

## GLOBAL STABILITY OF THE ENDEMIC EQUILIBRIUM OF MULTIGROUP SIR EPIDEMIC MODELS

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**ABSTRACT.** For a class of multigroup SIR epidemic models with varying subpopulation sizes, we establish that the global dynamics are completely determined by the basic reproduction number  $R_0$ . More specifically, we prove that, if  $R_0 \leq 1$ , then the disease-free equilibrium is globally asymptotically stable; if  $R_0 > 1$ , then there exists a unique endemic equilibrium and it is globally asymptotically stable in the interior of the feasible region. Our proof of global stability utilizes the method of global Lyapunov functions and results from graph theory.

**1 Introduction** Multigroup models have been proposed in the literature to describe the transmission dynamics of infectious diseases in heterogeneous host populations. Heterogeneity in host population can be the result of many factors. Individual hosts can be divided into groups according to different contact patterns such as those among children and adults for Measles and Mumps, or to distinct number of sexual partners for sexually transmitted diseases and HIV/AIDS. Groups can be geographical such as communities, cities, and countries, or epidemiological, to incorporate differential infectivity or co-infection of multiple strains of the disease agent. Multigroup models can also be used to investigate infectious diseases with multiple hosts such as West-Nile virus and vector borne diseases such as Malaria. For a recent survey of multigroup models, we refer the reader to [34].

A multigroup model is, in general, formulated by dividing the population of size  $N(t)$  into  $n$  distinct groups. For  $1 \leq k \leq n$ , the  $k$ -th group is further partitioned into three compartments: the susceptibles, infectious, and recovered, whose numbers of individuals at time  $t$  are

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denoted by  $S_k(t), I_k(t)$  and  $R_k(t)$ , respectively. For  $1 \leq i, j \leq n$ , the disease transmission coefficient between compartments  $S_i$  and  $I_j$  is denoted by  $\beta_{ij}$ , so that the new infection occurred in the  $k$ -th group is given by

$$(1.1) \quad \sum_{j=1}^n \beta_{kj} S_k I_j.$$

The form of incidence in (1.1) is bilinear. Other incidence forms have been used in the literature, depending on the assumptions made about the mixing among different groups. The matrix  $B = (\beta_{ij})_{n \times n}$  is the contact matrix, where  $\beta_{ij} \geq 0$ . Within the  $k$ -th group, it is assumed that natural death occurs in  $S_k, I_k$  and  $R_k$  compartments with rate constants  $d_k^S, d_k^I$  and  $d_k^R$ , respectively. Individuals in  $I_k$  suffer an additional death due to disease with rate constant  $\epsilon_k$ . The influx of individuals into the  $k$ -th group is given by a constant  $\Lambda_k$ , of which a fraction  $p_k$  is assumed to be immuned, and remaining fraction  $1 - p_k$  is susceptible. A simple immunization policy is considered where a fraction  $\theta_k$  of the compartment  $S_k$  is vaccinated. We assume that individuals in  $I_k$  recover with a rate constant  $\gamma_k$ , and once recovered they remain permanently immuned for the disease. Based on these assumptions, the following system of differential equations can be derived:

$$(1.2) \quad \begin{aligned} S'_k &= (1 - p_k)\Lambda_k - (d_k^S + \theta_k)S_k - \sum_{j=1}^n \beta_{kj} S_k I_j, \\ I'_k &= \sum_{j=1}^n \beta_{kj} S_k I_j - (d_k^I + \epsilon_k + \gamma_k)I_k, \\ R'_k &= p_k\Lambda_k + \theta_k S_k + \gamma_k I_k - d_k^R R_k, \end{aligned}$$

where  $k = 1, 2, \dots, n$ . The parameters in the model are summarized in the following list:

- $\beta_{ij}$  : transmission coefficient between compartments  $S_i$  and  $I_j$ ,
- $d_k^S, d_k^I, d_k^R$  : natural death rates of  $S, I, R$  compartments in the  $k$ -th group, respectively,
- $\Lambda_k$  : influx of individuals into the  $k$ -th group,
- $p_k$  : fraction of new individuals into the  $k$ -th group who are immuned,
- $\theta_k$  : fraction of individuals in  $S_k$  who are vaccinated,
- $\gamma_k$  : recovery rate of infectious individuals in the  $k$ -th group,
- $\epsilon_k$  : disease-caused death rate in the  $k$ -th group.

All parameter values are assumed to be nonnegative and  $d_k^S, d_k^I, d_k^R, \Lambda_k > 0$  for all  $k$ .

For each  $k$ , adding the three equations in (1.2), gives

$$\begin{aligned} (S_k + I_k + R_k)' &= \Lambda_k - d_k^S S_k - (d_k^I + \epsilon_k) I_k - d_k^R R_k \\ &\leq \Lambda_k - d_k^*(S_k + I_k + R_k), \end{aligned}$$

where  $d_k^* = \min\{d_k^S, d_k^I + \epsilon_k, d_k^R\}$ . Hence  $\limsup_{t \rightarrow \infty} (S_k + I_k + R_k) \leq \Lambda_k / d_k^*$ . Similarly, it follows from the first equation in (1.2) that

$$\limsup_{t \rightarrow \infty} S_k \leq \frac{(1 - p_k)\Lambda_k}{d_k^S + \theta_k}.$$

Observe that the variable  $R_k$  does not appear in the first two equations of (1.2). This allows us to consider first the following reduced system for  $S_k$  and  $I_k$

$$\begin{aligned} (1.3) \quad S_k' &= (1 - p_k)\Lambda_k - (d_k^S + \theta_k)S_k - \sum_{j=1}^n \beta_{kj} S_k I_j, \\ I_k' &= \sum_{j=1}^n \beta_{kj} S_k I_j - (d_k^I + \epsilon_k + \gamma_k)I_k, \end{aligned}$$

where  $k = 1, 2, \dots, n$ , in the feasible region

$$(1.4) \quad \Gamma = \left\{ (S_1, I_1, \dots, S_n, I_n) \in \mathbb{R}_+^{2n} \mid \begin{aligned} S_k &\leq \frac{\Lambda_k(1 - p_k)}{d_k^S + \theta_k}, \quad S_k + I_k \leq \frac{\Lambda_k}{d_k^*}, \quad k = 1, 2, \dots, n \end{aligned} \right\}.$$

Behaviors of  $R_k$  can then be determined from the last equation in (1.2). It can be verified that  $\Gamma$  in (1.4) is positively invariant with respect to (1.3). Let  $\overset{\circ}{\Gamma}$  denote the interior of  $\Gamma$ . Our results in this paper will be stated for system (1.3) in  $\Gamma$ , and can be translated straightforwardly to system (1.2).

System (1.3) always has the disease-free equilibrium

$$P_0 = (S_1^0, 0, S_2^0, 0, \dots, S_n^0, 0),$$

where

$$(1.5) \quad S_k^0 = \frac{\Lambda_k(1-p_k)}{d_k^S + \theta_k}, \quad k = 1, 2, \dots, n,$$

is the equilibrium of the  $S_k$  population in the absence of disease ( $I_1 = I_2 = \dots = I_n = 0$ ). An endemic equilibrium  $P^* = (S_1^*, I_1^*, S_2^*, I_2^*, \dots, S_n^*, I_n^*)$  of (1.3) belongs to  $\overset{\circ}{\Gamma}$ , namely,  $S_k^* > 0, I_k^* > 0, k = 1, 2, \dots, n$ . Set

$$(1.6) \quad R_0 = \rho(M_0),$$

where

$$(1.7) \quad M_0 = M(S_1^0, S_2^0, \dots, S_n^0) = \left( \frac{\beta_{ij} S_i^0}{d_i^I + \epsilon_i + \gamma_i} \right)_{n \times n},$$

and  $\rho$  denotes the spectral radius. As we will show,  $R_0$  is the key threshold parameter whose values completely characterize the global dynamics of (1.3). Furthermore,  $R_0$  is the basic reproduction number for system (1.3) (see, e.g., [35]).

