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Dynamics of Differential Equations on Invariant Manifolds

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The simplification resulting from reduction of dimension involved in the study of invariant manifolds of differential equations is often difficult to achieve in practice. Appropriate coordinate systems are difficult to find or are essentially local in nature thus complicating analysis of global dynamics. This paper develops an approach which avoids the selection of coordinate systems on the manifold. Conditions are given in terms compound linear differential equations for the stability of equilibria and periodic orbits. Global results include criteria for the nonexistence of periodic orbits and a discussion of the nature of limit sets. As an application, a global stability criterion is established for the endemic equilibrium in an epidemiological model.

Key Words: differential equations; invariant submanifolds; Bendixson conditions; periodic orbits; compound matrices; compound equations.

1. INTRODUCTION

Let $x \mapsto f(x)$ be a C^1 function with open domain in $\mathbf{R}^n$ and range in $\mathbf{R}^n$ and let $x(t) = \varphi_t(x)$ be the solution of

$$\dot{x} = f(x) \tag{1.1}$$

such that $x(0) = x$. If $x \mapsto g(x)$ is a $\mathbf{R}^m$-*valued* C^1 function with the same domain and Σ denotes the subset of $\mathbf{R}^n$ where $g(x) = 0$, then Σ is a

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manifold of dimension $n-m$ if $\mathrm{rk}\,\frac{\partial g}{\partial x}(x)=m$ when $g(x)=0$. It is an invariant manifold with respect to (1.1) if $g(x)=0$ implies $g(\varphi_t(x))=0$ for all t such that $\varphi_t(x)$ exists. For notational convenience, the situation where there is no invariant manifold in consideration will be denoted as the case $m=0$.

An important special case occurs when $g(\varphi_t(x))=g(x)$ for all x and every manifold $g(x)=c$ is invariant. The system (1.1) is then said to have m first integrals. In many scientific models, first integrals appear as conservation laws for quantities such as energy or population and provide important tools in analysis of the dynamics. The existence of first integrals effectively reduces the dimension of the system and the reduced problem may be studied by changes of variable. However, the changes of variable may be difficult to implement or may not be optimal for the study. This paper investigates the flow due to (1.1) on an invariant manifold without resort to a reduced system. Invariant manifolds also arise as the stable, unstable or centre manifolds associated with equilibria or other invariant structures. Frequently, only the existence of these manifolds and the nature of the dynamics nearby are known so techniques which analyze the dynamics in the manifold with incomplete information are desirable. Invariant manifolds may also arise from application of the LaSalle invariance principle and related results. For example, if $x \mapsto v(x)$ is C^1, real and such that $g(x)=v'_{(1.1)}(x)=\frac{\partial v}{\partial x}(x)f(x)$ satisfies $g(x)\leqslant 0$ in the domain of f, then every non-wandering point in general and every equilibrium, periodic orbit and omega limit set in particular, lies in the set where $g(x)=0$. This is an $(n-1)$-dimensional manifold if $\mathrm{rk}\,\frac{\partial g}{\partial x}(x)=1$. All of the interesting dynamics then occur in this manifold and it is useful if projects such as stability analysis, existence or non-existence of periodic orbits and so forth can be conducted without tedious calculations in coordinate systems on the manifold.

When $n=2$, it is well known that (1.1) has no non-equilibrium periodic solution whose orbit lies entirely in a simply connected region where $\mathrm{div}\, f\neq 0$. This is no longer true when $n>2$. However it is shown by Demidowitsch [2] that, if $n=3$ and (1.1) has a first integral, the Bendixson condition $\mathrm{div}\, f\neq 0$ in a simply connected region precludes periodic orbits there. In the case that $n>2$ and $m=0$, Muldowney [15] gives a generalization of Bendixson's criterion and shows that if the flow of (1.1) diminishes some measure of 2-dimensional surface area in a simply connected region, then the region does not contain a periodic orbit. It is shown by Li in the Ph.D. dissertation [7] that there is a relaxation in these conditions in the presence of first integrals. Essentially, if (1.1) has m independent first integrals and the flow decreases $(m+2)$-dimensional surface areas, then there are no periodic orbits. In the spirit of Demidowitsch, if (1.1) has $m=n-1$ first integrals, then $\mathrm{div}\, f\neq 0$ is still a valid Bendixson condition.

Li also investigates similar questions relative to invariant affine manifolds in [8] and discusses some biological implications in [7, 9]. M. Feckan pointed out to us in a private communication that the original proof of Demidowitsch in [2] contained a gap and a correction was given by Feckan. As we remarked earlier, Demidowitsch's result in [2] follows from Theorem 5.2 of the present paper.

The discussion in this paper is applicable to any invariant manifold but, when it is not associated with first integrals, some information on the dynamics near the manifold is required. When the function g is explicitly known, the required behaviour may be computed from g and f as shown in Section 3. Section 7 considers a 4-dimensional epidemiological model where the dynamics of interest occur in an invariant manifold of dimension 3. A new criterion for the global stability of the endemic equilibrium is established using the techniques developed in the earlier sections.

2. SOME LINEAR THEORY

Let $\mathbf{U}$ be an *n-dimensional* vector space over the real numbers with inner product $\langle\ ,\ \rangle$ and associated norm $\|\cdot\|$. Then, if $1 \leqslant k \leqslant n$, $\bigwedge^k \mathbf{U}$ denotes the kth exterior power of $\mathbf{U}$, the $\binom{n}{k}$-dimensional space of all real linear combinations of exterior k-products of the form $u^1 \wedge \cdots \wedge u^k$, where $u^i \in \mathbf{U}$, $i = 1, ..., k$. For elements $\alpha = u^1 \wedge \cdots \wedge u^k$, $\beta = v^1 \wedge \cdots \wedge v^k$ of $\bigwedge^k \mathbf{U}$, an inner product $\langle\ ,\ \rangle$ may be defined by

$$\langle \alpha, \beta \rangle = \det \langle u^i, v^j \rangle, \qquad i, j = 1, ..., k. \tag{2.1}$$

The definition may be extended to arbitrary elements $\alpha, \beta \in \bigwedge^k \mathbf{U}$ by (2.1) and linearity. In the present context an important consequence of (2.1) is that, when $\langle u^i, v^j \rangle = 0$, $i = m+1, ..., m+k$, $j = 1, ..., m$,

$$\begin{aligned} &\langle u^1 \wedge \cdots \wedge u^{m+k}, v^1 \wedge \cdots \wedge v^{m+k} \rangle \\ &\quad = \langle u^1 \wedge \cdots \wedge u^m, v^1 \wedge \cdots \wedge v^m \rangle \\ &\qquad \times \langle u^{m+1} \wedge \cdots \wedge u^{m+k}, v^{m+1} \wedge \cdots \wedge v^{m+k} \rangle. \end{aligned} \tag{2.2}$$

A norm on $\bigwedge^k \mathbf{U}$ is defined by $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$. Then $\langle \alpha, \beta \rangle \leqslant \|\alpha\| \, \|\beta\|$. A summary of the basic definitions and properties of $\bigwedge^k \mathbf{U}$ may be found in [20], Chapter V.

