

A Few Calculus Questions: Solutions

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The questions can be found accompanying this document at <https://sites.ualberta.ca/~pbucking/>.

1. (a) If $x = -7 + h$ where $h \neq 0$, then

$$f(x) = -\frac{1}{(x+7)^2} = -\frac{1}{h^2},$$

which approaches $-\infty$ as $h \rightarrow 0$. That is, $f(x) \rightarrow -\infty$ as $x \rightarrow -7$. In particular, $f(x) \rightarrow -\infty$ as $x \rightarrow -7^+$ and $f(x) \rightarrow -\infty$ as $x \rightarrow -7^-$.

- (b) Here, $x^2 - 4x + 4 = (x - 2)^2$, so

$$f(x) = \frac{1}{(x-2)(32x-65)}.$$

Therefore, if $x = 2 + h$ where $h \neq 0$, then

$$f(x) = \frac{1}{h(32h-1)},$$

which approaches $-\infty$ as $h \rightarrow 0^+$ and approaches ∞ as $h \rightarrow 0^-$. Thus,

- $f(x) \rightarrow -\infty$ as $x \rightarrow 2^+$,
- $f(x) \rightarrow \infty$ as $x \rightarrow 2^-$,
- $f(x)$ approaches neither ∞ nor $-\infty$ as $x \rightarrow 2$.

- (c) If $x = 5 + h$ where $h \neq 0$, then

$$f(x) = \frac{h}{|h|^3} = \begin{cases} \frac{1}{h^2} & \text{if } h > 0, \\ -\frac{1}{h^2} & \text{if } h < 0. \end{cases}$$

Therefore, $h/|h|^3$ approaches ∞ as $h \rightarrow 0^+$ and approaches $-\infty$ as $h \rightarrow 0^-$, so

- $f(x) \rightarrow \infty$ as $x \rightarrow 5^+$,
- $f(x) \rightarrow -\infty$ as $x \rightarrow 5^-$,
- $f(x)$ approaches neither ∞ nor $-\infty$ as $x \rightarrow 5$.

2. (a) If $x \notin \{-2, 3\}$, then

$$g(x) = \frac{(x+2)(x+6)}{(x+2)(x-3)} = \frac{x+6}{x-3}.$$

The only value of x where the denominator $x - 3$ is zero is $x = 3$, so the only possible vertical asymptote is the line $x = 3$. If $x = 3 + h$ where $h \neq 0$, then

$$g(x) = \frac{9 + h}{h} \rightarrow \begin{cases} \infty & \text{as } h \rightarrow 0^+, \\ -\infty & \text{as } h \rightarrow 0^-, \end{cases}$$

i.e., $g(x) \rightarrow \infty$ as $x \rightarrow 3^+$ and $g(x) \rightarrow -\infty$ as $x \rightarrow 3^-$, so $x = 3$ is a vertical asymptote but $g(x)$ approaches neither ∞ nor $-\infty$ as $x \rightarrow 3$.

(b) If $x \notin \{-7, 5\}$, then

$$g(x) = \frac{x + 9}{(x + 7)(x - 5)}.$$

The only values of x where the denominator $(x+7)(x-5)$ is zero are $x = -7, 5$, so the only possible vertical asymptotes are the lines $x = -7$ and $x = 5$.

If $x = -7 + h$ where $h \neq 0$, then

$$g(x) = \frac{2 + h}{h(-12 + h)} \rightarrow \begin{cases} -\infty & \text{as } h \rightarrow 0^+, \\ \infty & \text{as } h \rightarrow 0^-, \end{cases}$$

i.e., $g(x) \rightarrow -\infty$ as $x \rightarrow -7^+$ and $g(x) \rightarrow \infty$ as $x \rightarrow -7^-$, so $x = -7$ is a vertical asymptote but $g(x)$ approaches neither ∞ nor $-\infty$ as $x \rightarrow -7$.

If $x = 5 + h$ where $h \neq 0$, then

$$g(x) = \frac{14 + h}{(12 + 7)h} \rightarrow \begin{cases} \infty & \text{as } h \rightarrow 0^+, \\ -\infty & \text{as } h \rightarrow 0^-, \end{cases}$$

i.e., $g(x) \rightarrow \infty$ as $x \rightarrow 5^+$ and $g(x) \rightarrow -\infty$ as $x \rightarrow 5^-$, so $x = 5$ is a vertical asymptote but $g(x)$ approaches neither ∞ nor $-\infty$ as $x \rightarrow 5$.

(c) If $x \neq -3$, then

$$g(x) = 5 + \frac{x + 10}{(x + 3)^2}.$$

The only value of x where the denominator $x + 3$ is zero is $x = -3$, so the only possible vertical asymptote is the line $x = -3$. If $x = -3 + h$ where $h \neq 0$, then

$$g(x) = 5 + \frac{7 + h}{h^2} \rightarrow \infty \text{ as } h \rightarrow 0,$$

so $g(x) \rightarrow \infty$ as $x \rightarrow -3$. Thus, the line $x = -3$ is a vertical asymptote.

(d) First, $2x$ has a finite limit at every point, so we have only to worry about $7/(x - 8)$. The denominator $x - 8$ is zero only at $x = 8$, so the only possible vertical asymptote is the line $x = 8$. If $x = 8 + h$ where $h \neq 0$, then

$$g(x) = 2(8 + h) - \frac{7}{h} \rightarrow \begin{cases} -\infty & \text{as } h \rightarrow 0^+, \\ \infty & \text{as } h \rightarrow 0^-, \end{cases}$$

i.e., $g(x) \rightarrow -\infty$ as $x \rightarrow 8^+$ and $g(x) \rightarrow \infty$ as $x \rightarrow 8^-$, so $x = 8$ is a vertical asymptote but $g(x)$ approaches neither ∞ nor $-\infty$ as $x \rightarrow 8$.

3. (a) If $x \neq 1$, then

$$\frac{x - 4 + \frac{3}{x}}{1 - \frac{1}{x}} = \frac{x^2 - 4x + 3}{x - 1} = \frac{(x - 1)(x - 3)}{x - 1} = x - 3,$$

so

$$\lim_{x \rightarrow 1} \frac{x - 4 + \frac{3}{x}}{1 - \frac{1}{x}} = \lim_{x \rightarrow 1} (x - 3) = -2.$$

(b) If $x > -5$ and $x \neq 11$, then

$$\begin{aligned} \frac{\sqrt{x+5}-4}{x-11} &= \frac{(\sqrt{x+5}-4)(\sqrt{x+5}+4)}{(x-11)(\sqrt{x+5}+4)} = \frac{x+5-16}{(x-11)(\sqrt{x+5}+4)} \\ &= \frac{1}{\sqrt{x+5}+4}, \end{aligned}$$

so

$$\lim_{x \rightarrow 11} \frac{\sqrt{x+5}-4}{x-11} = \lim_{x \rightarrow 11} \frac{1}{\sqrt{x+5}+4} = \frac{1}{8}.$$

(c) If $x > 2$ and $x \neq 6$, then

$$\begin{aligned} \frac{2 - \sqrt{x-2}}{x^2 - x - 30} &= \frac{(2 - \sqrt{x-2})(2 + \sqrt{x-2})}{(x^2 - x - 30)(2 + \sqrt{x-2})} \\ &= \frac{4 - (x-2)}{(x-6)(x+5)(2 + \sqrt{x-2})} \\ &= -\frac{1}{(x+5)(2 + \sqrt{x-2})}, \end{aligned}$$

so

$$\lim_{x \rightarrow 6} \frac{2 - \sqrt{x-2}}{x^2 - x - 30} = -\lim_{x \rightarrow 6} \frac{1}{(x+5)(2 + \sqrt{x-2})} = -\frac{1}{44}.$$

