QUANTUM WALLED BRAUER-CLIFFORD SUPERALGEBRAS

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ABSTRACT. We introduce a new family of superalgebras, the quantum walled Brauer-Clifford superalgebras $BC_{r,s}(q)$. The superalgebra $BC_{r,s}(q)$ is a quantum deformation of the walled Brauer-Clifford superalgebra $BC_{r,s}$ and a super version of the quantum walled Brauer algebra. We prove that $BC_{r,s}(q)$ is the centralizer superalgebra of the action of $U_q(\mathfrak{gl}(n))$ on the mixed tensor space $V_q^{r,s} = V_q^{r} \otimes (V_q^*)^{\otimes s}$ where $n \geq r+s$, where $V_q = \mathbb{C}(q)^{(m|n)}$ is the natural representation of the quantum enveloping superalgebra $U_q(\mathfrak{gl}(n))$ and $V_q^*$ is its dual space. We also provide a diagrammatic realization of $BC_{r,s}(q)$ as the $(r,s)$-bead tangle algebra $BT_{r,s}(q)$. Finally, we define the notion of $q$-Schur superalgebras of type $Q$ and establish their basic properties.

INTRODUCTION

Schur-Weyl duality has been one of the most inspiring themes in representation theory, because it reveals many hidden connections between the representation theories of seemingly unrelated algebras. By the duality functor, one algebra appears as the centralizer of the other acting on a common representation space. Many interesting and important algebras have been constructed as centralizer algebras in this way.

For example, the group algebra $\mathbb{C} \Sigma_k$ of the symmetric group $\Sigma_k$ appears as the centralizer of the $\mathfrak{gl}(n)$-action on $V^{\otimes k}$, where $V = \mathbb{C}^n$ is the natural representation of the general linear Lie algebra $\mathfrak{gl}(n)$. Similarly, Hecke algebras, Brauer algebras, Birman-Murakami-Wenzl algebras and Hecke-Clifford superalgebras are the centralizer algebras of the action of corresponding Lie (super)algebras or quantum (super)algebras on the tensor powers of their natural representations.

There are further generalizations of Schur-Weyl duality on mixed tensor powers. Let $V^{r,*} = V^{r} \otimes (V^*)^{\otimes s}$ be the mixed tensor space of the natural representation $V$ of $\mathfrak{gl}(n)$ and its dual space $V^*$. The centralizer algebra of the $\mathfrak{gl}(n)$-action on $V^{r,*}$ is the walled Brauer algebra $B_{r,s}(n)$. The structure and properties of $B_{r,s}(n)$ were first investigated in [1, 12, 20]. By replacing $\mathfrak{gl}(n)$ by the quantum enveloping algebra $U_q(\mathfrak{gl}(n))$ and $V = \mathbb{C}^n$ by $V_q = \mathbb{C}(q)^n$, we obtain as the centralizer algebra the quantum walled Brauer algebra studied in [3, 4, 7, 11, 14]. Super versions of the above constructions have been investigated with the following substitutions: Replace $\mathfrak{gl}(n)$ by $\mathfrak{gl}(m|n)$, $\mathbb{C}^n$ by $\mathbb{C}^{(m|n)}$; $U_q(\mathfrak{gl}(n))$ by $U_q(\mathfrak{gl}(m|n))$; and $\mathbb{C}(q)^n$ by $\mathbb{C}(q)^{(m|n)}$ as in [15, 16].

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The Lie superalgebra \( q(n) \) is commonly referred to as the \textit{queer Lie superalgebra} because of its unique properties and the fact that it has no non-super counterpart. Its natural representation is the superspace \((\mathbb{Z}_2\text{-graded vector space}) \ V = \mathbb{C}^{n|n}\). The corresponding centralizer algebra \( \text{End}_{q(n)}(V^{\otimes r}) \) was studied by Sergeev in [19], and it is often referred to as the \textit{Sergeev algebra}. Using a modified version of a technique of Fadeev, Reshetikhin and Turaev, Olshanski introduced the \textit{quantum queer superalgebra} \( U_q(q(n)) \) and established an analogue of Schur-Weyl duality. That is, he showed that there is a surjective algebra homomorphism \( \rho^r_{n,q} : HC_r(q) \to \text{End}_{q(n)}(V^{\otimes r}) \), where \( HC_r(q) \) is the \textit{Hecke-Clifford superalgebra}, a quantum version of the Sergeev algebra. Moreover, \( \rho^r_{n,q} \) is an isomorphism when \( n \geq r \).

On the other hand, in [13] Jung and Kang considered a super version of the walled Brauer algebra. For the mixed tensor space \( V^{r,s} = V^{\otimes r} \otimes (V^*)^{\otimes s} \), one can ask, What is the centralizer of the \( q(n) \)-action on \( V^{r,s} \)? In order to answer this question, they introduced two versions of the \textit{walled Brauer-Clifford superalgebra}, (which is called the \textit{walled Brauer superalgebra} in [13]). The first is constructed using \((r,s)\)-superdiagrams, and the second is defined by generators and relations. The main results of [13] show that these two definitions are equivalent and that there is a surjective algebra homomorphism \( \rho^r_{n,q} : BC_{r,s}(q) \to \text{End}_{q(n)}(V^{r,s}) \), which is an isomorphism whenever \( n \geq r + s \).

The purpose of this paper is to combine the constructions in [18] and [13] to determine the centralizer algebra of the \( U_q(q(n)) \)-action on the mixed tensor space \( V^q_{r,s} := V^{q^r}_r \otimes (V^q_s)^{\otimes s} \). We begin by introducing the \textit{quantum walled Brauer-Clifford superalgebra} \( BC_{r,s}(q) \) via generators and relations. The superalgebra \( BC_{r,s}(q) \) contains the quantum walled Brauer algebra and the Hecke-Clifford superalgebra \( HC_r(q) \) as subalgebras.

We then define an action of \( BC_{r,s}(q) \) on the mixed tensor space \( V^q_{r,s} \) that supercommutes with the action of \( U_q(q(n)) \). As a result, there is a superalgebra homomorphism \( \rho^r_{n,q} : BC_{r,s}(q) \to \text{End}_{U_q(q(n))}(V^{r,s}) \). Actually, defining such an action is quite subtle and complicated, and we regard this as one of our main results (Theorem 3.7). We use the fact that \( BC_{r,s}(q) \) is the classical limit of \( BC_{r,s}(q) \) to show that the homomorphism \( \rho^r_{n,q} \) is surjective and that it is an isomorphism whenever \( n \geq r + s \) (Theorem 15).

We also give a diagrammatic realization of \( BC_{r,s}(q) \) as the \((r,s)\)-\textit{bead tangle algebra} \( BT_{r,s}(q) \). An \((r,s)\)-bead tangle is a portion of a planar knot diagram satisfying the conditions in Definition 4.1. The algebra \( BT_{r,s}(q) \) is a quantum deformation of the \((r,s)\)-\textit{bead diagram algebra} \( BD_{r,s} \), which is isomorphic to the walled Brauer-Clifford superalgebra \( BC_{r,s} \) (Theorem 2.5). Modifying the arguments in [10], we prove that the algebra \( BT_{r,s}(q) \) is isomorphic to \( BC_{r,s}(q) \) (Theorem 4.4), so that \( BC_{r,s}(q) \) can be regarded as a diagram algebra.

In the final section, we introduce \( q\text{-Schur superalgebras of type Q} \) and prove that the classical results for \( q\text{-Schur algebras} \) can be extended to this setting.

1. \textbf{The walled Brauer-Clifford superalgebras}

To begin, we recall the definition of the Lie superalgebra \( q(n) \) and its basic properties. Let \( I = \{ \pm i \mid i = 1, \ldots, n \} \), and set \( \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \). The superspace \( V = \mathbb{C}^{n|n} = \mathbb{C}^n \oplus \mathbb{C}^n \) has a standard basis \( \{ v_i \mid i \in I \} \). We say that the \textit{parity} of \( v_i \) equals \( |i| := |v_i| \), where \( |v_i| = 1 \) if \( i < 0 \) and \( |v_i| = 0 \) if \( i > 0 \).

The endomorphism algebra is \( \mathbb{Z}_2\text{-graded}, \text{End}_C(V) = \text{End}_C(V)_0 \oplus \text{End}_C(V)_1 \), and has a basis of matrix units \( E_{ij} \) with \( -n \leq i, j \leq n, \ ij \neq 0 \), where the parity of \( E_{ij} \) is \( |E_{ij}| = |i| + |j| \text{ (mod 2)} \). The general linear Lie superalgebra \( \mathfrak{gl}(n|n) \) is obtained from \( \text{End}_C(V) \) by using the \textit{supercommutator}

\[
[X, Y] = XY - (-1)^{|X||Y|}YX
\]
for homogeneous elements $X,Y$. The map $\iota : \mathfrak{gl}(n|n) \rightarrow \mathfrak{gl}(n|n)$ given by $E_{ij} \mapsto E_{-i,-j}$ is an involutive automorphism of $\mathfrak{gl}(n|n)$. Let $J = \sum_{a=1}^{n}(E_{a,-a} - E_{-a,a}) \in \mathfrak{gl}(n|n)$.

**Definition 1.1.** The *queer Lie superalgebra* $\mathfrak{q}(n)$ can be defined equivalently as either the centralizer of $J$ in $\mathfrak{gl}(n|n)$ (under the supercommutator product) or the fixed-point subalgebra of $\mathfrak{gl}(n|n)$ with respect to the involution $\iota$.

Identifying $V$ with the space of $(n|n)$ column vectors and $\{v_i \mid i \in I\}$ with the standard basis for the column vectors, we have $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, and $\mathfrak{q}(n)$ can be expressed in the matrix form as

$$\left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A,B \text{ are arbitrary } n \times n \text{ complex matrices} \right\}.$$  

Then $\mathfrak{q}(n)$ inherits a $\mathbb{Z}_2$-grading from $\mathfrak{gl}(n|n)$, and a basis for $\mathfrak{q}(n)_0$ is given by $E_{ab}^0 = E_{ab} + E_{-a,-b}$ and for $\mathfrak{q}(n)_1$ by $E_{ab}^1 = E_{a,-b} + E_{-a,b}$, where $1 \leq a, b \leq n$.

The superalgebra $\mathfrak{q}(n)$ acts naturally on $V$ by matrix multiplication on the column vectors, and $V$ is an irreducible representation of $\mathfrak{q}(n)$. The action on $V$ extends to one on $V^\otimes k$ by

$$g(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{j=1}^{k} (-1)^{|v_{i_j} + \cdots + |v_{i_j-1}|})g(v_{i_j} \otimes \cdots \otimes v_{i_{j-1}} \otimes gv_{i_{j}} \otimes v_{i_{j+1}} \otimes \cdots \otimes v_{i_k},$$

where $g$ is homogeneous. It also extends in a similar fashion to the mixed tensor space $V^{\otimes r,s} := V^{\otimes r} \otimes (V^*)^{\otimes s}$, where $V^*$ is the dual representation of $V$, and the action on $V^*$ is given by

$$(g\omega)(v) := (-1)^{|g|\omega} \omega(gv)$$

for homogeneous elements $g \in \mathfrak{q}(n), \omega \in V^*$, and $v \in V$. We assume $\{\omega_i \mid i \in I\}$ is the basis of $V^*$ dual to the standard basis $\{v_i \mid i \in I\}$ of $V$.

In an effort to construct the centralizer superalgebra $\text{End}_{\mathfrak{q}(n)}(V^{\otimes r,s})$, Jung and Kang [13] introduced the notion of the *walled Brauer-Clifford superalgebra* $\text{BC}_{r,s}$. The superalgebra $\text{BC}_{r,s}$ is generated by even generators $s_1, \ldots, s_{r-1}, s_{r+1}, \ldots, s_{r+s-1}, e_{r,r+1}$ and odd generators $c_1, \ldots, c_{r+s}$, which satisfy the following defining relations (for $i, j$ in the allowable range):

$$s_i^2 = 1, \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \quad s_is_j = s_js_i \quad (|i-j| > 1),$$

$$c_i^2 = 0, \quad e_{r+1,s} = s_re_{r+1} \quad (j \neq r-1, r+1),$$

$$c_i^2 = -1 \quad (1 \leq i \leq r), \quad c_i^2 = 1 \quad (r+1 \leq i \leq r+s), \quad c_ic_j = -cjc_i \quad (i \neq j),$$

$$s_is_{i+1}s_i = 0, \quad s_is_{i+1}s_is_{i+1} = 0, \quad e_{r,r+1} = e_{r+1,s} \quad (j \neq i, i+1),$$

$$c_ic_{i+1}s_i = c_{i+1}s_is_{i+1}, \quad e_{r,r+1}e_{r+1,r} = e_{r+1,r}e_{r+1,r+1},$$

$$e_{r,r+1}e_{r+1,r+1} = 0, \quad e_{r+1,r}e_{r,r+1} = 0 \quad (j \neq i, i+1).$$

For $1 \leq j \leq r-1$, $r+1 \leq k \leq r+s-1$, $1 \leq l \leq r$, and $r+1 \leq m \leq r+s$, the action of the generators of $\text{BC}_{r,s}$ on $V^{\otimes r,s}$ (which is on the right) is defined as follows:

$$(v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes \omega_{i_{r+1}} \otimes \cdots \otimes \omega_{i_{r+s}}) s_j = (-1)^{|i_j||i_{j+1}|}v_{i_1} \otimes \cdots \otimes v_{i_{j-1}} \otimes v_{i_{j+1}} \otimes v_{i_{j+2}} \otimes \cdots \otimes v_{i_{r+s}},$$

$$(v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes \omega_{i_{r+1}} \otimes \cdots \otimes \omega_{i_{r+s}}) s_k = (-1)^{|i_k||i_{k+1}|}v_{i_1} \otimes \cdots \otimes v_{i_{k-1}} \otimes v_{i_{k+1}} \otimes v_{i_{k+2}} \otimes \cdots \otimes v_{i_{r+s}},$$

$$(v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes \omega_{i_{r+1}} \otimes \cdots \otimes \omega_{i_{r+s}}) e_{r,r+1} = \delta_{i_r,i_{r+1}}(-1)^{|i_r|} \sum_{i=-n}^{n} v_{i_1} \otimes \cdots \otimes v_{i_{r-1}} \otimes v_{i_r} \otimes \omega_{i_{r+1}} \otimes \cdots \otimes \omega_{i_{r+s}}.$$
It was shown in [13, Sec. 3] that \(\dim \text{odd} (\text{resp. odd})\)

\[(v_1 \otimes \cdots \otimes v_r \otimes \omega_{r+1} \otimes \cdots \otimes \omega_{r+s}) \ c_l = (-1)^{|v_1|+\cdots+|v_{l-1}|} v_1 \otimes \cdots \otimes v_{l-1} \otimes J v_l \otimes v_{l+1} \otimes \cdots \otimes \omega_{r+s},\]

\[(v_1 \otimes \cdots \otimes v_i \otimes w_{r+1} \otimes \cdots \otimes w_{r+s}) \ c_m = (-1)^{|v_1|+\cdots+|v_{m-1}|} v_1 \otimes \cdots \otimes v_{m-1} \otimes J^T \omega_i \otimes \omega_{i+1} \otimes \cdots \otimes \omega_{r+s},\]

where \(J^T\) is the supertranspose of \(J\), and the supertranspose is given by \(E_{ij}^T := (-1)^{(|i|+|j|)|i|} E_{ji}\).

By direct calculation, it can be verified that this action of the generators gives rise to an action of \(BC_{r,s}\) on \(V^{r,s}\). Thus, there is an homomorphism of superalgebras,

\[\rho^{r,s}_n : BC_{r,s} \rightarrow \text{End}_C(V^{r,s})^{\text{op}}.\]

**Theorem 1.2.** [13]

(i) The actions of \(BC_{r,s}\) and \(q(n)\) on \(V^{r,s}\) supercommute with each other. Thus, there is an algebra homomorphism \(\rho^{r,s}_n : BC_{r,s} \rightarrow \text{End}_q(n)(V^{r,s})^{\text{op}}\).

(ii) The map \(\rho^{r,s}_n\) is surjective. Moreover, it is an isomorphism if \(n \geq r + s\).

**Definition 1.3.** [13] An \((r, s)\)-superdiagram is a graph with two rows of \(r + s\) vertices each such that the following conditions hold:

1. Each vertex is connected by an edge to exactly one other vertex.
2. There is a vertical wall which separates the first \(r\) vertices from the last \(s\) vertices in each row.
3. A vertical edge connects a vertex on the top row with one on the bottom row, and it cannot cross the wall. A horizontal edge connects vertices in the same row, and it must cross the wall.
4. Each edge may (or may not) be marked with an arrow, which is directed upward for a vertical edge and to the left for a horizontal edge, so that an unmarked vertical edge is denoted \(\bullet\) and an marked edge by \(\uparrow\).
5. No loops are permitted in an \((r, s)\)-superdiagram.