One of the earliest work on multigroup models is the seminal paper by Lajmanovich and Yorke [24] on a class of SIS multigroup models for the transmission dynamics of Gonorrhoea. A complete analysis of the global dynamics is established. The global stability of the unique endemic equilibrium is proved using a global Lyapunov function. Subsequently, much research has been done on multigroup models, see, e.g., [1, 2, 7, 11–17, 25, 26, 28, 29, 31, 32, 33] and references therein. Due to the large scale and complexity of multigroup models, progresses in the mathematical analysis of their global dynamics have been slow. In particular, the question of uniqueness and global stability of the endemic equilibrium, when the basic reproduction number  $R_0$  is greater than 1, has largely been open. Hethcote [15] proved global stability of the endemic equilibrium for multigroup SIR model without vital dynamics. Beretta and Capasso [2] derived sufficient conditions for global stability of the endemic equilibrium for multigroup SIR model with constant population in each group. Thieme [33] proved global stability of the endemic equilibrium of multigroup SEIRS models under certain restrictions. The most recent result on global stability is Lin and So [27] for a class of SIRS models with constant group sizes, in which they proved that the endemic equilibrium is globally asymptotically stable if the cross-group

contact rates are small or if the recovery rates in each group are small. The complete resolution of the global stability of endemic equilibrium when  $R_0 > 1$  has been a well-known open problem in mathematical epidemiology. On the other hand, results in the opposite direction also exist in the literature. For  $n$ -group SIR models with proportionate incidence, uniqueness of endemic equilibria may not hold when  $R_0 > 1$  (see [19, 34]). In light of these results, complete determination of the global dynamics of these models is essential for their application and further development. It is also crucial to determine how incidence forms or other epidemiological factors influence the uniqueness and global stability of endemic equilibria.

The aim of the present paper is to establish the global dynamics of the class of  $n$ -group SIR models with varying group sizes as defined in (1.2). In particular, we prove that, when  $R_0 > 1$ , the endemic equilibrium of the model is unique and globally asymptotically stable. For this class of multigroup models, our results completely resolve the open problem on the uniqueness and global stability of endemic equilibrium. Our proof relies on the use of a class of global Lyapunov functions. Lyapunov functions of this type has been used in the literature (e.g., see [8], [10], [18], [20]) and was recently rediscovered (e.g., see [22]) and successfully applied to several classes of epidemic models (see [11, 21]). It has the potential for a much wider applicability. The key to our analysis is a complete description of the complicated patterns exhibited in the derivative of the Lyapunov function, using graph theory.

In the next section, we quote some results from graph theory which will be used in the proof of our main results. In Section 3, our main results are stated. In Section 4, the global stability of the disease-free equilibrium when  $R_0 \leq 1$  is proved. In Section 5, the global stability of the endemic equilibrium when  $R_0 > 1$  is proved. In Section 6, we relate our definition of  $R_0$  to earlier definitions in the literature. Extension of our results to multigroup models with other forms of incidence are discussed in Section 7.

**2 Preliminaries** Let  $E = (e_{ij})_{n \times n}$ ,  $F = (f_{ij})_{n \times n}$  be *nonnegative matrices*, namely, all of their entries are nonnegative. We write  $E \geq F$  if  $e_{kj} \geq f_{kj}$  for all  $k$  and  $j$ , and  $E > F$  if  $E \geq F$  and  $E \neq F$ . For  $n > 1$ , an  $n \times n$  matrix  $E$  is *reducible* if, for some permutation matrix  $Q$ ,

$$QEQT = \begin{bmatrix} E_1 & 0 \\ E_2 & E_3 \end{bmatrix},$$

where  $E_1$  and  $E_3$  are square matrices. Otherwise,  $E$  is *irreducible*. The following properties of nonnegative matrices are standard (e.g., see [4]).

- P1. If  $E$  is nonnegative, then the spectral radius  $\rho(E)$  of  $E$  is an eigenvalue, and  $E$  has a nonnegative eigenvector corresponding to  $\rho(E)$ .
- P2. If  $E$  is nonnegative and irreducible, then  $\rho(E)$  is a simple eigenvalue, and  $E$  has a positive eigenvector  $x$  corresponding to  $\rho(E)$ .
- P3. If  $0 \leq E \leq F$ , then  $\rho(E) \leq \rho(F)$ . Moreover, if  $0 \leq E < F$  and  $E + F$  is irreducible, then  $\rho(E) < \rho(F)$ .
- P4. If  $E$  is nonnegative and irreducible, and  $F$  is diagonal and positive (namely, all of its entries are positive), then  $EF$  is irreducible.

Irreducibility of matrices can be easily tested using the associated directed graphs. A *directed graph*  $G_n$  is a set of  $n$  vertices and a set of directed arcs joining two vertices. A directed graph is *strongly connected* if any two distinct vertices are joined by an oriented path. The directed graph  $G(E)$  associated with an  $n \times n$  matrix  $E$  is a directed graph of  $n$  vertices,  $1, 2, \dots, n$ , such that there exists an arc  $(j, k)$  leading from  $j$  to  $k$  if and only if  $e_{kj} \neq 0$ . We have the following property.

- P5. Matrix  $E$  is irreducible if and only if  $G(E)$  is strongly connected.

In Figure 1, examples of reducible and irreducible matrices and their associated directed graphs are shown.

An *oriented cycle* in a directed graph is a simple oriented path from a vertex to itself. A *directed tree* is a connected directed graph with no cycles. A directed tree is said to be *rooted* at a vertex, called root, if every path between a non-root vertex and the root is oriented towards the root. We refer the reader to [23, 29] for more details.

Consider the linear system

$$(2.1) \quad \overline{B}v = 0,$$

where

$$(2.2) \quad \overline{B} = \begin{bmatrix} \sum_{l \neq 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n1} \\ -\bar{\beta}_{12} & \sum_{l \neq 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l \neq n} \bar{\beta}_{nl} \end{bmatrix},$$

and  $\bar{\beta}_{ij} \geq 0$ .

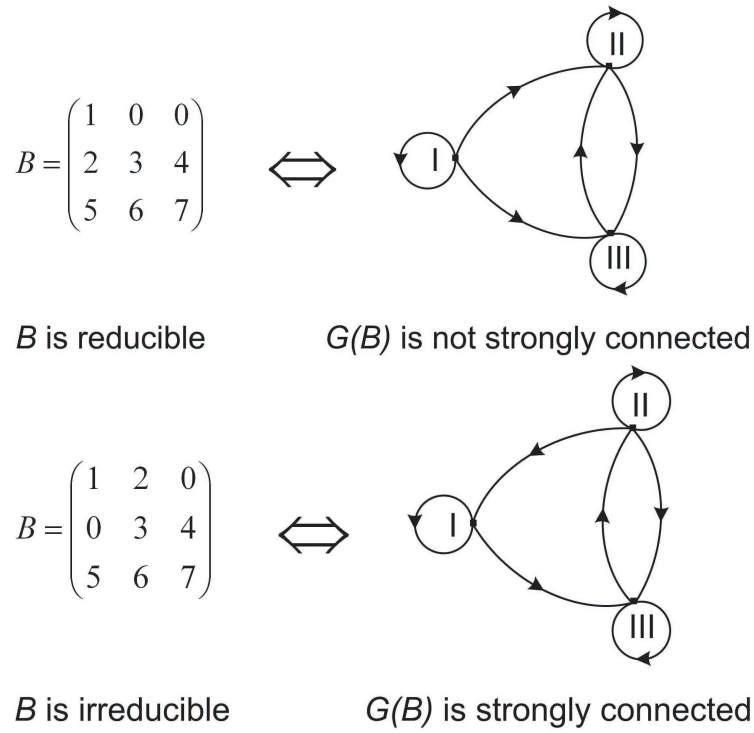


FIGURE 1: Examples of reducible and irreducible matrices and their associated directed graphs.

**Lemma 2.1.** *Assume that the matrix  $(\bar{\beta}_{ij})_{n \times n}$  is irreducible and  $n \geq 2$ . Then the followings hold.*

(1) *The solution space of system (2.1) has dimension 1.*

(2) *A basis of the solution space is given by*

$$(2.3) \quad (v_1, v_2, \dots, v_n) = (C_{11}, C_{22}, \dots, C_{nn}),$$

*where  $C_{kk}$  denotes the cofactor of the  $k$ -th diagonal entry of  $\bar{B}$ ,  $1 \leq k \leq n$ .*

(3) For all  $1 \leq k \leq n$ ,

$$(2.4) \quad C_{kk} = \sum_{T \in \mathbb{T}_k} \prod_{(j,h) \in E(T)} \bar{\beta}_{jh},$$

where  $\mathbb{T}_k$  is the set of all directed trees of  $n$  vertices rooted at the  $k$ -th vertex, and  $E(T)$  denotes the set of arcs in a directed tree  $T$ .