(d) If $x \in (-11, 0) \setminus \{-2\}$, then

$$\begin{aligned} \frac{\frac{3}{x} + \sqrt{\frac{1}{x} + \frac{11}{x^2}}}{4 + \frac{8}{x}} &= \frac{3 + x\sqrt{\frac{1}{x} + \frac{11}{x^2}}}{4x + 8} \\ &= \frac{\left(3 + x\sqrt{\frac{1}{x} + \frac{11}{x^2}}\right)\left(3 - x\sqrt{\frac{1}{x} + \frac{11}{x^2}}\right)}{(4x + 8)\left(3 - x\sqrt{\frac{1}{x} + \frac{11}{x^2}}\right)} \\ &= \frac{9 - x^2\left(\frac{1}{x} + \frac{11}{x^2}\right)}{(4x + 8)\left(3 - x\sqrt{\frac{1}{x} + \frac{11}{x^2}}\right)} \\ &= \frac{-(x+2)}{4(x+2)\left(3 - x\sqrt{\frac{1}{x} + \frac{11}{x^2}}\right)} \end{aligned}$$

$$= -\frac{1}{4} \frac{1}{3 - x\sqrt{\frac{1}{x} + \frac{11}{x^2}}},$$

so

$$\begin{aligned} \lim_{x \rightarrow -2} \frac{\frac{3}{x} + \sqrt{\frac{1}{x} + \frac{11}{x^2}}}{4 + \frac{8}{x}} &= -\frac{1}{4} \lim_{x \rightarrow -2} \frac{1}{3 - x\sqrt{\frac{1}{x} + \frac{11}{x^2}}} \\ &= -\frac{1}{4} \frac{1}{3 + 2\sqrt{\frac{9}{4}}} = -\frac{1}{4} \left(\frac{1}{6}\right) = -\frac{1}{24}. \end{aligned}$$

4. (a) First, multiplying numerator and denominator by $x \neq 0$, we obtain

$$\begin{aligned} f(x) &= \frac{3 + x\sqrt{\frac{1}{x} + \frac{11}{x^2}}}{4x + 8} = \frac{3 \pm |x|\sqrt{\frac{1}{x} + \frac{11}{x^2}}}{4x + 8} \\ &= \frac{3 \pm \sqrt{x^2\left(\frac{1}{x} + \frac{11}{x^2}\right)}}{4x + 8} \\ &= \frac{3 \pm \sqrt{x + 11}}{4x + 8}, \end{aligned}$$

the sign being plus if $x > 0$ and minus if $x < 0$. Hence, if $x > 0$, then

$$f(x) = \frac{3 + \sqrt{x + 11}}{4x + 8},$$

so

$$\lim_{x \rightarrow 0^+} f(x) = \frac{3 + \sqrt{11}}{8},$$

and if $x < 0$, then

$$f(x) = \frac{3 - \sqrt{x + 11}}{4x + 8},$$

so

$$\lim_{x \rightarrow 0^-} f(x) = \frac{3 - \sqrt{11}}{8}.$$

- (b) The limit $\lim_{x \rightarrow 0} f(x)$ does not exist, because the left and right limits are different.

5. (a)

$$\begin{aligned} \lim_{x \rightarrow 7} \frac{f_1(x)^3 f_2(x) + (3x + 1)f_2(x)}{f_1(x) + f_2(x)} &= \lim_{x \rightarrow 7} \frac{(f_1(x))^3 + 3x + 1}{f_1(x) + f_2(x)} \\ &= \frac{((-2)^3 + 22) \cdot 9}{-2 + 9} \\ &= \frac{14 \cdot 9}{7} \end{aligned}$$

$$= 2 \cdot 9$$

$$= 18.$$

(b)

$$\lim_{x \rightarrow 5} \frac{4g_1(x)^2 - 6g_1(x) + 5x}{g_2(x)^2 - 3g_2(x) + x^2} = \frac{4L^2 - 6L + 5 \cdot 5}{(2L)^2 - 3 \cdot 2L + 5^2}$$

$$= \frac{4L^2 - 6L + 25}{4L^2 - 6L + 25}$$

$$= 1.$$

6.

$$L + M = \lim_{x \rightarrow a} (f(x) + g(x)) + \lim_{x \rightarrow a} (f(x) - g(x))$$

$$= \lim_{x \rightarrow a} (f(x) + g(x) + f(x) - g(x))$$

$$= \lim_{x \rightarrow a} 2f(x)$$

$$= 2 \lim_{x \rightarrow a} f(x),$$

so

$$\lim_{x \rightarrow a} f(x) = \frac{1}{2}(L + M).$$

Similarly,

$$L - M = \lim_{x \rightarrow a} (f(x) + g(x)) - \lim_{x \rightarrow a} (f(x) - g(x))$$

$$= \lim_{x \rightarrow a} (f(x) + g(x) - f(x) + g(x))$$

$$= \lim_{x \rightarrow a} 2g(x)$$

$$= 2 \lim_{x \rightarrow a} g(x),$$

so

$$\lim_{x \rightarrow a} g(x) = \frac{1}{2}(L - M).$$

7. (a)

$$f'(x) = \frac{\ln(x) - 1}{\ln(x)^2}.$$

Further, $f'(x) = 0$ if and only if $\ln(x) - 1 = 0$ and $x \neq 1$, if and only if $x = e$.

(b) We offer three approaches.

Quotient rule immediately

$$f'(x) = \frac{\cos(x)(1 + \sin(x)) - \sin(x) \cos(x)}{(1 + \sin(x))^2}$$

$$= \frac{\cos(x)}{(1 + \sin(x))^2}.$$

Here, $f'(x) = 0$ if and only if $\cos(x) = 0$ and $\sin(x) \neq -1$, if and only if $x = 2n\pi + \frac{\pi}{2}$ for some $n \in \mathbb{Z}$.

Simplification and then quotient rule

Observe that $f(x) = 1 - \frac{1}{1 + \sin(x)}$. The constant term has zero derivative, so we may apply the quotient rule just to the second term:

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(\frac{-1}{1 + \sin(x)} \right) \\ &= \frac{\cos(x)}{(1 + \sin(x))^2}. \end{aligned}$$

Simplification and then chain rule

We again use the fact that $f(x) = 1 - \frac{1}{1 + \sin(x)}$, but this time we employ the chain rule for the quotient:

$$\begin{aligned} f'(x) &= -\frac{d}{dx} \left((1 + \sin(x))^{-1} \right) \\ &= (1 + \sin(x))^{-2} \cos(x) \\ &= \frac{\cos(x)}{(1 + \sin(x))^2}. \end{aligned}$$

(c)

$$f'(x) = \frac{1 - \ln(x)}{x^2}.$$

Also, $f'(x) = 0$ if and only if $1 - \ln(x) = 0$, if and only if $x = e$.