A basis in $\bigwedge^k \mathbf{U}$ is $\{e^{i_1} \wedge \cdots \wedge e^{i_k} : 1 \leqslant i_1 < \cdots < i_k \leqslant n\}$ ordered by the lexicographic order of the k-tuples $(i_1, ..., i_k)$ where $\{e^i : i = 1, ..., n\}$ is a basis in $\mathbf{U}$. If $\mathbf{W}$ is a real m-dimensional vector space, let A denote both a linear map from $\mathbf{U}$ to $\mathbf{W}$ and its matrix representation with respect to bases in these spaces. The *k-th multiplicative compound* or *k-th exterior power* $\bigwedge^k A = A^{(k)}$, when $1 \leqslant k \leqslant \min\{n, m\}$, and the *k-th additive compound* $A^{[k]}$, when $1 \leqslant k \leqslant m = n$ and $\mathbf{W} = \mathbf{U}$, are the linear maps from $\bigwedge^k \mathbf{U}$ to $\bigwedge^k \mathbf{W}$ and associated matrices defined by

$$\begin{aligned} A^{(k)}(u^1 \wedge \cdots \wedge u^k) &= (Au^1) \wedge \cdots \wedge (Au^k) \\ A^{[k]}(u^1 \wedge \cdots \wedge u^k) &= \sum_{i=1}^{k} u^1 \wedge \cdots \wedge (Au^i) \wedge \cdots \wedge u^k, \end{aligned} \tag{2.3}$$

if $u^1, ..., u^k \in \mathbf{U}$ and extended by linearity to $\bigwedge^k \mathbf{U}$. The terms multiplicative compound and additive compound are used since it can be readily seen from (2.3) that $(AB)^{(k)} = A^{(k)} B^{(k)}$ and $(A+B)^{[k]} = A^{[k]} + B^{[k]}$ when the matrices or maps A, B are compatible with respect to the multiplication and addition, respectively. Moreover, it can be easily checked from (2.3) that, if $\mathbf{W} = \mathbf{U}$,

$$A^{[k]} = D_t (I + t\,A)^{(k)}|_{t=0} = \lim_{t \to 0} \frac{1}{t} \left[(I + t\,A)^{(\mathrm{k})} - I^{(k)} \right], \tag{2.4}$$

where I is the identity on $\mathbf{U}$ and its compound $I^{(k)}$ is the identity on $\bigwedge^k \mathbf{U}$. Special cases of the compound matrices are when $k = 1, n : A^{(1)} = A^{[1]} = A$, $A^{(n)} = \det A$, $A^{[n]} = \operatorname{tr} A$. It follows readily from (2.3) that, if $\lambda_1, ..., \lambda_r$ are the eigenvalues of A, repeated according to multiplicity, with respect to an invariant r-dimensional subspace V of $\mathbf{U}$, then $\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$ and $\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$, are the eigenvalues of $\bigwedge^k A = A^{(k)}$ and $A^{[k]}$ respectively with respect to $\bigwedge^k V$, which is invariant for both of these operators. If α_i^2 are the eigenvalues of $A^* A$, then $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n \geqslant 0$ are the singular values of A, $\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k} \geqslant 0$ are the singular values of $\bigwedge^k A$, $\alpha_1 = \|A\|$ and $\alpha_1 \alpha_2 \cdots \alpha_k = \|\bigwedge^k A\|$.

We mention briefly the connection with differential equations. Let $t \mapsto A(t)$ be a continuous $n \times n$ real matrix-valued function on $\mathbf{R}$ and let $Y(t)$ be a matrix solution of the linear differential equation in $\mathbf{R}^n$

$$\dot{y} = A(t)\, y. \tag{2.5}$$

If $y^1, ..., y^k \in \mathbf{R}^n$, then $z(t) = Y^{(k)}(t)(y^1 \wedge \cdots \wedge y^k) = Y(t)\, y^1 \wedge \cdots \wedge Y(t)\, y^k)$ satisfies $\dot{z}(t) = \Sigma_{i=1}^k (Y(t)\, y^1) \wedge \cdots \wedge A(t)\; Y(t)\, y^i \wedge \cdots \wedge (Y(t)\, y^k) = A^{[k]}(t)$ $z(t)$, from (2.3) and (2.5). Thus $Z(t) = Y^{(k)}(t)$ is a matrix solution of

$$\dot{z} = A^{[k]}(t)\, z \tag{2.6}$$

which is called the kth *compound equation* of (2.5). It is clear from the preceding remarks that, if $Y(t)$ is a fundamental matrix for (2.5), then $Z(t) = Y^{(k)}(t)$ is a fundamental matrix for (2.6). Moreover $z(t) = y^1(t) \wedge \cdots \wedge y^k(t)$ is a solution of (2.6) if $y^1(t), ..., y^k(t)$ are solutions of (2.5). Thus, if we consider $y^1(t), ..., y^k(t)$ to be time varying oriented line segments in $\mathbf{R}^n$ whose evaluation in time is governed by (2.5), then we may regard $z(t)$ as the corresponding $\binom{n}{k}$-dimensional vector entity which is the oriented k-dimensional parallelopiped determined by this ordered set of line segments; $|z(t)|$ is a measure of the k-dimensional volume of this object if $|\cdot|$ is any norm in $\bigwedge^k \mathbf{R}^n$. The case $k=1$ is the equation (2.5) itself while $k=n$ is the well known Abel–Jacobi–Liouville equation $\dot{z} = \operatorname{tr} A(t)\, z$. It is an easy exercise to show that $(A^*)^{[k]} = (A^{[k]})^*$, where the asterisk denotes the transpose; thus the adjoint of equation (2.6) is the kth compound of the adjoint of (2.5).

A survey of compound matrices and compound equations with comprehensive references may be found in [15]. The treatment here is based largely on that of Fiedler [3].

3. SOME NONLINEAR THEORY

If $x \mapsto v(x)$ is a C^1 real-valued function on $\mathbf{R}^n$, let $v'_{(1.1)}(x) = \frac{\partial v}{\partial x}(x) f(x) = \langle \operatorname{grad} v(x), f(x) \rangle$ be the derivative of v with respect to (1.1). More generally, if the function is C^1 matrix-valued, let $v'_{(1.1)}$ denote the matrix obtained by replacing each entry in $v(x)$ by its derivative with respect to (1.1). For the smoothness requirement in the development of this paper, it is sufficient and henceforth assumed that the functions $x \mapsto v(x)$ and $x \mapsto v'_{(1.1)}(x)$ be both C^1. This condition is satisfied, in particular, when f is C^1 and v is C^2.

