Your Name (Please, PRINT!)

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Instructor's Name

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- Turn off and put away **ALL ELECTRONIC DEVICES**.
- This exam is closed book. You may not use any notes.
- In order to receive credit, you must show your work on the exam paper, with some explanation in English, if appropriate. Do not do computations in your head. Instead, write them out on the exam paper. **IF** a problem asks you to use a specific method, you **MUST** use this method. You will get zero credit for any other solution, even if it is correct.
- If you need more room, use the **back of the previous page** and indicate to the grader that you have done so.
- This exam consists of this title page and 9 pages with 8 problems.

Problem	Max Points	Score
1	10	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	10	
Total	50	

G O O D L U C K !

1. Circle **T**rue or **F**alse for the following questions. (No detailed explanations are required for this problem.)

(a) For a problem in standard form: corner points of the feasible set correspond to feasible basic solutions. [T] F

(b) For a problem in standard form: if all decision variables are strictly positive in a basic solution, then this solution is not degenerate. T [F]

If the number of constraints is bigger than the number of decision variables, more variables must be non-zero in a basic solution to conclude its non-degeneracy.

(c) The origin is always a non-degenerate vertex of the feasible set of a problem in standard form. T [F]

Only if it is a feasible point.

(d) The Consequence Theorem can only be applied to consistent linear systems. [T] F

(e) If x_1 is a free variable in a problem, the Complementary Slackness Theorem can never imply that $x_1 = 0$. [T] F

(f) The y vector in the revised simplex method is always dual feasible. T [F]

It is guaranteed only for optimal dictionaries.

(g) The y vector of an optimal revised dictionary is always non-negative. [T] F

(h) For a minimization linear programming problem "less than or equal to" inequalities correspond to non-positive dual decision variables. [T] F

(i) If the objective is changed in a problem in standard form, a dual feasible dictionary for the modified problem can be obtained from any optimal dictionary of the original problem. T [F]

A primal feasible (but not necessarily dual feasible) dictionary can be obtained from any optimal dictionary of the original problem.

(j) If the initial dictionary of a problem in standard form is infeasible, then it is dual feasible. T [F]

Only if all coefficients of the objective are non-positive.

2. For the problem

$$\begin{array}{l} \max\ 2x_1 - 6x_3 \\ -3x_1 + 4x_2 \leq 3 \\ -x_1 - 5x_2 + 4x_3 \leq 1 \\ x_1 - 5x_2 - x_3 \leq 3 \\ x_1, x_2, x_3 \geq 0 \end{array}$$

obtain **without using the simplex method** the (regular) dictionary corresponding to $x_1 = 3$, $x_2 = 0$, $x_3 = 0$.

You may find this formula useful:

$$\begin{array}{|rcl|} \hline x_B & = & B^{-1}b - B^{-1}A_N x_N \\ \hline z & = & c_B^T B^{-1} b + \left(c_N^T - c_B^T B^{-1} A_N\right) x_N \\ \hline \end{array}$$

First, we need to determine basic variables. One of them is x_1. The values of slack variables are $x_4 = 12$, $x_5 = 4$, $x_6 = 0$. So x_4 and x_5 are the other two basic variables. Let $x_B = (x_4,\, x_5,\, x_1)$, i.e. put x_4 and x_5 into their "natural" places to make inversion of B simpler. Let also $x_N = (x_2,\, x_3,\, x_6)$. Then we have:

$$B = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_N = \begin{pmatrix} 4 & 0 & 0 \\ -5 & 4 & 0 \\ -5 & -1 & 1 \end{pmatrix},$$

$$B^{-1}b = (12,\, 4,\, 3), \qquad c_B = (0,\, 0,\, 2), \qquad y = (0,\, 0,\, 2),$$

$$c_N = (0,\, -6,\, 0), \qquad c_N^T - y^T A_N = (10,\, -4,\, -2).$$

We also need the matrix product which we do not compute usually:

$$B^{-1}A_N = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ -5 & 4 & 0 \\ -5 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -11 & -3 & 3 \\ -10 & 3 & 1 \\ -5 & -1 & 1 \end{pmatrix}.$$