(d)

$$f'(x) = \frac{x^2 \sin(x) - 2x(1 - \cos(x))}{x^4} = \frac{x \sin(x) + 2 \cos(x) - 2}{x^3}.$$

(e) There is some small degree of simplification to be achieved by observing that $f(x) = \frac{\ln(x)}{\ln(2x)}$ and remembering that $\frac{d}{dx}(\ln(cx)) = 1/x$ if c is constant. Thus,

$$f'(x) = \frac{\frac{1}{x} \ln(2x) - \frac{1}{x} \ln(x)}{\ln(2x)^2} = \frac{\ln(2)}{x \ln(2x)^2}.$$

8. (a)

$$f'(x) = \sec^2(x^2 + \ln(x)) \left(2x + \frac{1}{x} \right).$$

(b)

$$f'(x) = -\csc^2\left(e^{3x} \sin(\ln(x))\right) \cdot \left(3e^{3x} \sin(\ln(x)) + e^{3x} \frac{\cos(\ln(x))}{x}\right).$$

(c)

$$\begin{aligned} f'(x) &= \sec\left(\ln\left(x^5 + \frac{x^2-1}{x^2+1}\right)\right) \tan\left(\ln\left(x^5 + \frac{x^2-1}{x^2+1}\right)\right) \\ &\quad \cdot \frac{1}{x^5 + \frac{x^2-1}{x^2+1}} \left(5x^4 + \frac{2x(x^2+1) - (x^2-1) \cdot 2x}{(x^2+1)^2}\right) \\ &= \sec\left(\ln\left(x^5 + \frac{x^2-1}{x^2+1}\right)\right) \tan\left(\ln\left(x^5 + \frac{x^2-1}{x^2+1}\right)\right) \\ &\quad \cdot \frac{1}{x^5 + \frac{x^2-1}{x^2+1}} \left(5x^4 + \frac{4x}{(x^2+1)^2}\right). \end{aligned}$$

(d)

$$\begin{aligned} f'(x) &= \frac{1}{2} \frac{1}{\sqrt{\csc\left(e^{\tan(5x^6-4x)}\right)}} \\ &\quad \cdot \left(-\csc\left(e^{\tan(5x^6-4x)}\right) \cot\left(e^{\tan(5x^6-4x)}\right)\right) \\ &\quad \cdot e^{\tan(5x^6-4x)} \sec^2(5x^6-4x)(30x^5-4). \end{aligned}$$

9. (a)

$$f'(e^3) = \frac{\ln(e^3) - 1}{\ln(e^3)^2} = \frac{2}{9},$$

so by the point-slope formula, the tangent line at the desired point has equation

$$\begin{aligned} y &= f(e^3) + f'(e^3)(x - e^3) \\ &= \frac{e^3}{3} + \frac{2}{9}(x - e^3) \\ &= \frac{2}{9}x + \frac{e^3}{3} - \frac{2}{9}e^3 \\ &= \frac{2}{9}x + \frac{e^3}{9}. \end{aligned}$$

(b)

$$f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}/2}{(3/2)^2} = \frac{2\sqrt{3}}{9},$$

so by the point-slope formula, the tangent line at the desired point has equation

$$\begin{aligned} y &= f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) \\ &= \frac{1/2}{3/2} + \frac{2\sqrt{3}}{9}\left(x - \frac{\pi}{6}\right) \\ &= \frac{2\sqrt{3}}{9}x + \frac{9 - \sqrt{3}\pi}{27}. \end{aligned}$$

(c) (i) Differentiating both sides of the equation gives

$$\begin{aligned} 290x^9 &= g(x)^5 + x \cdot 5g(x)^4g'(x) - 5x^4g(x) - x^5g'(x) \\ &= (5xg(x)^4 - x^5)g'(x) + g(x)^5 - 5x^4g(x), \end{aligned}$$

so we take $x = 1$ to obtain

$$\begin{aligned} 290 &= (5 \cdot 2^4 - 1)g'(1) + 2^5 - 5 \cdot 2 \\ &= 79g'(1) + 22, \end{aligned}$$

i.e.,

$$g'(1) = \frac{268}{79}.$$

(ii) We are looking for the equation of the tangent line to the graph of g at $(1, 2)$. By the point-slope formula, the equation is

$$\begin{aligned} y &= g(1) + g'(1)(x - 1) \\ &= 2 + \frac{268}{79}(x - 1) \\ &= \frac{268}{79}x - \frac{110}{79}. \end{aligned}$$

10. (a)

$$T(fg) = \frac{(fg)'}{fg} = \frac{f'g + fg'}{fg} = \frac{f'}{f} + \frac{g'}{g} = T(f) + T(g).$$

(b) For $k \in \{1, 2, 3, 4, 5\}$, define $f_k : \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$ by

$$f_k(x) = (x^2 + k)^k.$$

Then $j = f_1 f_2 f_3 f_4 f_5$, so

$$T(j) = T(f_1) + T(f_2) + T(f_3) + T(f_4) + T(f_5).$$

Now,

$$T(f_k)(x) = \frac{f'_k(x)}{f_k(x)} = \frac{k(x^2 + k)^{k-1} \cdot 2x}{(x^2 + k)^k} = 2x \frac{k}{x^2 + k},$$

so

$$T(j)(x) = 2x \left(\frac{1}{x^2+1} + \frac{2}{x^2+2} + \frac{3}{x^2+3} + \frac{4}{x^2+4} + \frac{5}{x^2+5} \right),$$

and so

$$\begin{aligned} j'(x) &= j(x) \cdot 2x \left(\frac{1}{x^2+1} + \frac{2}{x^2+2} + \frac{3}{x^2+3} + \frac{4}{x^2+4} + \frac{5}{x^2+5} \right) \\ &= 2x \left(\frac{1}{x^2+1} + \frac{2}{x^2+2} + \frac{3}{x^2+3} + \frac{4}{x^2+4} + \frac{5}{x^2+5} \right) \\ &\quad \cdot (x^2+1)(x^2+2)^2(x^2+3)^3(x^2+4)^4(x^2+5)^5. \end{aligned}$$

11. Differentiating both sides of the given equation, we find that

$$\begin{aligned} 0 &= \sin(f(x)) + x \cos(f(x))f'(x) + f'(x) \cos(x) - f(x) \sin(x) \\ &= (x \cos(f(x)) + \cos(x))f'(x) + \sin(f(x)) - f(x) \sin(x), \end{aligned}$$

so taking $x = \pi/3$ gives

$$0 = \left(\frac{\pi}{3} \frac{\sqrt{3}}{2} + \frac{1}{2} \right) f' \left(\frac{\pi}{3} \right) + \frac{1}{2} - \frac{\pi}{6} \frac{\sqrt{3}}{2},$$

i.e.,

$$f' \left(\frac{\pi}{3} \right) = \frac{\frac{\sqrt{3}\pi}{12} - \frac{1}{2}}{\frac{\sqrt{3}\pi}{6} + \frac{1}{2}} = \frac{\sqrt{3}\pi - 6}{2\sqrt{3}\pi + 6}.$$

Hence, the tangent line at $(\pi/3, \pi/6)$ has equation

$$\begin{aligned} y - \frac{\pi}{6} &= \frac{\sqrt{3}\pi - 6}{2\sqrt{3}\pi + 6} \left(x - \frac{\pi}{3} \right), \\ \text{i.e., } y &= \frac{\sqrt{3}\pi - 6}{2\sqrt{3}\pi + 6} x + \frac{\pi}{6} - \frac{\pi}{3} \frac{\sqrt{3}\pi - 6}{2\sqrt{3}\pi + 6}. \end{aligned}$$

12. (a) $x^2 + 2x - 8 = (x+1)^2 - 9$.