An \((r, s)\)-superdiagram which has an even (resp. odd) number of marked edges is regarded as even (resp. odd). Let \(\overrightarrow{B}_{r,s}\) be the superspace with a basis over \(C\) consisting of the \((r, s)\)-superdiagrams. It was shown in [13, Sec. 3] that \(\dim_C \overrightarrow{B}_{r,s} = (r + s)! 2^{r+s}\).

The superspace \(\overrightarrow{B}_{r,s}\) becomes an associative superalgebra under the multiplication

\[(1.3) \quad d_1 d_2 = (-1)^{A(d_1, d_2)} d_1 \ast d_2,\]

where the marked concatenation \(\ast\) and

\[(1.4) \quad A(d_1, d_2) = \ell(d_1, d_2) + \rho(d_1, d_2) + c(d_1, d_2) + p(d_1, d_2)\]

are defined in [13, Sec. 4].

We say \(\overrightarrow{B}_{r,s}\) is the \((r, s)\)-superdiagram algebra, or simply the superdiagram algebra. The identity element is the diagram with each vertex on the top row connected by an unmarked edge to the vertex directly below it on the bottom row.

**Theorem 1.4.** [13] The walled Brauer-Clifford superalgebra \(BC_{r,s}\) is isomorphic to the \((r, s)\)-superdiagram algebra \(\overrightarrow{B}_{r,s}\).

In the next section, we will give another diagrammatic realization of \(BC_{r,s}\). This requires some additional preparation. Let \(d = a_1 \cdots a_m\) be a monomial in \(\overrightarrow{B}_{r,s}\), where the \(a_i\) are \((r, s)\)-superdiagrams corresponding to generators of \(BC_{r,s}\). We define \(\overrightarrow{A}(a_1 \cdots a_m)\) inductively as follows:
Comparing (1), we obtain
\[ \tilde{A}(a_1 \cdots a_m) := \tilde{A}(a_1 \cdots a_{m-1}) + A(a_1 \cdots a_{m-1}, a_m) \pmod{2} \] for \( m \geq 2 \),
and \( \tilde{A}(a_i) = 0 \) for all \( i \), where the second summand is computed using (1.4). The next lemma will be used in the proof of Lemmas 2.3 and 2.4 below.

**Lemma 1.5.** Let \( d = a_1 \cdots a_m \in \tilde{B}_{r,s} \).

1. \( d = (-1)^\tilde{A}(a_1 \cdots a_m) a_1 \cdots a_m \).
2. \( \tilde{A}(a_1 \cdots a_m) = \tilde{A}(a_1 \cdots a_k) + \tilde{A}(a_{k+1} \cdots a_m) + A(a_1 \cdots a_k, a_{k+1} \cdots a_m) \) for \( 1 \leq k \leq n - 1 \).
3. If \( d = a_1 \cdots a_m = b_1 \cdots b_m' \), then \( \tilde{A}(a_1 \cdots a_m) = \tilde{A}(b_1 \cdots b_m') \). Therefore, the notation \( \tilde{A}(d) \) is unambiguous.

**Proof.** (1) The proof is by induction on \( m \), the case \( m = 1 \) being trivial. Now for \( m \geq 2 \),

\[
d = (a_1 \cdots a_{m-1})a_m = (-1)^{\tilde{A}(a_1 \cdots a_{m-1})}(a_1 \cdots a_{m-1})a_m \\
= (-1)^{\tilde{A}(a_1 \cdots a_{m-1})} + A(a_1 \cdots a_{m-1}, a_m) a_1 \cdots a_{m-1} \star a_m \\
= (-1)^{\tilde{A}(a_1 \cdots a_m)}a_1 \cdots a_{m-1} \star a_m.
\]

(2) Since \( \tilde{B}_{r,s} \) is associative, \( d = (a_1 \cdots a_k)(a_{k+1} \cdots a_m) \). Applying (1) gives the following string of equalities:

\[
d = (a_1 \cdots a_k)(a_{k+1} \cdots a_m) = (-1)^{\tilde{A}(a_1 \cdots a_k)} + \tilde{A}(a_{k+1} \cdots a_m) a_1 \cdots a_k(a_{k+1} \cdots a_m) \\
= (-1)^{\tilde{A}(a_1 \cdots a_k)} + \tilde{A}(a_{k+1} \cdots a_m) + A(a_1 \cdots a_k, a_{k+1} \cdots a_m) a_1 \cdots a_m.
\]

Comparing (1), we obtain \( \tilde{A}(a_1 \cdots a_m) = \tilde{A}(a_1 \cdots a_k) + \tilde{A}(a_{k+1} \cdots a_m) + A(a_1 \cdots a_k, a_{k+1} \cdots a_m) \pmod{2} \).

(3) If \( d = a_1 \cdots a_m = b_1 \cdots b_m' \), then

\[
d = (-1)^{\tilde{A}(a_1 \cdots a_m)}a_1 \cdots a_m = (-1)^{\tilde{A}(b_1 \cdots b_m')}b_1 \cdots b_m'
\]

by (1). The linearly independence of the \((r,s)\)-superdiagrams then implies \( \tilde{A}(a_1 \cdots a_m) = \tilde{A}(b_1 \cdots b_m') \pmod{2} \).

2. The \((r,s)\)-bead diagram algebra \( BD_{r,s} \)

In this section, we give a new diagrammatic realization of the walled Brauer-Clifford superalgebra \( BC_{r,s} \).

**Definition 2.1.** An \((r,s)\)-bead diagram is a graph consisting of two rows with \( r + s \) vertices in each row such that the following conditions hold:

1. Each vertex is connected by a strand to exactly one other vertex.
2. Each strand may (or may not) have finitely many numbered beads. The bead numbers on a diagram start with 1 and are distinct consecutive positive integers.
3. There is a vertical wall which separates the first \( r \) vertices from the last \( s \) vertices in each row.
4. A vertical strand connects a vertex on the top row with one on the bottom row, and it cannot cross the wall. A horizontal strand connects vertices in the same row, and it must cross the wall.
5. No loops are permitted in an \((r,s)\)-bead diagram.
The following diagram is an example of (3, 2)-bead diagram.

\[ d = \]

\[
\begin{array}{c}
\text{3} \\
\text{2} \\
\text{1} \\
\text{3} \\
\end{array}
\]

Beads can slide along a given strand, but they cannot jump to another strand nor can they interchange positions on a given strand. For example, the following diagrams are the same as (2, 1)-bead diagrams.

\[
\begin{array}{c}
\text{3} \\
\text{2} \\
\text{1} \\
\end{array}
= \begin{array}{c}
\text{3} \\
\text{1} \\
\text{2} \\
\end{array}
= \begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\end{array}
= \cdots ,
\end{array}
\]

while the following diagrams are considered to be different

\[
\begin{array}{c}
\text{2} \\
\text{1} \\
\end{array}
\neq \begin{array}{c}
\text{1} \\
\text{2} \\
\end{array},
\end{array}
\]

An \((r, s)\)-bead diagram having an even number of beads is regarded as even (resp. odd). Let \(\mathbb{BD}_{r,s}\) be the superspace with basis consisting of the \((r, s)\)-bead diagrams. We define a multiplication on \(\mathbb{BD}_{r,s}\). For \((r, s)\)-bead diagrams \(d_1, d_2\), we put \(d_1\) under \(d_2\) and identify the top vertices of \(d_1\) with the bottom vertices of \(d_2\). If there is a loop in the middle row, we say \(d_1 d_2 = 0\). If there is no loop in the middle row, we add the largest bead number in \(d_1\) to each bead number in \(d_2\), so that if \(m\) is the largest bead number in \(d_1\), then a bead numbered \(i\) in \(d_2\) is now numbered \(m + i\) in \(d_1 d_2\). Then we concatenate the diagrams. For example, if

\[
\begin{array}{c}
\text{4} \\
\text{2} \\
\text{1} \\
\end{array}
, \quad \text{and} \quad \begin{array}{c}
\text{2} \\
\text{1} \\
\end{array}
,
\end{array}
\]

then

\[
\begin{array}{c}
\text{5} \\
\text{4} \\
\text{3} \\
\text{5} \\
\end{array}
= \begin{array}{c}
\text{2} \\
\text{4} \\
\text{3} \\
\text{2} \\
\end{array}
.
\end{array}
\]

Observe that \(\mathbb{BD}_{r,s}\) is closed under this product. If the number of beads in \(d_1\) (resp. \(d_2\)) is \(m_1\) (resp. \(m_2\)) and \(d_1 d_2 \neq 0\), then the number of beads in \(d_1 d_2\) is \(m_1 + m_2\). Hence, the multiplication
respects the \( \mathbb{Z}_2 \)-grading. Let \( d_1, d_2, d_3 \in \widehat{BD}_{r,s} \). Note that the connections in \((d_1d_2)d_3\) and \(d_1(d_2d_3)\) are the same. The strands where the beads are placed and the bead numbers are also the same in \((d_1d_2)d_3\) and \(d_1(d_2d_3)\). Therefore, \((d_1d_2)d_3 = d_1(d_2d_3)\). The identity element of \( BD_{r,s} \) is the diagram with no beads such that each top vertex is connected to the corresponding bottom vertex.

For \( 1 \leq i \leq r - 1, \ r + 1 \leq j \leq r + s - 1, \ 1 \leq k \leq r, \ r + 1 \leq l \leq r + s \), let \( BD'_{r,s} \) be the subalgebra of \( \widehat{BD}_{r,s} \) generated by the following diagrams:

\[
\begin{align*}
    s_i &:= \quad \cdots \quad i \quad i + 1 \quad \cdots \quad , \quad s_j := \quad \cdots \quad j \quad j + 1 \quad \cdots \quad , \\
    e_{r,r+1} &:= \quad \cdots \quad r \quad r + 1 \quad r + s \quad \cdots \quad , \\
    c_k &:= \quad \cdots \quad k \quad \quad \cdots \quad , \quad c_l := \quad \cdots \quad l \quad \cdots \quad .
\end{align*}
\]

Suppose now that \( R \) is the ideal of \( BD'_{r,s} \) corresponding to the following homogeneous relations,

\[
(2.1) \quad c_k^2 = -1 \quad (1 \leq k \leq r), \quad c_l^2 = 1 \quad (r + 1 \leq l \leq r + s), \quad \text{and} \quad c_i c_j = -c_j c_i \quad (1 \leq i \neq j \leq r + s),
\]

and let \( BD_{r,s} \) be the quotient superalgebra \( BD'_{r,s}/R \). We say that \( BD_{r,s} \) is the \((r,s)\)-bead diagram algebra, or simply the bead diagram algebra. For simplicity, we identify cosets in \( BD_{r,s} \) with their diagram representatives. We identify the following diagrams in \( BD_{r,s} \):

\[
\begin{align*}
    c_k^2 &= \quad \cdots \quad k \quad \quad \cdots \quad , \\
    c_l^2 &= \quad \cdots \quad l \quad \quad \cdots \quad ,
\end{align*}
\]

and

\[
\begin{align*}
    c_i c_j &= \quad \cdots \quad i \quad \quad \cdots \quad j \quad \quad \cdots \quad = - \quad \cdots \quad i \quad \quad \cdots \quad j \quad \quad \cdots \quad = -c_j c_i .
\end{align*}
\]

We will prove that the superalgebra \( BC_{r,s} \) is isomorphic to \( BD_{r,s} \). By Theorem 1.4, it is enough to show that \( \overline{B}_{r,s} \) is isomorphic to \( BD_{r,s} \) and for this purpose, we define a map from \( BD_{r,s} \) to \( \overline{B}_{r,s} \). For an \((r,s)\)-bead diagram \( d \), we denote by \( \overline{d} \) the diagram obtained by keeping the same connections.
between vertices and replacing a strand in \( d \) having an even (resp. odd) number of beads with an unmarked edge (resp. marked edge). For example, if

\[
d = \begin{array}{ccc}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array}
\]

then \( \tilde{d} = \begin{array}{ccc}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array} \).

For an \((r,s)\)-bead diagram \( d \in \mathcal{BD}_{r,s} \), the diagram \( \tilde{d} \in \mathcal{B}_{r,s} \) is well-defined. Now we define \( \ell_1(d), \ell_2(d), p_1(d), p_2(d), c(d) \) for each \((r,s)\)-bead diagram \( d \) as follows. Here we will assume the vertices on the top (and on the bottom) are numbered \( 1, 2, \ldots, r + s \) from left to right.

1. **The top vertex of a vertical strand or the left vertex of a horizontal strand is said to be good.**
2. **Assume the bead numbers on the vertical strands and on the top row horizontal strands** are \( 1 \leq \vartheta_1 < \vartheta_2 < \cdots < \vartheta_p \). Let \( a_j \) be the good vertex of the vertical strand or of the top row horizontal strand with bead \( \vartheta_j \). Then we obtain a sequence \( a_1 \cdots a_p \). Let \( \ell_1(d) := \text{Inv}(a_1 \cdots a_p) = |\{(j,k) \mid j < k, a_j > a_k\}| \).
3. **Assume the bead numbers along the bottom row horizontal strands** are \( 1 \leq \eta_1 < \eta_2 < \cdots < \eta_q \). Let \( b_j \) be the good vertex of the bottom horizontal strand with the bead \( \eta_j \). Then we obtain a sequence \( b_1 \cdots b_q \). Let \( \ell_2(d) := \text{Inv}(b_1 \cdots b_q) = |\{(j,k) \mid j < k, b_j > b_k\}| \).
4. **Let** \( p_1(d) \) **be the number of pairs with the same entry in** \( \{1, \ldots, r\} \) **in the sequence** \( a_1 \cdots a_p \).
5. **Let** \( p_2(d) \) **be the number of pairs with the same entry in** \( \{1, \ldots, r\} \) **in the sequence** \( b_1 \cdots b_q \).
6. **For each bead** \( \vartheta_j \), its **passing number** counts the number of beads \( \vartheta_k \) with \( \vartheta_k < \vartheta_j \) when \( \vartheta_j \) moves to the good vertex of its strand. Let \( p_1(d) \) **be the sum of the passing numbers for all** \( \vartheta_1, \ldots, \vartheta_p \).
7. **For each bead** \( \eta_j \), its **passing number** counts the number of beads \( \eta_k \) such that \( \eta_k > \eta_j \) when \( \eta_j \) moves to the good vertex of its strand. Let \( p_2(d) \) **be the sum of passing numbers for all** \( \eta_1, \ldots, \eta_q \).
8. **Let** \( c(d) \) **be the sum of these numbers for all beads** \( \vartheta_1, \ldots, \vartheta_p \), \( \eta_1, \ldots, \eta_q \).
9. **We define** \( \beta(d) := \ell_1(d) + \ell_2(d) + p_1(d) + p_2(d) + c(d) \) **mod 2.**

**Example 2.2.** If \( d = \begin{array}{ccc}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array} \), then we get \( \ell_1(d) = \text{Inv}(2121) = 3 \) by beads \( 1, 3, 4 \) and \( 5 \). The other numbers are \( \ell_2(d) = \text{Inv}(1) = 0, p_1(d) = 2 \) and \( p_2(d) = 0 \). The beads \( 1, 2 \) yield \( c(d) = 1 \), and we get \( p_1(d) = p_2(d) = 0 \). In conclusion, \( \beta(d) = 0 \).

Now we have a well-defined map \( \phi_{r,s} : \mathcal{BD}_{r,s} \to \mathcal{B}_{r,s} \) given by

\[
d \mapsto (-1)^\beta(d) \tilde{d}.
\]

Note that the number \( k \) of strands having an odd number of beads determines the parity in \( \mathcal{BD}_{r,s} \), so that \( k \) is even if and only if \( d \) is even in \( \mathcal{BD}_{r,s} \). It is also true that the number \( k \) of marked edges determines the parity in \( \mathcal{B}_{r,s} \), and when \( d \) has \( k \) strands with an odd number of beads, then \( \tilde{d} \) has \( k \) marked edges. Hence, the map \( \phi_{r,s} \) preserves the parity and is an even linear map.
Now we consider the restriction $\phi'_{r,s} : \text{BD}'_{r,s} \rightarrow \overline{B}_{r,s}$ of $\phi_{r,s}$. We will show that $\phi'_{r,s}$ is a superalgebra homomorphism. Let $d = x_1 x_2 \cdots x_m$ be a monomial in $\text{BD}'_{r,s}$, where the $x_i$ are generators of $\text{BD}'_{r,s}$, and set $x'_i = \phi'_{r,s}(x_i) \in \overline{B}_{r,s}$ so that

$$s'_i = s_i, \quad s'_j = s_j, \quad e'_{r,r+1} = e_{r,r+1}, \quad c'_k = c_k, \quad c'_l = c_l,$$

and $d' = x'_1 \cdots x'_m \in \overline{B}_{r,s}$. Then using the marked concatenation $*$ in [13, Sec. 4], we have $\overline{d} = x'_1 \ast \cdots \ast x'_m$, since $x_i$ and $x'_i$ have the same connection structures, and a strand with an even (resp. odd) number of beads in $d$ becomes an unmarked (resp. marked) edge in $\overline{d}$, as well as an unmarked (resp. marked) edge in $x'_1 \ast \cdots \ast x'_m$. We denote $x'_1 \ast \cdots \ast x'_m$ by $(d'_{x_1 \cdots x_m})^\ast$.

