Algebraic Number Theory
MATH 512

Assignment 6

1. Let L be the field in question **3** of Assignment 5.

(a) Find the degree of the extension $\mathfrak{k}_L/\mathbb{F}_7$.

(b) Show that for any $a \in \mathbb{Z}$ that is not square mod 7, and any $\gamma \in L$ with $\gamma^2 = a$, we have $\mathcal{O}_L = \mathbb{Z}_7[\gamma]$.

2. Let p be a prime number, r a positive integer, and $\zeta = \zeta_{p^r}$ a primitive p^rth root of unity in a fixed algebraic closure of $\mathbb{Q}_p$. The aim of this exercise is to compute the ramification groups of the extension $L/\mathbb{Q}_p$ where $L = \mathbb{Q}_p(\zeta_{p^r})$. You may assume that $L/\mathbb{Q}_p$ is totally ramified of degree $\phi(p^r)$ where ϕ is Euler's ϕ-function, and that $\zeta - 1$ is a uniformizer of L.

For an integer a not divisible by p, let $\sigma_a \in G = \mathrm{Gal}(L/\mathbb{Q}_p)$ be given by $\sigma_a : \zeta \mapsto \zeta^a$.

(a) Explain why $G_0 = G$.

(b) Take $a \in \mathbb{Z}$ not divisible by p and not congruent to 1 mod p^r, let v be the normalized valuation on L, and let v_p be the p-adic valuation of $\mathbb{Q}$ (i.e. $v_p(p) = 1$). Show that $v(\sigma_a(\zeta) - \zeta) = p^{m_a}$ where $m_a = v_p(a-1)$.

(c) Let k be an integer with $1 \le k \le r$, and take $n \in \mathbb{Z}$ with $p^{k-1} \le n < p^k$. With a as in part (b), show that $\sigma_a \in G_n$ if and only if $m_a \ge k$.

(d) Conclude that $G_n = \{\sigma_a \mid a \equiv 1 \bmod p^k\}$.

3. Let K be a local field, π a uniformizer of K, and $f, g \in \mathcal{F}_\pi$. You may assume that there is a unique power series $\mathbf{1}_{f,g}(x) \in \mathcal{O}_K[\![x]\!]$ congruent to x mod deg 2 such that $\mathbf{1}_{f,g} \circ [a]_g = [a]_f \circ \mathbf{1}_{f,g}$ for all $a \in \mathcal{O}_K$.

(a) Show that if $a \in \mathcal{O}_K$ and $\alpha \in \mathfrak{p}_{K_\mathrm{s}}$, then $a \cdot_g \alpha = 0$ implies $a \cdot_f \mathbf{1}_{f,g}(\alpha) = 0$.

(b) Deduce that for $a \in \mathcal{O}_K$, the sets $\{\alpha \in \mathfrak{p}_{K_\mathrm{s}} \mid a \cdot_f \alpha = 0\}$ and $\{\alpha \in \mathfrak{p}_{K_\mathrm{s}} \mid a \cdot_g \alpha = 0\}$ are in canonical bijection.

4. Let K be a local field, let π be a uniformizer of K, and let $f(x) = \pi x + x^q \in \mathcal{F}_\pi$, where $q = \#\mathfrak{k}_K$.

(a) Show that for any $\alpha \in \mathfrak{p}_{K_\mathrm{s}}$, $\pi \cdot_f \alpha = f(\alpha)$.

(b) Show that for any $\beta \in \mathfrak{p}_{K_\mathrm{s}}$, the equation $\pi \cdot_f \alpha = \beta$ has q *distinct* solutions α in K_s, and that they in fact all lie in $\mathfrak{p}_{K_\mathrm{s}}$.

(c) Let $n \ge 1$. Using the above in conjunction with question **3**, deduce that $\{\alpha \in \mathfrak{p}_{K_\mathrm{s}} \mid \pi^n \cdot_g \alpha = 0\}$ has cardinality q^n for any $g \in \mathcal{F}_\pi$.