(b) If $y \in [7, \infty)$ and $x \in (-\infty, -4]$, then

$$\begin{aligned} f(x) = y &\iff \sqrt{(x+1)^2 - 9} = y - 7 \\ &\iff (x+1)^2 - 9 = (y-7)^2 \\ &\iff (x+1)^2 = y^2 - 14y + 58 \\ &\iff x+1 = -\sqrt{y^2 - 14y + 58} \quad \text{since } x \leq -4 \\ &\iff x = -1 - \sqrt{y^2 - 14y + 58}. \end{aligned}$$

This is the unique solution for x in terms of y , so f is invertible and

$$f^{-1}(y) = -1 - \sqrt{y^2 - 14y + 58}.$$

(c) (i) By the formula just found,

$$\begin{aligned}(f^{-1})'(y) &= -\frac{1}{2}(y^2 - 14y + 58)^{-1/2}(2y - 14) \\ &= -\frac{y - 7}{\sqrt{y^2 - 14y + 58}} \\ &= -\frac{y - 7}{\sqrt{(y - 7)^2 + 9}},\end{aligned}$$

so

$$(f^{-1})'(11) = -\frac{4}{5}.$$

(ii)

$$f'(x) = \frac{1}{2}(x^2 + 2x - 8)^{-1/2}(2x + 2) = \frac{x + 1}{\sqrt{x^2 + 2x - 8}},$$

so

$$(f^{-1})'(11) = \frac{1}{f'(f^{-1}(11))} = \frac{1}{f'(-6)} = \frac{1}{-5/4} = -\frac{4}{5}.$$

(d) The slope of f^{-1} at 11 is $(f^{-1})'(11) = -4/5$. Hence, using the point-slope formula, we find that the tangent line to the graph of f^{-1} at $x = 11$ has equation

$$\begin{aligned}y - f^{-1}(11) &= (f^{-1})'(11)(x - 11), \\ \text{i.e., } y &= f^{-1}(11) + (f^{-1})'(11)(x - 11) \\ &= -6 - \frac{4}{5}(x - 11) \\ &= -\frac{4}{5}x + \frac{44}{5} - 6 \\ &= -\frac{4}{5}x + \frac{14}{5}.\end{aligned}$$

13. (a)

$$\begin{aligned}\int_{1/2}^{1/\sqrt{2}} \left(\frac{1}{\sqrt{1-x^2}} - \frac{2}{x} \right) dx &= \left[\arcsin(x) - 2 \ln(x) \right]_{1/2}^{1/\sqrt{2}} \\ &= \frac{\pi}{4} - 2 \ln(1/\sqrt{2}) - \frac{\pi}{6} + 2 \ln(1/2) \\ &= \frac{\pi}{12} + \ln(2) - 2 \ln(2) \\ &= \frac{\pi}{12} - \ln(2).\end{aligned}$$

(b)

$$\int_2^6 \frac{3x^2 + 2x + 1}{x^3} dx = \int_2^6 \left(\frac{3}{x} + \frac{2}{x^2} + \frac{1}{x^3} \right) dx$$

$$\begin{aligned}
&= \left[3 \ln(x) - \frac{2}{x} - \frac{1}{2x^3} \right]_2^6 \\
&= 3(\ln(6) - \ln(2)) - 2 \left(\frac{1}{6} - \frac{1}{2} \right) - \frac{1}{2} \left(\frac{1}{36} - \frac{1}{4} \right) \\
&= 3 \ln(3) + \frac{2}{3} + \frac{1}{9} \\
&= 3 \ln(3) + \frac{7}{9}.
\end{aligned}$$

(c)

$$\begin{aligned}
\int_{-1}^1 (4 \sin(x) + x^6) dx &= \left[-4 \cos(x) + \frac{1}{7} x^7 \right]_{-1}^1 \\
&= -4(\cos(1) - \cos(-1)) + \frac{1}{7}(1 - (-1)) \\
&= \frac{2}{7}
\end{aligned}$$

because $\cos(1) = \cos(-1)$.

(d)

$$\begin{aligned}
\int_{-7\pi/6}^{7\pi/6} (5 \cos(x) - x^7) dx &= \left[5 \sin(x) - \frac{1}{8} x^8 \right]_{-7\pi/6}^{7\pi/6} \\
&= -\frac{5}{2} - \frac{5}{2} \\
&= -5.
\end{aligned}$$

(e) First,

$$\begin{aligned}
\frac{\cos(x)}{1 + \cos(x)} + \frac{\cos(x)}{1 - \cos(x)} &= \frac{\cos(x)(1 - \cos(x) + 1 + \cos(x))}{1 - \cos^2(x)} \\
&= \frac{2 \cos(x)}{\sin^2(x)} \\
&= 2 \csc(x) \cot(x),
\end{aligned}$$

so

$$\begin{aligned}
\int_{\pi/6}^{\pi/2} \left(\frac{\cos(x)}{1 + \cos(x)} + \frac{\cos(x)}{1 - \cos(x)} \right) dx &= 2 \int_{\pi/6}^{\pi/2} \csc(x) \cot(x) dx \\
&= -2 \left[\csc(x) \right]_{\pi/6}^{\pi/2} \\
&= -2(1 - 2) \\
&= 2.
\end{aligned}$$

14. (a) (i) If

$$G(y) = \int_0^y e^t \sin^2(t^2 + 1) dt,$$

then

$$G'(y) = e^y \sin^2(y^2 + 1) \quad \text{and} \quad F(x) = G(\ln(x)),$$

so

$$\begin{aligned} F'(x) &= G'(\ln(x)) \frac{1}{x} = x \sin^2(\ln(x)^2 + 1) \frac{1}{x} \\ &= \sin^2(\ln(x)^2 + 1). \end{aligned}$$

(ii) Note that the point $1/2$ is in the domain of the integrand, so we may define

$$G(y) = \int_{1/2}^y \ln(t) \arcsin(t) \cos(t) dt.$$

One may use, instead of $1/2$, any t in the domain of the continuous function

$$\begin{aligned} (0, 1] &\rightarrow \mathbb{R} \\ t &\mapsto \ln(t) \arcsin(t) \cos(t), \end{aligned}$$

i.e., any $t \in (0, 1]$. Now,

$$G'(y) = \ln(y) \arcsin(y) \cos(y),$$

and

$$\begin{aligned} F(x) &= \int_{x^4}^{1/2} \ln(t) \arcsin(t) \cos(t) dt + \int_{1/2}^{x^2} \ln(t) \arcsin(t) \cos(t) dt \\ &= \int_{1/2}^{x^2} \ln(t) \arcsin(t) \cos(t) dt - \int_{1/2}^{x^4} \ln(t) \arcsin(t) \cos(t) dt \\ &= G(x^2) - G(x^4), \end{aligned}$$

so

$$\begin{aligned} F'(x) &= G'(x^2) \cdot 2x - G'(x^4) \cdot 4x^3 \\ &= \ln(x^2) \arcsin(x^2) \cos(x^2) \cdot 2x \\ &\quad - \ln(x^4) \arcsin(x^4) \cos(x^4) \cdot 4x^3 \\ &= 4x \ln(x) \arcsin(x^2) \cos(x^2) - 16x^3 \ln(x) \arcsin(x^4) \cos(x^4) \\ &= 4x \ln(x) \left(\arcsin(x^2) \cos(x^2) - 4x^2 \arcsin(x^4) \cos(x^4) \right). \end{aligned}$$

(b) For any $a \in \mathbb{R}$, if we define

$$G(x) = \int_a^x \cos(\cos(\cos(t))) dt,$$

then $G'(x) = \cos(\cos(\cos(x)))$ for all $x \in \mathbb{R}$. If we take $a = 1$, then we further have $G(1) = 0$. Thus,

$$G(x) = \int_1^x \cos(\cos(\cos(t))) dt.$$

15.

$$\begin{aligned} f'(x) &= \frac{1}{\left(\frac{1+x}{1-x}\right)^2 + 1} \frac{(1-x) + (1+x)}{(1-x)^2} \\ &= \frac{2}{(1+x)^2 + (1-x)^2} \\ &= \frac{2}{1 + 2x + x^2 + 1 - 2x + x^2} \\ &= \frac{1}{1 + x^2}. \end{aligned}$$

This is also the derivative of $\arctan(x)$, so there is $C \in \mathbb{R}$ such that

$$\arctan\left(\frac{1+x}{1-x}\right) = \arctan(x) + C \quad \text{for all } x < 1.$$

(There is also such a constant for $x > 1$, but it is a different constant.) To find C , we may evaluate both sides of the equation at $x = 0$. The left-hand side in this case is

$$\arctan(1) = \frac{\pi}{4},$$

and the right-hand side is

$$\arctan(0) + C = C,$$

so $C = \frac{\pi}{4}$.