**Lemma 2.3.**

1. The number $\beta(d)$ equals $\tilde{A}(d'_{x_1 \cdots x_m}) \pmod{2}$ for all $(r,s)$-bead diagrams $d = x_1 \cdots x_m \in \text{BD}'_{r,s}$.
2. When $d = x_1 \cdots x_m = y_1 \cdots y_n$ for generators $x_i, y_j$ of $\text{BD}'_{r,s}$, we obtain $d'_{x_1 \cdots x_m} = d'_{y_1 \cdots y_n}$.

(Hence, the notation $d'$, which we adopt for these expressions, is well defined.)

**Proof.** (1) We use induction on the number of beads in $d$. Since we will fix a presentation $d = x_1 \cdots x_m$ with generators $x_i$ in demonstrating (1), we omit the subscripts in $d'_{x_1 \cdots x_m}$.

If there is no bead in $d$, then $\beta(d) = \tilde{A}(d') = 0$. If there is only one bead in $d$, then $\beta(d) = 0$. Let $d = d_1 c_i d_2$, where there are no $c_k$'s in $d_1, d_2$. By Lemma 1.5 (2), we observe

$$\tilde{A}(d') = \tilde{A}(d'_1 c_i d'_2) = \tilde{A}(d'_1) + \tilde{A}(c_i d'_2) + \tilde{A}((d'_1)^\ast, c_i \ast (d'_2)^\ast).$$

Since there are also no $c_k$'s in $d'_1, d'_2$, we have

$$\tilde{A}(d'_1) = \tilde{A}(d'_2) = \tilde{A}(c_i) = \tilde{A}(c_i d'_2) = 0 \text{ and } A((d'_1)^\ast, c_i \ast (d'_2)^\ast) = A(c_i, (d'_2)^\ast) = 0,$$

and hence $\tilde{A}(d') = 0$.

Suppose that the number of beads in $d$ is greater than or equal to 2. Let $d = d_1 c_i d_2$, where there are no $c_k$'s in $d_2$. Here, we suppose that $c_i$ creates a bead with the largest bead number $m$ in $d$. Let $j$ be the good vertex of the strand with the bead $\overline{i}$. $j$.

First, we consider the case that the bead $\overline{i}$ is on a vertical strand or a top horizontal strand. We know that the numbers $\ell_2(d), p_2(d), p_2(d)$ depend only on the beads along the bottom row horizontal strands. We observe $\ell_1(d) = \ell_1(d_1 d_2 c_j)$ and $\rho_1(d) = \rho_1(d_1 d_2 c_j)$ and $c(d) = c(d_1 d_2 c_j)$. Hence, we obtain

$$\beta(d) = \beta(d_1 d_2 c_j) + p(m),$$

where $p(m)$ is the passing number caused by a bead $\overline{i}$.

Let us compare $\beta(d_1 d_2 c_j)$ with $\beta(d_1 d_2)$. When we calculate $\ell_1(d_1 d_2 c_j)$, we obtain the sequence $a_1 \cdots a_p$. We know

$$\ell_1(d_1 d_2 c_j) = \text{Inv}(a_1 \cdots a_p) = \text{Inv}(a_1 \cdots a_p) + \nu = \ell_1(d_1 d_2) + \nu,$$

where $\nu$ is the number of $a_i$'s such that $a_i > j$. If the strand with $\overline{i}$ in $d_1 d_2 c_j$ is a vertical strand to the left of the wall or a horizontal strand such that there are odd number of beads along the strand containing $\overline{i}$, we obtain $\rho_1(d_1 d_2 c_j) = \rho_1(d_1 d_2)$. If the strand with $\overline{i}$ in $d_1 d_2 c_j$ is a vertical strand to the right of the wall, then $\rho_1(d_1 d_2 c_j) = \rho_1(d_1 d_2)$. If the strand with $\overline{i}$ in $d_1 d_2 c_j$ is a vertical strand on the left side of the wall or a horizontal strand such that there are even number of beads on the strand containing $\overline{i}$, then we obtain

$$\rho_1(d_1 d_2 c_j) = \rho_1(d_1 d_2) + 1.$$
The other numbers $p_1, c, \ell_2, \rho_2, p_2$ are the same in $\beta(d_1 d_2 c_j)$ and $\beta(d_1 d_2)$. In conclusion, we have
\begin{equation}
(2.2) \quad \beta(d) = \beta(d_1 d_2 c_j) + p(m) = \beta(d_1 d_2) + \nu + (0 \text{ or } 1) + p(m).
\end{equation}

Now we consider $\widetilde{A}(d')$. By Lemma 1.5 (2), we get
\begin{equation}
\widetilde{A}(d') = \widetilde{A}(d'_1 c_i d'_2) = \widetilde{A}(d'_1) + \widetilde{A}(c_i d'_2) + A((d'_1)^*, c_i * (d'_2)^*).
\end{equation}

Since there are no $c_i$'s in $d'_2$, we have $\widetilde{A}(c_i d'_2) = 0$. Let us compare $A((d'_1)^*, c_i * (d'_2)^*)$ with $A((d'_1)^*, (d'_2)^*)$. When we calculate $\ell_1((d'_1)^*, c_i * (d'_2)^*)$, we get a subsequence of $a_1 \cdots a_{p_j}$ obtained by deleting some pairs of the same numbers that results from the marked concatenation in $(d'_1)^*$. Therefore, we get
\begin{equation}
\ell_1((d'_1)^*, c_i * (d'_2)^*) = \ell_1((d'_1)^*, (d'_2)^*) + \nu \pmod{2}.
\end{equation}

If an edge containing $c_i$ in $(d'_1)^* \ast c_i \ast (d'_2)^*$ is a vertical edge in the left side of the wall or a horizontal edge such that there are odd number of marks (containing the mark caused by $c_i$), then we obtain
\begin{equation}
\rho_1((d'_1)^*, c_i * (d'_2)^*) = \rho_1((d'_1)^*, (d'_2)^*).
\end{equation}

If an edge containing $c_i$ in $(d'_1)^* \ast c_i \ast (d'_2)^*$ is a vertical edge to the right of the wall, then $\rho_1((d'_1)^*, c_i * (d'_2)^*) = \rho_1((d'_1)^*, (d'_2)^*)$. If an edge containing $c_i$ in $(d'_1)^* \ast c_i \ast (d'_2)^*$ is a vertical edge to the left of the wall or a horizontal edge such that there are even number of marks (containing a mark caused by $c_i$), then we obtain
\begin{equation}
\rho_1((d'_1)^*, c_i * (d'_2)^*) = \rho_1((d'_1)^*, (d'_2)^*) + 1.
\end{equation}

Comparing $\rho_1((d'_1)^*, c_i * (d'_2)^*)$ with $\rho_1((d'_1)^*, (d'_2)^*)$, we get
\begin{equation}
\rho_1((d'_1)^*, c_i * (d'_2)^*) = \rho_1((d'_1)^*, (d'_2)^*) + p(m) \pmod{2}.
\end{equation}

The other numbers $c, \ell_2, \rho_2, p_2$ are the same in $A((d'_1)^*, c_i * (d'_2)^*)$ and $A((d'_1)^*, (d'_2)^*)$.

In conclusion, we have
\begin{equation}
(2.3) \quad \widetilde{A}(d') = \widetilde{A}(d'_1 c_i d'_2) = \widetilde{A}(d'_1) + \widetilde{A}(c_i d'_2) + A((d'_1)^*, c_i * (d'_2)^*)
\end{equation}
\begin{equation}
= \widetilde{A}(d'_1) + \widetilde{A}(d'_2) + A((d'_1)^*, (d'_2)^*) + \nu + (0 \text{ or } 1) + p(m)
\end{equation}
\begin{equation}
= \widetilde{A}(d'_1 d'_2) + \nu + (0 \text{ or } 1) + p(m) \pmod{2}.
\end{equation}

One can check that if $\rho_1(d_1 d_2 c_j) = \rho_1(d_1 d_2) + a$ ($a \in \{0, 1\}$), then $\rho_1((d'_1)^*, c_i * (d'_2)^*) = \rho_1((d'_1)^*, (d'_2)^*) + a$. By the induction hypothesis, we obtain $\beta(d) = \widetilde{A}(d') \pmod{2}$ from (2.2) and (2.3).

In a similar manner, one can show that $\beta(d) = \widetilde{A}(d')$ when the bead $\emptyset$ lies on a bottom row horizontal strand.

(2) By part (1), we obtain $\beta(d) = \widetilde{A}(d'_{x_1 \cdots x_m}) = \widetilde{A}(d'_{y_1 \cdots y_n})$. We observe that
\begin{equation}
x'_1 \cdots x'_m = (-1)^{\widetilde{A}(d'_{x_1 \cdots x_m})} x'_1 \cdots x'_m = (-1)^{\widetilde{A}(d'_{x_1 \cdots x_m})} \tilde{d}
\end{equation}
and similarly $y'_1 \cdots y'_n = (-1)^{\widetilde{A}(d'_{y_1 \cdots y_n})} \tilde{d}$. Hence $d'_{x_1 \cdots x_m} = x'_1 \cdots x'_m = y'_1 \cdots y'_n = d'_{y_1 \cdots y_n}$. \hfill \Box

Lemma 2.4. The map $\phi'_{r,s}$ is a superalgebra homomorphism.
Proof. Since $\phi_{r,s}$ is an even linear map, so is $\phi'_{r,s}$. Let $d_1, d_2$ be $(r, s)$-bead diagrams. By Lemma 2.3 (1),
\[
\beta(d_1) = \tilde{A}(d'_1), \quad \beta(d_2) = \tilde{A}(d'_2), \quad \beta(d_1 d_2) = \tilde{A}((d_1 d_2)'') = \tilde{A}(d'_1 d'_2).
\]
Note that $\tilde{d}_1 * \tilde{d}_2 = \tilde{d}_1 \tilde{d}_2$. Using Lemma 1.5, we obtain
\[
\phi'_{r,s}(d_1) \phi'_{r,s}(d_2) = (-1)^{\beta(d_1) + \beta(d_2)} \tilde{d}_1 \tilde{d}_2 = (-1)^{\beta(d_1) + \beta(d_2)} \tilde{A}(d'_1 d'_2) \tilde{d}_1 * \tilde{d}_2
\]
\[
= (-1)^{\tilde{A}((d'_1, (d'_2)) \tilde{d}_1 \tilde{d}_2}
\]
\[
= (-1)^{\tilde{A}(d'_1) + \tilde{A}(d'_2) + A((d'_1), (d'_2))} \tilde{d}_1 \tilde{d}_2
\]
\[
= (-1)^{\beta(d_1 d_2)} \tilde{d}_1 \tilde{d}_2 = (-1)^{\beta(d_1)} \tilde{d}_1 \tilde{d}_2
\]
\[
= \phi'_{r,s}(d_1 d_2).
\]

We now prove the main theorem of this section.

**Theorem 2.5.** The walled Brauer-Clifford superalgebra $BC_{r,s}$ is isomorphic to the $(r, s)$-bead diagram algebra $BD_{r,s}$ as associative superalgebras.

Proof. Note that $\phi'_{r,s}(s_i) = s_i, \phi'_{r,s}(s_j) = s_j, \phi'_{r,s}(e_{r,r+1}) = e_{r,r+1}, \phi'_{r,s}(c_k) = c_k$, and $\phi'_{r,s}(e_l) = e_l$. It is easy to verify directly that
\[
\phi'_{r,s}(c_k^2) = -\phi'_{r,s}(1), \quad \phi'_{r,s}(c_k^2) = \phi'_{r,s}(1) \quad \text{and} \quad \phi'_{r,s}(e_l) = e_l.
\]
So there is a well-defined surjective homomorphism $\overline{\psi}_{r,s} : BD_{r,s} \rightarrow \overline{BC}_{r,s}$. Since we know a presentation for $\overline{BC}_{r,s}$ in Theorem 1.4, we can define an algebra homomorphism $\psi_{r,s} : \overline{BC}_{r,s} \rightarrow BD_{r,s}$ specified by
\[
\psi_{r,s}(s_i) = s_i, \quad \psi_{r,s}(s_j) = s_j, \quad \psi_{r,s}(e_{r,r+1}) = e_{r,r+1}, \quad \psi_{r,s}(c_k) = c_k, \quad \text{and} \quad \psi_{r,s}(e_l) = e_l.
\]
The corresponding defining relations of $\overline{BC}_{r,s}$ are all satisfied in $BD_{r,s}$. Since $\psi_{r,s} \overline{\psi}_{r,s} = \text{id}_{BD_{r,s}}$ and $\overline{\psi}_{r,s} \psi_{r,s} = \text{id}_{BC_{r,s}}$, the map $\overline{\psi}_{r,s}$ is an isomorphism, which proves our claim.

\[ \square \]

3. The Quantum Walled Brauer-Clifford Superalgebra $BC_{r,s}(q)$

Let $q$ be an indeterminate and $\mathbb{C}(q)$ be the field of rational functions in $q$. Set $V_q = \mathbb{C}(q) \otimes \mathbb{C} V = \mathbb{C}(q) \otimes \mathbb{C}$. $C(n|n)$. Corresponding to any $X = \sum_k Y_k \otimes Z_k \in (\text{End}_{\mathbb{C}(q)}(V_q))^\otimes 2$, let $X^{12} = \sum_k Y_k \otimes Z_k \otimes \text{id}$, $X^{13} = \sum_k Y_k \otimes \text{id} \otimes Z_k$, and $X^{23} = \sum_k \text{id} \otimes Y_k \otimes Z_k$ in $(\text{End}_{\mathbb{C}(q)}(V_q))^\otimes 3$, where $\text{id} = \text{id}_{V_q}$.

Let $\xi = q - q^{-1}$ and define $S = \sum_{i,j \in I} S_{ij} \otimes E_{ij} \in (\text{End}_{\mathbb{C}(q)}(V_q))^\otimes 2$ by
\[
S = \sum_{i,j \in I} q^{|d_{ij} + d_{i,-j} - (1-2j)|} E_{ii} \otimes E_{jj} + \xi \sum_{i,j \in I, i < j} (-1)^{|i|}(E_{ji} + E_{j,-i-1}) \otimes E_{ij}.
\]

S is known to satisfy the quantum Yang-Baxter equation $S^{12} S^{13} S^{23} = S^{23} S^{13} S^{12}$. In [18], Olshanski constructed a quantization of $\mathfrak{f}(q(n))$ of $q(n)$ in terms of $S$.

**Definition 3.1.** [18] The quantum queer superalgebra $\mathfrak{U}_q(q(n))$ is the unital associative superalgebra over $\mathbb{C}(q)$ generated by elements $u_{ij}$ with $i \leq j$ and $i, j \in I = \{\pm i \mid i = 1, \ldots, n\}$, which satisfy the following relations:
\[
u_{ii} u_{i,-i,-i} = 1 = u_{i,-i,-i} u_{ii}, \quad U^{12} U^{13} S^{23} = S^{23} U^{13} U^{12},
\]
where $U = \sum_{i,j \in I, i \leq j} u_{ij} \otimes E_{ij}$, and the last equality holds in $\mathfrak{U}_q(q(n)) \otimes \mathbb{C}(q) (\text{End}_{\mathbb{C}(q)}(V_q))^\otimes 2$. The $\mathbb{Z}_2$-degree of $u_{ij}$ is $|i| + |j|$. 
By the construction, the assignment $u_{ijk} \mapsto S_{ij}$ is a representation of $\mathcal{U}_q(q(n))$ on $V_q$ (see [18, Sec. 4]). The superalgebra $\mathcal{U}_q(q(n))$ is a Hopf superalgebra with coproduct $\Delta(U) = U^{13}U^{23} \in (\mathcal{U}_q(q(n)))^{\otimes 2} \otimes \mathbb{C}(q) \text{End}_{\mathbb{C}(q)}(V_q)$, or more explicitly, $\Delta(u_{ijk}) = \sum_{k \in \mathbb{Z}} (-1)^{(i+k)(|l|+|j|)} u_{ik} \otimes u_{kj}$. The counit is given by $\varepsilon(U) = 1$ and the antipode by $U \mapsto U^{-1}$.

Let $V^r_s \otimes \mathbb{C}(q)(V^r_s) \otimes \mathbb{C}(q)$ be the mixed tensor space of $V_q$ and $V^r_s$. Then $V^r_s \otimes \mathbb{C}(q)(V^r_s)$ is a representation of $\mathcal{U}_q(q(n))$ via the coproduct and antipode mappings. To describe the structure of the centralizer superalgebra $\text{End}_{\mathcal{U}_q(q(n))}(V^r_s)$, we introduce the quantum walled Brauer-Clifford superalgebra $\mathbb{BC}_{r,s}(q)$.

