(x)$ for each of the following functions F:

(*i*) $F : (0,\infty) \to \mathbb{R},\ F(x) = \displaystyle\int_{0}^{\ln(x)} e^t \sin^2(t^2+1)\, dt$

(*ii*) $F : (0,1] \to \mathbb{R},\ F(x) = \displaystyle\int_{x^4}^{x^2} \ln(t) \arcsin(t) \cos(t)\, dt$

(b) Find a differentiable function $G : \mathbb{R} \to \mathbb{R}$ such that

$$G'(x) = \cos(\cos(\cos(x))) \quad \text{for all } x \in \mathbb{R}$$
$$\text{and} \quad G(1) = 0.$$

Express your answer in the form of an integral.

15. Consider the function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ defined by

$$f(x) = \arctan\left(\frac{1+x}{1-x} \right).$$

Find $f'(x)$, and deduce from your calculation that

$$\arctan\left(\frac{1+x}{1-x} \right) = \arctan(x) + \frac{\pi}{4} \quad \text{for all } x < 1.$$

Now use the equality above to find $\arctan(2+\sqrt{3})$. What is the relationship between $\arctan(\frac{1+x}{1-x})$ and $\arctan(x)$ for $x > 1$?

16. (a) Evaluate each of the following definite integrals via a change of variables, simplifying your answers as much as possible:

(*i*) $\displaystyle\int_6^{12} \frac{7}{2x-9}\,dx$

(*ii*) $\displaystyle\int_1^{4} \frac{x}{\sqrt{65-x^2}}\,dx$

(*iii*) $\displaystyle\int_{-1}^{0} \frac{2x-1}{x^2-2x+1}\,dx$ (try $y = x - 1$)

(b) Evaluate each of the following indefinite integrals via a change of variables:

(*i*) $\displaystyle\int \frac{8}{3x^2+5}\,dx$

(*ii*) $\displaystyle\int \frac{1}{e^x+1}\,dx$

Hint: There is a change of variables that avoids the use of partial fractions. Play around with some options. Multiplying the top and bottom of the integrand by e^{-x} *may help you find a good change of variables.*

(*iii*) $\displaystyle\int \frac{1}{(x+1)^2}\csc^2\left(\frac{x-1}{x+1}\right)dx$

17. Find the indefinite integral

$$\int x \arctan(x)\,dx$$

via integration by parts. *Hint: Make a clever choice of antiderivative of* x*.*

18. (a) Let $x \in [0, \sqrt{2}]$. Evaluate

$$\int_0^x \frac{1}{\sqrt{2-t^2}}\,dt$$

in two different ways:

(*i*) by making the change of variables $u = t/\sqrt{2}$, and

(*ii*) by making the change of variables $u = 1 - t^2$.

(b) Using your answers to part (a), deduce that

$$\arcsin(1-x^2) = \frac{\pi}{2} - 2\arcsin\left(\frac{x}{\sqrt{2}}\right) \quad \text{for all } x \in [0, \sqrt{2}].$$

19. For each of the following functions, find its points and values of global extremum. Explain your answers.

(a) $f : [-1, 1] \to \mathbb{R},\ f(x) = x^2 - x + 1$

(b) $f : [-4, 4] \to \mathbb{R},\ f(x) = x^3 + 3x^2 - 9x - 1$

(c) $f : [\frac{1}{2}, 3] \to \mathbb{R}$, $f(x) = \ln(x) - x + 2$

(d) $f : [-\pi, 2\pi] \to \mathbb{R}$, $f(x) = \frac{1}{2}x + \cos(x)$

20. Sketch the graph of the function f given by

$$f(x) = \frac{x^3 - 9x^2 + 27x - 23}{x^2 - 6x + 9},$$

showing first the work involved in determining the domain, axis intercepts, extrema, inflection points, regions of upward and downward concavity, and asymptotes.

21. Show that there are two and only two real numbers x such that

$$x^2 + \frac{9}{5}x - \frac{1}{2} = \arctan(x),$$

justifying any inequalities you claim. If you wish, you may take advantage of the factorization

$$10x^3 + 9x^2 + 10x + 4 = (2x + 1)(5x^2 + 2x + 4).$$

22. In this question, use only the Mean Value Theorem as directed.

(a) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x + \cos(x) - 1$, and suppose first that $0 < x \leq \frac{\pi}{2}$. By applying the Mean Value Theorem to f on the interval $[0, x]$, show that $f(x) > 0$.