16. (a) (i)

$$\begin{aligned} \int_6^{12} \frac{7}{2x-9} dx &= \frac{1}{2} \int_3^{15} \frac{7}{u} du \quad (u = 2x - 9, du = 2 dx) \\ &= \frac{7}{2} [\ln(u)]_3^{15} \\ &= \frac{7}{2} (\ln(15) - \ln(3)) \\ &= \frac{7}{2} \ln(5). \end{aligned}$$

(ii)

$$\int_1^4 \frac{x}{\sqrt{65-x^2}} dx = -\frac{1}{2} \int_{64}^{49} \frac{1}{\sqrt{t}} dt \quad (t = 65 - x^2, dt = -2x dx)$$

$$\begin{aligned}
&= -[\sqrt{t}]_{64}^{49} \\
&= 8 - 7 \\
&= 1.
\end{aligned}$$

(iii)

$$\begin{aligned}
\int_{-1}^0 \frac{2x-1}{x^2-2x+1} dx &= \int_{-1}^0 \frac{2(x-1)+1}{(x-1)^2} dx \\
&= \int_{-2}^{-1} \frac{2y+1}{y^2} dy \quad (y = x-1, dy = dx) \\
&= \int_{-2}^{-1} \left(\frac{2}{y} + \frac{1}{y^2} \right) dy \\
&= \left[2 \ln(-y) - \frac{1}{y} \right]_{-2}^{-1} \\
&= -2 \ln(2) - \left(-1 - \left(-\frac{1}{2} \right) \right) \\
&= \frac{1}{2} - 2 \ln(2).
\end{aligned}$$

(b) (i)

$$\begin{aligned}
\int \frac{8}{3x^2+5} dx &= \frac{8}{5} \int \frac{1}{\frac{3}{5}x^2+1} dx \\
&= \frac{8}{5} \sqrt{\frac{5}{3}} \int \frac{1}{y^2+1} dy \quad (y = \sqrt{\frac{3}{5}}x, dy = \sqrt{\frac{3}{5}} dx) \\
&= \frac{8}{5} \sqrt{\frac{5}{3}} \arctan(y) + C \\
&= \frac{8}{\sqrt{15}} \arctan\left(\sqrt{\frac{3}{5}}x\right) + C.
\end{aligned}$$

(ii)

$$\begin{aligned}
\int \frac{1}{e^x+1} dx &= \int \frac{e^{-x}}{1+e^{-x}} dx \\
&= - \int \frac{1}{t} dt \quad (t = 1 + e^{-x}, dt = -e^{-x} dx) \\
&= -\ln(t) + C \\
&= -\ln(1 + e^{-x}) + C.
\end{aligned}$$

(iii) Here, we will make the change of variables $u = \frac{x-1}{x+1}$, for which the corresponding change of differentials is

$$du = \frac{(x+1) - (x-1)}{(x+1)^2} dx = \frac{2}{(x+1)^2} dx.$$

Hence,

$$\begin{aligned} \int \frac{1}{(x+1)^2} \csc^2\left(\frac{x-1}{x+1}\right) dx &= \frac{1}{2} \int \csc^2(u) du \\ &= -\frac{1}{2} \cot(u) + C \\ &= -\frac{1}{2} \cot\left(\frac{x-1}{x+1}\right) + C. \end{aligned}$$

17. We choose the antiderivative $\frac{1}{2}(x^2 + 1)$ of x , so that

$$\begin{aligned} \int x \arctan(x) dx &= \frac{1}{2}(x^2 + 1) \arctan(x) - \int \frac{1}{2}(x^2 + 1) \frac{1}{x^2 + 1} dx \\ &= \frac{1}{2}(x^2 + 1) \arctan(x) - \int \frac{1}{2} dx \\ &= \frac{1}{2}(x^2 + 1) \arctan(x) - \frac{1}{2}x + C. \end{aligned}$$

18. (a) (i) Making the change of variables $u = t/\sqrt{2}$, we have

$$\begin{aligned} \int_0^x \frac{1}{\sqrt{2-t^2}} dt &= \int_0^{x/\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2-2u^2}} du \\ &= \int_0^{x/\sqrt{2}} \frac{1}{\sqrt{1-u^2}} du \\ &= [\arcsin(u)]_0^{x/\sqrt{2}} \\ &= \arcsin\left(\frac{x}{\sqrt{2}}\right). \end{aligned}$$

(ii) Under the change of variables $u = 1 - t^2$, $t = \sqrt{1-u}$ because the fact that $x \geq 0$ implies that $t \geq 0$ in the integral. Therefore, the relationship between the differentials is

$$dt = -\frac{1}{2} \frac{1}{\sqrt{1-u}} du.$$

Hence,

$$\begin{aligned} \int_0^x \frac{1}{\sqrt{2-t^2}} dt &= -\frac{1}{2} \int_1^{1-x^2} \frac{1}{\sqrt{1+u}} \frac{1}{\sqrt{1-u}} du \\ &= \frac{1}{2} \int_{1-x^2}^1 \frac{1}{\sqrt{1-u^2}} du \\ &= \frac{1}{2} [\arcsin(u)]_{1-x^2}^1 \\ &= \frac{1}{2} \left(\frac{\pi}{2} - \arcsin(1-x^2) \right). \end{aligned}$$

(b) We simply equate the two expressions found in part (a):

$$\arcsin\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{2}\left(\frac{\pi}{2} - \arcsin(1 - x^2)\right),$$

i.e., $\arcsin(1 - x^2) = \frac{\pi}{2} - 2 \arcsin\left(\frac{x}{\sqrt{2}}\right).$

19. Before beginning, we note that every function is continuous and has a domain that is a closed interval, so we can find points and values of global extrema by considering critical points alone. Further, each function is differentiable on the interior of its domain, so the critical points are the end-points and the points where the derivative is zero.

(a) $f'(x) = 2x - 1$, so $f'(x) = 0$ if and only if $x = \frac{1}{2}$. The critical points are therefore $-1, \frac{1}{2}, 1$ with values

$$f(-1) = 3, \quad f\left(\frac{1}{2}\right) = \frac{3}{4}, \quad f(1) = 1,$$

so the value of global maximum is 3 and occurs at -1 , and the value of global minimum is $\frac{3}{4}$ and occurs at $\frac{1}{2}$.

(b) $f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x + 3)(x - 1)$, so $f'(x) = 0$ if and only if $x \in \{-3, 1\}$. The critical points are therefore $-4, -3, 1, 4$, and we have the following table:

x	$f(x)$
-4	19
-3	26
1	-6
4	75

Thus, the value of global maximum is 75, occurring at 4, and the value of global minimum is -6 , occurring at 1.

