# Algebraic Number Theory
# MATH 512

## Assignment 4

**1.** By reducing mod 9, prove the first case of Fermat with $p = 3$.

In the next exercise, you may assume that if $\zeta$ is a primitive cubic root of unity in $\bar{\mathbb{Q}}$, then the ring $\mathbb{Z}[\zeta]$ has uniqueness of factorization.

**2.** Prove Fermat with $p = 3$ as follows:

(a) Assume that there are non-zero integers $x$, $y$ and $z$ with $x^3 + y^3 = z^3$. Deduce the existence of pairwise coprime integers $x, y, z$ with $3 \nmid x, y, z$ and a positive integer $m$ such that

$$x^3 + y^3 = (3^m z)^3. \tag{1}$$

(The new $x, y, z$ may be different from the previous $x, y, z$. From now on, $x, y, z$ will refer to the integers appearing in (1).)

(b) Let $\alpha = x + y$, $\beta = x + \zeta y$ and $\gamma = x + \zeta^2 y$ where $\zeta$ is a primitive cubic root of unity in $\bar{\mathbb{Q}}$. Show that $\pi = 1 - \zeta$ divides $\beta$ and $\gamma$ in $\mathbb{Z}[\zeta]$. ($\pi$ also divides $\alpha$, but we will be more precise about this in (c).)

(c) As a consequence of (b), we may write $\beta = (1 - \zeta)\beta'$ and $\gamma = (1 - \zeta^2)\gamma'$ with $\beta', \gamma' \in \mathbb{Z}[\zeta]$. Show that $\pi$ divides neither $\beta'$ nor $\gamma'$, and conclude that $\alpha = 3^{3m-1}\alpha'$ with $\alpha' \in \mathbb{Z}[\zeta]$. Show further that $\alpha'$, $\beta'$ and $\gamma'$ are pairwise coprime.

(d) Because of (c) and uniqueness of factorization in $\mathbb{Z}[\zeta]$, we may write

$$\begin{aligned} \alpha' &= u_1\delta_1^3 \\ \beta' &= u_2\delta_2^3 \\ \gamma' &= u_3\delta_3^3 \end{aligned}$$

with $u_i \in \mathbb{Z}[\zeta]^\times$ and $\delta_i \in \mathbb{Z}[\zeta]$ for $i = 1, 2, 3$. Show that we may assume $u_1 = u_2 = u_3 = 1$. *(Hint: What are the units in $\mathbb{Z}[\zeta]$?)*

(e) Show that we may assume $\delta_1 \in \mathbb{Z}$.

(f) Write $\delta_2 = a + \zeta b$ with $a, b \in \mathbb{Z}$. Show that $a$ and $b$ are coprime, so that $a$, $b$ and $a - b$ are pairwise coprime.

(g) Show that $x + y = 9ab(a - b)$, and conclude that $ab(a - b) = 3^{3m-3}\delta_1^3$.

(h) Show that there are pairwise coprime integers $r, s, t$ such that some permutation of $(a, b, a - b)$ is equal to $(r^3, s^3, 3^{3m-3}t^3)$.

(i) Using (h), show that there are pairwise coprime integers $r_1$, $s_1$ and $t_1$ such that $r_1^3 + s_1^3 = (3^{m-1}t_1)^3$.

(j) There is a crucial step required in order to complete the induction. Explain what that step is, and prove that the step is justified.

**3.** Let $v = |\cdot|$ be a non-archimedean absolute value on a number field $K$. Let $\mathcal{O}_v = \{x \in K \mid |x| \leq 1\}$, which is a ring by the non-archimedean property of $|\cdot|$. Show that $\mathcal{O}_K \subseteq \mathcal{O}_v$. (Do not use Theorem 56, since the statement $\mathcal{O}_K \subseteq \mathcal{O}_v$ is assumed in the proof of that theorem.)