**Definition 3.2.** The quantum walled Brauer-Clifford superalgebra $\mathbb{BC}_{r,s}(q)$ is the associative superalgebra over $\mathbb{C}(q)$ generated by even elements $t_1, t_2, \ldots, t_{r-1}, t_1^*, t_2^*, \ldots, t_{s-1}^*$ and odd elements $c_1, c_2, \ldots, c_r, c_1^*, c_2^*, \ldots, c_s^*$ satisfying the following defining relations (for $i, j$ in the allowable range):

$$
t_i^2 - (q + q^{-1}) t_i - 1 = 0, \ t_it_{i+1}t_i = t_{i+1}t_it_{i+1}, \quad (t_i^*)^2 - (q + q^{-1}) t_i^* - 1 = 0, \ t_i^*t_{i+1}^*t_i^* = t_{i+1}^*t_i^*t_{i+1},
$$

$$
t_i, t_j = t_jt_i, \quad t_i^2 = t_i, \quad t_i^* t_i^* = t_i^* t_i^* \quad (|i-j| > 1),
$$

$$
e^2 = 0, \ et_{r-1}e = e, \ et_j = t_j e \quad (j \neq r-1), \quad et_i e = e, \ et_j = t_j e \quad (j \neq 1),
$$

$$
et_{r-1}^{-1}t_i^* et_{r-1}^{-1} = t_{r-1}^{-1}t_i^* et_{r-1}^{-1} e,
$$

$$
c_i^2 = -1, \ c_i c_j = -c_j c_i \quad (i \neq j), \quad c_i c_i^* = -c_i^* c_i, \quad (c_i^*)^2 = 1, \quad c_i^* c_j^* = -c_j^* c_i^* \quad (i \neq j),
$$

$$
t_i c_i = c_{i+1} t_i, \quad t_i c_j = c_j t_i \quad (j \neq i, i+1), \quad t_i c_i^* = c_{i+1} t_i^*, \quad t_i c_j^* = c_j t_i^* \quad (j \neq i, i+1),
$$

$$
t_i c_j^* = c_j^* t_i, \quad e e = c_i e, \quad c_j e = ec_j \quad (j \neq r), \quad t_i c_j = c_i t_j^*, \quad ec_r = ec_i^*, \quad c_j e = ec_j^* \quad (j \neq 1),
$$

$$
e c_e e = 0.
$$

**Definition 3.3.** (i) The (finite) Hecke-Clifford superalgebra $\mathbb{HC}_r(q)$ in [18] is the associative superalgebra over $\mathbb{C}(q)$ generated by the even elements $t_1, t_2, \ldots, t_{r-1}$ and the odd elements $c_1, c_2, \ldots, c_r$ with the following defining relations (for allowable $i, j$):

$$
t_i^2 - (q + q^{-1}) t_i - 1 = 0, \ t_it_{i+1}t_i = t_{i+1}t_it_{i+1}, \quad t_it_j = t_jt_i \quad (|i-j| > 1),
$$

$$
c_i^2 = -1, \ c_i c_j = -c_j c_i \quad (i \neq j),
$$

$$
t_i c_i = c_{i+1} t_i, \quad t_i c_j = c_j t_i \quad (j \neq i, i+1),
$$

(ii) The quantum walled Brauer algebra $\mathbb{H}_{r,s}(q)$ in [11] is the associative algebra over $\mathbb{C}(q)$ generated by the elements $t_1, t_2, \ldots, t_{r-1}, t_1^*, t_2^*, \ldots, t_{s-1}^*$ and $e$ which satisfy the first four lines in (3.3).

**Remark 3.4.** The relations in the first three lines in (3.3) appear in Definition 2.1 of [11]. In line 4 of (3.3), we have the one relation

$$
et_{r-1}^{-1}t_i^* et_{r-1}^{-1} = t_{r-1}^{-1}t_i^* et_{r-1}^{-1} e.
$$

instead of the following two relations of [11]:

$$
et_{r-1}^{-1}t_i^* et_{r-1} = et_{r-1}^{-1}t_i^* et_{r-1}, \quad t_{r-1} et_{r-1}^{-1} t_i^* = et_{r-1}^{-1} t_i^* et_{r-1}.
$$

The relations in (3.6) can be derived using (3.5) and various identities of (3.3) (especially the fact that $t_{r-1}$ and $t_i^*$ commute) in the following way:

$$
et_{r-1}^{-1}t_i^* e = (et_{r-1}^{-1} e) t_{r-1}^{-1} t_i^* e = e(t_i^*)^{-1} t_i^* t_{r-1}^{-1} t_i^* e
$$

$$
= e(t_i^*)^{-1} (t_i^* t_{r-1}^{-1} et_{r-1}^{-1} t_i^* e) = e(t_i^*)^{-1} (et_{r-1}^{-1} t_i^* t_{r-1}^{-1} t_i^*)
$$

$$
= (e(t_i^*)^{-1}) e(t_i^*)^{-1} et_{r-1}^{-1} t_i^* e = et_{r-1}^{-1} t_i^* et_{r-1}^{-1} t_i^* = et_{r-1}^{-1} t_i^* et_{r-1}^{-1} t_i^* = t_{r-1}^{-1} t_i^* et_{r-1}^{-1} t_i^* e,
$$

implying both relations in (3.6).
The following simple expression will be useful in several calculations henceforth.

**Lemma 3.5.** With the conventions \((-1)^{[0]} = 0, \ [j] = 1 \) for any integer \( j < 0, \) and \([j] = 0 \) for \( j > 0 \), we have

\[
(3.7) \quad \xi \sum_{i<j<k} (-1)^{|i|} q^{2j(1-2|i|)} = q^{(2k-1)(1-2|k|)} - q^{(2i+1)(1-2|i|)}
\]

for any nonzero integers \( i < k \), where \( \xi = q - q^{-1} \) as above.

**Proof.** This can be checked by considering the three cases \( 0 < i < k, \ i < k < 0 \) and \( i < 0 < k \). \( \square \)

In order to construct an action of \( \mathfrak{BC}_{r,s}(q) \) on the mixed tensor space \( V_q^{r,s} \), we will need a number of \( \mathfrak{U}_q(q(n)) \)-module homomorphisms. Note that \( \mathbb{C}(q) \) becomes a \( \mathfrak{U}_q(q(n)) \)-module by sending \( U \) to the identity map in \( \text{End}_{\mathbb{C}(q)}(\mathbb{C}(q) \otimes \mathbb{C}(q) V_q) \).

**Lemma 3.6.** There are \( \mathfrak{U}_q(q(n)) \)-module homomorphisms \( \cap : \mathbb{C}(q) \to V_q \otimes V_q^* \) and \( \cup : V_q \otimes V_q^* \to \mathbb{C}(q) \) given by

\[
\cap(1) = \sum_{i \in I} v_i \otimes \omega_i, \quad \cup(v_i \otimes \omega_j) = (-1)^{|i|} q^{2(i-2|i|)-(2n+1)} \delta_{ij}.
\]

**Proof.** Since \( \cap \) is the canonical map \( \mathbb{C}(q) \to V_q \otimes V_q^* \), we have

\[
(3.8) \quad (X \otimes \text{id}) \cap = (\text{id} \otimes X^T) \cap
\]

for any \( X \in \text{End}_{\mathbb{C}(q)}(V_q) \), where \(^T\) denotes the supertranspose. Since \( \cap \) is even, it follows that

\[
((S^{23})^{-1})^{T_2} (\cap \otimes \text{id}) = (S^{13})^{-1} (\cap \otimes \text{id})
\]

in \( \text{Hom}_{\mathbb{C}(q)}(\mathbb{C}(q) \otimes V_q, V_q \otimes V_q^* \otimes V_q) \), where \( T_2 \) indicates taking the supertranspose on the second factor. Thus

\[
S^{13} ((S^{23})^{-1})^{T_2} (\cap \otimes \text{id}) = (\cap \otimes \text{id}).
\]

The action of \( \mathfrak{U}_q(q(n)) \) on \( V_q \otimes V_q^* \) (resp. on \( \mathbb{C}(q) \)) is defined by sending \( U \) to \( S^{13} ((S^{23})^{-1})^{T_2} \) (resp. to \( \text{id} \)), so this shows that \( \cap \) is a \( \mathfrak{U}_q(q(n)) \)-module homomorphism.

To check that \( \cup \) is a homomorphism, we require an explicit expression for \( S^{13} ((S^{23})^{-1})^{T_2} \). We have

\[
S^{-1} = \sum_{i,j \in I} q^{-(\delta_{ij} + \delta_{i,-j})(1-2|j|)} E_{ii} \otimes E_{jj} - \xi \sum_{i,j \in I, i < j} (-1)^{|i|} (E_{ji} + E_{-j,-i}) \otimes E_{ij}.
\]

If we identify \( V_q \) with \( V_q^* \) via \( v_i \mapsto \omega_i \), then \( (S^{-1})^{T_2} \) becomes identified with an endomorphism \( S^* \) of \( V_q \otimes V_q \) given by

\[
(3.9) \quad S^* = \sum_{i,j \in I} q^{-(\delta_{ij} + \delta_{i,-j})(1-2|j|)} E_{ii} \otimes E_{jj} - \xi \sum_{i,j \in I, i < j} (-1)^{|i||j|} (E_{ij} + E_{-i,-j}) \otimes E_{ij}.
\]
Therefore, identifying $V_q \otimes V_q^*$ with $V_q \otimes V_q$, we have that the action on $V_q \otimes V_q$ is defined by sending $U$ to

$$S^{13}(S^*)^{23} = \sum_{i,j,k \in \mathbb{I}} q^{(\delta_{ij} + \delta_{i,j} - \delta_{j,k} - \delta_{k,i}) (1-2|j|)} E_{ii} \otimes E_{kk} \otimes E_{jj}$$

$$- \xi \sum_{i \in \mathbb{I}, j,k \in \mathbb{I}, j \neq k} (-1)^{|j|+|j|} q^{(\delta_{i-j} + \delta_{i,j}) (1-2|j|)} E_{ii} \otimes \left( (-1)^{|j|+|k|} E_{jk} + E_{-j,-k} \right) \otimes E_{jk}$$

$$+ \xi \sum_{j,k \in \mathbb{I}, j \neq k} (-1)^{|j|} q^{(\delta_{j-k} + \delta_{j,k}) (1-2|k|)} \left( E_{jk} + E_{-j,-k} \right) \otimes E_{ii} \otimes E_{jk}$$

$$- \xi^2 \sum_{i,j,k \in \mathbb{I}, j < i < k} (-1)^{|j|+|j|+|j|} \left( E_{ij} + E_{i,j} \right) \otimes \left( (-1)^{|k|} E_{ik} + (-1)^{|i|} E_{-i,-k} \right) \otimes E_{jk}.$$

The map $\cup$ can be identified with the map $q^{-(2n+1)} \sum_{i \in \mathbb{I}} q^{2(|i|+|j|)} \omega_i \otimes \omega_j$, and $\omega_E E_{ij} = \delta_{k,i} \omega_j$. Moreover, direct calculations show

$$q^{2n+1} (\cup \otimes \text{id}) S^{13}(S^*)^{23} = q^{2n+1} (\cup \otimes \text{id})$$

$$= -\xi \sum_{j < k} (-1)^{|j|+|j|} q^{2(|j|+|j|)} \left( (-1)^{|j|+|k|} \omega_j \otimes \omega_k + \omega_{-j} \otimes \omega_{-k} \right) \otimes E_{jk}$$

$$+ \xi \sum_{j < k} (-1)^{|j|} q^{2(|j|+|j|)} \left( (-1)^{|k|+|j|} \omega_j \otimes \omega_k + (-1)^{|j|+|j|} \omega_{-j} \otimes \omega_{-k} \right) \otimes E_{jk}$$

$$- \xi^2 \sum_{j < i < k} (-1)^{|j|+|j|+|j|} q^{2(|i|+|j|)} \left( (-1)^{|k|+|j|} \omega_j \otimes \omega_k + (-1)^{|j|+|j|} \omega_{-j} \otimes \omega_{-k} \right) \otimes E_{jk}$$

$$= \xi \sum_{j < k} \left( q^{(2k-1)(1-2|k|)} - q^{2(2j+1)(1-2|j|)} - \xi \sum_{k > i > j} (-1)^{|i|} q^{2(|i|+|j|)} \right)$$

$$\left( (-1)^{|k|+|j|} \omega_j \otimes \omega_k + (-1)^{|j|+|j|} \omega_{-j} \otimes \omega_{-k} \right) \otimes E_{jk}$$

$$= 0 \text{ by (3.7)}.$$

Therefore $(\cup \otimes \text{id}) S^{13}(S^*)^{23} = (\cup \otimes \text{id})$, so $\cup$ defines a $\mathfrak{U}_q(q(n))$-module homomorphism.

**Theorem 3.7.** There is an action of $BC_{r,s}(q)$ on $V_q^r s$ which supercommutes with the action of $\mathfrak{U}_q(q(n))$, such that the action of each generator is given by

$$t_i \mapsto \text{id} \otimes (i-1) \otimes PS \otimes \text{id} \otimes (r+s-i), \quad c_i \mapsto \text{id} \otimes (i-1) \otimes \Phi \otimes \text{id} \otimes (r+s-i),$$

$$t^*_i \mapsto \text{id} \otimes (r+i-1) \otimes P^T S^T \otimes \text{id} \otimes (s-i), \quad c^*_i \mapsto \text{id} \otimes (r+i-1) \otimes \Phi^T \otimes \text{id} \otimes (s-i),$$

$$e \mapsto \text{id} \otimes (r-1) \otimes \cap \cup \otimes \text{id} \otimes (s-1),$$

where $\text{id} = \text{id}_{V_q}$,

$$(3.10) \quad P = \sum_{i,j \in \mathbb{I}} (-1)^{|j|} E_{ij} \otimes E_{ji} \in \text{End}(V_q \otimes V_q), \quad \Phi = \sum_{i \in \mathbb{I}} (-1)^{|i|} E_{i,-i} \in \text{End}(V_q),$$
and $^T$ is the supertranspose. Explicitly, identifying $V_q$ with $V_q^*$ as above, we have

$$PS = \sum_{i,j \in I} (-1)^{\delta_{ij}} q^{(\delta_{ij} + \delta_{i,j})} E_{ij} \otimes E_{ij},$$

$$P^T S^T = \sum_{i,j \in I} (-1)^{\delta_{ij}} q^{(\delta_{ij} + \delta_{i,j})} E_{ij} \otimes E_{ij} + \xi \sum_{i,j \in I, i < j} (E_{ii} \otimes E_{jj} - E_{i,-i} \otimes E_{j,-j}),$$

$$\phi^T = \sum_{i \in I} E_{i,-i} \cap \cup = q^{-2n+1} \sum_{i,j \in I} (-1)^{\delta_{ij}} q^{2(\delta_{ij})} E_{ij} \otimes E_{ij}.$$  

(3.11)

Proof. The fact that the actions of $t_i$ and $c_i$ are $\mathcal{U}_q(q(n))$-module endomorphisms satisfying the relations of $HC_r(q)$ is shown in [18]. Consider the linear map given by the cyclic permutation $\sigma: V_q^{\otimes s} \to V_q^{\otimes s}, v_1 \otimes \cdots \otimes v_n \mapsto (-1)^{\sum_{i<j} |v_i| |v_j|} v_n \otimes \cdots \otimes v_1$. Conjugating by $\sigma$, we obtain another action of $HC_s(q)$ on $V_q^{\otimes s}$ specified by

$$t_i \mapsto id^{\otimes (s-1-i)} \otimes SP \otimes id^{\otimes (i-1)}, \quad c_i \mapsto id^{\otimes (s-1-i)} \otimes \phi \otimes id^{\otimes (i-1)}.$$  

These maps are also $\mathcal{U}_q(q(n))$-module endomorphisms (even though $\sigma$ is not). Applying the anti-automorphism of $HC_s(q)$ that sends $t_i$ to $t_{s-i}$ and $c_i$ to $c_{s+1-i}$, we see that

$$t_i^* \mapsto id^{\otimes (i-1)} \otimes P^T S^T \otimes id^{\otimes (s-1-i)}, \quad c_i^* \mapsto id^{\otimes (i-1)} \otimes \phi^T \otimes id^{\otimes (s-i)}$$

satisfy the required relations.

Since $\cap$ and $\cup$ are $\mathcal{U}_q(q(n))$-module homomorphisms, the same is true of $e$. We have

$$\cup(id \otimes \phi^T) = q^{-2n+1} \sum_{i \in I} q^{2i (1-2|\omega_i|)} \omega_{-i} \otimes \omega_i = \cup(\phi \otimes id).$$

Thus, $\cup(id \otimes \phi^T) = \cup(\phi \otimes id)$, so $ec_r = ec_r^*$. Also $(id \otimes \phi^T) \cap = (\phi \otimes id) \cap$ by (3.8), so $c_r e = c_r^* e$.

