Algebraic Number Theory
MATH 512

Assignment 5

Throughout, if K is a local field then $\mathfrak{k}_K$ denotes its residue field.

1. Let $\{a_n\}_n$ be a sequence of elements in a local field, and assume that for all n, $a_n \neq -1$. Show that the product $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if $a_n \to 0$. (Recall that an infinite product of non-zero elements is said to converge if the sequence of partial products has a *non-zero* limit.)

2. Let p be an odd prime, take $c \in \mathbb{F}_p^\times$, and assume c is not a square in $\mathbb{F}_p$. Show that if $a, b \in \mathbb{Z}$ both represent the class c, and if $\alpha^2 = a$ and $\beta^2 = b$, then $\mathbb{Q}_p(\alpha) = \mathbb{Q}_p(\beta)$.

3. Let L be the extension of $\mathbb{Q}_7$ obtained by adjoining the roots of the polynomial $x^3 - x + 1$.

(a) Show that $L/\mathbb{Q}_7$ is quadratic.

(b) Show that in fact $L = \mathbb{Q}_7(\sqrt{a})$ for any rational integer a that is not square mod 7.

4. Let K be a local field of residue characteristic p. The exercise below shows that U_K^1 can be viewed as a $\mathbb{Z}_p$-module.

(a) Show that for each $n \geq 1$, U_K^n/U_K^{n+1} is a finite p-group. *(Hint: Consider $U_K^n \to \mathfrak{k}_K$ given by $1 + a\pi^n \mapsto a \bmod \mathfrak{p}_K$, where π is a uniformizer.)*

(b) Deduce that U_K^1/U_K^n is a finite p-group for all $n \geq 1$, and is therefore a $\mathbb{Z}_p$-module.

(c) Show that the natural map $U_K^1 \to \varprojlim_n U_K^1/U_K^n$ is a group isomorphism. (Therefore U_K^1 inherits the $\mathbb{Z}_p$-structure from $\varprojlim_n U_K^1/U_K^n$.)

5. Let K be a local field of characteristic 0, and fix a positive integer m. We denote by $(U_K)^m$ the subgroup of mth powers in U_K. In this exercise, **do not** use the p-adic logarithm or Theorem 70. However, use of the equality $K^\times = \langle \pi \rangle \times U_K$ for a uniformizer π is permitted.

(a) For a positive integer n, find a sufficient condition on n, in terms of m, such that $U_K^n \subseteq (U_K)^m$.

(b) Using part (a), show that $U_K/(U_K)^m$ is finite. (See also question **4**.)

(c) Using Kummer theory, deduce that any characteristic 0 local field admits only finitely many abelian extensions of given exponent (in a fixed algebraic closure). *Be careful!*