(4) For all  $1 \leq k \leq n$ ,

$$C_{kk} > 0.$$

*Proof.* Since the sum of each column in  $\bar{B}$  equals zero, we have

$$(2.5) \quad C_{jk} = C_{lk}, \quad 1 \leq j, k, l \leq n,$$

where  $C_{jk}$  denotes the cofactor of the  $(j, k)$  entry of  $\bar{B}$ . Since  $\bar{B}$  is singular, we know that  $(C_{11}, C_{12}, \dots, C_{1n})$  is a solution of system (2.1). Therefore, by (2.5),  $(C_{11}, C_{22}, \dots, C_{nn})$  is also a solution of system (2.1).

For  $1 \leq k \leq n$ , in the  $k$ -th column of  $\bar{B}$ , the diagonal entry,  $\sum_{l \neq k} \bar{\beta}_{kl}$ , equals the negative of the sum of nondiagonal entries. By a result on directed graphs in [29, p. 47, Theorem 5.5], we obtain

$$C_{kk} = \sum_{T \in \mathbb{T}_k} \prod_{(j,h) \in E(T)} \bar{\beta}_{jh}.$$

Since  $(\bar{\beta}_{ij})$  is irreducible, its associated directed graph is strongly connected, by P5. Thus, for each  $k$ , at least one term in  $\sum_{T \in \mathbb{T}_k} \prod_{(j,h) \in E(T)} \bar{\beta}_{jh}$  is positive. Therefore,  $C_{kk} > 0$  for  $k = 1, 2, \dots, n$ . Since  $C_{11}$  is a  $(n-1)$  minor of  $\bar{B}$ , we know  $\text{rank}(\bar{B}) = n-1$ , and the solution space of (2.1) has dimension 1, completing the proof of Lemma 2.1.  $\square$

As an illustration of  $C_{kk}$ , let  $n = 3$  and  $\mathbb{T}_1$  be the set of all directed trees rooted at the first vertex. Then, as shown in Figure 2,  $\mathbb{T}_1 = \{T_1^1, T_1^2, T_1^3\}$ , and  $E(T_1^1) = \{(3, 2), (2, 1)\}$ ,  $E(T_1^2) = \{(2, 1), (3, 1)\}$ ,  $E(T_1^3) = \{(2, 3), (3, 1)\}$ . Therefore,

$$(2.6) \quad C_{11} = \sum_{T_1^i \in \mathbb{T}_1} \prod_{(j,h) \in E(T_1^i)} \bar{\beta}_{jh} = \bar{\beta}_{32}\bar{\beta}_{21} + \bar{\beta}_{31}\bar{\beta}_{21} + \bar{\beta}_{23}\bar{\beta}_{31}.$$

A *unicyclic graph* is a directed graph that is obtained from a collection of directed rooted trees by joining their roots to form an oriented cycle.

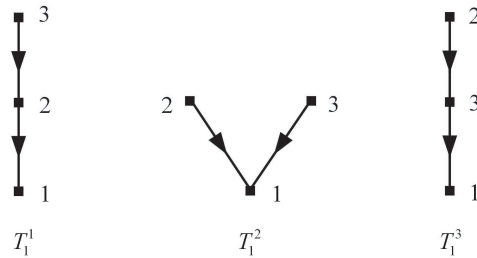


FIGURE 2: All directed trees with three vertices and rooted at 1.

For  $1 \leq l \leq n$ , let  $D(n, l)$  denote the number of unicyclic graphs with  $n$  vertices whose cycle has length  $l$ . Then

$$(2.7) \quad D(n, l) = \binom{n}{l} n^{n-l-1} l!$$

and

$$(2.8) \quad n^n = \sum_{l=1}^n D(n, l)l.$$

For proofs of these relations, we refer the reader to ([3, Chapter 2]). A unicyclic graph with an oriented cycle of length  $l$  can produce  $l$  directed rooted trees by deleting different arcs in the oriented cycle, and the vertex from which the deleted arc starts becomes the root. Conversely, a unicyclic graph is obtained from a directed rooted tree by adding a directed arc from the root to any non-root vertex, see Figure 3.

**3 Main results** System (1.3) is said to be uniformly persistent [6, 36] in  $\overset{\circ}{\Gamma}$  if there exists constant  $c > 0$  such that

$$\liminf_{t \rightarrow \infty} S_k(t) > c \quad \text{and} \quad \liminf_{t \rightarrow \infty} I_k(t) > c, \quad k = 1, \dots, n,$$

provided  $(S_1(0), I_1(0), \dots, S_n(0), I_n(0)) \in \overset{\circ}{\Gamma}$ .

Let  $R_0$  be defined in (1.6). We have the following results.

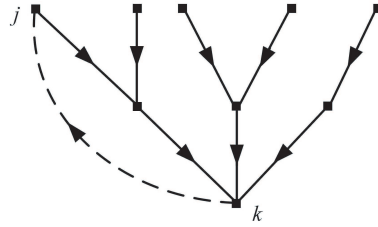


FIGURE 3: Forming a unicyclic graph from a directed tree rooted at vertex  $k$  by adding a directed arc from  $k$  to  $j$ .

**Proposition 3.1.** *Assume  $B = (\beta_{ij})$  is irreducible. Then the following hold.*

- (1) *If  $R_0 \leq 1$ , then  $P_0$  is the unique equilibrium of (1.3) and it is globally stable in  $\Gamma$ .*
- (2) *If  $R_0 > 1$ , then  $P_0$  is unstable and system (1.3) is uniformly persistent in  $\overset{\circ}{\Gamma}$ .*

Results like Proposition 3.1 are known in the literature, at least for some special classes of model (1.3) (see [15, 27, 34]). We will provide a proof in the next section for completeness and to demonstrate our derivation of  $R_0$ .

Uniform persistence of (1.3), together with uniform boundedness of solutions in  $\overset{\circ}{\Gamma}$  (follows from the positive invariance of the bounded region  $\Gamma$ ), implies the existence of an equilibrium of (1.3) in  $\overset{\circ}{\Gamma}$  (see Theorem D.3 in [32] or Theorem 2.8.6 in [5]).

**Corollary 3.2.** *Assume  $B = (\beta_{ij})$  is irreducible. If  $R_0 > 1$ , then (1.3) has at least one endemic equilibrium.*

Denote the endemic equilibrium by

$$P^* = (S_1^*, I_1^*, S_2^*, I_2^*, \dots, S_n^*, I_n^*),$$

$S_k^* > 0$  and  $I_k^* > 0$  for  $k = 1, 2, \dots, n$ . We have the following main result on the uniqueness and global stability of  $P^*$  when  $R_0 > 1$ .

**Theorem 3.3.** *Assume that  $B = (\beta_{ij})$  is irreducible. If  $R_0 > 1$ , then there exists a unique endemic equilibrium  $P^*$ , and  $P^*$  is globally asymptotically stable in  $\overset{\circ}{\Gamma}$ .*

Proposition 3.1 and Theorem 3.3 completely determine the global dynamics of (1.3) and those of (1.2). They establish the basic reproduction number  $R_0$  as a sharp threshold parameter. Biologically, our results imply that, if  $R_0 \leq 1$ , then the disease always dies out from all groups; if  $R_0 > 1$ , then the disease always persists at the unique endemic equilibrium level in all groups, irrespective of the initial conditions.

The proof of Proposition 3.1 will be given in the next section, and that of Theorem 3.3 in Section 5.

**4 Proof of Proposition 3.1** Let  $S = (S_1, S_2, \dots, S_n)$  and  $S^0 = (S_1^0, S_2^0, \dots, S_n^0)$ . Then  $M_0 = M(S^0)$ . For  $1 \leq k \leq n$ ,  $0 \leq S_k \leq S_k^0$ , we have  $0 \leq M(S) \leq M(S_0) = M_0$ . If  $S \neq S^0$ , then  $M(S) < M_0$ . On the other hand, since  $B$  is irreducible, we know  $M(S)$  and  $M_0$  are irreducible. Furthermore,  $M(S) + M_0$  is also irreducible. Therefore,  $\rho(M(S)) < \rho(M_0)$  provided  $S \neq S^0$ , by P3 of Section 2.

If  $R_0 = \rho(M_0) \leq 1$  and  $S \neq S^0$ , then  $\rho(M(S)) < 1$ , and

$$M(S)I = I$$

has only the trivial solution  $I = 0$ . Thus  $P_0$  is the only equilibrium of system (1.3) in  $\Gamma$  if  $R_0 \leq 1$ .