Finally, we put everything together:

$$\begin{array}{|rcl|} \hline x_4 & = & 12 + 11x_2 + 3x_3 - 3x_6 \\ x_5 & = & 4 + 10x_2 - 3x_3 - x_6 \\ x_1 & = & 3 + 5x_2 + x_3 - x_6 \\ \hline z & = & 6 + 10x_2 - 4x_3 - 2x_6 \\ \hline \end{array}$$

3. Using Revised Simplex Method find *all optimal solutions* of the following problem:

$$\begin{aligned}&\max\ 5x_1 + 4x_2 + 5x_3\\ &\ \ 2x_1 + \ \ x_2 + 3x_3 \le 1\\ &\ \ 3x_1 - 4x_2 + 5x_3 \le 4\\ &x_1, x_2, x_3 \ge 0\end{aligned}$$

Since we do not have negative constant terms in inequalities, the first dictionary includes only slack variables and $x_B = (x_4,\, x_5)$. Then

x_B	B	B^{-1}	$B^{-1}b$	$B^{-1}a_?$	Ratio	Leaving
x_4	1 0	1 0	1			
x_5	0 1	0 1	4			

and we compute for $x_N = (x_1,\, x_2,\, x_3)$

$$c_B = (0,\, 0)\,, \qquad y = (0,\, 0)\,, \qquad c_N = (5,\, 4,\, 5)\,, \qquad c_N^T - y^T A_N = (5,\, 4,\, 5)\,.$$

Let x_1 enter, then x_4 must leave:

x_B	B	B^{-1}	$B^{-1}b$	$B^{-1}a_1$	Ratio	Leaving
x_4	1 0	1 0	1	2	$\frac{1}{2}$	x_4
x_5	0 1	0 1	4	3	$\frac{4}{3}$	

The next dictionary is

x_B	B	B^{-1}	$B^{-1}b$	$B^{-1}a_?$	Ratio	Leaving
x_1	2 0	$\frac{1}{2}$ 0	$\frac{1}{2}$			
x_5	3 1	$-\frac{3}{2}$ 1	$\frac{5}{2}$			

and we compute for $x_N = (x_2,\, x_3,\, x_4)$

$$c_B = (5,\, 0)\,, \qquad y = \left(\frac{5}{2},\, 0\right), \qquad c_N = (4,\, 5,\, 0)\,, \qquad c_N^T - y^T A_N = \left(\frac{3}{2},\, -\frac{5}{2},\, -\frac{5}{2}\right).$$

The only option for the entering variable is x_2 and the only option for leaving is x_1:

x_B	B	B^{-1}	$B^{-1}b$	$B^{-1}a_2$	Ratio	Leaving
x_1	2 0	$\frac{1}{2}$ 0	$\frac{1}{2}$	$\frac{1}{2}$	1	x_1
x_5	3 1	$-\frac{3}{2}$ 1	$\frac{5}{2}$	$-\frac{11}{2}$	None	

The next dictionary is

x_B	B	B^{-1}	$B^{-1}b$	$B^{-1}a_?$	Ratio	Leaving
x_2	1 0	1 0	1			
x_5	−4 1	4 1	8			

and we compute for $x_N = (x_1,\, x_3,\, x_4)$

$$c_B = (4,\, 0)\,, \qquad y = (4,\, 0)\,, \qquad c_N = (5,\, 5,\, 0)\,, \qquad c_N^T - y^T A_N = (-3,\, -7,\, -4)\,.$$

Since all objective coefficients are strictly negative, the only optimal solution is the basic solution of this dictionary: $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, which gives the optimal value 4.