(c) $f'(x) = \frac{1}{x} - 1$, so $f'(x) = 0$ if and only if $x = 1$. The critical points are therefore $\frac{1}{2}, 1, 3$. Note also that $f'(x)$ is negative if $x < 1$ and is positive if $x > 1$, so $x = 1$ is the unique point of global maximum, and the value of global maximum is $f(1) = 1$.

It remains to decide which of $f\left(\frac{1}{2}\right)$ and $f(3)$ is the lesser. For this, consider

$$f\left(\frac{1}{2}\right) - f(3) = \ln\left(\frac{1}{2}\right) - \frac{1}{2} - \ln(3) + 3 = \frac{5}{2} - \ln(6).$$

Now, $e^2 > (2.7)^2 = (2 + 0.7)^2 = 4 + 2.8 + 0.49 > 6$, so $\ln(6) < \ln(e^2) = 2$. Therefore, continuing from the above, we have

$$f\left(\frac{1}{2}\right) - f(3) > \frac{5}{2} - 2 = \frac{1}{2} > 0,$$

Therefore,

$$f(x) = \frac{(x-3)(x^2 - 6x + 9) + 4}{x^2 - 6x + 9} = x - 3 + \frac{4}{(x-3)^2},$$

so

$$f'(x) = 1 - \frac{8}{(x-3)^3},$$
$$f''(x) = \frac{24}{(x-3)^4}.$$

Domain: The denominator is zero when $x = 3$, and the numerator is non-zero at that point, so f is not defined at 3. It is defined everywhere else, so the domain is $\mathbb{R} \setminus \{3\} = (-\infty, 3) \cup (3, \infty)$.

Axis intercepts: The y -intercept is $f(0) = -\frac{23}{9}$, and for the x -intercepts we solve

$$f(x) = 0 \iff x - 3 + \frac{4}{(x-3)^2} = 0$$
$$\iff (x-3)^3 + 4 = 0$$
$$\iff x = 3 - 4^{1/3}.$$

Extrema:

$$f'(x) = 0 \iff 1 - \frac{8}{(x-3)^3} = 0$$
$$\iff (x-3)^3 = 8$$
$$\iff x = 5,$$

so $x = 5$ is a potential point of extremum. The same steps show that $f'(x) > 0$ if and only if

$$\frac{8}{(x-3)^3} < 1. \tag{1}$$

When $x < 3$, (1) holds unconditionally. When $x > 3$, it holds if and only if $(x-3)^3 > 8$, if and only if $x > 5$. We summarize thus:

	$x < 3$	$3 < x < 5$	$x > 5$
f'	+	-	+
f	\nearrow	\searrow	\nearrow

Therefore, $x = 5$ is a point of local minimum. Another way to see that this is a local minimum rather than a local maximum is to note that $f''(5) = \frac{24}{2^4} > 0$.

Inflection points: $f''(x) > 0$ for all x in the domain of f , so f has no inflection points.

Regions of upward and downward concavity: Because f'' is everywhere positive, all regions of the graph are concave up.

Asymptotes: From our description of $f(x)$ obtained via long division of polynomials,

$$f(x) - (x - 3) = \frac{4}{(x - 3)^2},$$

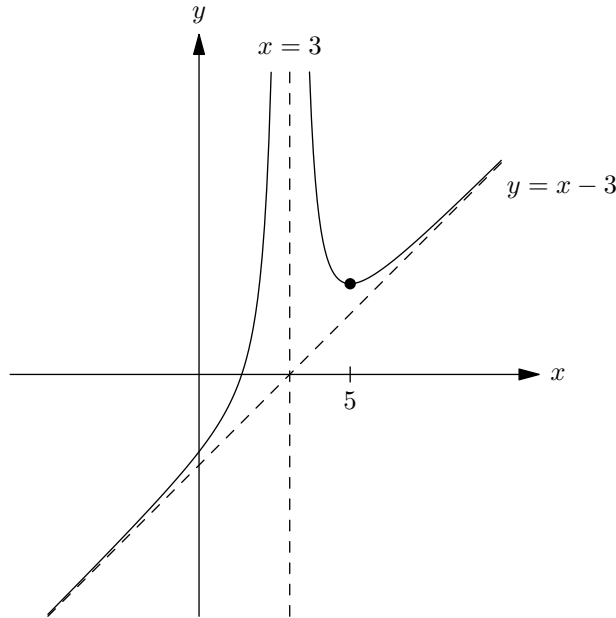
and this tends to zero as $x \rightarrow \infty$ and also as $x \rightarrow -\infty$, so the line $y = x - 3$ is a slant asymptote in both directions. Consequently, there are no horizontal asymptotes.

The line $x = 3$ is a vertical asymptote, and in fact $f(x) \rightarrow \infty$ as $x \rightarrow 3$ (in the two-sided sense), because for $h \neq 0$,

$$f(3 + h) = h + \frac{4}{h^2} \rightarrow \infty \quad \text{as } h \rightarrow 0.$$

There are no other vertical asymptotes, because there are no other points where the denominator is zero.

Here is the graph, with the asymptotes shown as dashed lines and the point of local minimum indicated by a dot:



21. We offer two solutions.

Solution 1: Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = x^2 + \frac{9}{5}x - \frac{1}{2} - \arctan(x),$$

which is differentiable. Observe that

$$f'(x) = 2x + \frac{9}{5} - \frac{1}{x^2 + 1},$$

so $f'(x) = 0$ if and only if

$$\begin{aligned} 0 &= (10x + 9)(x^2 + 1) - 5 \\ &= 10x^3 + 9x^2 + 10x + 4 \\ &= (2x + 1)(5x^2 + 2x + 4), \end{aligned}$$

the last equality by the fact given in the question. The quadratic factor has discriminant

$$2^2 - 4 \cdot 5 \cdot 4 = 4 - 80 = -76 < 0,$$

so the quadratic factor is never zero. Therefore, $f'(x) = 0$ if and only if $x = -\frac{1}{2}$.

Let us examine $f(-\frac{1}{2})$, then:

$$\begin{aligned} f\left(-\frac{1}{2}\right) &= \frac{1}{4} - \frac{9}{10} - \frac{1}{2} + \arctan\left(\frac{1}{2}\right) \\ &= \arctan\left(\frac{1}{2}\right) - \frac{23}{20} \\ &< \arctan(1) - \frac{23}{20} \\ &= \frac{\pi}{4} - \frac{23}{20} \\ &< 1 - \frac{23}{20} \\ &< 0. \end{aligned}$$

Now, $x^2 + \frac{9}{5}x - \frac{1}{2} \rightarrow \infty$ as $x \rightarrow \pm\infty$, while $-\frac{\pi}{2} < \arctan(x) < \frac{\pi}{2}$ for all x , so $f(x) \rightarrow \infty$ as $x \rightarrow \pm\infty$. Therefore, there are $a_1 < -\frac{1}{2}$ and $a_2 > -\frac{1}{2}$ such that $f(a_1) > 0$ and $f(a_2) > 0$, so by the Intermediate Value Theorem, there are $x_1 \in (a_1, -\frac{1}{2})$ and $x_2 \in (-\frac{1}{2}, a_2)$ such that

$$f(x_1) = f(x_2) = 0.$$

Of course, x_1 and x_2 are not equal, because they lie on opposite sides of $-\frac{1}{2}$.