PROPOSITION 3.1. *Let $x \mapsto g(x)$ be a $\mathbf{R}^m$-valued function in $\mathbf{R}^n$ such that $\operatorname{rk} \frac{\partial g}{\partial x}(x) = m$ if $g(x) = 0$. Then $\Sigma = \{x : g(x) = 0\}$ is an invariant manifold if and only if there is a continuous $m \times m$ matrix-valued function $N(x)$ such that*

$$g'_{(1.1)}(x) = N(x)\, g(x). \tag{3.1}$$

Proof. It is clear that (3.1) is a sufficient condition for the invariance of Σ. To show its necessity, we will construct $N(x)$ satisfying (3.1) if Σ is invariant. Let $h(x) = g'_{(1.1)}(x)$. Then (3.1) is $h(x) = N(x)\, g(x)$ and any such function N must then satisfy

$$\frac{\partial h}{\partial x}(y) = N(y) \frac{\partial g}{\partial x}(y) \tag{3.2}$$

if $y \in \Sigma$ since $g(y) = 0$ implies $h(y) = 0$. Moreover, since $\frac{\partial g}{\partial x}(y)$ has full rank if $y \in \Sigma$, (3.2) uniquely determines $y \mapsto N(y)$ as a continuous function on Σ. If $x \notin \Sigma$, let $N(x) = (n_{ij})$ be the $m \times m$ matrix be defined by

$$n_{ij}(x) = h_i(x)\, g_j(x) / \|g(x)\|^2. \tag{3.3}$$

Clearly (3.1) is satisfied for all x; it remains to show that $x \mapsto N(x)$ is continuous. The restrictions of this function to Σ and to its complement Σ^c are both continuous, so it remains to show that it is continuous at $y \in \Sigma$ with respect to its full domain. We begin by showing that $N(x)$ is bounded if x is in some neighbourhood of any compact subset Σ_0 of Σ. First $N(x)$, $x \in \Sigma$, is bounded if x is near Σ_0, from the continuity of the restriction to Σ. If $x_0 \in \Sigma$, let $\mathcal{N}_{x_0} = \{u \in \mathbf{R}^n : u = \frac{\partial g}{\partial x}^*(x_0)\, c,\ c \in \mathbf{R}^m\}$. If x is near Σ_0, then $x - x_0 \in \mathcal{N}_{x_0}$ for some $x_0 \in \Sigma$. Since $h(x_0) = g(x_0) = 0$, there exists a constant c_1 such that $\|h(x)\| \leqslant c_1 \|x - x_0\|$, $\|g(x)\| \leqslant c_1 \|x - x_0\|$. If $\frac{\partial g}{\partial x}(x_0)|_{\mathcal{N}_{x_0}}$ denotes the restriction to $\mathcal{N}_{x_0}$ of the differential operator represented by $\frac{\partial g}{\partial x}(x_0)$ then, since $\frac{\partial g}{\partial x}(x_0)|_{\mathcal{N}_{x_0}}$ is invertible, there is a constant $c_2 > 0$ such that $\|g(x)\| \geqslant c_2 \|x - x_0\|$ when x is near x_0 and thus, from (3.3), there is a constant K such that

$$\|N(x)\| \leqslant K. \tag{3.4}$$

The constants c_1, c_2 and hence K may be chosen independent of x_0 for any compact $\Sigma_0 \subset \Sigma$. Now, if $y \in \Sigma$, $h, g \in C^1$ implies that $h(x) = \frac{\partial h}{\partial x}(y)(x - y) + o(\|x - y\|)$, $g(x) = \frac{\partial g}{\partial x}(y)(x - y) + o(\|x - y\|)$, when $x \to y$. Then $h(x) = N(x)\, g(x)$ and (3.4) imply

$$\frac{\partial}{\partial x}(y)(x - y) = N(x) \frac{\partial g}{\partial x}(y)(x - y) + o(\|x - y\|)$$

which together with (3.2) implies

$$(N(x) - N(y)) \frac{\partial g}{\partial x}(y)(x - y) = o(\|x - y\|)$$

and therefore $\lim_{x \to y} N(x) = N(y)$ since $\operatorname{rk} \frac{\partial g}{\partial x}(y) = m$. Thus N is continuous at $y \in \Sigma$ as asserted. ∎

For an invariant manifold Σ, a matrix $N(x)$ in (3.1) may be evident from the original system (1.1). For example, if $g(x)$ is a first integral, we have $N(x) = 0$. If (1.1) satisfies a logistic law $p'_{(1.1)}(x) = a(x)\, p(x)(1 - p(x))$ on a population $p(x)$, we could choose $g(x) = p(x)$, $N(x) = a(x)(1 - p(x))$ for the invariant manifold $p(x) = 0$ and $g(x) = p(x) - 1$, $N(x) = -a(x)\, p(x)$ for $p(x) = 1$. If $m = 1$, then $N(x) = g'_{(1.1)}(x)/g(x)$ if $x \notin \Sigma$, $N(x) = \langle \operatorname{grad} g'_{(1.1)}(x), \operatorname{grad} g(x) \rangle / \|\operatorname{grad} g(x)\|^2$, if $x \in \Sigma$. In general when $m > 1$, $N(x)$ is not unique if $x \notin \Sigma$, however $N(y)$, $y \in \Sigma$, is uniquely determined by (3.2).

If we consider g as a vector Lyapunov function, since $v(t) = g(\varphi_t(x))$ is a solution of the m-dimensional system

$$\dot{v} = N(\varphi_t(x))\, v, \tag{3.5}$$

it follows that (3.5) has a close bearing on the stability characteristics of the manifold Σ. The focus of this paper, however, is on the dynamics in Σ itself. Throughout we will be concerned with the trace of the unique matrix $N(x)$ determined by (3.2) with $h(x) = g'_{(1.1)}(x)$, $x \in \Sigma$.

DEFINITION. *If $m > 0$, let $x \mapsto v(x)$ be the real-valued function on Σ defined by*

$$v(x) = \operatorname{tr} N(x),$$

where $N(x)$ is defined by $\frac{\partial}{\partial x} g'_{(1.1)}(x) = N(x) \frac{\partial g}{\partial x}(x)$, $x \in \Sigma$. If $m = 0$, let $v(x) = 0$ and Σ be the domain of f in $\mathbf{R}^n$.

The matrix $Y(t) = \frac{\partial \varphi_t}{\partial x}(x)$ is a fundamental matrix for the linearization with respect to the solution $\varphi_t(x)$ of (1.1),

$$\dot{y} = \frac{\partial f}{\partial x}(\varphi_t(x))\, y, \tag{3.6}$$

with $Y(0) = I$. Now let $t \mapsto U(t)$ be a C^1 $m \times m$ matrix-valued function and consider the $n \times m$ matrix $W(t) = \frac{\partial g}{\partial x}^*(\varphi(t))\, U(t)$, where $\varphi(t) = \varphi_t(x) \in \Sigma$. Then

$$\dot{W}^*(t) = \dot{U}^*(t) \frac{\partial g}{\partial x}(\varphi(t)) + U^*(t) \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right)(\varphi(t))\, f(\varphi(t)). \tag{3.7}$$

From (3.2)

$$\frac{\partial}{\partial x} g'_{(1.1)}(x) = N(x) \frac{\partial g}{\partial x}(x), \tag{3.8}$$

if $g(x) = 0$. But $g'_{(1.1)}(x) = \frac{\partial g}{\partial x}(x)\, f(x)$ implies

$$\frac{\partial}{\partial x} g'_{(1.1)}(x) = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right)(x)\, f(x) + \frac{\partial g}{\partial x}(x) \frac{\partial f}{\partial x}(x). \tag{3.9}$$

Together (3.7), (3.8), (3.9) yield

$$\begin{aligned} \dot{W}^*(t) &= \dot{U}^*(t) \frac{\partial g}{\partial x}(\varphi(t)) + U^*(t)\left[N(\varphi(t)) \frac{\partial g}{\partial x}(\varphi(t)) - \frac{\partial g}{\partial x}(\varphi(t)) \frac{\partial f}{\partial x}(\varphi(t))\right] \\ &= -U^*(t) \frac{\partial g}{\partial x}(\varphi(t)) \frac{\partial f}{\partial x}(\varphi(t)), \end{aligned}$$

provided that $U(t)$ is a solution matrix of the adjoint equation of (3.5), $\dot{U}^*(t) + U^*(t)\, N(\varphi(t)) = 0$. We can deduce the following proposition.