4. While solving the following problem via Revised Simplex Method

$$\begin{aligned}
\max\ & 5x_1 - 4x_2 - 5x_3 + 5x_4 + x_5 - 4x_6 \\
& x_1 - x_2 - 2x_3 - 3x_4 + x_5 \le 0 \\
& 2x_1 - x_2 - 3x_3 + 2x_5 - x_6 \le 4 \\
& 2x_1 - x_2 + x_3 + 2x_4 - 2x_5 - 3x_6 \le 5 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \ge 0
\end{aligned}$$

you have encountered the dictionary

x_B	B			B^{-1}			$B^{-1}b$	$B^{-1}a_3$	Ratio	Leaving
x_1	1	-1	0	-1	1	0	4	-1	None	
x_2	2	-1	0	-2	1	0	4	1	4	
x_9	2	-1	1	0	-1	1	1	4	$\frac{1}{4}$	x_9

Using the "updating technique" find B and B^{-1} of the next dictionary.

We see that x_3 is entering and x_9 is leaving. This means that B_{new} is obtained from B_{old} by replacing the last column with $(-2, -3, 1)$. To find B^{-1}_{new}, we compute $E^{-1}B^{-1}_{\text{old}}$, where E is the identity matrix with the last column replaced by $B^{-1}_{\text{old}}a_3$:

$$E = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}.$$

The inverse of this matrix is easy to compute:

$$E^{-1} = \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

Now

$$E^{-1}B^{-1}_{\text{old}} = \begin{pmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{3}{4} & \frac{1}{4} \\ -2 & \frac{5}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

and putting everything together we get

x_B	B			B^{-1}			$B^{-1}b$	$B^{-1}a_?$	Ratio	Leaving
x_1	1	-1	-2	-1	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{17}{4}$			
x_2	2	-1	-3	-2	$\frac{5}{4}$	$-\frac{1}{4}$	$\frac{15}{4}$			
x_3	2	-1	1	0	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$			

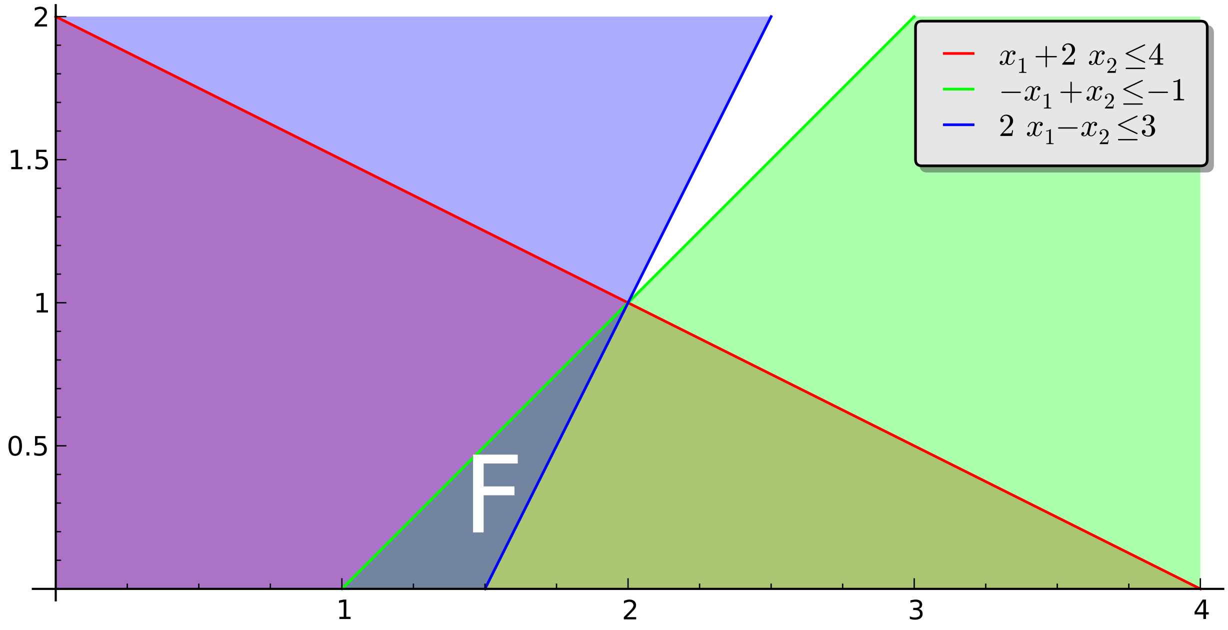
5. Find a non-degenerate vertex, a degenerate vertex, and a basic solution that does not correspond to a vertex of the feasible set of the following problem:

$$\begin{aligned} \max\ & 2x_1 - x_2 \\ & x_1 + 2x_2 \le 4 \\ & -x_1 + x_2 \le -1 \\ & 2x_1 - x_2 \le 3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

For each of your examples explain why it has the required property.

It may be helpful, although not necessary, to plot the feasible set.

Following the hint, let's plot the feasible set first:



It is now easy to see that $(1, 0)$ is a non-degenerate vertex of the feasible set, corresponding to $x_2 = x_4 = 0$, i.e. $x_B = (x_1, x_3, x_5) = (1, 3, 1)$. Since all basic coordinates are non-zero, it is indeed non-degenerate. Another possible answer is $(3/2, 0)$ with $x_2 = x_5 = 0$, i.e. $x_B = (x_1, x_3, x_4) = (3/2, 5/2, 1/2)$.

A degenerate vertex (the only one) is located at $(2, 1)$, corresponding to all slack variables equal to zero. It comes from, for example, $x_B = (x_1, x_2, x_3) = (2, 1, 0)$. Since one of the basic variables is zero, it is a degenerate solution.

Finally, an infeasible basic solution that does not correspond to a vertex is the origin $(0, 0)$ with $x_B = (x_3, x_4, x_5) = (4, -1, 3)$. Since some of the basic variables are negative, it is infeasible. Other possible examples: $(4, 0)$, $(0, 2)$, $(0, -1)$, and $(0, -3/2)$ (the last two are not visible on this plot - they are intersections of green and blue lines with the vertical axis).

6. Find *the optimal value* of the following problem **without converting it to standard form**:

$$\begin{aligned} \min\ & -11\,x_1 - 27\,x_2 + 16\,x_3 \\ & x_1 + 3\,x_2 - 5\,x_3 \leq 1 \\ & -2\,x_1 - 5\,x_2 + 3\,x_3 \geq -3 \\ & 2\,x_1 + 4\,x_2 + x_3 \leq -5 \end{aligned}$$

Since all variables in this problem are free, the dual one has constraints $A^T y = c$ with sign restrictions $y_1, y_3 \leq 0$ and $y_2 \geq 0$. We solve this system using row reduction:

$$\left(\begin{array}{rrr|r} 1 & -2 & 2 & -11 \\ 3 & -5 & 4 & -27 \\ -5 & 3 & 1 & 16 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & -2 & 2 & -11 \\ 0 & 1 & -2 & 6 \\ 0 & -7 & 11 & -39 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & -3 & 3 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

The solution is $y_1 = y_3 = -1$ and $y_2 = 4$, which does satisfy the sign restrictions. Since it is the only solution, it is the only dual feasible point, thus it is automatically the only dual optimal point. The optimal value (which is the same for primal and dual problems) is $1 \cdot (-1) - 3 \cdot 4 - 5 \cdot (-1) = -8$.

(We are not asked to find primal optimal solutions, but it we want to - by complementary slackness all constraints of the primal must be satisfied as equalities at optimal points, so we can also just apply the row reduction.)