Let  $(\omega_1, \omega_2, \dots, \omega_n)$  be a left eigenvector of  $M_0$  corresponding to  $\rho(M_0)$ , i.e.,

$$(\omega_1, \omega_2, \dots, \omega_n) \rho(M_0) = (\omega_1, \omega_2, \dots, \omega_n)M_0.$$

Since  $M_0$  is irreducible, we know  $\omega_k > 0$  for  $k = 1, 2, \dots, n$ , by P2 of Section 2. Set

$$L = \sum_{k=1}^n \frac{\omega_k}{d_k^I + \epsilon_k + \gamma_k} I_k.$$

Differentiation gives

$$\begin{aligned} L' &= \sum_{k=1}^n \frac{\omega_k}{d_k^I + \epsilon_k + \gamma_k} I_k' = (\omega_1, \omega_2, \dots, \omega_n) [M(S)I - I] \\ &\leq (\omega_1, \omega_2, \dots, \omega_n) [M_0I - I] \end{aligned}$$

$$= [\rho(M_0) - 1](\omega_1, \omega_2, \dots, \omega_n)I \leq 0, \quad \text{if } R_0 \leq 1.$$

If  $R_0 = \rho(M_0) < 1$ , then  $L' = 0 \Leftrightarrow I = 0$ . If  $R_0 = 1$ , then  $L' = 0$  implies

$$(4.1) \quad (\omega_1, \omega_2, \dots, \omega_n)M(S)I = (\omega_1, \omega_2, \dots, \omega_n)I.$$

If  $S \neq S^0$ , then

$$(\omega_1, \omega_2, \dots, \omega_n)M(S) < (\omega_1, \omega_2, \dots, \omega_n)M_0 = (\omega_1, \omega_2, \dots, \omega_n).$$

Thus (4.1) has only the trivial solution  $I = 0$ . Therefore,  $L' = 0 \Leftrightarrow I = 0$  or  $S = S^0$  provided  $R_0 \leq 1$ . It can be verified that the only compact invariant subset of the set where  $L' = 0$  is the singleton  $\{P_0\}$ . By LaSalle's Invariance Principle [25],  $P_0$  is globally asymptotically stable in  $\Gamma$  if  $R_0 \leq 1$ .

If  $R_0 = \rho(M_0) > 1$  and  $I \neq 0$ , we know that

$$\begin{aligned} & (\omega_1, \omega_2, \dots, \omega_n)M_0 - (\omega_1, \omega_2, \dots, \omega_n) \\ &= [\rho(M_0) - 1](\omega_1, \omega_2, \dots, \omega_n) > 0, \end{aligned}$$

and thus  $L' = (\omega_1, \omega_2, \dots, \omega_n)[M(S)I - I] > 0$  in a neighborhood of  $P_0$  in  $\overset{\circ}{\Gamma}$ , by continuity. This implies  $P_0$  is unstable. Using a uniform persistence result from [9] and a similar argument as in the proof of Proposition 3.3 of [26], we can show that, when  $R_0 > 1$ , the instability of  $P_0$  implies the uniform persistence of (1.3). This completes the proof of Proposition 3.1.

**5 Proof of Theorem 3.3** In this section, we prove that the endemic equilibrium  $P^*$  is globally asymptotically stable when  $R_0 > 1$ . In particular, this implies that the endemic equilibrium is unique in the region  $\overset{\circ}{\Gamma}$  when it exists.

Let  $(S_1^*, I_1^*, S_2^*, I_2^*, \dots, S_n^*, I_n^*) \in \overset{\circ}{\Gamma}$  be an endemic equilibrium. Set

$$(5.1) \quad \bar{\beta}_{ij} = \beta_{ij}S_i^*I_j^*, \quad 1 \leq i, j \leq n, \quad n \geq 2,$$

and

$$(5.2) \quad \bar{B} = \begin{bmatrix} \sum_{l \neq 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n1} \\ -\bar{\beta}_{12} & \sum_{l \neq 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l \neq n} \bar{\beta}_{nl} \end{bmatrix}.$$

Then, by Lemma 2.1, a basis for the solution space of the linear system

$$(5.3) \quad \overline{B}v = 0$$

can be written as

$$(5.4) \quad (v_1, \dots, v_n) = (C_{11}, \dots, C_{nn}),$$

where  $C_{kk}$  denotes the cofactor of the  $k$ -th diagonal entry of  $\overline{B}$ ,  $1 \leq k \leq n$ . By the irreducibility of  $B$ , we know that  $(\beta_{ij})$  is irreducible and  $v_k = C_{kk} > 0$ ,  $k = 1, \dots, n$ , by Lemma 2.1.

For  $n = 1$ , i.e., the case of single group SIR model, Theorem 3.3 is well known (e.g., see [14]). We only consider the case  $n \geq 2$ . Let  $v_1, \dots, v_n$  be chosen as in (5.4). Set

$$(5.5) \quad V = \sum_{k=1}^n v_k (S_k - S_k^* \ln S_k + I_k - I_k^* \ln I_k).$$

Differentiating  $V$  and using the equilibrium equations

$$(5.6) \quad (1 - p_k)\Lambda_k = (d_k^S + \theta_k)S_k^* + \sum_{j=1}^n \beta_{kj}S_k^*I_j^*,$$

and

$$(5.7) \quad (d_k^I + \epsilon_k + \gamma_k)I_k^* = \sum_{j=1}^n \beta_{kj}S_k^*I_j^*,$$

we obtain

$$\begin{aligned} V' &= \sum_{k=1}^n v_k \left( S_k' - \frac{S_k^*}{S_k} S_k' + I_k' - \frac{I_k^*}{I_k} I_k' \right) \\ &= \sum_{k=1}^n v_k \left[ (1 - p_k)\Lambda_k - (d_k^S + \theta_k)S_k - \sum_{j=1}^n \beta_{kj}S_k I_j \right. \\ &\quad \left. - (1 - p_k)\Lambda_k \frac{S_k^*}{S_k} + (d_k^S + \theta_k)S_k^* + \sum_{j=1}^n \beta_{kj}S_k^* I_j \right. \\ &\quad \left. + \sum_{j=1}^n \beta_{kj}S_k I_j - (d_k^I + \epsilon_k + \gamma_k)I_k \right. \\ &\quad \left. - \sum_{j=1}^n \beta_{kj}S_k I_j \frac{I_k^*}{I_k} + (d_k^I + \epsilon_k + \gamma_k)I_k^* \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^n v_k \left[ -S_k^*(d_k^S + \theta_k) \left( \frac{S_k^*}{S_k} + \frac{S_k}{S_k^*} - 2 \right) \right. \\
&\quad + \left( \sum_{j=1}^n \beta_{kj} S_k^* I_j - (d_k^I + \epsilon_k + \gamma_k) I_k \right) \\
&\quad \left. + \left( 2 \sum_{j=1}^n \beta_{kj} S_k^* I_j^* - \sum_{j=1}^n \beta_{kj} I_j^* \frac{(S_k^*)^2}{S_k} - \sum_{j=1}^n \beta_{kj} S_k I_j \frac{I_k^*}{I_k} \right) \right].
\end{aligned}$$

Since

$$\frac{S_k^*}{S_k} + \frac{S_k}{S_k^*} - 2 \geq 0,$$

we have

$$(5.8) \quad -S_k^*(d_k^S + \theta_k) \left( \frac{S_k^*}{S_k} + \frac{S_k}{S_k^*} - 2 \right) \leq 0,$$

and the equal sign holds if and only if  $S_k = S_k^*$ . Next, we claim

$$(5.9) \quad \sum_{k=1}^n v_k \left( \sum_{j=1}^n \beta_{kj} S_k^* I_j - (d_k^I + \epsilon_k + \gamma_k) I_k \right) = 0,$$

for all  $(I_1, I_2, \dots, I_n) \in \mathbb{R}_+^n$ . To see this, we note that

$$(5.10) \quad \sum_{k=1}^n v_k \sum_{j=1}^n \beta_{kj} S_k^* I_j = \sum_{j=1}^n v_j \sum_{k=1}^n \beta_{jk} S_j^* I_k = \sum_{k=1}^n \left( \sum_{j=1}^n \beta_{jk} S_j^* v_j \right) I_k.$$

It suffices to show

$$\sum_{j=1}^n \beta_{jk} S_j^* v_j = v_k (d_k^I + \epsilon_k + \gamma_k), \quad k = 1, 2, \dots, n.$$