PROPOSITION 3.2. *Let* $w^i(t) = \frac{\partial g}{\partial x}^* (\varphi_t(x))\, U(t)\, e^i$, *where* $x \in \Sigma$, $\{e^i : i = 1, ..., m\}$ *is the canonical basis in* $\mathbf{R}^m$ *and* $\dot{U}(t) = -N^*(\varphi_t(x))\, U(t)$, $U(0) = I$. *Let* $\mathcal{T}_x$ *denote the tangent space to* Σ *at* x. *Then*

(a) *each* $w^i(t)$ *is orthogonal to* $\mathcal{T}_{\varphi_t(x)}$ *and* $\{w^i(t) : i = 1, ..., m\}$ *spans a m-dimensional solution subspace of the adjoint equation of* (3.6), $\dot{w} = -\frac{\partial f}{\partial x}^* (\varphi_t(x))w$, *which is normal to* $\mathcal{T}_{\varphi_t(x)}$.

(b) *If* $y(t)$ *is a solution of* (3.6) *and* $y(0) \in \mathcal{T}_x$, *then* $y(t) \in \mathcal{T}_{\varphi_t(x)}$ *for all* t *such that* $\varphi_t(x)$ *exists.*

Part (b) is of course well known; see for example Wiggins [21] page 48. We wish however to emphasize the relationship with (3.5) and observe that it also follows from part (a) since $\langle y(t), w(t) \rangle$ is constant if $w(t)$ is any solution of the adjoint equation of (3.6). Throughout the remainder of this paper, $\frac{\partial \varphi_t}{\partial x}(x)$ will denote not only the Jacobian matrix but also the differential of the map $x \mapsto \varphi_t(x)$ that it represents. Part (*b*) of Proposition 3.2 shows that $\frac{\partial \varphi_t}{\partial x}(x)\mathcal{T}_x = \mathcal{T}_{\varphi_t(x)}$. We denote by $\frac{\partial \varphi_t}{\partial x}(x)|_{\mathcal{T}_x}$ the restriction of the differential $\frac{\partial \varphi_t}{\partial x}(x)$ to $\mathcal{T}_x$. Let $\gamma_+(x) = \{\varphi_t(x) : t \geqslant 0\}$.

PROPOSITION 3.3. *Let* $U(t)$, $w^i(t)$, $i = 1, ..., m$, *be as in Proposition* 3.2. *Then*

$$\|w^1(t) \wedge \cdots \wedge w^m(t)\| \leqslant C \exp\left(-\int_0^t v(\varphi_s(x))\, ds\right) \|w^1(0) \wedge \cdots \wedge w^m(0)\|,$$

where $C = \sup\left\{ \|\bigwedge^m \frac{\partial g}{\partial x}^* (y)\| / \|\bigwedge^m \frac{\partial g}{\partial x}^* (x)\| : y \in \gamma_+(x) \right\}$.

Proof. Since

$$w^1(t) \wedge \cdots \wedge w^m(t) = \bigwedge^m W(t)\, (e^1 \wedge \cdots \wedge e^m)$$

$$= \bigwedge^m \frac{\partial g}{\partial x}^* (\varphi_t(x)) \det U(t),$$

we have

$$\|w^1(t) \wedge \cdots \wedge w^m(t)\|$$

$$= \left\| \bigwedge^m \frac{\partial g}{\partial x}^* (\varphi_t(x)) \right\| \exp - \left(\int_0^t v(\varphi_s(x) \right) ds$$

$$\leqslant C \exp\left(-\int_0^t v(\varphi_s(x))\, ds\right) \|w^1(0) \wedge \cdots \wedge w^m(0\|. \quad \blacksquare$$

PROPOSITION 3.4. *Under the conditions of Proposition* 3.3

$$\left\| \bigwedge^{k} \left(\frac{\partial \varphi_t}{\partial x}(x) \Big|_{\mathcal{T}_x} \right) \right\| \leqslant C \exp\left(-\int_0^t \nu(\varphi_s(x))\, ds \right) \left\| \bigwedge^{m+k} \frac{\partial \varphi_t}{\partial x}(x) \right\|.$$

Proof. Consider solutions $y^i(t)$ of (3.6), $i = 1, ..., m+k$, such that $y^i(0) = w^i(0)$, $i = 1, ..., m$ and $y^i(0) \in \mathcal{T}_x$, $i = m+1, ..., m+k$. Since $\langle y^i(t), w^j(t) \rangle = \langle y^i(0), w^j(0) \rangle = 0$, $i = m+1, ..., m+k$, $j = 1, ..., m$, we may choose $u^i = y^i(t)$, $i = 1, ..., m+k$, $v^i = w^i(t)$, $i = 1, ..., m$, $v^i = y^i(t)$, $i = m+1, ..., m+k$ in (2.2) to find

$$\begin{aligned} &\langle y^1(t) \wedge \cdots \wedge y^{m+k}(t), w^1(t) \wedge \cdots \wedge w^m(t) \wedge y^{m+1}(t) \wedge \cdots \wedge y^{m+k}(t) \rangle \\ &\quad = \langle y^1(t) \wedge \cdots \wedge y^m(t),\ w^1(t) \wedge \cdots \wedge w^m(t) \rangle \\ &\qquad \times \langle y^{m+1}(t) \wedge \cdots \wedge y^{m+k}(t),\ y^{m+1}(t) \wedge \cdots \wedge y^{m+k}(t) \rangle \\ &\quad = \|w^1(0) \wedge \cdots \wedge w^m(0)\|^2 \, \|y^{m+1}(t) \wedge \cdots \wedge y^{m+k}(t)\|^2 \end{aligned} \tag{3.10}$$

from (2.2), since $\langle y^i(t), w^j(t) \rangle = \langle y^i(0), w^j(0) \rangle = \langle w^i(0), w^j(0) \rangle$, $i, j = 1, ..., m$. We also have