We have

$$\cup(\phi \otimes id) \cap (1) = q^{-2n+1} \left( \sum_{j \in I} q^{2j (1-2|\omega_j|)} \omega_j \otimes \omega_j \right) \left( \sum_{i \in I} v_i \otimes v_i \right)$$

$$= q^{-2n+1} \sum_{i \in I} (-1)^{|\omega_i|} q^{2(1-2|\omega_i|)} = 0$$

and

$$\cup(\phi \otimes id) \cap (1) = q^{-2n+1} \left( \sum_{j \in I} q^{2(1-2|\omega_j|)} \omega_{-i} \otimes \omega_i \right) \left( \sum_{j \in I} v_j \otimes v_j \right) = 0.$$  

so $e^2 = ec_r e = 0$. Now using $q^{2n+1} \cup (E_{ij} \otimes id) \cap (1) = (-1)^{|i|} q^{2(1-2|\delta_{ij}|)} \delta_{ij}$ and identifying $V_q$ with $V_q \otimes \mathbb{C}(q)$, we have

$$q^{2n+1}(id \cup)(PS \otimes id)(id \otimes \cap) = \sum_{j \in I} q^{(2j+1)(1-2|\delta_{ij}|)} E_{jj} + \xi \sum_{i,j \in I, j < i} (-1)^{|i|} q^{2(1-2|\delta_{ij}|)} E_{jj}$$

$$= \sum_{j \in I} \left( q^{(2j+1)(1-2|\delta_{ij}|)} + \xi \sum_{i,j \in I, j < i} (-1)^{|i|} q^{2(1-2|\delta_{ij}|)} \right) E_{jj}$$

$$= q^{2n+1} id \quad \text{by (3.7)}.$$
Thus \((id \otimes \cap \cup)(PS \otimes id)(id \otimes \cap \cup) = (id \otimes \cap \cup)\), so \(et_{r-1} e = e\). Similarly,

\[ q^{2n+1}(\cap \otimes id)(id \otimes P^T S^T)(\cap \otimes id) = \sum_j q^{2(j+1)(1-2|j|)} E_{jj} + \xi \sum_{i>j} (-1)^{|i|} q^{2i(1-2|i|)} E_{jj} = q^{2n+1} id. \]

Thus \((\cap \cup \otimes id)(id \otimes P^T S^T)(\cap \otimes id) = (\cap \cup \otimes id)\), so \(et^* e = e\).

Identifying \(V_q \otimes V^*_q\) with \(V_q \otimes \mathcal{C}(q) \otimes V^*_q\), we have

\[
(P \otimes id_{V^*_q \otimes V_q})(id_{V_q} \otimes \cap \cup \otimes id_{V^*_q}) \cap (1) = (P \otimes id) \left( \sum_{i,j} v_{ij} \otimes v_i \otimes \omega_i \otimes \omega_j \right)
= \sum_{i,j} (-1)^{|i||j|} v_{ij} \otimes v_i \otimes \omega_i \otimes \omega_j.
\]

The above is the canonical map \(\mathcal{C}(q) \to V_q \otimes V_q \otimes V^*_q \otimes V^*_q\), so by the same reasoning that led to (3.8), we know

\[
(SP \otimes id)(P \otimes id)(id \otimes \cap \cup \otimes id) \cap = (id \otimes P^T S^T)(P \otimes id)(id \otimes \cap \cup \otimes id) \cap.
\]

Thus

(3.12) \( (PS \otimes id)(id \otimes \cap \cup \otimes id) \cap = (id \otimes P^T S^T)(id \otimes \cap \cup \otimes id) \cap. \)

To prove the corresponding expression for \(\cup\), we must explicitly compute the following:

\[
q^{4n+2} \cup (id \otimes \cup \otimes id)(PS \otimes id)
= q^{4n+2} \cup (id \otimes \cup \otimes id) \left( \sum_{i,j} (-1)^{|i|} q^{(\delta_{ij}+\delta_{-i,j})(1-2|j|)} E_{ji} \otimes E_{ij} \otimes id \otimes id \right)
+ \xi \sum_{i>j} (E_{jj} \otimes E_{ii} - (-1)^{|i|+|j|} E_{ji} \otimes E_{-i,-j}) \otimes id \otimes id
= \sum_{i,j} (-1)^{|i||j|} \left( q^{(\delta_{ij}+\delta_{i,j})(1-2|j|)+2(1-2|i|)} \omega_i \otimes \omega_j \otimes \omega_i \otimes \omega_j \right)
+ \xi \sum_{i>j} q^{2i(1-2|i|)+2j(1-2|j|)} \left( \omega_j \otimes \omega_i \otimes \omega_i \otimes \omega_j + (-1)^{|j|} \omega_{-j} \otimes \omega_i \otimes \omega_{-i} \otimes \omega_j \right),
\]

\[
q^{4n+2} \cup (id \otimes \cup \otimes id)(id \otimes P^T S^T)
= q^{4n+2} \cup (id \otimes \cup \otimes id) \left( \sum_{i,j} (-1)^{|i|} q^{(\delta_{ij}+\delta_{-i,j})(1-2|j|)} id \otimes id \otimes E_{ji} \otimes E_{ij} \right)
+ \xi \sum_{i>j} id \otimes id \otimes (E_{ii} \otimes E_{jj} - E_{i,-i} \otimes E_{-j,j})
= q^{2n+1} \cup \left( \sum_{i,j} (-1)^{|i|} q^{(\delta_{ij}+\delta_{i,j})(1-2|j|)} id \otimes \omega_j \otimes \omega_i \otimes E_{ij} \right)
+ \xi \sum_{i>j} q^{2i(1-2|i|)} (id \otimes \omega_i \otimes \omega_i \otimes E_{jj} - id \otimes \omega_i \otimes \omega_{-i} \otimes E_{-j,j})
= q^{4n+2} \cup (id \otimes \cup \otimes id)(PS \otimes id).
\]

Thus,

(3.13) \( \cup(id \otimes \cup \otimes id)(PS \otimes id) = \cup(id \otimes \cup \otimes id)(id \otimes P^T S^T). \)
Finally, using
\[ S^{-1}P = \sum_{i,j} (-1)^{ij} q^{-\delta_{ij} + 3 \delta_{i,j} - 2} E_{ij} \otimes E_{ij} - \xi \sum_{i > j} \left( E_{ii} \otimes E_{jj} + (-1)^{i+j} E_{i,j} \otimes E_{j,i} \right) \]
and
\[ q^{2n+1} \cup (E_{ij} \otimes E_{kl}) \cap = \delta_{ik} \delta_{jl} (-1)^{i+j} q^{2(1-2|i|)} \]
we have (with the help of (3.7))
\[ q^{2n+1} (\text{id} \otimes \cup \otimes \text{id}) (S^{-1}P \otimes P^T S^T) (P \otimes \text{id}) \]
\[ = \sum_{i,j} q^{2j(1-2|i|)} (-1)^{i+j} E_{ij} \otimes E_{ij} \]
\[ + \xi \sum_{i > j} \left( q^{2i(1-2|i|)} - q^{2j(1-2|i|)} \right) \left( E_{ii} \otimes E_{jj} + (-1)^{i+j} E_{i,j} \otimes E_{j,i} \right) \]
\[ - \xi^2 \sum_{i > k > j} (-1)^{|j|} q^{2k(1-2|k|)} \left( E_{ii} \otimes E_{jj} + (-1)^{|i|} E_{i,j} \otimes E_{j,i} \right) \]
\[ = q^{2n+1} \cap \cup. \]
Combining this with (3.13) gives
\[ (\text{id} \otimes \cup \otimes \text{id}) (S^{-1}P \otimes P^T S^T) (P \otimes \text{id}) \]
\[ = (\text{id} \otimes \cup \otimes \text{id}) \cup (\text{id} \otimes \cup \otimes \text{id}) (P \otimes \text{id}) \]
\[ = (\text{id} \otimes \cup \otimes \text{id}) (S^{-1}P \otimes P^T S^T) (\text{id} \otimes \cup \otimes \text{id}) (P \otimes P^T S^T). \]
Thus, \( et_{r-1}^{-1} t^*_r e = et_{r-1}^{-1} t^*_r e \). Similarly combining with (3.12) shows that
\[ (P \otimes \text{id}) (\text{id} \otimes \cup \otimes \text{id}) (S^{-1}P \otimes P^T S^T) (\text{id} \otimes \cup \otimes \text{id}) \]
\[ = (P \otimes \text{id}) (\text{id} \otimes \cup \otimes \text{id}) \cup (\text{id} \otimes \cup \otimes \text{id}) \]
\[ = (\text{id} \otimes P^T S^T) (\text{id} \otimes \cup \otimes \text{id}) (S^{-1}P \otimes P^T S^T) (\text{id} \otimes \cup \otimes \text{id}). \]
Hence, \( t_{r-1}^{-1} t^*_r e t_{r-1}^{-1} t^*_r e \), and it follows that \( et_{r-1}^{-1} t^*_r e t_{r-1}^{-1} t^*_r e \).

**Proposition 3.8.** The walled Brauer-Clifford superalgebra \( \mathcal{BC}_{r,s} \) is the classical limit of the quantum walled Brauer-clifford superalgebra \( \mathcal{BC}_{r,s}(q) \).

**Proof.** To see this, let \( \mathcal{R} = \mathbb{C}[q, q^{-1}]/(q-1) \) be the localization of \( \mathbb{C}[q, q^{-1}] \) at the ideal generated by \( q-1 \). Let \( \mathcal{BC}_{r,s}(\mathcal{R}) \) be the \( \mathcal{R} \)-subalgebra of \( \mathcal{BC}_{r,s}(q) \) generated by \( t_1, \ldots, t_{r-1}, c_1, \ldots, c_r, t^*_1, \ldots, t^*_s, c_1^*, \ldots, c_r^* \), e. Let \( V_{\mathcal{R}} = V \otimes_{\mathbb{C}} \mathcal{V} \) and set \( V_{\mathcal{R}}^{r,s} = \mathcal{R} \otimes_{\mathbb{C}} V_{r,s} \).

It follows from [13, Thm. 5.1] that there is a natural epimorphism from the walled Brauer-Clifford superalgebra \( \mathcal{BC}_{r,s} \) onto \((\mathcal{R}/(q-1)\mathcal{R}) \otimes_{\mathcal{R}} \mathcal{BC}_{r,s}(\mathcal{R}) \cong \mathcal{BC}_{r,s}(\mathcal{R})/(q-1)\mathcal{BC}_{r,s}(\mathcal{R}) \) and hence a natural epimorphism
\[ \pi : \mathcal{BC}_{r,s} \rightarrow \mathcal{BC}_{r,s}(\mathcal{R})/(q-1)\mathcal{BC}_{r,s}(\mathcal{R}). \]
We want to argue that \( \pi \) is an isomorphism.

Let \( \rho_{n,\mathcal{R}} : \mathcal{BC}_{r,s} \rightarrow \text{End}_{\mathcal{C}}(V_{r,s})^{\text{op}} \) be the representation given just before Theorem 1.2. The action of \( \mathcal{BC}_{r,s}(q) \) on \( V_{\mathcal{R}}^{r,s} \) defined in Theorem 3.7 restricts to a representation \( \rho_{n,\mathcal{R}}^{r,s} : \mathcal{BC}_{r,s}(\mathcal{R}) \rightarrow \text{End}_{\mathcal{R}}(V_{\mathcal{R}}^{r,s}) \). Let \( \sigma_{n,\mathcal{R}}^{r,s} \) be the homomorphism
\[ \mathcal{BC}_{r,s}(\mathcal{R})/(q-1)\mathcal{BC}_{r,s}(\mathcal{R}) \rightarrow \text{End}_{\mathcal{R}}(V_{\mathcal{R}}^{r,s})/(q-1)\text{End}_{\mathcal{R}}(V_{\mathcal{R}}^{r,s}). \]
Since \( V_{\mathcal{R}}^{r,s} \) is a free \( \mathcal{R} \)-module, the algebra \( \text{End}_{\mathcal{R}}(V_{\mathcal{R}}^{r,s}) \) is also free; thus, it is possible to identify \( \text{End}_{\mathcal{R}}(V_{\mathcal{R}}^{r,s})/(q-1)\text{End}_{\mathcal{R}}(V_{\mathcal{R}}^{r,s}) \) with \( \text{End}_{\mathcal{C}}(V_{\mathcal{R}}^{r,s}) \).
Let $j$ be the anti-involution of $BC_{r,s}$ which fixes each generator. Upon the previous identification, the composite $\overline{\rho}_{n,R}^{r,s} \circ \pi$ is equal to $\rho_{n,R}^{r,s} \circ j$, as can be checked from the action of the generators on the mixed tensor space given in Theorem 3.7. (Setting $q = 1$ in the formula for $S$ in (3.1) gives the identity map). When $n \geq r + s$, the map $\overline{\rho}_{n,R}^{r,s}$ is known to be injective by [13, Thm. 4.5]. Therefore, if $n \geq r + s$, then $\overline{\rho}_{n,R}^{r,s} \circ \pi$ must also be injective, hence so is $\pi$. In conclusion, $\pi$ is an isomorphism.

In the proof of Theorem 5.1 in [13], a vector space basis of the walled Brauer-Clifford superalgebra $BC_{r,s}$ is constructed. In this section, we obtain a basis of $BC_{r,s}(q)$ which specializes to the one in loc. cit. when $q \to 1$ (in a suitable sense). Our basis is inspired by the basis of the quantum walled Brauer algebra $H^n_{r,s}(q)$ constructed in Section 2 of [11] (see Corollary 3.13 below).

**Definition 3.9.** [11] A monomial $n$ in normal form in the generators $t_1, t_2, \ldots, t_{r-1}$ is a product of the form $n = p_1 p_2 \cdots p_{r-1}$, where $p_i = t_i^{r_i} \cdot t_{i-1}^{r_{i-1}} \cdots t_1^{r_1}$ for some $j$ with $1 \leq j \leq i + 1$. (If $j = i + 1$, then $p_i = 1$.) A monomial $n^*$ in normal form in the generators $t_1^*, t_2^*, \ldots, t_{s-1}^*$ is a product of the form $n^* = p_1^* p_2^* \cdots p_{s-1}^*$, where $p_i^* = t_i^* \cdot t_{i-1}^* \cdots t_1^*$ for some $j$ with $1 \leq j \leq i + 1$. (If $j = i + 1$, then $p_i^* = 1$.)

**Definition 3.10.** Suppose that $I = (i_1, \ldots, i_a)$ with $1 \leq i_1 < \cdots < i_a \leq r$, $J = (j_1, \ldots, j_a)$ with $0 \leq j_k \leq s - 1$ for $k = 1, \ldots, a$, and if $k_1 \neq k_2$, then $j_{k_1} \neq j_{k_2}$. Let $\bar{I} \subseteq I$ and $\bar{J} \subseteq \{1, 2, \ldots, r + s\} \setminus J$.

A monomial $m$ in normal form in $BC_{r,s}(q)$ is one of the form

$$m = c_{\bar{I}} \left( \prod_{k=1,\ldots,a} t_i^{j_k} \cdot t_{i-1}^{j_{k-1}} \cdots t_1^{j_1} \right) c_{\bar{J}} n^* n^*,$$

where

1. $n$ is a monomial in normal form in the generators $t_1, t_2, \ldots, t_{r-1}$;
2. $n^*$ is a monomial in normal form in the generators $t_1^*, t_2^*, \ldots, t_{s-1}^*$;
3. the product is arranged from $k = 1$ to $k = a$ from left to right;
4. $c_{\bar{I}}$ is the product of $c_i$ over $i \in \bar{I}$ in increasing order, and $c_{\bar{J}}$ is defined similarly.
5. Moreover, it is required that if $n = p_1 p_2 \cdots p_{r-1}$ and $p_i = t_i^{r_i} \cdot t_{i-1}^{r_{i-1}} \cdots t_1^{r_1}$, then $\pi^{-1}(i_1) < \cdots < \pi^{-1}(i_a)$ where $\pi = \sigma_1 \sigma_2 \cdots \sigma_{r-1}$ and $\sigma_i$ is the cycle $(i + 1 \ i \ i - 1 \ \cdots \ j \ j + 1)$. 

**Theorem 3.11.** The set $\mathcal{B}$ of monomials $m$ in normal form is a basis of $BC_{r,s}(q)$ over $\mathbb{C}(q)$.

**Proof.** As in Step 1 of [13, Thm. 5.1], there are relations analogous to those labelled (1)-(8), except that words of strictly smaller length need to be added on one side of each equality. By using induction on the length of words, we can verify that the set $\mathcal{B}$ spans $BC_{r,s}(q)$ over $\mathbb{C}(q)$.

Since $BC_{r,s}(\mathbb{R})$ is a finitely generated torsion-free $\mathbb{R}$-module, it is free over $\mathbb{R}$. Now by a standard argument in abstract algebra (cf. [9, Chap. 4, Thm. 5.11]), it follows that $\mathcal{B}$ is linearly independent over $\mathbb{C}(q)$.