In fact, by (5.2), (5.3) and (5.7), we have

$$\begin{bmatrix} \beta_{11} S_1^* I_1^* & \beta_{21} S_2^* I_1^* & \cdots & \beta_{n1} S_n^* I_1^* \\ \beta_{12} S_1^* I_2^* & \beta_{22} S_2^* I_2^* & \cdots & \beta_{n2} S_n^* I_2^* \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1n} S_1^* I_n^* & \beta_{2n} S_2^* I_n^* & \cdots & \beta_{nn} S_n^* I_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^n \beta_{1j} S_1^* I_j^* v_1 \\ \sum_{j=1}^n \beta_{2j} S_2^* I_j^* v_2 \\ \vdots \\ \sum_{j=1}^n \beta_{nj} S_n^* I_j^* v_n \end{bmatrix} = \begin{bmatrix} (d_1^I + \epsilon_1 + \gamma_1) I_1^* v_1 \\ (d_2^I + \epsilon_2 + \gamma_2) I_2^* v_2 \\ \vdots \\ (d_n^I + \epsilon_n + \gamma_n) I_n^* v_n \end{bmatrix},$$

and thus

$$\begin{bmatrix} \beta_{11} S_1^* & \beta_{21} S_2^* & \cdots & \beta_{n1} S_n^* \\ \beta_{12} S_1^* & \beta_{22} S_2^* & \cdots & \beta_{n2} S_n^* \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1n} S_1^* & \beta_{2n} S_2^* & \cdots & \beta_{nn} S_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} (d_1^I + \epsilon_1 + \gamma_1) v_1 \\ (d_2^I + \epsilon_2 + \gamma_2) v_2 \\ \vdots \\ (d_n^I + \epsilon_n + \gamma_n) v_n \end{bmatrix}.$$

Using inequality (5.8), identity (5.9), and notation  $\bar{\beta}_{kj}$  as given in (5.1), we have

$$\begin{aligned} (5.11) \quad V' &\leq \sum_{k=1}^n v_k \left( 2 \sum_{j=1}^n \bar{\beta}_{kj} - \sum_{j=1}^n \bar{\beta}_{kj} \frac{S_k^*}{S_k} - \sum_{j=1}^n \bar{\beta}_{kj} \frac{I_j S_k I_k^*}{I_k S_k^* I_j^*} \right) \\ &= \sum_{k,j=1}^n v_k \bar{\beta}_{kj} \left( 2 - \frac{S_k^*}{S_k} - \frac{I_j S_k I_k^*}{I_k S_k^* I_j^*} \right). \end{aligned}$$

Denote

$$\begin{aligned} (5.12) \quad H_n &= H_n(S_1, I_1, S_2, I_2, \dots, S_n, I_n) \\ &= \sum_{k,j=1}^n v_k \bar{\beta}_{kj} \left( 2 - \frac{S_k^*}{S_k} - \frac{I_j S_k I_k^*}{I_k S_k^* I_j^*} \right). \end{aligned}$$

In the following we show  $H_n \leq 0$  for all  $(S_1, I_1, S_2, I_2, \dots, S_n, I_n) \in \mathring{\Gamma}$ . Since the proof for the general case is rather complicated, we first give detailed proofs for  $n = 2$  and  $n = 3$ . Then major steps of the proof for arbitrary  $n \geq 2$  is given.

**Case  $n = 2$ :** We have

$$H_2 = H_2(S_1, I_1, S_2, I_2) = \sum_{k,j=1}^2 v_k \bar{\beta}_{kj} \left( 2 - \frac{S_k^*}{S_k} - \frac{I_j S_k I_k^*}{I_k S_k^* I_j^*} \right).$$

From (2.3) we obtain  $v_1 = \bar{\beta}_{21}$  and  $v_2 = \bar{\beta}_{12}$ . Expanding  $H_2$  gives

$$(5.13) \quad H_2 = \bar{\beta}_{21}\bar{\beta}_{11} \left( 2 - \frac{S_1^*}{S_1} - \frac{I_1 S_1 I_1^*}{I_1 S_1^* I_1^*} \right) + \bar{\beta}_{21}\bar{\beta}_{12} \left( 2 - \frac{S_1^*}{S_1} - \frac{I_2 S_1 I_1^*}{I_1 S_1^* I_2^*} \right) \\ + \bar{\beta}_{12}\bar{\beta}_{21} \left( 2 - \frac{S_2^*}{S_2} - \frac{I_1 S_2 I_2^*}{I_2 S_2^* I_1^*} \right) + \bar{\beta}_{12}\bar{\beta}_{22} \left( 2 - \frac{S_2^*}{S_2} - \frac{I_2 S_2 I_2^*}{I_2 S_2^* I_2^*} \right).$$

Write the subindices of  $\bar{\beta}_{ij}$ 's in (5.13) in the form

$$(5.14) \quad \left\{ \begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right\}, \quad \left\{ \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right\}, \quad \left\{ \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{cc} 1 & 2 \\ 2 & 2 \end{array} \right\},$$

respectively. Each expression in (5.14) defines a transformation from row 1 to row 2. Observe that each transformation in (5.14) possesses one cycle of length 1 or 2. Moreover, both terms in (5.13) corresponding to the transformations with a 2-cycle (i.e., a cycle with length 2) have the same coefficients. Therefore, the sum  $H_2$  can be naturally grouped according to the length of cycles appearing in the subindices of  $\bar{\beta}_{ij}$ 's.

$$H_2 = \bar{\beta}_{21}\bar{\beta}_{11} \left( 2 - \frac{S_1^*}{S_1} - \frac{S_1}{S_1^*} \right) + \bar{\beta}_{12}\bar{\beta}_{22} \left( 2 - \frac{S_2^*}{S_2} - \frac{S_2}{S_2^*} \right) \\ + \bar{\beta}_{21}\bar{\beta}_{12} \left( 4 - \frac{S_1^*}{S_1} - \frac{I_2 S_1 I_1^*}{I_1 S_1^* I_2^*} - \frac{S_2^*}{S_2} - \frac{I_1 S_2 I_2^*}{I_2 S_2^* I_1^*} \right).$$

Note that

$$\bar{\beta}_{21}\bar{\beta}_{11} \left( 2 - \frac{S_1^*}{S_1} - \frac{S_1}{S_1^*} \right) \leq 0, \\ \bar{\beta}_{12}\bar{\beta}_{22} \left( 2 - \frac{S_2^*}{S_2} - \frac{S_2}{S_2^*} \right) \leq 0,$$

and

$$(5.15) \quad \bar{\beta}_{21}\bar{\beta}_{12} \left( 4 - \frac{S_1^*}{S_1} - \frac{I_2 S_1 I_1^*}{I_1 S_1^* I_2^*} - \frac{S_2^*}{S_2} - \frac{I_1 S_2 I_2^*}{I_2 S_2^* I_1^*} \right) \leq 0,$$

we obtain  $H_2(S_1, I_1, S_2, I_2) \leq 0$  for all  $(S_1, I_1, S_2, I_2) \in \overset{\circ}{\Gamma}$ . From (5.8), we know  $V' = 0$  if and only if  $S_k = S_k^*$ ,  $k = 1, 2$ , and  $H_2 = 0$ . Moreover, irreducibility of matrix  $B$ , or equivalently, the strong connectedness of the directed graph  $G(B)$ , implies  $\bar{\beta}_{21}\bar{\beta}_{12} > 0$ . Consequently, we obtain from (5.15)

$$V' = 0 \iff S_k = S_k^*, \quad I_k = aI_k^*, \quad k = 1, 2,$$

where  $a$  is an arbitrary positive number.