7. Determine if $-x_1 + 2x_2 - 2x_3 \leq 3$ is a consequence of the linear system

$$\begin{aligned} x_1 - x_2 + x_3 &= 5 \\ -x_1 + x_2 - x_3 &\leq -1 \\ 2x_1 - x_2 + x_3 &\leq 3 \end{aligned}$$

First we determine if this system is consistent or not. We try to find y_1, y_2, and y_3 such that 1) $y_2, y_3 \geq 0$; 2) $A^T y = 0$; 3) $by < 0$. We start with the second condition by row reducing A^T:

$$\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that any point with $y_1 = y_2 \geq 0$ and $y_3 = 0$ satisfies 2) and 1). To satisfy 3) at the same time we need $5y_1 - y_1 = 4y_1 < 0$, which is incompatible with sign restrictions. Therefore, this linear system is consistent.

Now we apply the Consequence Theorem. Our goal is to find y_1, y_2, and y_3 such that 1) $y_2, y_3 \geq 0$; 2) $A^T y = a_0$; 3) $by \leq b_0$, where $a_0 = 3$ and $b_0 = (-1, 2, -2)$. We use row reduction to satisfy 2):

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ -1 & 1 & -1 & 2 \\ 1 & -1 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

If we let y_2 to be a parameter, the solution is $y_1 = -3 + y_2$ and $y_3 = 1$. We want $y_2 \geq 0$ and to satisfy 3) we need

$$5 \cdot (y2 - 3) - 1 \cdot y2 + 3 \cdot 1 = 4y_2 - 12 \leq 3,$$

which is clearly the case for $y_2 = 0$. Therefore, the given inequality is a consequence of this linear system.

8. Consider the problem

$$\begin{aligned} \max\ -3x_1 + 2x_2 - x_3 & \\ 3x_2 - x_3 &\le 2 \\ -4x_1 + x_2 &\le 3 \\ -4x_1 - 3x_2 - 4x_3 &\le 0 \\ x_1, x_2, x_3 \ge 0 & \end{aligned}$$

Its final revised dictionary and related quantities (for $x_N = (x_1,\ x_3,\ x_4)$) are

x_B	B	B^{-1}	$B^{-1}b$	$B^{-1}a_?$	Ratio	Leaving
x_2	$3\ \ 0\ \ 0$	$\frac{1}{3}\ \ 0\ \ 0$	$\frac{2}{3}$			
x_5	$1\ \ 1\ \ 0$	$-\frac{1}{3}\ \ 1\ \ 0$	$\frac{7}{3}$			
x_6	$-3\ \ 0\ \ 1$	$1\ \ 0\ \ 1$	2			

$$c_B = (2,\ 0,\ 0), \qquad y = \left(\frac{2}{3},\ 0,\ 0\right), \qquad c_N = (-3,\ -1,\ 0), \qquad c_N^T - y^T A_N = \left(-3,\ -\frac{1}{3},\ -\frac{2}{3}\right).$$

(a) Constant terms of the constraints are changed to $(3, 0, -2)$. **Using the above dictionary** obtain a "close to optimal" dictionary for the new problem, state what kind of the simplex method can be applied to this dictionary, and determine entering and leaving variables.

You do not have to compute the next dictionary.

When we change the constant terms of constraints, only $B^{-1}b$ is changing in the revised dictionary, so we get

x_B	B	B^{-1}	$B^{-1}b$	$B^{-1}a_?$	Ratio	Leaving
x_2	$3\ \ 0\ \ 0$	$\frac{1}{3}\ \ 0\ \ 0$	1			
x_5	$1\ \ 1\ \ 0$	$-\frac{1}{3}\ \ 1\ \ 0$	-1			
x_6	$-3\ \ 0\ \ 1$	$1\ \ 0\ \ 1$	1			

which is infeasible, but it is dual feasible, since the objective coefficients of this dictionary are the same as before. This means that we can apply the dual simplex method. The leaving variable is x_5 as the only variable with negative value. To compute ratios, we need to get its line, which is the second line of B^{-1} times A_N:

$$\left(-\frac{1}{3},\ 1,\ 0\right) \begin{pmatrix} 0 & -1 & 1 \\ -4 & 0 & 0 \\ -4 & -4 & 0 \end{pmatrix} = \left(-4,\ \frac{1}{3},\ -\frac{1}{3}\right).$$

We compute ratios for the first and last non-basic variables: 3/4 and 2, so the first one, x_1, must be the entering variable.