(b) Now suppose that $x > \frac{\pi}{2}$. By applying the Mean Value Theorem to f on the interval $[\frac{\pi}{2}, x]$ and using part (a), show that $f(x) > 0$.

We have thus shown that $x + \cos(x) - 1 > 0$ for all $x > 0$.

23. Water is being poured into a container in such a way that the height of the water at time t is $H(t) = \beta(1 - e^{-3t})^{1/3}$, where β is some constant.

(a) Show that the height of the water approaches (but never quite reaches) some maximum height H_∞. What is H_∞?

(b) At what rate is the height of the water increasing when it is at height $H_0 = 0.95H_\infty$? Express your answer in terms of β alone.

(c) By inverting the function H, find the time $T(h)$ at which the height of the water is h.

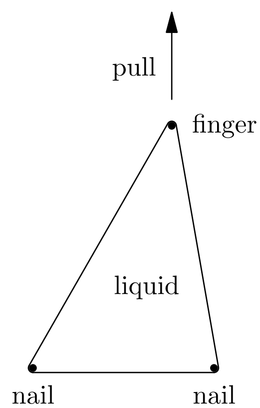
(d) Assume now that $\beta = 0.4$. When the height of the water is H_0, approximately how long does it take for the height to increase by 0.001? Calculate your answer numerically, to four significant figures, in two ways:

(i) by computing $T(H_0 + 0.001) - T(H_0)$ directly, using your answer to part (c).

(ii) by differentiating both sides of the equation $T(H(t)) = t$ with respect to t and using your answer to part (b).

(e) Explain why your answer to part (d)(ii) is slightly less than your answer to part (d)(i).

24. A fixed volume V of liquid is enclosed in a rubber band resting on a table. No liquid is able to leak out from within the band, so you may assume that V is constant throughout. The band loops around two nails a distance b apart, and you pull it taut with your finger to create a triangle, not necessarily isosceles. You then begin to stretch the rubber band by pulling at the corner in a direction perpendicular to the line joining the nails. Here is a view from above:



(a) If, at some point in time, the area enclosed by the triangle is A and you are pulling the corner of the triangle at a speed s, at what rate is the height of the enclosed liquid decreasing? Give your answer in terms of b, V, A, and s. You may assume that the nails and your finger have negligible size, so that the rubber band is in the shape of an ideal triangle whose base has length b.

(b) If, in the previous part,

$$b = 5\,\text{cm}, \quad V = 12\,\text{cm}^3, \quad A = 30\,\text{cm}^2, \quad s = 5\,\text{mm}\,\text{s}^{-1},$$

find the rate of decrease of the height of the liquid in $\text{mm}\,\text{s}^{-1}$. Leave your answer as a fraction.

25. This question uses the inverse-square law, which states that the light intensity experienced at a distance x from a light source is proportional to the luminosity of the source times x^{-2}.

It is the dead of night in Gormenghast Castle, and Steerpike, on the run, is trying to hide in a long, narrow corridor that is dimly lit by two lights, one at each end.

(a) Steerpike knows that his best chance of hiding is to be in the spot between the lights where the combined light intensity is least. Where should he hide? You may assume that the luminosity of the first light is R times the luminosity of the second, and that the lights are a distance 2Δ apart, the first being at position $X = -\Delta$ and the second at position $X = \Delta$. Express your answer in terms of Δ and R.

(b) Before Steerpike makes his dash, he notices that the first light is actually getting brighter moment by moment, in such a way that the ratio of the luminosities at time t is $R(t) = 6(t + 1)$. The place X where the combined light intensity is least is therefore changing over time. If Steerpike, by moving at just the right speed, is to remain always in the spot of least light intensity, how fast should he be moving when he is at position X? Your answer should involve Δ and X but not t. *Hint: Express R in terms of X first.*

(c) How fast should he be moving as he passes the midpoint between the two lights? Express your answer in terms of Δ.

26. Consider an isosceles triangle whose perimeter is L and whose interior angles are θ, θ, and $\pi - 2\theta$. We take L to be fixed and allow θ to vary.

(a) If $A(\theta)$ is the area of the triangle for a given θ, express $A(\theta)$ in terms of L and θ alone. *Hint: Draw a diagram and use trigonometry.*

(b) By extending $A : (0, \frac{\pi}{2}) \to \mathbb{R}$ in the obvious way to a function on the closed interval $[0, \frac{\pi}{2}]$, find its points and values of global extremum, and thus find the angle θ that maximizes the area for the given perimeter L. Take care to show that your chosen θ is indeed a point of global maximum and not some other point of extremum.