**Case  $n = 3$ :** We have

$$(5.16) \quad H_3 = H_3(S_1, I_1, S_2, I_2, S_3, I_3) \\ = \sum_{k,j=1}^3 v_k \bar{\beta}_{kj} \left( 2 - \frac{S_k^*}{S_k} - \frac{I_j S_k I_k^*}{I_k S_k^* I_j^*} \right).$$

From (2.3) we obtain

$$(5.17) \quad v_1 = \bar{\beta}_{32} \bar{\beta}_{21} + \bar{\beta}_{31} \bar{\beta}_{21} + \bar{\beta}_{23} \bar{\beta}_{31}, \\ v_2 = \bar{\beta}_{31} \bar{\beta}_{12} + \bar{\beta}_{13} \bar{\beta}_{32} + \bar{\beta}_{12} \bar{\beta}_{32}, \\ v_3 = \bar{\beta}_{12} \bar{\beta}_{23} + \bar{\beta}_{21} \bar{\beta}_{13} + \bar{\beta}_{13} \bar{\beta}_{23}.$$

Substituting expressions of  $v_k$  in (5.17) into  $H_3$ , we observe that  $H_3$  is the sum of  $3^3 = 27$  terms of forms

$$(5.18) \quad \bar{\beta}_{rl} \bar{\beta}_{lk} \bar{\beta}_{kj} \left( 2 - \frac{S_k^*}{S_k} - \frac{I_j S_k I_k^*}{I_k S_k^* I_j^*} \right),$$

or

$$(5.19) \quad \bar{\beta}_{rk} \bar{\beta}_{lk} \bar{\beta}_{kj} \left( 2 - \frac{S_k^*}{S_k} - \frac{I_j S_k I_k^*}{I_k S_k^* I_j^*} \right),$$

where  $\{r, l, k\}$  is a permutation of  $\{1, 2, 3\}$ , and  $1 \leq j \leq 3$ . Write the subindices of  $\bar{\beta}_{ij}$ 's in (5.18) and (5.19) in the form of transformations

$$(5.20) \quad \left\{ \begin{matrix} r & l & k \\ l & k & j \end{matrix} \right\} \quad \text{and} \quad \left\{ \begin{matrix} r & l & k \\ k & k & j \end{matrix} \right\},$$

respectively. When  $j = k, l$  or  $r$ , both transformations in (5.20) possesses cycles of length 1, 2 or 3. The terms in  $H_3$  will be grouped together according to the length of cycles appearing in (5.20).

When  $j = k$ , both transformations in (5.20) have a 1 cycle  $\{ \begin{smallmatrix} * & * & k \\ * & * & k \end{smallmatrix} \}$ , and accordingly, the terms in (5.18) and (5.19) satisfy

$$\bar{\beta}_{rl} \bar{\beta}_{lk} \bar{\beta}_{kk} \left( 2 - \frac{S_k^*}{S_k} - \frac{I_k S_k I_k^*}{I_k S_k^* I_k^*} \right) = \bar{\beta}_{rl} \bar{\beta}_{lk} \bar{\beta}_{kk} \left( 2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*} \right) \leq 0,$$

and

$$\bar{\beta}_{rk}\bar{\beta}_{lk}\bar{\beta}_{kk}\left(2 - \frac{S_k^*}{S_k} - \frac{I_k S_k I_k^*}{I_k S_k^* I_k^*}\right) = \bar{\beta}_{rk}\bar{\beta}_{lk}\bar{\beta}_{kk}\left(2 - \frac{S_k^*}{S_k} - \frac{S_k}{S_k^*}\right) \leq 0.$$

When  $j = r$ , the first transformation in (5.20) produces two distinct 3-cycle patterns  $\left\{\begin{smallmatrix} r & l & k \\ l & k & r \end{smallmatrix}\right\}$  and  $\left\{\begin{smallmatrix} r & k & l \\ k & l & r \end{smallmatrix}\right\}$ . There are 6 terms in  $H_3$  of a 3-cycle form, three of them correspond to each cycle pattern, and thus have the same coefficients  $\bar{\beta}_{rl}\bar{\beta}_{lk}\bar{\beta}_{kr}$  or  $\bar{\beta}_{rk}\bar{\beta}_{kl}\bar{\beta}_{lr}$ . These six terms can be divided into two groups and each has a sum of form

$$\begin{aligned} (5.21) \quad & \bar{\beta}_{rl}\bar{\beta}_{lk}\bar{\beta}_{kr}\left(2 - \frac{S_k^*}{S_k} - \frac{I_r S_k I_k^*}{I_k S_k^* I_r^*}\right) + \bar{\beta}_{lk}\bar{\beta}_{kr}\bar{\beta}_{rl}\left(2 - \frac{S_r^*}{S_r} - \frac{I_l S_r I_r^*}{I_r S_r^* I_l^*}\right) \\ & + \bar{\beta}_{kr}\bar{\beta}_{rl}\bar{\beta}_{lk}\left(2 - \frac{S_l^*}{S_l} - \frac{I_k S_l I_l^*}{I_l S_l^* I_k^*}\right) \\ & = \bar{\beta}_{rl}\bar{\beta}_{lk}\bar{\beta}_{kr}\left(6 - \frac{S_k^*}{S_k} - \frac{I_r S_k I_k^*}{I_k S_k^* I_r^*} - \frac{S_r^*}{S_r} \right. \\ & \quad \left. - \frac{I_l S_r I_r^*}{I_r S_r^* I_l^*} - \frac{S_l^*}{S_l} - \frac{I_k S_l I_l^*}{I_l S_l^* I_k^*}\right) \leq 0. \end{aligned}$$

When  $j = r$ , the second transformation in (5.20) has a 2-cycle  $\left\{\begin{smallmatrix} r & * & k \\ k & * & r \end{smallmatrix}\right\}$ . Also, when  $j = l$ , both transformations in (5.20) have a 2 cycle  $\left\{\begin{smallmatrix} * & l & k \\ * & k & l \end{smallmatrix}\right\}$ . There are altogether 12 terms in  $H_3$  corresponding to 2-cycle patterns. Each 2-cycle pattern corresponds to 2 terms in  $H_3$  with the same coefficients (products of  $\bar{\beta}$ 's). These 12 terms can be grouped into 6 pairs and each has a sum of form

$$\begin{aligned} (5.22) \quad & \bar{\beta}_{rk}\bar{\beta}_{lk}\bar{\beta}_{kr}\left(2 - \frac{S_k^*}{S_k} - \frac{I_r S_k I_k^*}{I_k S_k^* I_r^*}\right) \\ & + \bar{\beta}_{kr}\bar{\beta}_{lk}\bar{\beta}_{rk}\left(2 - \frac{S_r^*}{S_r} - \frac{I_k S_r I_r^*}{I_r S_r^* I_k^*}\right) \\ & = \bar{\beta}_{rk}\bar{\beta}_{lk}\bar{\beta}_{kr}\left(4 - \frac{S_k^*}{S_k} - \frac{I_r S_k I_k^*}{I_k S_k^* I_r^*} - \frac{S_r^*}{S_r} - \frac{I_k S_r I_r^*}{I_r S_r^* I_k^*}\right) \leq 0, \end{aligned}$$

or

$$(5.23) \quad \bar{\beta}_{rl}\bar{\beta}_{lk}\bar{\beta}_{kl}\left(2 - \frac{S_k^*}{S_k} - \frac{I_l S_k I_k^*}{I_k S_k^* I_l^*}\right) + \bar{\beta}_{rl}\bar{\beta}_{kl}\bar{\beta}_{lk}\left(2 - \frac{S_l^*}{S_l} - \frac{I_k S_l I_l^*}{I_l S_l^* I_k^*}\right)$$

$$= \bar{\beta}_{rl}\bar{\beta}_{lk}\bar{\beta}_{kl} \left( 4 - \frac{S_k^*}{S_k} - \frac{I_l S_k I_k^*}{I_k S_k^* I_l^*} - \frac{S_l^*}{S_l} - \frac{I_k S_l I_l^*}{I_l S_l^* I_k^*} \right) \leq 0.$$

In summary, each term in  $H_3$  corresponds to a transformation in (5.20) which possesses a unique cycle of length 1, 2 or 3. The number of transformations in (5.20) with an  $l$ -cycle is given by  $D(n, l) \times l$ ,  $l = 1, 2, 3$ . In particular, by (2.7), the number of 1-cycles in (5.20) is  $D(3, 1) \times 1 = 9$ , the number of 2-cycles is  $D(3, 2) \times 2 = 12$ , and the number of 3-cycles is  $D(3, 3) \times 3 = 6$ . Therefore, by (2.8),

$$3^3 = 27 = D(3, 1) \times 1 + D(3, 2) \times 2 + D(3, 3) \times 3.$$

This shows that all terms in  $H_3$  are accounted for in our grouping according to cycle patterns and lengths in (5.20). Therefore, we have shown  $H_3(S_1, I_1, S_2, I_2, S_3, I_3) \leq 0$  for all  $(S_1, I_1, S_2, I_2, S_3, I_3) \in \overset{\circ}{\Gamma}$ . From (5.8) we know  $V' = 0$  if and only if  $S_k = S_k^*$ ,  $k = 1, 2, 3$ , and  $H_3 = 0$ . We claim that if  $S_k = S_k^*$ ,  $k = 1, 2, 3$ , then

$$(5.24) \quad H_3 = 0 \iff I_k = aI_k^*, \quad k = 1, 2, 3,$$

where  $a$  is an arbitrary positive number. It suffices to show that  $H_3 = 0$  implies

$$(5.25) \quad \frac{I_k}{I_k^*} = \frac{I_r}{I_r^*}, \quad 1 \leq k, r \leq 3.$$