(b) Consider the problem (the same as on the previous page)

$$\begin{aligned} \max\ & -3\,x_1 + 2\,x_2 - x_3 \\ & \qquad\quad 3\,x_2 - \quad x_3 \le 2 \\ & -4\,x_1 + \quad x_2 \qquad\qquad \le 3 \\ & -4\,x_1 - 3\,x_2 - 4\,x_3 \le 0 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Its final revised dictionary and related quantities (for $x_N = (x_1,\, x_3,\, x_4)$) are

x_B	B			B^{-1}			$B^{-1}b$	$B^{-1}a_?$	Ratio	Leaving
x_2	3	0	0	$\frac{1}{3}$	0	0	$\frac{2}{3}$			
x_5	1	1	0	$-\frac{1}{3}$	1	0	$\frac{7}{3}$			
x_6	−3	0	1	1	0	1	2			

$$c_B = (2,\, 0,\, 0)\,, \qquad y = \left(\frac{2}{3},\, 0,\, 0\right), \qquad c_N = (-3,\, -1,\, 0)\,, \qquad c_N^T - y^T A_N = \left(-3,\, -\frac{1}{3},\, -\frac{2}{3}\right).$$

A new variable is added to the problem with constraint coefficients $(-3,\, 5,\, 1)$ and the objective coefficient 1. **Using the above dictionary** obtain a "close to optimal" dictionary for the new problem, state what kind of the simplex method can be applied to this dictionary, and determine entering and leaving variables.

You do not have to compute the next dictionary.

When a variable is added to the problem, it becomes a new non-basic variable. So "the table" of the revised dictionary remains the same, while c_N and objective coefficients get one more entry in the end.

While this is true, we will instead "insert" the new variable as x_4 shifting indices of all slack variables by one. New entries then will be inserted in the third place, since x_4 will be the third in x_B.

The new problem is

$$\begin{aligned} \max\ & -3\,x_1 + 2\,x_2 - x_3 + x_4 \\ & \qquad\quad 3\,x_2 - \quad x_3 - 3\,x_4 \le 2 \\ & -4\,x_1 + \quad x_2 \qquad\quad + 5\,x_4 \le 3 \\ & -4\,x_1 - 3\,x_2 - 4\,x_3 + \quad x_4 \le 0 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

Its final revised dictionary and related quantities (for $x_N = (x_1,\, x_3,\, x_4,\, x_5)$) are

x_B	B			B^{-1}			$B^{-1}b$	$B^{-1}a_?$	Ratio	Leaving
x_2	3	0	0	$\frac{1}{3}$	0	0	$\frac{2}{3}$			
x_6	1	1	0	$-\frac{1}{3}$	1	0	$\frac{7}{3}$			
x_7	−3	0	1	1	0	1	2			

$$c_B = (2,\, 0,\, 0)\,, \qquad y = \left(\frac{2}{3},\, 0,\, 0\right), \qquad c_N = (-3,\, -1,\, 1,\, 0)\,, \qquad c_N^T - y^T A_N = \left(-3,\, -\frac{1}{3},\, 3,\, -\frac{2}{3}\right).$$

This dictionary is not optimal, but it is still feasible, so we can apply the regular simplex method. The only option for the entering variables is x_4, and by considering ratios we see that x_6 has to leave:

x_B	B			B^{-1}			$B^{-1}b$	$B^{-1}a_4$	Ratio	Leaving
x_2	3	0	0	$\frac{1}{3}$	0	0	$\frac{2}{3}$	−1	None	
x_6	1	1	0	$-\frac{1}{3}$	1	0	$\frac{7}{3}$	6	$\frac{7}{18}$	x_6
x_7	−3	0	1	1	0	1	2	−2	None	