**Corollary 3.12.** The dimension of $BC_{r,s}(q)$ over $\mathbb{C}(q)$ is $(r + s)! 2^{r+s}$.

**Proof.** This follows from Theorem 3.11, Step 2 in the proof [13, Thm. 5.1], and [11, Lem. 1.7].

**Corollary 3.13.** The subalgebra of $BC_{r,s}(q)$ generated by $t_1, \ldots, t_{r-1}, c_1, \ldots, c_r$ (resp. by $t_1^*, \ldots, t_{s-1}^*$ and $c_1^*, \ldots, c_r^*$) is isomorphic to the finite Hecke-Clifford superalgebra $HC_{r}(q)$ (resp. to $HC_{s}(q)$). The subalgebra generated by $t_1, \ldots, t_{r-1}, t_1^*, \ldots, t_{s-1}^*$, and $e$ is isomorphic to the quantum walled Brauer algebra $H^0_{r,s}(q)$ in [11].
Proof. The first assertion follows from the fact that the set \( \{ c_i n \} \), as \( \tilde{I} \) ranges over the subsets of \( I = \{ 1, \ldots, r \} \) and \( n \) ranges over the monomials in normal form in \( t_1, t_2, \ldots, t_{r-1} \), is a basis of \( HC_r(q) \) over \( \mathbb{C}(q) \). For the second assertion, one can show that the set \( \mathcal{B}' \) consisting of the elements

\[
\left( \prod_{k=1, \ldots, a} t_{jk}^* t_{jk-1}^* \cdots t_{1}^* t_{r-2}^* t_{r-1} t_{r-2} t_{r-1} t_{r-2-1} \cdots t_{t_k}^* t_{j_k}^* t_{j_k-1}^* \right) n^*,
\]

where \( n \) is a monomial in normal form in \( t_1, t_2, \ldots, t_{r-1} \) satisfying the condition (5) in Definition 3.10, and \( n^* \) is a monomial in normal form in \( t_1^*, t_2^*, \ldots, t_{r-1}^* \), spans \( H_{r,s}^0(q) \) over \( \mathbb{C}(q) \). Since \( \dim_{\mathbb{C}(q)} H_{r,s}^0(q) = (r+s)! \) and \( |\mathcal{B}'| \leq (r+s)! \), the set \( \mathcal{B}' \) is a basis of \( H_{r,s}^0(q) \) over \( \mathbb{C}(q) \). \( \square \)

We will frequently deduce properties of \( BC_{r,s}(q) \) from the corresponding properties of \( BC_{r,s} \) using the following well-known facts about specialization, which we prove here for convenience.

**Lemma 3.14.** Suppose \( R \) is a Noetherian local integral domain whose maximal ideal is generated by a single element \( x \in R \). Let \( \psi : A \to B \) be a homomorphism of finitely generated \( R \)-modules, and consider the corresponding induced homomorphism

\[
\overline{\psi} : A/xA \to B/xB, \quad \overline{\psi}(a + xA) = \psi(a) + xB.
\]

(i) If \( \overline{\psi} \) is surjective, then \( \psi \) is surjective.

(ii) If \( B \) is torsion free and \( \overline{\psi} \) is injective, then \( \psi \) is injective, and its cokernel is also torsion free.

**Proof.** (i) Let \( C \) be the cokernel of \( \psi \). The right exact sequence \( A \to B \to C \) induces a right exact sequence \( A/xA \xrightarrow{\overline{\psi}} B/xB \to C/xC \). By assumption, \( \overline{\psi} \) is surjective, so \( C/xC = 0 \). Thus \( C = 0 \) by Nakayama’s lemma.

(ii) Let \( K \) be the kernel of \( \psi \). If \( k \in K \), then \( \psi(k) + xA \) is in the kernel of \( \overline{\psi} \), which is zero by assumption. Thus \( k \in xA \), so \( k = xa \) for some \( a \in A \). Thus \( x\psi(a) = \psi(k) = 0 \), so \( \psi(a) = 0 \) since \( B \) is torsion free. Thus, \( a \in K \), so \( K = xK \). Again by Nakayama’s lemma, \( K = 0 \), so \( \psi \) is injective. Finally, choose an \( (R/xR) \)-basis \( X \) of \( A/xA \) and extend it to a basis \( X \sqcup Y \) of \( B/xB \) (here we are identifying \( \overline{\psi}(X) \) with \( X \) by injectivity). By lifting these basis elements arbitrarily to \( A \) and \( B \), we obtain a commutative diagram

\[
\begin{array}{ccc}
R^X & \to & R^X \oplus R^Y \\
\downarrow & & \downarrow \\
A & \to & B \to B/A
\end{array}
\]

where the top two modules are free over \( R \), and the vertical maps induce isomorphisms of \( (R/xR) \)-vector spaces. By what we’ve shown so far, \( R^X \to A \) is surjective and \( R^X \oplus R^Y \to B \) is an isomorphism. Therefore \( B/A \cong R^Y \) is free, and in particular, is torsion free. \( \square \)

We now show that \( BC_{r,s}(q) \) gives the centralizer of the action of \( \mathfrak{U}_q(\mathfrak{g}(n)) \) on \( V_q^{r,s} \). We deduce this from the corresponding result in the classical case, which is proven in [13].

**Theorem 3.15.** Let \( \rho_{n,q}^{r,s} : BC_{r,s}(q) \to \text{End}_{\mathfrak{U}_q(\mathfrak{g}(n))}(V_q^{r,s}) \) be the representation of \( BC_{r,s}(q) \) coming from Theorem 3.7. Then \( \rho_{n,q}^{r,s} \) is surjective, and when \( n \geq r + s \), it is an isomorphism.
\[ \text{Proof.} \] Let \( \pi : BC_{r,s} \xrightarrow{\sim} (\mathcal{R}/(q-1)\mathcal{R}) \otimes_{\mathcal{R}} BC_{r,s}(\mathcal{R}) \) be the isomorphism established in Proposition 3.8. Consider the following elements of \( U_q(q(n)) \) for \( i, j \in I = \{ \pm i \mid i = 1, \ldots, n \} \) with \( i \leq j \):

\[
\tilde{u}_{ij} = (q - 1)^{-1} \begin{cases} 
    u_{ij} - 1 & \text{if } i = j, \\
    u_{ij} & \text{if } i \neq j.
\end{cases}
\]

Let

\[ \tilde{U} = \sum_{i \leq j} \tilde{u}_{ij} \otimes E_{ij} \in U_q(q(n)) \otimes_{\mathcal{C}} \text{End}_{\mathcal{C}}(V). \]

By (3.1), the action of \( \tilde{u}_{ij} \) on \( V_q \) lies in \( \text{End}_{\mathcal{R}}(V) \subseteq \text{End}_{\mathcal{C}}(V) \). Moreover, under the surjection \( \text{End}_{\mathcal{R}}(V) \to \text{End}_{\mathcal{C}}(V) \) given by evaluation at \( q = 1 \), the action of \( \tilde{u}_{ij} \) maps to the action of the following element of \( q(n) \):

\[
u_{ij} = \begin{cases} 
(-1)^{|i|} E_{ij}^0 & \text{if } i = j, \\
(-1)^{|i|} 2 \mathbf{E}_{(1)^{i} j \cdot (-1)^{|i|} i} & \text{if } i < j.
\end{cases}
\]

Similarly, by (3.9), the action of \( \tilde{u}_{ij} \) on \( V^*_q \) lies in \( \text{End}_{\mathcal{R}}(V^*_q) \) and maps to the action of \( u_{ij} \) on \( V^* \) by evaluation at \( q = 1 \). Finally, since the coproduct on \( U_q(q(n)) \) sends

\[ \Delta(\tilde{U}) = \tilde{U}^{13} + \tilde{U}^{23} + (q - 1) \tilde{U}^{13} \tilde{U}^{23}, \]

the corresponding statements extend to the action of \( \tilde{u}_{ij} \) on \( V^*_q \).

Now let \( \text{End}_{\mathcal{R}}(V^r_{q,n}) \) denote the space of endomorphisms in \( \text{End}_{\mathcal{R}}(V^r_{q,n}) \) which supercommute with the action of \( \tilde{u}_{ij} \) for all \( i \leq j \). We will show that the \( \mathcal{R} \)-module homomorphism

\[ \psi : BC_{r,s}(\mathcal{R}) \to \text{End}_{\mathcal{R}}(V^r_{q,n}) \]

is surjective, and an isomorphism if \( n \geq r + s \). Note that if \( X \in \text{End}_{\mathcal{R}}(V^r_{q,n}) \) is such that \( (q - 1)X \) supercommutes with \( u_{ij} \), then \( X \) also supercommutes with \( u_{ij} \). Therefore, the induced homomorphism

\[ (\mathcal{R}/(q - 1)\mathcal{R}) \otimes_{\mathcal{R}} \text{End}_{\mathcal{R}}(V^r_{q,n}) \to \text{End}_{\mathcal{C}}(V^r_{q,n}) \]

is injective. Moreover since the elements \( \{ u_{ij} \mid i \leq j \} \) generate \( q(n) \), this map factors through \( \text{End}_{\mathcal{R}}(V^r_{q,n}) \). We obtain the following diagram.

\[
\begin{array}{ccc}
BC_{r,s} & \xrightarrow{\pi} & (\mathcal{R}/(q - 1)\mathcal{R}) \otimes_{\mathcal{R}} BC_{r,s}(\mathcal{R}) \\
\text{End}_{\mathcal{R}}(V^r_{q,n}) & \xrightarrow{id \otimes \psi} & (\mathcal{R}/(q - 1)\mathcal{R}) \otimes_{\mathcal{R}} \text{End}_{\mathcal{R}}(V^r_{q,n}) \\
& & \xrightarrow{\pi} \text{End}_{\mathcal{C}}(V^r_{q,n})
\end{array}
\]

Now Theorem 3.5 of [13] shows that the homomorphism \( \rho_{r,s} : BC_{r,s} \to \text{End}_{\mathcal{R}}(V^r_{q,n}) \) given by the \( BC_{r,s} \)-module action is surjective, and also injective for \( n \geq r + s \). It follows that \( (\mathcal{R}/(q - 1)\mathcal{R}) \otimes_{\mathcal{R}} \text{End}_{\mathcal{R}}(V^r_{q,n}) \to \text{End}_{\mathcal{R}}(V^r_{q,n}) \) is an isomorphism for all \( n \), so \( (\mathcal{R}/(q - 1)\mathcal{R}) \otimes_{\mathcal{R}} \psi \) is surjective for all \( n \) and injective for \( n \geq r + s \). Since \( \text{End}_{\mathcal{R}}(V^r_{q,n}) \) is torsion free, we conclude by Lemma 3.14 that

\[ BC_{r,s}(\mathcal{R}) \to \text{End}_{\mathcal{R}}(V^r_{q,n}) \]

is surjective for all \( n \) and injective for \( n \geq r + s \). Finally, since \( \tilde{u}_{ij} \) generate \( U_q(q(n)) \), we have

\[ \mathcal{C}(q) \otimes_{\mathcal{R}} \text{End}_{\mathcal{R}}(V^r_{q,n}) = \text{End}_{\mathcal{C}}(U_q(q(n)))V^r_{q,n}. \]

Therefore, tensoring by \( \mathcal{C}(q) \), we obtain the desired result. \( \square \)

**Remark 3.16.** The following question is left open: Does \( U_q(q(n)) \) surject onto \( \text{End}_{BC_{r,s}(q)}(V^r_{q,n}) \)?
4. The \((r, s)\)-bead tangle algebras \(\mathcal{BT}_{r,s}(q)\)

In this section, we introduce a diagrammatic realization of the quantum walled Brauer-Clifford superalgebra \(\mathcal{BC}_{r,s}(q)\) given in Definition 3.2.

**Definition 4.1.** An \((r, s)\)-bead tangle is a portion of a planar knot diagram in a rectangle \(R\) with the following conditions:

1. The top and bottom boundaries of \(R\) each have \(r + s\) vertices in some standard position.
2. There is a vertical wall that separates the first \(r\) vertices from the last \(s\) vertices on the top and bottom boundaries.
3. Each vertex must be connected to exactly one other vertex by an arc.
4. Each arc may (or may not) have finitely many numbered beads. The bead numbers in the tangle start with 1 and are distinct consecutive positive integers.
5. A vertical arc connects a vertex on the top boundary to a vertex on the bottom boundary of \(R\), and it cannot cross the wall. A horizontal arc connects two vertices on the same boundary of \(R\), and it must cross the wall.
6. An \((r, s)\)-bead tangle may have finitely many loops.

The following is an example of \((3, 2)\)-bead tangle.

![Example of (3, 2)-bead tangle](image)

We want to stress that an \((r, s)\)-bead tangle is in the plane, not in 3-dimensional space. We consider a bead as a point on the arc. Two \((r, s)\)-bead tangles are regularly isotopic if they are related by a finite sequence of the Reidemeister moves II, III together with isotopies fixing the boundaries of \(R\).

Reidemeister move II:

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{reidemeister_2.png}
\end{array}
\]

Reidemeister move III:

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{reidemeister_3.png}
\end{array}
\]

We observe that there are isotopies fixing the boundaries of the rectangles between the following tangles:

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{example_tangles.png}
\end{array}
\]
Therefore, moving a bead along a non-crossing arc or an over-crossing arc gives tangles that are regularly isotopic. We want to emphasize that the following are not regularly isotopic:

* * * and * * * * * * * *.

We identify an \((r,s)\)-bead tangle with its regular isotopy class, and denote by \(\widetilde{BT}_{r,s}\) the set of \((r,s)\)-bead tangles (up to regular isotopy). The \((r,s)\)-bead tangle in which there are even (resp. odd) number of beads is regarded as even (resp. odd).

Now we define a multiplication on \(\widetilde{BT}_{r,s}\). For \((r,s)\)-bead tangles \(d_1, d_2\), we place \(d_1\) under \(d_2\) and identify the top row of \(d_1\) with the bottom row of \(d_2\). We add the largest bead number in \(d_1\) to each bead number in \(d_2\), as we did for \((r,s)\)-bead diagrams, and then concatenate the tangles. For example, if

\[
\begin{align*}
d_1 &= \includegraphics{example1}, \\
d_2 &= \includegraphics{example2},
\end{align*}
\]

then,

\[
\begin{align*}
d_1d_2 &= \includegraphics{example3}.
\end{align*}
\]

We observe \(\widetilde{BT}_{r,s}\) is closed under this product, and it is \(\mathbb{Z}_2\)-graded. The shape of the arcs are the same in \((d_1d_2)d_3\) and \(d_1(d_2d_3)\). Moreover, the locations of the beads and the bead numbers are also the same in \((d_1d_2)d_3\) and \(d_1(d_2d_3)\). It follows that the multiplication on \(\widetilde{BT}_{r,s}\) is associative. Hence \(\widetilde{BT}_{r,s}\) is a monoid with identity element

\[
\begin{align*}
\includegraphics{example4}.
\end{align*}
\]

For \(1 \leq i \leq r - 1, \ 1 \leq j \leq s - 1, \ 1 \leq k \leq r, \ 1 \leq l \leq s\), we define the following \((r,s)\)-bead tangles:
From Reidemeister move II, we obtain the following elements in $\tilde{\text{BT}}_{r,s}$:

\[
\sigma^{-1}_i = \begin{bmatrix}
\cdots & i & i+1 & \cdots & \cdots
\end{bmatrix}, \quad \sigma^*_j = \begin{bmatrix}
\cdots & j & j+1 & \cdots & \cdots
\end{bmatrix},
\]

\[
h := \begin{bmatrix}
\cdots & 1 & r & r+1 & r+s & \cdots
\end{bmatrix},
\]

\[
c_k := \begin{bmatrix}
\cdots & 1 & \cdots & \cdots & \cdots
\end{bmatrix}, \quad c^*_l := \begin{bmatrix}
\cdots & 1 & \cdots & \cdots & \cdots
\end{bmatrix}.
\]

Now we consider the submonoid $\text{BT}'_{r,s}$ of $\tilde{\text{BT}}_{r,s}$ generated by $\sigma_i^{\pm 1}, (\sigma_j^*)^{\pm 1}, h, c_k, c^*_l$. We denote by $\text{BT}'_{r,s}(q)$ the monoid algebra of $\text{BT}'_{r,s}$ over $\mathbb{C}(q)$ and define the algebra $\text{BT}_{r,s}(q)$ to be the quotient of $\text{BT}'_{r,s}(q)$ by the following relations (for allowable $i, j$):

\[
\sigma^{-1}_i = \sigma_i - (q - q^{-1}), \quad (\sigma^*_j)^{-1} = \sigma^*_j - (q - q^{-1}),
\]

\[
h \sigma_{r-1} h = h, \quad h^2 = 0, \quad h \sigma^*_r h = h, \quad h c_r h = 0,
\]

\[
c_i^2 = -1, \quad c_i c_j = -c_j c_i (i \neq j), \quad c_i c_j^* = -c_j^* c_i, \quad (c_i^*)^2 = 1, \quad c_i^* c_j^* = -c_i^* c_j^* (i \neq j).
\]

For simplicity, we identify the coset of a diagram in $\text{BT}_{r,s}(q)$ with the diagram itself. Note that we get extra terms when a bead moves along an under-crossing arc. That is, we have

\[
\begin{bmatrix}
1
\end{bmatrix} \quad + \quad \begin{bmatrix}
1
\end{bmatrix} (q - q^{-1})
\]

\[
\begin{bmatrix}
1
\end{bmatrix} \quad + \quad \begin{bmatrix}
1
\end{bmatrix} (q - q^{-1})
\]

\[
\begin{bmatrix}
1
\end{bmatrix} \quad + \quad \begin{bmatrix}
1
\end{bmatrix} (q - q^{-1})
\]

\[
\begin{bmatrix}
1
\end{bmatrix} \quad + \quad \begin{bmatrix}
1
\end{bmatrix} (q - q^{-1})(\begin{bmatrix}
1
\end{bmatrix} - \begin{bmatrix}
1
\end{bmatrix})
\]
which is equivalent to \( c_i \sigma_i = \sigma_i c_{i+1} + (q - q^{-1})(c_i - c_{i+1}) \). Similarly, we obtain \( c_{i+1} \sigma_{i}^{-1} = \sigma_{i}^{-1} c_i + (q - q^{-1})(c_i - c_{i+1}) \). We call \( BT_{r,s}(q) \) the \((r,s)\)-bead tangle algebra or simply the bead tangle algebra.