If  $\beta_{kr} = 0$ , for some  $k \neq r$ , then, by the irreducibility of  $B = (\beta_{ij})$ , or equivalently, the strong connectedness of the  $G(B)$ , we necessarily have  $\bar{\beta}_{kl}\bar{\beta}_{lr} \neq 0$  for  $l \neq k, r$ . Therefore, either a 3-cycle  $\left\{ \begin{smallmatrix} k & l & r \\ l & r & k \end{smallmatrix} \right\}$  exists or both 2-cycles  $\left\{ \begin{smallmatrix} k & l & * \\ l & k & * \end{smallmatrix} \right\}$  and  $\left\{ \begin{smallmatrix} l & r & * \\ r & l & * \end{smallmatrix} \right\}$  exist. In either case, (5.25) follows from  $H_3 = 0$ , and from relations (5.21), (5.22) and (5.23). If all  $\beta_{ij} \neq 0$ ,  $i \neq j$ , then  $\bar{\beta}_{kl}\bar{\beta}_{lr} \neq 0$  for  $l \neq k, r$ , and the same argument shows that (5.25) holds. We thus obtain

$$V' = 0 \iff S_k = S_k^*, \quad I_k = aI_k^*, \quad k = 1, 2, 3,$$

where  $a$  is an arbitrary positive number.

**Case  $n \geq 2$ :** We have

$$(5.26) \quad \begin{aligned} H_n &= H_n(S_1, I_1, S_2, I_2, \dots, S_n, I_n) \\ &= \sum_{k,j=1}^n v_k \bar{\beta}_{kj} \left( 2 - \frac{S_k^*}{S_k} - \frac{I_j S_k I_k^*}{I_k S_k^* I_j^*} \right). \end{aligned}$$

Here  $v_k = C_{kk}$  as given in (2.4) is a sum of  $n^{n-2}$  terms, each of which is a product of  $n - 1$   $\bar{\beta}_{ij}$ 's whose subindices can be represented by all arcs in a directed tree  $T$  rooted at the  $k$ -th vertex, by Lemma 2.1(3). Therefore,  $v_k \bar{\beta}_{kj}$  is the sum of  $n^{n-2}$  terms. Each term in  $v_k \bar{\beta}_{kj}$  is a product of  $n$   $\bar{\beta}_{ij}$ 's whose subindices define transformations as in (5.14) and (5.20), and are represented by directed arcs in a unicyclic graph  $Q$ . From our discussion at the end of Section 2, we know  $Q$  is formed by adding an oriented arc  $(k, j)$  to the directed tree  $T$ . Each unicyclic graph  $Q$  has a unique cycle  $CQ$  of length  $1 \leq l \leq n$ . Furthermore, there are  $l$  terms in  $H_n$ , whose coefficients correspond to all  $l$  rotations of the same  $l$ -cycle and thus are the same, are naturally grouped together. We can show, as in the cases of  $n = 2, 3$ , the sum of these  $l$  terms is nonpositive. More specifically, we can group all terms in  $H_n$  according to the length of cycles, and further group the terms of the same cycle length according to their cycle patterns and using (2.4), as shown in the following.

$$\begin{aligned}
 (5.27) \quad H_n &= \sum_{k,j=1}^n v_k \bar{\beta}_{kj} \left( 2 - \frac{S_k^*}{S_k} - \frac{I_j S_k I_k^*}{I_k S_k^* I_j^*} \right) \\
 &= \sum_{l=1}^n \left[ \sum_{Q \in \mathcal{D}(n,l)} \sum_{(r,m) \in E(CQ)} \left( \prod_{(j,h) \in E(Q)} \bar{\beta}_{jh} \right) \right. \\
 &\quad \left. \times \left( 2 - \frac{S_r^*}{S_r} - \frac{S_r I_m I_r^*}{S_r^* I_r I_m^*} \right) \right] \\
 &= \sum_{l=1}^n \left[ \sum_{Q \in \mathcal{D}(n,l)} \left( \prod_{(j,h) \in E(Q)} \bar{\beta}_{jh} \right) \right. \\
 &\quad \left. \times \sum_{(r,m) \in E(CQ)} \left( 2 - \frac{S_r^*}{S_r} - \frac{S_r I_m I_r^*}{S_r^* I_r I_m^*} \right) \right],
 \end{aligned}$$

where  $\mathcal{D}(n, l)$  presents the set of all unicyclic graphs of  $n$  vertices with an oriented cycle of length  $l$ ,  $CQ$  is the oriented cycle with length  $l$  in a unicyclic graph  $Q \in \mathcal{D}(n, l)$ , and  $E(CQ)$  and  $E(Q)$  represent the sets of arcs in  $CQ$  and  $Q$ , respectively. Since the cardinality of  $E(Q)$  is  $n$ , the coefficient of each term in (5.27),  $\prod_{(j,h) \in E(Q)} \bar{\beta}_{jh}$ , is a product of  $n$   $\bar{\beta}_{ij}$ 's. The cardinality of the set  $\mathcal{D}(n, l)$  is

$$(5.28) \quad D(n, l) = \binom{n}{l} n^{n-l-1} l!,$$

and the cardinality of  $E(CQ)$  is the length  $l$  of the cycle  $CQ$ . By the identity (2.8)

$$(5.29) \quad n^n = \sum_{l=1}^n D(n, l)l,$$

we see that all terms in  $H_n$  are accounted for in our grouping (5.27). For the oriented cycle  $CQ$  in any  $Q \in \mathcal{D}(n, l)$ , we have

$$(5.30) \quad \sum_{(r,m) \in E(CQ)} \left( 2 - \frac{S_r^*}{S_r} - \frac{S_r I_m I_r^*}{S_r^* I_r I_m^*} \right) \leq 0.$$

By (5.27) and (5.30), we know  $H_n(S_1, I_1, S_2, I_2, \dots, S_n, I_n) \leq 0$  for all  $(S_1, I_1, S_2, I_2, \dots, S_n, I_n) \in \overset{\circ}{\Gamma}$ . Therefore, we have

$$V' \leq 0.$$

We claim that if  $S_k = S_k^*$ ,  $1 \leq k \leq n$ , then

$$(5.31) \quad H_n = 0 \iff I_k = aI_k^*, \quad k = 1, 2, \dots, n,$$

where  $a$  is an arbitrary positive number. It suffices to show that  $I_k/I_k^* = I_r/I_r^*$  when  $\bar{\beta}_{kr} \neq 0$ . By the irreducibility of  $(\beta_{ij})$ , there exists  $1 \leq m_1, m_2, \dots, m_s \leq n$ ,  $0 \leq s \leq n - 2$  such that  $k, r, m_1, \dots, m_s$  are distinct, and the product  $\bar{\beta}_{kr} \bar{\beta}_{r m_1} \bar{\beta}_{m_1 m_2} \cdots \bar{\beta}_{m_s k} \neq 0$ . Furthermore, there exists a unicyclic graph  $Q \in \mathcal{D}(n, l)$  such that  $E(CQ) = \{(k, r), (r, m_1), \dots, (m_s, k)\}$  and  $\prod_{(j,h) \in E(Q)} \bar{\beta}_{jh} \neq 0$ . Therefore, from

(5.27) and (5.30), we know  $I_k/I_k^* = I_r/I_r^*$  if  $H_n = 0$ .

From (5.8) and (5.31) we know that  $V' = 0 \iff S_k = S_k^*$ ,  $I_k = aI_k^*$ ,  $k = 1, 2, \dots, n$ . Substituting  $S_k = S_k^*$  and  $I_k = aI_k^*$  into the first equation of system (1.3), we obtain

$$(5.32) \quad 0 = (1 - p_k)\Lambda_k - (d_k^S + \theta_k)S_k^* - a \sum_{j=1}^n \beta_{kj} S_k^* I_j^*.$$

Since the right-hand-side of (5.32) is strictly decreasing in  $a$ , by (5.6), we know (5.32) holds if and only if  $a = 1$ , namely, at  $P^*$ . Therefore, the only compact invariant subset of the set where  $V' = 0$  is the singleton  $\{P^*\}$ . By LaSalle's Invariance Principle,  $P^*$  is global asymptotically stable in  $\overset{\circ}{\Gamma}$  if  $R_0 > 1$ . This completes the proof of Theorem 3.3.