$$\begin{aligned} &\langle y^1(t) \wedge \cdots \wedge y^{m+k}(t), w^1(t) \wedge \cdots \wedge w^m(t) \wedge y^{m+1}(t) \wedge \cdots \wedge y^{m+k}(t) \rangle \\ &\quad \leqslant \|y^1(t) \wedge \cdots \wedge y^{m+k}(t)\| \\ &\qquad \times \|w^1(t) \wedge \cdots \wedge w^m(t) \wedge y^{m+1}(t) \wedge \cdots \wedge y^{m+k}(t)\| \\ &\quad = \|y^1(t) \wedge \cdots \wedge y^{m+k}(t)\| \\ &\qquad \times \|w^1(t) \wedge \cdots \wedge w^m(t)\| \, \|y^{m+1}(t) \wedge \cdots \wedge y^{m+k}(t)\| \end{aligned} \tag{3.11}$$

again from (2.2), since $\langle y^i(t), w^j(t) \rangle = \langle y^i(0), w^j(0) \rangle = 0$, $i = m+1, ..., m+k$, $j = 1, ..., m$. From (3.10) and (3.11), we conclude

$$\begin{aligned} &\|w^1(0) \wedge \cdots \wedge w^m(0)\|^2 \, \|y^{m+1}(t) \wedge \cdots \wedge y^{m+k}(t)\| \\ &\qquad \leqslant \|w^1(t) \wedge \cdots \wedge w^m(t)\| \, \|y^1(t) \wedge \cdots \wedge y^{m+k}(t)\|. \end{aligned} \tag{3.12}$$

Noting that $y^i(t) = \frac{\partial \varphi_t}{\partial x}(x)\, y^i(0)$, $i = 1, ..., m$, and $y^i(t) = \frac{\partial \varphi_t}{\partial x}(x)|_{\mathcal{T}_x}\, y^i(0)$, $i = m+1, ..., m+k$, and also that

$$\begin{aligned} &\|y^1(t) \wedge \cdots \wedge y^{m+k}(t)\| \\ &\quad \leqslant \left\| \bigwedge^{m+k} \frac{\partial \varphi_t}{\partial x}(x) \right\| \, \|y^1(0) \wedge \cdots \wedge y^{m+k}(0)\| \\ &\quad = \left\| \bigwedge^{m+k} \frac{\partial \varphi_t}{\partial x}(x) \right\| \, \|w^1(0) \wedge \cdots \wedge w^m(0)\| \, \|y^{m+1}(0) \wedge \cdots \wedge y^{m+k}(0)\|, \end{aligned} \tag{3.13}$$

we deduce from Proposition 3.3 and (3.11), (3.12) that

$$\left\| \bigwedge^{k} \left(\frac{\partial \varphi_t}{\partial x}(x)\Big|_{\mathcal{T}_x} \right) (y^{m+1}(0) \wedge \cdots \wedge y^{m+k}(0)) \right\|$$

$$\leqslant C \exp\left(-\int_0^t v(\varphi_s(x))\, ds \right) \left\| \bigwedge^{m+k} \frac{\partial \varphi_t}{\partial x}(x) \right\| \|y^{m+1}(0) \wedge \cdots \wedge y^{m+k}(0)\|. \tag{3.14}$$

If we revisit the formula (2.2) we may observe that, if we fix $u^1, ..., u^m$ and $v^1, ..., v^m$ and replace $u^{m+1} \wedge \cdots \wedge u^{m+k}$ and $v^{m+1} \wedge \cdots \wedge v^{m+k}$ by any linear combination of such expressions, still requiring that $\langle u^i, v^j \rangle = 0$, $i = m+1, ..., m+k$, $j = 1, ..., m$, the corresponding formally extended equation (2.2) is still valid. Consequently, the expression $y^{m+1}(t) \wedge \cdots \wedge y^{m+k}(t)$ may be replaced throughout the preceding argument by any linear combination of such terms to show that $y^{m+1}(0) \wedge \cdots \wedge y^{m+k}(0)$ in (3.14) may be replaced by any element $\alpha \in \bigwedge^k \mathcal{T}_x$ thus proving Proposition 3.4. ∎

The preceding considerations show that bounds on the growth, subject to the dynamics of (3.6), of *k-volumes* in $\mathcal{T}_{\varphi_t(x)}$ may be inferred from bounds on the growth of general $(m+k)$-*volumes* by using the Liouville equation from (3.5) to form the growth rate of projections of the $(m+k)$-*volumes* onto the m-dimensional space normal to Σ at $\varphi_t(x)$. Finally, Proposition 3.5 relates $v(x) = \operatorname{tr} N(x)$ to the spectrum of $\frac{\partial f}{\partial x}(x)$.

PROPOSITION 3.5. *Let $\lambda_1(x_0), ..., \lambda_n(x_0)$ be the eigenvalues of $\frac{\partial f}{\partial x}(x_0)$. If $x_0 \in \Sigma$ and $\lambda_{m+1}(x_0), ..., \lambda_n(x_0)$ are the eigenvalues which correspond to the invariant subspace $\mathcal{T}_{x_0}$ of $\frac{\partial f}{\partial x}(x_0)$, then*

$$v(x_0) = \lambda_1(x_0) + \cdots + \lambda_m(x_0).$$

Proof. Without loss of generality, the leading $m \times m$ submatrix of $\frac{\partial g}{\partial x}(x_0)$ is nonsingular. We may thus define a local coordinate system (v, w) near x_0 in $\mathbf{R}^n$ by $v_i = g_i(x)$, $i = 1, ..., m$, and $(w_1, ..., w_{n-m}) = (x_{m+1}, ..., x_n)$. In these coordinates the equation (1.1) has the form

$$\dot{v} = N(x)\, v, \qquad \dot{w} = M(v, w). \tag{3.15}$$

If $(0, w_0)$ are the (v, w)-*coordinates* of $x_0 \in \Sigma$, then $\lambda_{m+1}(x_0), ..., \lambda_n(x_0)$ are the eigenvalues of $\frac{\partial M}{\partial w}(0, w_0)$ and $\lambda_1(x_0), ..., \lambda_m(x_0)$ are the eigenvalues of $N(x_0)$ so that $v(x_0) = \lambda(x_0) + \cdots + \lambda_m(x_0)$ as asserted. ∎

4. EQUILIBRIA AND PERIODIC ORBITS

Suppose that $x_0 \in \Sigma$. If $f(x_0)=0$, the equilibrium is *stable hyperbolic* with respect to the dynamics on Σ if every eigenvalue λ_j of $\frac{\partial f}{\partial x}(x_0)$ corresponding to the invariant subspace $\mathscr{T}_{x0}$ satisfies $\operatorname{Re}\lambda_j<0$; see Szlenk [19] page 58. Then all orbits in Σ near x_0 are attracted to x_0 exponentially in time. Similarly $x_0 \in \Sigma$ is ω-*periodic* if $\varphi(t+\omega)=\varphi(t)$ for some minimal $\omega>0$, where $\varphi(t)=\varphi_t(x_0)$. Then x_0 is a fixed point of the diffeomorphism $x \mapsto \varphi_\omega(x)$. Since $\frac{\partial \varphi_\omega}{\partial x}(x_0)\,\mathscr{T}_{x0}=\mathscr{T}_{\varphi_\omega(x_0)}=\mathscr{T}_{x0}$, $\mathscr{T}_{x0}$ is an invariant subspace of $\frac{\partial \varphi_\omega}{\partial x}(x_0)$. Moreover $\dot\varphi(0)\in\mathscr{T}_{x0}$ and $\frac{\partial \varphi_\omega}{\partial x}(x_0)\,\dot\varphi(0)=\dot\varphi(\omega)=\dot\varphi(0)$ so $\mu_n=1$ is an eigenvalue of $\frac{\partial \varphi_\omega}{\partial x}(x_0)|_{\mathscr{T}x0}$. If all remaining eigenvalues μ_j of this matrix satisfy $|\mu_j|<1$, then the periodic orbit $\gamma_+(x_0)$ is *stable hyperbolic*; see for example Szlenk [19], Section 1.9. Each orbit in Σ near $\gamma_+(x_0)$ is then attracted exponentially to $\gamma_+(x_0)$ with a certain phase with respect to $\varphi(t)$; see Coppel [1] page 82 and Hartman [5] page 255.