In [10], Kauffman introduced the algebra of tangles and showed that it is isomorphic to the Birman-Murakami-Wenzl algebra. To show that \( BT_{r,s}(q) \) is isomorphic to \( BC_{r,s}(q) \), we will follow the outline of the argument given in [10, Thm. 4.4].

Let \( F'_{r,s} \) be the monoid generated by \( t_{1}^{ \pm 1}, t_{2}^{ \pm 1}, \ldots, t_{r-1}^{ \pm 1}, (t_{1}^{*})^{ \pm 1}, (t_{2}^{*})^{ \pm 1}, \ldots, (t_{s-1}^{*})^{ \pm 1}, e, c_{1}, c_{2}, \ldots, c_{r} \) and \( c_{1}^{*}, c_{2}^{*}, \ldots, c_{s}^{*} \) with the following defining relations (for \( i, j \) in the allowable range):

\[
\begin{align*}
(4.2) & \quad t_{i} t_{i}^{-1} = t_{i}^{-1} t_{i} = 1, \\
(4.3) & \quad t_{i} t_{i+1} t_{i} = t_{i+1} t_{i} t_{i+1}, \\
(4.4) & \quad t_{i} t_{j} = t_{j} t_{i} \quad (|i - j| > 1), \\
(4.5) & \quad t_{i}^{*} t_{j} = t_{j}^{*} t_{i}, \\
(4.6) & \quad e t_{j} = t_{j} e \quad (j \neq r - 1), \\
(4.7) & \quad e t_{r-1}^{\pm 1} t_{i}^{\pm 1} = e t_{r-1}^{\pm 1} t_{i}^{\pm 1} = e t_{r-1}^{\pm 1} t_{i}^{\pm 1}, \\
(4.8) & \quad e t_{r-1} (t_{1}^{*})^{-1} e = e t_{r-1} (t_{1}^{*})^{-1} e (t_{1}^{*})^{-1} t_{r-1} \\
& \quad = e t_{r-1} (t_{1}^{*})^{-1} e (t_{1}^{*})^{-1} t_{r-1}, \\
(4.9) & \quad e t_{r-1}^{\pm 1} t_{i}^{\pm 1} e = e t_{r-1} (t_{1}^{*})^{-1} e, \\
(4.10) & \quad t_{i} c_{i} = c_{i+1} t_{i}, \\
(4.11) & \quad t_{i} c_{j} = c_{j} t_{i} \quad (j \neq i, i + 1), \\
(4.12) & \quad t_{i} c_{j}^{*} = c_{j}^{*} t_{i}, \\
(4.13) & \quad c_{r} e = c_{1}^{*} e, \\
(4.14) & \quad c_{j} e = e c_{j} \quad (j \neq r),
\end{align*}
\]

We define a monoid homomorphism \( \varphi_{r,s} : F'_{r,s} \to BT'_{r,s} \) by

\[
\varphi_{r,s}(t_{i}^{\pm 1}) = \sigma_{i}^{\pm 1}, \quad \varphi_{r,s}(t_{1}^{*})^{\pm 1} = (\sigma_{1}^{*})^{\pm 1}, \quad \varphi_{r,s}(e) = h, \quad \varphi_{r,s}(c_{i}) = c_{i}, \quad \text{and} \quad \varphi_{r,s}(c_{j}^{*}) = c_{j}^{*}.
\]

By direct computations, one can check that \( \sigma_{i}^{\pm 1}, (\sigma_{1}^{*})^{\pm 1}, h, c_{k}, c_{j}^{*} \) satisfy the corresponding defining relations (4.2) - (4.14) in \( BT'_{r,s}(q) \). Moreover, \( \varphi_{r,s} \) is surjective.

**Theorem 4.2.** The monoid \( F'_{r,s} \) is isomorphic to \( BT'_{r,s} \) as monoids.

**Proof.** It is suffices to show that \( \varphi_{r,s} \) is injective. Assume that \( \varphi_{r,s}(d) = \varphi_{r,s}(d') \in BT'_{r,s} \). This means that \( \varphi_{r,s}(d') \) can be obtained from \( \varphi_{r,s}(d) \) by a finite sequence of Reidemeister moves II, III and vice versa.

If \( \varphi_{r,s}(d') \) can be obtained from \( \varphi_{r,s}(d) \) without moving beads, then modifying the proof of [10, Thm. 4.4], we can show that \( d' \) can be obtained from \( d \) using relations (4.2) - (4.9).

We consider the various cases in which we need to move the beads.

**Case 1:** A bead moves along a non-crossing arc. We have the following five cases.
Observe that the above are equivalent to the following relations for $j \neq i, i+1, l \neq r, m \neq 1$, and allowable values of $k$:

\[
\begin{align*}
    c_j\sigma_i &= \sigma_i c_j, & c_j\sigma_k^* &= \sigma_k^* c_j, & c_j\sigma_i^{-1} &= \sigma_i^{-1} c_j, & c_j(\sigma_k^*)^{-1} &= (\sigma_k^*)^{-1} c_j, \\
    c_j^*\sigma_i^* &= \sigma_i^* c_j^*, & c_j^*\sigma_k &= \sigma_k c_j^*, & c_j^*(\sigma_i^*)^{-1} &= (\sigma_i^*)^{-1} c_j^*, & c_j^*(\sigma_k^{-1})^{-1} &= (\sigma_k^{-1})^{-1} c_j^*.
\end{align*}
\]

Hence $d$ and $d'$ are related in $F'_{r,s}$.

**Case 2:** A bead moves along an over-crossing arc. In the following two cases,

the corresponding relations are equivalent to

\[
\begin{align*}
    c_{i+1}\sigma_i &= \sigma_i c_{i+1}, & c_{i+1}\sigma_i^* &= \sigma_i^* c_{i+1}, & c_i\sigma_i^{-1} &= \sigma_i^{-1} c_{i+1}, & c_i(\sigma_i^*)^{-1} &= (\sigma_i^*)^{-1} c_{i+1},
\end{align*}
\]

which implies that $d$ and $d'$ are related.

The remaining cases appear as a mixture of **Case 1** and **Case 2**. For instance, a crossing of a horizontal and a vertical arc is a combination of the following moves.
which can be written as $c_2 \sigma_1 h = \sigma_1 c_1 h = \sigma_1 hc_1$.

In conclusion, when $\varphi_{r,s}(d')$ is obtained from $\varphi_{r,s}(d)$ by moving beads, $d$ can be transformed to $d'$ by the corresponding relations in (4.10) - (4.14).

Let $F'_{r,s}(q) = \mathbb{C}(q) F'_{r,s}$ be the associated monoid algebra, and let $R$ be the two-sided ideal of $F'_{r,s}(q)$ corresponding to the following relations:

$$
\begin{align*}
t_i^{-1} &= t_i - (q - q^{-1}), & (t_j^*)^{-1} &= t_j^* - (q - q^{-1}), \\
et_{r-1} e &= e, & e^2 &= 0, & et_i^* e &= e, & ec_r e &= 0, \\
c_i^2 &= -1, & c_i c_j &= -c_j c_i \, (i \neq j), & c_i c_j^* &= -c_j^* c_i, & \quad (c_i^*)^2 &= 1, & c_i^* c_j^* &= -c_i^* c_j^* \, (i \neq j).
\end{align*}
$$

We consider $(t_i)_{\pm 1}, (t_j^*)_{\pm 1}, h$ as the even generators and $c_i, c_j^*$ as the odd generators.

We denote by $F_{r,s}(q)$ the quotient superalgebra $F'_{r,s}(q)/R$. For ease of notation, we also use $t_i, t_j^*, h, c_k$ and $c_i^*$ for the generators of $F_{r,s}(q)$. We note that the relations in (4.1) correspond to the relations in (4.15) via the map $\varphi_{r,s}$. By the definitions of $F_{r,s}(q)$ and $BT_{r,s}(q)$ and Theorem 4.2, we obtain the following corollary.

**Corollary 4.3.** The superalgebra $F_{r,s}(q)$ is isomorphic to $BT_{r,s}(q)$ as associative superalgebras.

One can check that relations (4.2) - (4.14) and (4.15) include the corresponding relations (3.3) if we map $t_i, t_j, e, c_k$ and $c_i^*$ to $t_i, t_j, h, c_k$ and $c_i^*$, respectively. Using the relations in (4.15) and Remark 3.4, we obtain that the relations corresponding to (4.7) - (4.9) are also satisfied in $BC_{r,s}(q)$. It follows that $F_{r,s}(q)$ is isomorphic to $BC_{r,s}(q)$ as associative superalgebras. Therefore, we obtain the following main result of this section.

**Theorem 4.4.** The quantum walled Brauer-Clifford superalgebra $BC_{r,s}(q)$ is isomorphic to the $(r, s)$-bead tangle algebra $BT_{r,s}(q)$ as associative superalgebras.

**Corollary 4.5.** The dimension of $BT_{r,s}(q)$ over $\mathbb{C}(q)$ is $(r + s)! 2^{r+s}$.

**Remark 4.6.** Since $BC_{r,s}$ is the classical limit of $BC_{r,s}(q)$, by Theorem 2.5 and Theorem 4.4, we conclude that $BD_{r,s}$ is the classical limit of $BT_{r,s}(q)$.

5. **The $q$-Schur superalgebra of type $Q$ and its dual**

There are two equivalent ways to define the $q$-Schur algebra $S_q(n; \ell)$ associated to $U_q(\mathfrak{gl}(n))$: either as the image of $U_q(\mathfrak{gl}(n))$ in $\text{End}_{\mathbb{C}(q)}((\mathbb{C}(q)^n)^{\otimes \ell})$ or as $\text{End}_{H\ell(q)}((\mathbb{C}(q)^n)^{\otimes \ell})$, where $H\ell(q)$ is the Hecke algebra (the subalgebra of $H\ell(q)$ generated by the $t_i, i = 1, \ldots, \ell - 1$). Analogous definitions can be considered in our quantum super context, but we are not able to prove that they are equivalent. Therefore, in order to develop a viable theory, we have settled on the following definition.
Definition 5.1. The $q$-Schur superalgebra of type $Q$, denoted $S_q(n;r,s)$, is $\text{End}_{\mathcal{B}_r,s}(V_q^{r,s})$.

Even when $s = 0$, this superalgebra had not been studied until the recent paper [6]. In this case, it follows from [18, Thm. 5.3] that the next result holds, but we don’t know if it is true for arbitrary $s \geq 1$.

Proposition 5.2. $S_q(n;r,0)$ is equal to the image of $\text{U}_q(q(n))$ in $\text{End}_{C}(V_q^0)$.

There is a third point of view on $q$-Schur algebras adopted for instance in [5], which is as duals of certain homogeneous subspaces of quantum matrix algebras. Super analogues of quantum matrix superalgebras were first introduced in [17] and more general superalgebras were studied in [8], where bases were constructed using quantum minors and indexed by standard bitableaux. In this section, we obtain similar results for a quantum matrix superalgebra of type $Q$.

Let

$$\delta_{i<j} = \begin{cases} 1 & \text{if } i < j, \\ 0 & \text{otherwise} \end{cases}$$

and $\delta_{i\pm j} = \delta_{ij} + \delta_{i,-j}$. Also recall that $\xi = q - q^{-1}$.

Definition 5.3. We denote by $A_q(n)$ the associative unital algebra over $C(q)$ generated by $x_{ab}$ and $\bar{x}_{ab}$ for $1 \leq a, b \leq n$, subject to the following relations for any $1 \leq a, b, c, d \leq n$ with $a \leq c$:

- $q^{\delta_{ac}}x_{ab}x_{cd} = q^{\delta_{BD}}x_{cd}x_{ab} + \xi\delta_{b<d}x_{cb}x_{ad} + \xi\bar{x}_{cb}\bar{x}_{ad}$,
- $q^{\delta_{ac}}x_{ab}\bar{x}_{cd} = q^{-\delta_{BD}}\bar{x}_{cd}x_{ab} - \xi\delta_{d<b}\bar{x}_{cb}\bar{x}_{ad}$,
- $q^{\delta_{ac}}\bar{x}_{ab}x_{cd} = q^{\delta_{BD}}x_{cd}\bar{x}_{ab} + \xi\delta_{b<d}x_{cb}\bar{x}_{ad}$,
- $q^{\delta_{ac}}\bar{x}_{ab}\bar{x}_{cd} = -q^{-\delta_{BD}}\bar{x}_{cd}\bar{x}_{ab} + \xi\delta_{d<b}\bar{x}_{cb}\bar{x}_{ad}$.

We define a $\mathbb{Z}$-grading on $A_q(n)$ by declaring each generator to have degree 1. We call $A_q(n)$ the quantum matrix superalgebra of type $Q$.

Remark 5.4. The quotient of $A_q(n)$ by the two-sided ideal generated by the odd elements is isomorphic to the quantum matrix algebra as presented for instance in Section 1.3 of [2] with $v = q^{-1}$.

The algebra $A_q(n)$ can be viewed as a $q$-deformation of the algebra of polynomial functions on the space $M_n(Q)$ of $(2n \times 2n)$-matrices of type $Q$ inside $M_{n|n}(C)$. A $q$-deformation of the algebra of polynomial functions on $M_{n|n}(C)$ was first given in [17].

Lemma 5.5. The algebra $A_q(n)$ is isomorphic to the unital associative algebra over $C(q)$ generated by elements $x_{ij}$ with $i, j \in I = \{\pm 1, \ldots, \pm n\}$, which satisfy the relations $x_{ij} = x_{-i,-j}$ and

$$S^{23}X^{12}X^{13} = X^{13}X^{12}S^{23}$$

where $S^{23}$ is the same matrix used in Definition 3.1.

Proof. This follows from relations (5.3) and (5.4) below and from the proof of Theorem 5.7. \qed

Corollary 5.6. The algebra $A_q(n)$ is a bialgebra with coproduct $\Delta$ given by

$$\Delta(x_{ij}) = \sum_{k=-n}^{n} (-1)^{(|i|+|k|)(|j|+|k|)}x_{ik} \otimes x_{kj}.$$
Theorem 5.7. Let $A_q(n, r)$ denote the degree $r$ component of $A_q(n)$. There is a vector space isomorphism $A_q(n, r) \xrightarrow{\cong} \operatorname{End}_{HC_r(q)}(V_q^\otimes r)^*$. Explicitly, let $\{E_{ij}\}$ denote the basis of $\operatorname{End}_{HC_r(q)}(V_q)^*$ dual to the natural basis $\operatorname{End}_{HC_r(q)}(V_q)$. Define a map $A_q(n, 1) \rightarrow \operatorname{End}_{HC_r(q)}(V_q^\otimes r)^*$ by $x_{ab} \mapsto E_{ab}^\vee$, $\bar{x}_{ab} \mapsto E_{a,-b}$. This extends to the map $A_q(n, r) \rightarrow \operatorname{End}_{HC_r(q)}(V_q^\otimes r)^*$ via the (super) identification

$$(\operatorname{End}_{HC_r(q)}(V_q^\otimes r)^*)^{\otimes r} \cong \operatorname{End}_{HC_r(q)}(V_q^\otimes r)^*$$

Proof. Let $F_q(n)$ denote the free algebra generated by $\varepsilon_{ij}$ for $i, j \in I = \{\pm 1, \pm 2, \ldots, \pm n\}$, and let $F_q(n, r)$ denote the degree $r$ component of $F_q(n)$ where each generator has degree 1. Sending $\varepsilon_{ij}$ to $E_{ij}^\vee$, we obtain a vector space isomorphism $F_q(n, r) \cong \operatorname{End}_{HC_r(q)}(V_q^\otimes r)^*$. An element of $\operatorname{End}_{HC_r(q)}(V_q^\otimes r)^*$ will lie in the subspace $\operatorname{End}_{HC_r(q)}(V_q^\otimes r)$ if its coefficients satisfy certain relations. We can obtain $\operatorname{End}_{HC_r(q)}(V_q^\otimes r)^*$ by quotienting $F_q(n)$ by the same relations.