To demonstrate that x_1 and x_2 are the only solutions, we take any $x \in \mathbb{R} \setminus \{x_1, x_2\}$ and show that $f(x) \neq 0$. If $x \leq -\frac{1}{2}$, then we can consider the restriction of f to $[w_1, -\frac{1}{2}]$ where $w_1 < x, x_1$. Because f' is non-zero on $(w_1, -\frac{1}{2})$, f is injective on $[w_1, -\frac{1}{2}]$, so the fact that $f(x_1) = 0$ implies that $f(x) \neq 0$. Similarly, if $x > -\frac{1}{2}$, we consider the restriction of f to $[-\frac{1}{2}, w_2]$ where $w_2 > x, x_2$. This time, f' is non-zero on $(-\frac{1}{2}, w_2)$, so the same reasoning shows that $f(x) \neq 0$.

For uniqueness, we may instead appeal to the Mean Value Theorem directly. Suppose that $x \in \mathbb{R} \setminus \{x_1, x_2\}$. We saw in the existence portion of our answer that

$f(-\frac{1}{2}) \neq 0$, so assume that $x \neq -\frac{1}{2}$. If $x < -\frac{1}{2}$, then by the Mean Value Theorem, there is c between x and x_1 such that

$$f'(c) = \frac{f(x) - f(x_1)}{x - x_1} = \frac{f(x)}{x - x_1},$$

so $f(x) = f'(c)(x - x_1)$. Because c lies between x and x_1 , it is not equal to $-\frac{1}{2}$, so $f'(c) \neq 0$. Thus, $f(x) \neq 0$. If, instead, $x > -\frac{1}{2}$, then there is c between x and x_2 such that

$$f'(c) = \frac{f(x) - f(x_2)}{x - x_2} = \frac{f(x)}{x - x_2},$$

and then the same reasoning shows again that $f(x) \neq 0$.

Solution 2: Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = x^2 + \frac{9}{5}x - \frac{1}{2} - \arctan(x),$$

which is differentiable. Observe that

$$f'(x) = 2x + \frac{9}{5} - \frac{1}{x^2 + 1}$$

and hence that

$$\begin{aligned} f''(x) &= 2 + \frac{2x}{(x^2 + 1)^2} \\ &= 2 \frac{(x^2 + 1)^2 + x}{(x^2 + 1)^2} \\ &\geq 2 \frac{x^2 + 1 + x}{(x^2 + 1)^2} \quad \text{because } x^2 + 1 \geq 1 \\ &= 2 \frac{x^2 + x + 1}{(x^2 + 1)^2}. \end{aligned}$$

Because the polynomial $x^2 + x + 1$ has discriminant $-3 < 0$ and has a positive leading coefficient, it takes only positive values, so $f''(x) > 0$ for all $x \in \mathbb{R}$. Therefore, f' is monotone increasing on \mathbb{R} , so it is injective. Suppose, then, that $x_1, x_2 \in \mathbb{R}$ are solutions with $x_1 < x_2$. We show that they are the only solutions, in fact. By the Mean Value Theorem, there is $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = 0,$$

so by the injectivity of f' , it follows that $f'(\tilde{c}) \neq 0$ for all $\tilde{c} \in \mathbb{R} \setminus \{c\}$. Let $x \in \mathbb{R} \setminus \{x_1, x_2\}$. If $x \leq c$, then there is c_1 strictly between x and x_1 such that

$$f'(c_1) = \frac{f(x) - f(x_1)}{x - x_1} = \frac{f(x)}{x - x_1}.$$

But $f'(c_1) \neq 0$ because $c_1 \neq c$, so $f(x) \neq 0$. Similarly, if $x > c$, then there is c_2 strictly between x and x_2 such that

$$f'(c_2) = \frac{f(x) - f(x_2)}{x - x_2} = \frac{f(x)}{x - x_2},$$

and then the same reasoning shows that $f(x) \neq 0$.

To see that two solutions do exist, we can use the same argument as in the existence portion of our first solution.

22. (a) The function is differentiable, so we may apply the Mean Value Theorem to it on any closed interval. In particular, if $x \in (0, \frac{\pi}{2}]$, there is $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0},$$

i.e., $1 - \sin(c) = \frac{f(x)}{x}$.

Because $0 < c < \frac{\pi}{2}$, we know that $1 - \sin(c) > 0$, so $f(x)/x > 0$, and so $f(x) > 0$.

- (b) Let $x > \frac{\pi}{2}$. Then by the Mean Value Theorem, there is $c \in (\frac{\pi}{2}, x)$ such that

$$1 - \sin(c) = \frac{f(x) - f(\frac{\pi}{2})}{x - \frac{\pi}{2}}.$$

But $1 - \sin(c) \geq 0$, so $f(x) - f(\frac{\pi}{2}) \geq 0$, i.e.,

$$f(x) \geq f(\frac{\pi}{2}) > 0 \quad \text{by part (a).}$$

23. (a) The limit as $t \rightarrow \infty$ of $H(t) = \beta(1 - e^{-3t})^{1/3}$ is β . The height β is never attained, because $1 - e^{-3t} < 1$ for all t .
- (b) We need to find $H'(t_0)$, where t_0 is the time at which the height of the water is $H_0 = 0.95H_\infty = 0.95\beta$. Computing the derivative of the function H , we obtain

$$H'(t_0) = \beta e^{-3t_0} (1 - e^{-3t_0})^{-2/3}. \quad (2)$$

It remains to eliminate t_0 . Observe that

$$0.95\beta = H(t_0) = \beta(1 - e^{-3t_0})^{1/3},$$

i.e., $0.95 = (1 - e^{-3t_0})^{1/3}$, from which we obtain

$$\begin{aligned} 1 - e^{-3t_0} &= 0.95^3 \\ \text{and } e^{-3t_0} &= 1 - 0.95^3. \end{aligned}$$

Hence, (2) gives

$$H'(t_0) = \frac{1 - 0.95^3}{0.95^2} \beta.$$

- (c) For a given height h , $H(t) = h$ if and only if

$$\beta(1 - e^{-3t})^{1/3} = h,$$

$$\begin{aligned} \Leftrightarrow e^{-3t} &= 1 - \frac{h^3}{\beta^3}, \\ \Leftrightarrow t &= -\frac{1}{3} \ln \left(1 - \frac{h^3}{\beta^3} \right). \end{aligned}$$

Therefore,

$$T(h) = -\frac{1}{3} \ln \left(1 - \frac{h^3}{\beta^3} \right). \quad (3)$$

- (d) (i) Making the substitutions $h = 0.95 \times 0.4 + 0.001$ and $h = 0.95 \times 0.4$ in (3), we find

$$T(H_0 + 0.001) - T(H_0) \approx 0.01625.$$

- (ii) Differentiating both sides of $T(H(t)) = t$ with respect to t and then evaluating at $t = t_0$ gives $H'(t_0)T'(H_0) = 1$, so

$$T'(H_0) = \frac{1}{H'(t_0)} = \frac{0.95^2}{(1 - 0.95^3)\beta}$$

by part (b). Hence,

$$\begin{aligned} T(H_0 + \delta) - T(H_0) &= \delta \frac{T(H_0 + \delta) - T(H_0)}{\delta} \\ &\approx \delta \cdot T'(H_0) \quad \text{because } \delta \text{ is small} \\ &= \delta \frac{0.95^2}{(1 - 0.95^3)\beta}, \end{aligned} \quad (4)$$

which evaluates to approximately 0.01582 when $\delta = 0.001$ and $\beta = 0.4$.

- (e) The approximation made at (4) assumes that the rate at which the height of the water increases between heights H_0 and $H_0 + 0.001$ remains constant, at the rate $H'(t_0)$. We are essentially dividing the instantaneous rate of change of height, $H'(t_0)$, into the overall change in height of 0.001. However, in reality, the rate of increase in height tails off over time, i.e., H' is a decreasing function, so the true time taken to achieve the change in height is a little longer.