We will prove the following theorems.

THEOREM 4.1. *Let $x_0\in\Sigma$ be an equilibrium of* (1.1).

(a) *A sufficient condition for x_0 to be stable hyperbolic with respect to the dynamics of* (1.1) *on Σ is that*

$$\dot{z}=\left(\frac{\partial f^{[m+1]}}{\partial x}(\mathrm{x}_0)-\nu(\mathrm{x}_0)\,I\right)z \tag{4.1}$$

be asymptotically stable.

(b) *The sufficient condition of* (a) *is also necessary if the system $\dot u=-N^*(x_0)\,u$ is stable.*

THEOREM 4.2. *Let $x_0\in\Sigma$ be ω-periodic with $\varphi(t)=\varphi_t(x_0)$.*

(a) *A sufficient condition for $\gamma_+(x_0)$ to be stable hyperbolic with respect to the dynamics of* (1.1) *on Σ is that*

$$\dot{z}=\left(\frac{\partial f^{[m+2]}}{\partial x}(\varphi(t))-\nu(\varphi(\mathrm{t}))\,I\right)z \tag{4.2}$$

be asymptotically stable.

(b) *The sufficient condition of* (a) *is also necessary if the system $\dot u=-N^*(\varphi(t))\,u$ is stable.*

Remark. When g is a first integral, $N(x)=0$ and the condition (a) of each of these theorems is both necessary and sufficient for the hyperbolic stability considered.

These theorems provide a mechanism for testing the stability of equilibria and periodic orbits with respect to the dynamics of (1.1) on Σ without the use of any particular coordinate system. The matrix $\frac{\partial f^{[k]}}{\partial x}$, $k=m+1, m+2$, are concrete entities and $v(x_0)$ may always be calculated if x_0, $g(x)$ are known explicitly. Even this information is not always necessary; for example if $g(x)$ is a system of first integrals then $v(x)=0$ for all x. More generally, if it is known that Σ does not attract nearby orbits, it can often be inferred that $v(x) \geqslant 0$ in (4.1), (4.2).

If we consider $V(z)=|z|$ as a Lyapunov function in (4.1), $k=m+1$ and (4.2), $k=m+2$, we find that $\dot{V} \leqslant (\mu(\frac{\partial f^{[k]}}{\partial x}) - v)\, V$, where $\mu(A) = \lim_{h\to 0} \frac{1}{h}(|I+hA|-1)$ is the Lozinskiĭ measure of the square matrix A and $|\cdot|$ denotes both the vector norm and the matrix norm it induces; see Coppel [1] page 41. When $|\cdot|$ is the l^∞, l^1 or l^2 norm on $\mathbf{R}^{\binom{n}{k}}$, $\mu(\frac{\partial f^{[k]}}{\partial x})$ is, respectively, the expression (i), (ii), or (iii),

$$\begin{aligned}
&\text{(i)} \quad \sup_{(i)} \left\{ \frac{\partial f_{i_1}}{\partial x_{i_1}} + \cdots + \frac{\partial f_{i_k}}{\partial x_{i_k}} + \Sigma_{j \notin (i)} \left(\left| \frac{\partial f_j}{\partial x_{i_1}} \right| + \cdots + \left| \frac{\partial f_j}{\partial x_{i_k}} \right| \right) \right\},\\
&\text{(ii)} \quad \sup_{(i)} \left\{ \frac{\partial f_{i_1}}{\partial x_{i_1}} + \cdots + \frac{\partial f_{i_k}}{\partial x_{i_k}} + \Sigma_{j \notin (i)} \left(\left| \frac{\partial f_{i_1}}{\partial x_j} \right| + \cdots + \left| \frac{\partial f_{i_k}}{\partial x_j} \right| \right) \right\}, \qquad (4.3)\\
&\text{(iii)} \quad \alpha_1 + \cdots + \alpha_k,
\end{aligned}$$

where the suprema are taken over all *k-tuples* $(i) = (i_1, ..., i_k)$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$, and $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$ are the eigenvalues of $\frac{1}{2}(\frac{\partial f}{\partial x}^* + \frac{\partial f}{\partial x})$. Thus we find the following corollaries.

COROLLARY 4.3. *An equilibrium* $x_0 \in \Sigma$ *is asymptotically stable with respect to the flow of* (1.1) *on* Σ *if*

$$\mu\left(\frac{\partial f^{[m+1]}}{\partial x}(x_0)\right) - v(x_0) < 0.$$

COROLLARY 4.3. *An* ω*-periodic solution* $\varphi(t) \in \Sigma$ *is orbitally asymptotically stable with asymptotic phase with respect to the flow of* (1.1) *on* Σ *if*

$$\int_0^\omega \left[\mu\left(\frac{\partial f^{[m+2]}}{\partial x}(\varphi(s))\right) - v(\varphi(s)) \right] < 0.$$

Proof of Theorem 4.1. The eigenvalues of $\frac{\partial f^{[m+1]}}{\partial x}(x_0)$ are $\lambda_{i_1} + \cdots + \lambda_{i_{m+1}}$, $1 \leqslant i_1 < \cdots < i_{m+1} \leqslant n$, where $\lambda_1, ..., \lambda_n$ are the eigenvalues of $\frac{\partial f}{\partial x}(x_0)$. The asymptotic stability of the constant coefficient system (4.1) is therefore equivalent to $\mathrm{Re}(\lambda_{i_1} + \cdots + \lambda_{i_{m+1}}) - v(x_0) < 0$. In particular, since