The generator $c_k$ of $HC_r(q)$ acts on the $k$th tensor factor via the map $\Phi$ (3.10), and the generator $t_k$ of $HC_r(q)$ acts on the $k$th and $(k + 1)$st factors via the map $PS$ (3.11). We compute the supercommutator of each of these maps with an arbitrary endomorphism.

$$\left[\Phi, \sum_{ij} a_{ij} E_{ij}\right] = \sum_{ij} (-1)^{|i|} (a_{-i,-j} - a_{ij}) E_{i,-j}.$$

$$\left[PS, \sum_{ijkl} a_{ijkl} E_{ij} \otimes E_{kl}\right] = \sum_{ijkl} a_{ijkl} \left[(-1)^{|i|+|j|+|k|+|l|} q^{\delta_{i+k}(-1)^{|k|}} E_{kj} \otimes E_{il}
+ \xi \delta_{k<i} E_{ij} \otimes E_{kl} + \xi \delta_{-i<k} (-1)^{|i|+|k|} E_{-i,j} \otimes E_{-k,l}
- (-1)^{|i|+|j|+|k|+|l|} q^{\delta_{i+k}(-1)^{|i|}} E_{il} \otimes E_{kj}
- \xi \delta_{j<l} E_{ij} \otimes E_{kl} - \xi \delta_{<l} (-1)^{|j|+|k|} E_{i,-j} \otimes E_{K,-l}\right]
= \sum_{ijkl} \left[(-1)^{|i|+|j|+|k|+|l|} q^{\delta_{i+k}(-1)^{|k|}} a_{ijkl}
+ \xi \delta_{k<i} a_{kjl} - \xi \delta_{k<-i} (-1)^{|i|+|j|} a_{-k,j,-i,l}
- (-1)^{|j|+|k|+|l|} q^{\delta_{i+k}(-1)^{|i|}} a_{klji}
- \xi \delta_{j<l} a_{kjl} + \xi \delta_{<l} (-1)^{|i|+|j|} a_{k,-l,i,-l}\right] E_{kj} \otimes E_{il}.$$

Therefore, $\operatorname{End}_{HC_r(q)}(V_q^\otimes r)^*$ is the degree $r$ component of $A_q(n)' = F_q(n)/\langle R(ij), R(ijk)\rangle$ where we have factored out by the ideal generated by the elements

$$(3.3) \quad R(ij) = \varepsilon_{-i,-j} - \varepsilon_{ij},$$

$$(3.4) \quad R(ijkl) = q^{\delta_{i+k}(-1)^{|k|}} (-1)^{|i|+|j|} |\varepsilon_{ij}\varepsilon_{kl}
- q^{\delta_{j+l}(-1)^{|l|}} (-1)^{|i|+|j|} |\varepsilon_{kl}\varepsilon_{ij}
+ \xi (-1)^{|k|+|j|+|l|} (\delta_{k<i} - \delta_{j<l}) \varepsilon_{kj}\varepsilon_{il}
+ \xi (-1)^{|i|+|j|+|k|+|l|} (\delta_{k<i} \varepsilon_{-k,j}\varepsilon_{-l,i} - \delta_{j<l} \varepsilon_{k,j}\varepsilon_{i,-l}).$$

The element $R(ijk)$ was obtained from the right-hand side of (5.2) by multiplying it by $(-1)^{|i|+|j|}$ and by replacing each $a_{ijkl}$ by $(-1)^{|i|+|j|+|k|+|l|} \varepsilon_{ij}\varepsilon_{kl}$. It remains to show that $A_q(n)' \cong A_q(n)$. Let $F_q(n)'$ be the free algebra generated by $x_{ab}$ and $\bar{x}_{ab}$ for $1 \leq a, b \leq n$. For convenience, we define
elements $x_{ij}$ and $\bar{x}_{ij}$ in $F_q(n)'$ for all $i, j \in I$ such that

$$x_{-i,j} = \bar{x}_{ij} = x_{i,-j}.$$  

Clearly there is an isomorphism $F_q(n)/(R(ij)) \cong F_q(n)'$ sending $\varepsilon_{ij}$ to $x_{ij}$. Let $R(ijkl)'$ be the image of $R(ijkl)$ in $F_q(n)'$. Note that

$$R(-i, -j, k, l)' = (-1)^{|i||j|} q^{\delta_{j+1,-1}(-1)^{|i|}} x_{ij} x_{kl} - (-1)^{|k||i|+|j|} q^{\delta_{j,-1}(-1)^{|i|}} x_{kl} x_{ij}$$

$$+ (-1)^{|k||i|+|j|} q^{\delta_{j,-1}(-1)^{|i|}} x_{kl} x_{ij} + (-1)^{|k||i|+|j|} q^{\delta_{j,-1}(-1)^{|i|}} x_{kl} x_{ij}$$

Similarly, using $q^{\delta_{j+1,-1}(-1)^{|i|}} - q^{\delta_{j,-1}(-1)^{|i|}} = \delta_{j+1,-1}(1)^{|i|} \xi$, we have

$$(-1)^{|i||j|} R(i, j, -k, -l)' = (-1)^{|i||j|} q^{\delta_{j+1,-1}(-1)^{|i|}} x_{ij} x_{kl} + (-1)^{|k||i|+|j|} q^{\delta_{j,-1}(-1)^{|i|}} x_{kl} x_{ij}$$

$$- (-1)^{|k||i|+|j|} q^{\delta_{j,-1}(-1)^{|i|}} x_{kl} x_{ij} - (-1)^{|k||i|+|j|} q^{\delta_{j,-1}(-1)^{|i|}} x_{kl} x_{ij}$$

Observe also that $q^{\delta_{j+1,-1}(-1)^{|i|}} + 1 = \delta_{j+1,-1}(1)^{|i|} \xi$. Therefore

$$(-1)^{|i||j|} q^{\delta_{j+1,-1}(-1)^{|i|}} R(ijkl)' + (-1)^{|i||k|} q^{\delta_{j,-1}(-1)^{|i|}} R(klij)'$$

$$= (-1)^{|i||j|} \left( q^{\delta_{j+1,-1}(-1)^{|i|}} - q^{\delta_{j,-1}(-1)^{|i|}} \right) x_{ij} x_{kl}$$

$$+ (-1)^{|k||i|+|j|} q^{\delta_{j+1,-1}(-1)^{|i|}} \xi (\delta_{k,-i} - \delta_{k,l}) x_{kl} x_{ij}$$

$$+ (-1)^{|k||i|+|j|} q^{\delta_{j,-1}(-1)^{|i|}} (\delta_{k,-i} - \delta_{k,l}) x_{kl} x_{ij}$$

$$+ (-1)^{|i||j|+|k|} q^{\delta_{j,-1}(-1)^{|i|}} (\delta_{i,-k} - \delta_{l,-j}) x_{kl} x_{ij}$$

$$+ (-1)^{|i||j|+|k|} q^{\delta_{j,-1}(-1)^{|i|}} (\delta_{i,-k} - \delta_{l,-j}) x_{kl} x_{ij}$$

$$= (-1)^{|k||i|+|j|} q^{\delta_{j+1,-1}(-1)^{|i|}} \xi (\delta_{k,-i} + \delta_{k,-j}) x_{kl} x_{ij}$$

$$+ (-1)^{|i||j|+|k|} q^{\delta_{j,-1}(-1)^{|i|}} (\delta_{i,-k} - \delta_{l,-j}) x_{kl} x_{ij}$$

$$+ \xi (\delta_{i,-k} - \delta_{l,-j}) x_{kl} x_{ij}$$

On the other hand, note that if $\delta_1, \delta_2 \in \{0, 1\}$ then $(\delta_1 - \delta_2)(1 - \delta_1 - \delta_2) = \delta_1 - \delta_2 - \delta_1^2 + \delta_2^2 = 0$, so
\[(1 - \delta_i < k - \delta_j < l) R(kji) = (1 - \delta_i < k - \delta_j < l) \times \]
\[\left( -1 \right)^{|l|(|k| + |j|)} q^\delta_i < k (-1)^{|i|} x_{kj} x_{il} - \left( -1 \right)^{|i|(|k| + |j|)} q^\delta_j < l (-1)^{|l|} x_{il} x_{kj} \right] \]
\[+ \left( -1 \right)^{|i|(|k| + |j|)} q^\delta_i < k (-1)^{|i|} x_{ij} x_{kl} \]
\[= \left( -1 \right)^{|l|(|k| + |j|)} q^\delta_i < k (-1)^{|i|} x_{kl} x_{ij} \]
\[+ \left( -1 \right)^{|i|(|k| + |j|)} q^\delta_i < k (-1)^{|i|} x_{ij} x_{kl} \right].
\]

Similarly
\[R(k, -j, i, -l) = \left( -1 \right)^{|l|(|k| + |j|)} q^\delta_i < k (-1)^{|i|} x_{kj} x_{il} - \left( -1 \right)^{|i|(|k| + |j|)} q^\delta_j < l (-1)^{|l|} x_{il} x_{kj} \]
\[= \left( -1 \right)^{|l|(|k| + |j|)} q^\delta_i < k (-1)^{|i|} x_{ij} x_{kl} \]
\[= \left( -1 \right)^{|l|(|k| + |j|)} q^\delta_i < k (-1)^{|i|} x_{ij} x_{kl} \text{ or } (-1)^{|i|(|k| + |j|)} q^\delta_j < l (-1)^{|l|} x_{il} x_{kj} \right].
\]

Thus,
\[-\left( -1 \right)^{|i|(|k| + |j|)} q^\delta_i < k (-1)^{|i|} R(ijkl) + \left( -1 \right)^{|l|(|k| + |j|)} q^\delta_i < k (-1)^{|i|} R(klji) \]
\[= \xi (-1)^{|k|(|j| + |l|)} (1 - \delta_i < k - \delta_j < l) R(kji) \]
\[= \xi (-1)^{|k|(|j| + |l|)} \left( \delta_i < k - \delta_j < l \right) R(k, -j, i, -l) \text{ or } (-1)^{|i|(|k| + |j|)} q^\delta_j < l (-1)^{|l|} x_{il} x_{kj}.
\]

These dependencies amongst the \(R(ijkl)\) imply that
\[\text{span}_{C(q)} \{ R(ijkl) \mid i, j, k, l \in \mathbb{I} \} = \text{span}_{C(q)} \{ R(ijkl) \mid i, j, k, l \in \mathbb{I} \text{ with } 0 < i \leq k \}.
\]

Note that if \(0 < i \leq k\), then \(R(ijkl)\) simplifies to
\[R(ijkl) = q^\delta_i < k (-1)^{|j|} x_{ij} x_{kl} - q^\delta_j < l (-1)^{|l|} x_{kl} x_{ij} \]
\[= (-1)^{|i|(|k| + |j|)} \xi \delta_j < l x_{kj} x_{il} - (-1)^{|i|(|k| + |j|)} \xi \delta_j < l x_{kl} x_{ij}.
\]

By considering the four possibilities for \((|j|, |l|)\), we obtain the four relations in the definition of \(A_q(n)\). Thus
\[A_q(n) = F_q(n)/\langle R(ijkl) \mid i, j, k, l \in \mathbb{I} \text{ with } 0 < i \leq k \rangle = A_q(n).
\]

It is also possible to prove an analogue of Theorem 5.7 when \(s \neq 0\) using the coalgebra \(A_q(n; r, s)\) that we define immediately below. Set \(x_{ab} = x_{ab} \text{ and } \bar{x}_{ab} = \sqrt{-1} x_{ab}\). The relations in the following definition are super analogues of those in Lemma 4.1 of [4].

**Definition 5.8.** Abbreviate \(x_{ab} \otimes 1\) and \(1 \otimes x_{ab}^\ast\) in \(A_q(n) \otimes_{C(q)} A_q(n)\) by \(x_{ab}\) and \(x_{ab}^\ast\), respectively. Then \(\tilde{A}_q(n)\) is defined to be the quotient of \(A_q(n) \otimes_{C(q)} A_q(n)\) by the two-sided ideal generated by the following:

if \(b \neq d\),
\[\sum_{e=1}^{n} q^{2e} x_{eb} x_{ed}^\ast = 0 = \sum_{e=1}^{n} q^{2e} \bar{x}_{eb} \bar{x}_{ed}^\ast, \sum_{e=1}^{n} q^{2e} x_{eb} \bar{x}_{ed}^\ast = 0 = \sum_{e=1}^{n} q^{2e} \bar{x}_{eb} x_{ed}^\ast.
\]

if \(a \neq c\),
\[\sum_{e=1}^{n} q^{2e} x_{ae} x_{ce}^\ast = 0 = \sum_{e=1}^{n} q^{2e} \bar{x}_{ae} \bar{x}_{ce}^\ast, \sum_{e=1}^{n} q^{2e} x_{ae} \bar{x}_{ce}^\ast = 0 = \sum_{e=1}^{n} q^{2e} \bar{x}_{ae} x_{ce}^\ast.
\]
The relations (5.5)-(5.7) can be deduced by considering the cases $U$.

Remark 5.11. $A_q(n; r, s)$ is defined as the subspace of $\bar{A}_q(n)$ spanned by monomials in the generators of bidegree $(r, s)$; that is, of degree $r$ in the generators $x_{ab}$, $\bar{x}_{ab}$ and of degree $s$ in the generators $x_{ab}^*$, $\bar{x}_{ab}^*$.

Theorem 5.10. There is a vector space isomorphism $A_q(n; r, s) \cong \text{End}_{\mathcal{BC}_{r,s}(q)}(V_q^{r,s})$.

Proof. Most of the necessary computations are already contained in the proof of Theorem 5.7. Recall that $\mathcal{H} = q^{-(2n+1)} \sum_{i,j \in I} (-1)^{|i||j|} q^{2|1-2||j|} E_{ij} \otimes E_{ij}$. We only need to explain where the new relations (5.5)-(5.7) in Definition 5.8 come from, and for this we have to compute the following commutator:

$$\left[ \bigcap \bigcup_{ijkl} a_{ijkl} E_{ij} \otimes E_{kl} \right] = q^{-(2n+1)} \sum_{ijkl} \left( \sum_p a_{ijpl} q^{2p(1-2||j|)} (-1)^{|i||j|+|j||l|} \right) (-1)^{|i||j|} E_{ij} \otimes E_{il}$$

$$-q^{-(2n+1)} \sum_{ijkl} \left( \sum_p a_{ipkj} q^{2p(1-2||j|)} (-1)^{|i||k|+|k||l|} \right) (-1)^{|i||l|} E_{il} \otimes E_{kl}$$

This leads to the relation

$$\delta_{ik} \sum_p (-1)^{|p||l|+|j||l|} q^{2p(1-2||j|)} x_{pj}^* x_{pl}^* = \delta_{lj} \sum_p (-1)^{|i||k|+|i||l|} q^{2p(1-2||j|)} x_{ip}^* x_{kp}^*.$$

The relations (5.5)-(5.7) can be deduced by considering the cases $i \neq k$ and $j = l$; $i = k$ and $j \neq l$; $i = k$ and $j = l$; and also the various possibilities for the signs of $i, j, k, l$.

Remark 5.11. The algebra $A_q(n; r, s)$ could possibly be used to prove the open problem of showing the surjectivity of the map $\mathcal{U}_q(q(n)) \rightarrow \text{End}_{\mathcal{BC}_{r,s}(q)}(V_q^{r,s})$ (see Remark 3.16). The following line of reasoning was applied in [4] to establish a similar surjectivity result for $\mathcal{U}_q(\mathfrak{gl}(n))$ over a quite general base ring. First, it might be possible to obtain a homomorphism $\tau : S_q(n; n') \rightarrow S_q(n; r, s)$ for some $n'$ (possibly $n' = r + s$) via some embedding of the mixed tensor space into $\mathbb{C}(n|n')$. The surjectivity of the map $\mathcal{U}_q(q(n)) \rightarrow \text{End}_{\mathcal{BC}_{r,s}(q)}(V_q^{r,s})$ then would follow from the surjectivity of $\tau$, which is equivalent to the injectivity of $\tau^* : S_q(n; r, s)^* \rightarrow S_q(n; n')^*$. As suggested by [4], the injectivity of $\tau^*$ could perhaps be shown by constructing bases of $A_q(n; r, s)$ and $A_q(n; n')$ using super analogues of bideterminants. For the quantum general linear supergroup, this was accomplished in [8].

Remark 5.12. The algebra $A_q(n)$ is a bialgebra, so it can be enlarged to a Hopf algebra, the so-called Hopf envelope of $A_q(n)$. This is explained in [17] in the context of the quantum general linear supergroup attached to $\mathfrak{gl}(m|n)$. Moreover, it is proved in loc. cit. that the Hopf envelope of $A_q(m|n)$, the quantized algebra of functions on the space of super matrices of size $(m|n) \times (m|n)$, is isomorphic to the localization of $A_q(m|n)$ with respect to the quantum Berezinian, a super analogue of the quantum determinant. This localization is the quantized algebra of functions $\mathbb{C}_q[\mathfrak{GL}_{m|n}]$. This raises the following question: is the Hopf envelope of $A_q(n)$ isomorphic to the localization of $A_q(n)$ with respect to an appropriate super version of the quantum determinant? Such a localization could be thought of as a quantized algebra of functions for the supergroup of type $Q_n$. 
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