**6 Basic reproduction number  $R_0$**  We have derived our threshold parameter

$$(6.1) \quad R_0 = \rho \left( \begin{bmatrix} \frac{\beta_{11}S_1^0}{d_1^l + \epsilon_1 + \gamma_1} & \cdots & \frac{\beta_{1n}S_1^0}{d_1^l + \epsilon_1 + \gamma_1} \\ \vdots & \ddots & \vdots \\ \frac{\beta_{n1}S_n^0}{d_n^l + \epsilon_n + \gamma_n} & \cdots & \frac{\beta_{nn}S_n^0}{d_n^l + \epsilon_n + \gamma_n} \end{bmatrix} \right)$$

from the stability analysis of the disease-free equilibrium  $P_0$  using the method of Lyapunov functions. We have shown that  $R_0$  plays the role expected of the basic reproduction number, namely, if  $R_0 \leq 1$  the disease always dies out, and if  $R_0 > 1$ , the disease persists. In [35], a method of deriving the basic reproduction number for epidemic models in heterogeneous populations is proposed. Apply the method of [35] to our model (1.3), we can derive the basic reproduction number as

$$(6.2) \quad R_0 = \rho \left( \begin{bmatrix} \frac{\beta_{11}S_1^0}{d_1^l + \epsilon_1 + \gamma_1} & \cdots & \frac{\beta_{1n}S_1^0}{d_n^l + \epsilon_n + \gamma_n} \\ \vdots & \ddots & \vdots \\ \frac{\beta_{n1}S_n^0}{d_1^l + \epsilon_1 + \gamma_1} & \cdots & \frac{\beta_{nn}S_n^0}{d_n^l + \epsilon_n + \gamma_n} \end{bmatrix} \right),$$

where the matrix  $\left( \frac{\beta_{kj}S_k^0}{d_j^l + \epsilon_j + \gamma_j} \right)$  is called the next generation matrix in [7] (see Example 4.2 in [35]). Biological interpretation of  $R_0$  in (6.2) as the basic reproduction number is given in [7, 35]. It is straightforward to verify that two expressions of  $R_0$  in (6.1) and (6.2) are equivalent.

**7 Other incidence forms** Our analysis in Sections 4 and 5 completely determines the global dynamics of multigroup SIR model (1.2). The same analysis applies to models with certain proportionate incidence forms.

Consider an  $n$ -group SIR model

$$(7.1) \quad \begin{aligned} S_k' &= dN_k(1 - p_k) - (d + \theta_k)S_k - \sum_{j=1}^n \lambda_{kj} \frac{S_k}{N_k} I_j, \\ I_k' &= \sum_{j=1}^n \lambda_{kj} \frac{S_k}{N_k} I_j - (d + \gamma_k)I_k, \\ R_k' &= dN_k p_k + \theta_k S_k + \gamma_k I_k - dR_k, \end{aligned}$$

for  $k = 1, 2, \dots, n$ . In this model, the incidence is of proportionate form. It can be verified that the total population in each group  $N_k = S_k + I_k + R_k$  is a constant. Consequently, system (7.1) reduces to system (1.2) if we choose  $\Lambda_k = dN_k$  and  $\beta_{ij} = \lambda_{ij}/N_i$ . Therefore, the global dynamics of (7.1) is completely determined by Proposition 3.1 and Theorem 3.3.

Another class of  $n$ -group SIR models that can be reduced to the form of (1.2) is the following.

$$\begin{aligned}
 (7.2) \quad S'_k &= bNq_k(1 - p_k) - (d + \theta_k)S_k - \sum_{j=1}^n \beta_{kj} \frac{S_k}{N} I_j, \\
 I'_k &= \sum_{j=1}^n \beta_{kj} \frac{S_k}{N} I_j - (d + \gamma_k)I_k, \\
 R'_k &= bNp_kq_k + \theta_k S_k + \gamma_k I_k - dR_k,
 \end{aligned}$$

for  $k = 1, 2, \dots, n$ , where  $N = \sum_{k=1}^n (S_k + I_k + R_k)$  denotes the total population size,  $b$  and  $d$  respectively represent the birth and death rate constants,  $q_k$  is the ratio of new birth to enter the  $k$ -th group, and  $\sum_{k=1}^n q_k = 1$ . The differences between (7.2) and (1.2) are in the influx of susceptibles and the incidence forms.

The total population size  $N(t)$  can be determined by the differential equation

$$(7.3) \quad N' = (b - d)N,$$

which is derived by adding the equations in (7.2). Let  $s_k = S_k/N$ ,  $i_k = I_k/N$  and  $r_k = R_k/N$  denote the fractions of the classes  $S_k, I_k$  and  $R_k$  in the total population, respectively. It can be verified that  $s_k, i_k$  and  $r_k$  satisfy the following system of differential equations

$$\begin{aligned}
 (7.4) \quad s'_k &= bq_k(1 - p_k) - (b + \theta_k)s_k - \sum_{j=1}^n \beta_{kj} s_k i_j, \\
 i'_k &= \sum_{j=1}^n \beta_{kj} s_k i_j - (b + \gamma_k)i_k, \\
 r'_k &= bp_kq_k + \theta_k s_k + \gamma_k i_k - br_k.
 \end{aligned}$$

System (7.4) is a special case of system (1.2). From (1.6), we obtain the

basic reproduction number for system (7.4)

$$(7.5) \quad R_0 = \rho \left( \begin{bmatrix} \frac{bq_1\beta_{11}(1-p_1)}{(b+\gamma_1)(b+\theta_1)} & \cdots & \frac{bq_1\beta_{1n}(1-p_1)}{(b+\gamma_1)(b+\theta_1)} \\ \vdots & \ddots & \vdots \\ \frac{bq_n\beta_{n1}(1-p_n)}{(b+\gamma_n)(b+\theta_n)} & \cdots & \frac{bq_n\beta_{nn}(1-p_n)}{(b+\gamma_n)(b+\theta_n)} \end{bmatrix} \right).$$

The feasible region for (7.4) is given as follows:

$$(7.6) \quad \Delta = \{ (s_1, i_1, r_1, s_2, i_2, r_2, \dots, s_n, i_n, r_n) \in \mathbb{R}_+^{3n} \mid s_k + i_k + r_k = q_k, k = 1, 2, \dots, n \}.$$

System (7.4) always has the disease-free equilibrium

$$P_0 = (s_1^0, 0, r_1^0, s_2^0, 0, r_2^0, \dots, s_n^0, 0, r_n^0),$$

where

$$s_k^0 = \frac{bq_k(1-p_k)}{b+\theta_k}, \quad r_k^0 = \frac{q_k(bp_k+\theta_k)}{b+\theta_k}, \quad k = 1, 2, \dots, n.$$

An endemic equilibrium  $P^* = (s_1^*, i_1^*, r_1^*, s_2^*, i_2^*, r_2^*, \dots, s_n^*, i_n^*, r_n^*)$  of (7.4) belongs to  $\overset{\circ}{\Delta}$ , namely,  $s_k^* > 0, i_k^* > 0, r_k^* > 0$  for  $k = 1, 2, \dots, n$ . Applying Proposition 3.1 and Theorem 3.3 to system (7.4), we obtain the following result.

**Theorem 7.1.** *Assume  $B = (\beta_{ij})$  is irreducible. Then the following results hold for system (7.4).*

- (1) *If  $R_0 \leq 1$ , then the disease-free equilibrium  $P_0$  is globally asymptotically stable in  $\Delta$ .*
- (2) *If  $R_0 > 1$ , then  $P_0$  is unstable. There exists a unique endemic equilibrium  $P^*$ , and  $P^*$  is globally asymptotically stable in  $\overset{\circ}{\Delta}$ .*

Biologically, Theorem 7.1 implies that, if the basic reproduction number  $R_0 \leq 1$ , then the diseases dies out in the sense that infectious fractions go to zero from all the groups, and if  $R_0 > 1$ , then any initial infections will lead to endemic disease in the sense that the infectious fractions tend to a positive constant in all the groups.

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