$\nu(x_0)=\lambda_1+\cdots+\lambda_m$ from Proposition 3.5, Re $\lambda_j=\mathrm{Re}(\lambda_1+\cdots+\lambda_m+\lambda_j)-\nu(x_0)<0$, $j=m+1, ..., n$, if (4.1) is asymptotically stable. Thus all eigenvalues λ_j of $\frac{\partial f}{\partial x}(x_0)$ corresponding to the invariant subspace $\mathscr{T}_{x_0}$ satisfy Re $\lambda_j<0$ and x_0 is stable hyperbolic with respect to the flow on Σ as asserted in (a); see Szlenk [19] page 58, Theorem 1.7.2. To prove part (b), note that $-\lambda_1, ..., -\lambda_m$ are the eigenvalues of $-N^*(x_0)$ and that Re $\lambda_i \geqslant 0$, $i=1,..., m$, if the system in (b) is stable. Therefore, $\mathrm{Re}(\lambda_{i_1}+\cdots+\lambda_{i_{m+1}})-\nu(x_0)=\mathrm{Re}(\lambda_{i_1}+\cdots+\lambda_{i_{m+1}}-\lambda_1-\cdots-\lambda_m)<0$ if Re $\lambda_j<0$, $j=m+1, ..., n$, establishing the asymptotic stability of (4.1) as asserted when x_0 is stable hyperbolic with respect to the flow on Σ. ∎

Proof of Theorem 4.2. First we note that $\dot{y}=\frac{\partial f}{\partial x}(\varphi(t))\, y$ is an *ω-periodic* system. The eigenvalues $\mu_1, ..., \mu_n$ of $\frac{\partial \varphi_\omega}{\partial x}(x_0)$ are the Floquet multipliers of this system. As remarked previously, $\mathscr{T}_{x_0}$ is an invariant subspace of $\frac{\partial \varphi_\omega}{\partial x}(x_0)$; the multipliers $\mu_{m+1}, ..., \mu_n$ corresponding to this subspace are thus the eigenvalues of $\frac{\partial \varphi_\omega}{\partial x}(x_0)|_{\mathscr{T}_{x_0}}$. Since $\mu_n=1$, we must show that the asymptotic stability of (4.2) implies $|\mu_j|<1$, $j=m+1, ..., n-1$ to deduce the hyperbolic stability of $\gamma_+(x_0)$. With $U(t)$ as in Proposition 3.2, $U^*(t)\,\frac{\partial g}{\partial x}(\varphi(t))\,\frac{\partial \varphi_t}{\partial x}(x_0)=\frac{\partial g}{\partial x}(\varphi(0))$ from that proposition. Since $x_0=\varphi(0)=\varphi(\omega)$,

$$\frac{\partial g}{\partial x}(x_0)\,\frac{\partial \varphi_\omega}{\partial x}(x_0)=U^{*-1}(\omega)\,\frac{\partial g}{\partial x}(x_0). \tag{4.4}$$

Referred to an orthogonal basis $\{u^1, ..., u^n\}$ of $\mathbf{R}^n$ where $\{u^{n-m}, ..., u^n\}$ span $\mathscr{T}_{x_0}$, the matrices in (4.4) have the form

$$\frac{\partial g}{\partial x}(x_0)=[G_{m\times m}\, 0_{m\times(n-m)}], \qquad \frac{\partial \varphi_w}{\partial x}(x_0)=\begin{bmatrix} A_{m\times m} & 0_{m\times(n-m)} \\ C_{(n-m)\times m} & B_{(n-m)\times(n-m)} \end{bmatrix}$$

since the row space of $\frac{\partial g}{\partial x}(x_0)$ is orthogonal to $\mathscr{T}_{x_0}$. The eigenvalues of A are $\mu_1, ..., \mu_m$ and those of B are $\mu_{m+1}, ..., \mu_n$ and (4.4) implies

$$\mu_1\cdots\mu_m=\det U^{*-1}(\omega)=\exp\left(\int_0^\omega \nu(\varphi)\right). \tag{4.5}$$

Now the system (4.2) is ω-periodic. Its Floquet multipliers are eigenvalues of

$$\bigwedge^{m+2}\frac{\partial \varphi_\omega}{\partial x}(x_0)\exp\left(-\int_0^\omega \nu(\varphi)\right),$$

which are $\alpha(i)=\mu_{i_1}\cdots\mu_{i_{m+2}}\exp(-\int_0^\omega \nu(\varphi))=\mu_{i_1}\cdots\mu_{i_{m+2}}/\mu_1\cdots\mu_m$, $1\leqslant i_1<\cdots<i_{m+2}\leqslant n$, from (4.5). The asymptotic stability of (4.2) implies

$|\alpha(i)| < 1$. In particular, since $\mu_n = 1$, $|\mu_j| = |\mu_1 \cdots \mu_m \mu_j \mu_n / \mu_1 \cdots \mu_m| < 1$, $j = m+1, ..., n-1$, which implies part (a) of Theorem 4.2. To prove part (b), note that the hyperbolic stability of $\gamma_+(x_0)$ implies $|\mu_j| < 1$, $j = m+1, ..., n-1$, which in turn implies $|\alpha(i)| < 1$, and hence the asymptotic stability of (4.2) as asserted, when $\dot{u} = -N^*(\varphi(t))\, u$ is stable. This follows from the fact that the Floquet multipliers of this system are $1/\mu_1, ..., 1/\mu_m$ and its stability implies $|1/\mu_j| \leqslant 1$ and therefore $|\mu_j| \geqslant 1$, $j = 1, ..., m$. ∎

Theorem 4.2 reduces, when $m = 0$, to a result of Muldowney [14] which in turn generalizes a result of Poincaré that, when $n = 2$, an ω-periodic solution $\varphi(t)$ of (1.1) is orbitally asymptotically stable with asymptotic phase if $\int_0^\omega \operatorname{div} f(\varphi(t))\, dt < 0$, which is equivalent to the asymptotic stability of $\dot{z} = \frac{\partial f^{[2]}}{\partial x}(\varphi(t))\, z$ since $\frac{\partial f^{[2]}}{\partial x} = \operatorname{div} f$ when $n = 2$; see Coppel [1] page 85. The present theorem is motivated by a result of Li [8] in which $x \mapsto g(x)$ is an affine function and the $m \times m$ matrix $N(x) = \alpha(x)\, I$ where $\alpha(x)$ is real and, in the notation of this paper, $v(x) = m\alpha(x)$.

5. BENDIXSON CONDITIONS

Suppose $\mathscr{D} \subset \mathbf{R}^n$. Let $\mathscr{U}$ be the euclidean unit ball in $\mathbf{R}^2$ and $\bar{\mathscr{U}}$, $\partial\mathscr{U}$ be its closure and boundary, respectively. A function $\phi \in \operatorname{lip}(\bar{\mathscr{U}} \to \mathscr{D})$ will be described as a *rectifiable* 2-*surface* in $\mathscr{D}$; a function $\psi \in \operatorname{lip}(\partial\mathscr{U} \to \mathscr{D})$ is a *closed rectifiable curve* in $\mathscr{D}$ and will be called *simple* if it is one-to-one. The sets $\phi(\bar{\mathscr{U}})$, $\psi(\partial\mathscr{U}) \subset \mathscr{D}$ are the trace of ϕ and ψ respectively. A curve ψ will be said to be *invariant* with respect to (1.1) if its trace $\psi(\partial\mathscr{U})$ is an invariant set. Nonequilibrium periodic orbits, homoclinic and heteroclinic cycles are examples of the traces of simple closed curves which are invariant.