24. (a) Let x be the perpendicular distance of the moving corner of the triangle to the line between the nails. Then the area of the triangle is

$$A = \frac{1}{2}bx.$$

Also, if the height of the enclosed liquid is h , then

$$h = \frac{V}{A}.$$

Considering all quantities as functions of time, except V and b , which are constant, we differentiate both sides of the equation $h = V/A$ to obtain, via the chain rule,

$$h' = -\frac{V}{A^2}A' = -\frac{V}{A^2} \cdot \frac{1}{2}bx' = -\frac{bV}{2A^2}x'.$$

Thus, if the corner is moving with speed s , then the height is decreasing at the rate $\frac{bVs}{2A^2}$.

- (b) With the quantities given, working in distance units of centimetres and time units of seconds, we have

$$\frac{bVs}{2A^2} = \frac{5(12)(0.5)}{2(30^2)} = \frac{5 \cdot 12}{4 \cdot 30^2} = \frac{15}{30^2} = \frac{1}{2 \cdot 30} = \frac{1}{60}.$$

Converting to millimetres, we see that the rate of decrease is $\frac{1}{6} \text{ mm s}^{-1}$.

25. (a) Let the two lights be A and B with luminosities a and b respectively, so that $R = a/b$. By the inverse-square law, there is a positive constant λ such that the light intensity at position X experienced from source A is $\lambda a/(X+\Delta)^2$ and the light intensity, at the same point, experienced from source B is $\lambda b/(X-\Delta)^2$. The combined light intensity is therefore

$$f(X) = \lambda \left(\frac{a}{(X+\Delta)^2} + \frac{b}{(X-\Delta)^2} \right).$$

We wish to determine where the function f is minimum on the interval $(-\Delta, \Delta)$. Differentiating, we find that

$$f'(X) = -2\lambda \left(\frac{a}{(X+\Delta)^3} + \frac{b}{(X-\Delta)^3} \right),$$

so $f'(X) = 0$ if and only if

$$\begin{aligned} \frac{a}{(\Delta+X)^3} &= \frac{b}{(\Delta-X)^3} \\ \iff \frac{R}{(\Delta+X)^3} &= \frac{1}{(\Delta-X)^3} & (5) \\ \iff \frac{R^{1/3}}{\Delta+X} &= \frac{1}{\Delta-X} \\ \iff R^{1/3}\Delta - R^{1/3}X &= \Delta + X \\ \iff X &= \frac{R^{1/3}-1}{R^{1/3}+1}\Delta. \end{aligned}$$

Further, the same sequence of steps, but with inequalities instead of equalities, shows that $f'(X) > 0$ if $X > \frac{R^{1/3}-1}{R^{1/3}+1}\Delta$, and $f'(X) < 0$ if $X < \frac{R^{1/3}-1}{R^{1/3}+1}\Delta$, so the function f does indeed have a local minimum at $\frac{R^{1/3}-1}{R^{1/3}+1}\Delta$. Alternatively, one may calculate

$$f''(X) = 6\lambda \left(\frac{a}{(X+\Delta)^4} + \frac{b}{(X-\Delta)^4} \right)$$

to see that it is positive for all $X \in (-\Delta, \Delta)$, which again shows that the point $X = (R^{1/3} - 1)\Delta/(R^{1/3} + 1)$ is a local minimum. Because f has no other critical points in $(-\Delta, \Delta)$, this must be a global minimum.

(b) From (5), we find that

$$R = \left(\frac{\Delta + X}{\Delta - X} \right)^3. \quad (6)$$

Hence, if we regard R and X both as functions of time t and differentiate (6) with respect to t , then the chain rule gives us

$$R' = X' \cdot 6\Delta \frac{(\Delta + X)^2}{(\Delta - X)^4}. \quad (7)$$

Here, we have used that

$$\frac{d}{dx} \left(\frac{\Delta + x}{\Delta - x} \right)^3 = 6\Delta \frac{(\Delta + x)^2}{(\Delta - x)^4}.$$

But $R(t) = 6(t + 1)$, so R' is constantly 6, and rearranging (7) then gives

$$X' = \frac{(\Delta - X)^4}{\Delta(\Delta + X)^2}.$$

(c) The midpoint between the lights is $X = 0$, and then $X' = \Delta^4/\Delta^3 = \Delta$.

26. (a) Let the base of the triangle have length $L - 2x$, so that the other two sides each have length x . By trigonometry, the base is $2x \cos(\theta)$ and the height $x \sin(\theta)$, so

$$L - 2x = 2x \cos(\theta), \quad \text{i.e.} \quad x = \frac{L}{2(\cos(\theta) + 1)},$$

and

$$\begin{aligned} A(\theta) &= \frac{1}{2}x \cos(\theta) \cdot x \sin(\theta) \\ &= \frac{1}{2}x^2 \sin(2\theta) \\ &= \frac{1}{2} \frac{L^2}{4(\cos(\theta) + 1)^2} \sin(2\theta) \\ &= \frac{L^2}{8} \frac{\sin(2\theta)}{(\cos(\theta) + 1)^2}. \end{aligned}$$

- (b) Define $A : [0, \frac{\pi}{2}] \rightarrow \mathbb{R}$ by the same formula as in part (a). The function A is continuous, and its domain is a closed interval, so it has a value of global maximum and a value of global minimum, found by considering the critical points. Two of these are the end-points, 0 and $\frac{\pi}{2}$, but other potential critical

points are numbers $\theta \in (0, \frac{\pi}{2})$ where the derivative is either zero or undefined. In fact, A is differentiable on all of $(0, \frac{\pi}{2})$, and

$$\begin{aligned} A'(\theta) &= \frac{L^2}{8} \frac{2 \cos(2\theta)(\cos(\theta) + 1)^2 - \sin(2\theta) \cdot 2(\cos(\theta) + 1)(-\sin(\theta))}{(\cos(\theta) + 1)^4} \\ &= \frac{L^2}{4} \frac{\cos(2\theta) + \cos(2\theta) \cos(\theta) + \sin(2\theta) \sin(\theta)}{(\cos(\theta) + 1)^3} \\ &= \frac{L^2}{4} \frac{\cos(2\theta) + \cos(\theta)}{(\cos(\theta) + 1)^3} \\ &= \frac{L^2}{4} \frac{2 \cos^2(\theta) - 1 + \cos(\theta)}{(\cos(\theta) + 1)^3} \\ &= \frac{L^2}{4} \frac{(2 \cos(\theta) - 1)(\cos(\theta) + 1)}{(\cos(\theta) + 1)^3} \\ &= \frac{L^2}{4} \frac{2 \cos(\theta) - 1}{(\cos(\theta) + 1)^2}, \end{aligned}$$

so $A'(\theta) = 0$ if and only if $\cos(\theta) = \frac{1}{2}$, if and only if $\theta = \frac{\pi}{3}$. Thus, the critical points are 0 , $\frac{\pi}{3}$, and $\frac{\pi}{2}$, and the values of A at those points are

$$A(0) = 0, \quad A\left(\frac{\pi}{3}\right) = \frac{L^2}{8} \frac{\sqrt{3}/2}{9/4} = \frac{\sqrt{3}}{36} L^2, \quad A\left(\frac{\pi}{2}\right) = 0.$$

The global maximum value is therefore $\frac{\sqrt{3}}{36} L^2$, occurring at $\theta = \frac{\pi}{3}$, and the global minimum value is 0 , occurring at the end-points. This shows that the area is maximized when $\theta = \frac{\pi}{3}$.