If $\psi(\partial\mathscr{U})$ is in a simply connected subset of Σ, then the set $\mathscr{S}(\psi, \Sigma)$ of rectifiable 2-*surfaces* ϕ such that $\phi|_{\partial\mathscr{U}}$ is one to one and $\phi(\partial\mathscr{U}) = \psi(\partial\mathscr{U})$ is nonempty; see [11]. If $x \mapsto a(x) \geqslant 0$ is a continuous real-valued function on Σ, we will consider the functional $\mathscr{A}$ on $\mathscr{S}(\psi, \Sigma)$ defined by

$$\mathscr{A}\phi = \int_{\phi(\bar{\mathscr{U}})} a = \int_{\bar{\mathscr{U}}} a \circ \phi \left\| \frac{\partial \phi}{\partial u_1} \wedge \frac{\partial \phi}{\partial u_2} \right\|,$$

where ϕ is the map $(u_1, u_2) \mapsto \phi(u_1, u_2)$.

Proposition 5.1. *Suppose that ψ is a simple closed rectifiable curve in Σ. If $a(x_0) > 0$ for some $x_0 \in \psi(\partial\mathscr{U})$, then there exists $\delta > 0$ such that $\mathscr{A}\phi \geqslant \delta$ for all $\phi \in \mathscr{S}(\psi, \Sigma)$.*

Proof (*sketch*). First assume that $\psi(\partial\mathscr{U})$ contains a line segment. Specifically we may suppose that $x_0 \in l \subset \psi(\partial\mathscr{U})$, where l is an open segment of the e^1-axis in $\mathbf{R}^n$. Choose $\varepsilon > 0$ sufficiently small that the only points of $\psi(\partial\mathscr{U})$ in $B(x_0, \varepsilon)$ are those in l and $a(x) \geqslant a_0 > 0$ if $x \in B(x_0, \varepsilon)$. Consider the surface in $\mathbf{R}^2$, $u \mapsto \rho(u) = (\phi_1(u), (\sum_{i=2}^n \phi_i(u)^2)^{1/2})$, $u \in \overline{\mathscr{U}}$; this can be considered as a rigid rotation of the surface ϕ into a half plane with l in its boundary. Let $\mathscr{U}_\varepsilon \subset \phi^{-1}(B(x_0, \varepsilon))$. A degree theoretic argument like that in the proof of Lemma 2 of [12], page 462, shows that $\rho(\overline{\mathscr{U}}_\varepsilon)$ contains a half disc of radius ε. Consequently $\frac{1}{2}\pi\varepsilon^2 \leqslant \text{area } \rho(\overline{\mathscr{U}}_\varepsilon) = \int_{\overline{\mathscr{U}}_\varepsilon} \| \frac{\partial\rho}{\partial u_1} \wedge \frac{\partial\rho}{\partial u_2} \|$. The Cauchy–Bunyakowski–Schwarz inequality implies $a_0 \int_{\overline{\mathscr{U}}_\varepsilon} \| \frac{\partial\rho}{\partial u_1} \wedge \frac{\partial\rho}{\partial u_2} \| \leqslant \int_{\overline{\mathscr{U}}_\varepsilon} a \circ \phi \, \| \frac{\partial\phi}{\partial u_1} \wedge \frac{\partial\phi}{\partial u_2} \| \leqslant \mathscr{A}\phi$ and therefore $\frac{1}{2}a_0\pi\varepsilon^2 \leqslant \mathscr{A}\phi$. If $\psi(\partial U)$ contains no line segment, we may still use this argument by approximating ψ by a function which is affine near $\psi^{-1}(x_0)$. ∎

Now consider the autonomous system of dimension $n + \binom{n}{m+2}$

$$\dot{x} = f(x), \qquad \dot{z} = \frac{\partial f^{[m+2]}}{\partial x}(x)\, z \tag{5.1}$$

-and a locally lipschitzian function $(x, z) \mapsto V(x, z)$ such that, if $x \in \Sigma$, $z \in \mathbf{R}^{\binom{n}{m+2}}$,

$$a(x)\, \|z\| \leqslant V(x, z) \leqslant b(x)\, \|z\|, \tag{5.2}$$

where $x \mapsto a(x)$, $b(x)$ are non-negative continuous functions. Suppose that $x \mapsto \tilde{\mu}(x)$ is continuous and

$$V'_{(5.1)}(x, z) \leqslant \tilde{\mu}(x)\, V(x, z), \tag{5.3}$$

where $V'_{(5.1)}(x, z) = \limsup_{h \to 0^+} \frac{1}{h}[V(x + hf(x), z + h\, \frac{\partial f^{[m+2]}}{\partial x}(x)\, z) - V(x, z)]$. The system (1.1) will be said to satisfy a *Bendixson condition* on a set $\mathscr{D} \subset \Sigma$ if $\varphi_t(x)$ exists for all $t \geqslant 0$, $x \in \mathscr{D}$ and there exists a function V satisfying (5.2), (5.3) and the following two conditions hold uniformly with respect to x in each compact subset of $\mathscr{D}$:

$$\left\| \bigwedge^m \frac{\partial g^*}{\partial x}(y) \right\| \Big/ \left\| \bigwedge^m \frac{\partial g^*}{\partial x}(x) \right\|, \ y \in \gamma_+(x), \text{ is bounded and}$$

$$\lim_{t \to \infty} \int_0^t (\tilde{\mu}(\varphi_s(x)) - \nu(\varphi_s(x))\, ds = -\infty. \tag{5.4}$$

The Bendixson condition will be said to be *global* on $\mathscr{D}$ if $a(x)>0$, for all $x \in \mathscr{D}$, in (5.2). The solutions of (5.1) are $(x(t), z(t)) = (\varphi_t(x), \wedge^{m+2} \frac{\partial \varphi_t}{\partial x}(x)\, z)$, where $(x(0), z(0)) = (x, z)$. When (1.1) satisfies a Bendixson condition on $\mathscr{D}$, (5.2) and (5.3) imply that, when $x \in \mathscr{D}$,

$$a(\varphi_t(x)) \left\| \bigwedge^{m+2} \frac{\partial \varphi_t}{\partial x}(x)\, z \right\| \leqslant b(x)\, \|z\| \exp\left(\int_0^t \tilde{\mu}(\varphi_s(x))\, ds \right). \tag{5.5}$$

From (5.5) and Proposition 3.4, it follows that

$$a(\varphi_t(x)) \left\| \bigwedge^{2} \left(\frac{\partial \varphi_t}{\partial}(x) \Big|_{\mathscr{T}_x} \right) \right\| \leqslant Cb(x) \exp\left(\int_0^t (\tilde{\mu}(\varphi_s(x)) - v(\varphi_s(x)))\, ds \right). \tag{5.6}$$

Thus (5.4) implies $\lim_{t \to \infty} a(\varphi_t(x))\, \|\wedge^2 (\frac{\partial \varphi_t}{\partial x}(x)|_{\mathscr{T}_x})\| = 0$ uniformly with respect to x in each compact subset of $\mathscr{D}$ when (1.1) satisfies a Bendixson condition on $\mathscr{D}$.

THEOREM 5.2. *Suppose that* (1.1) *satisfies a Bendixson condition in a simply connected relatively open set* $\mathscr{D} \subset \Sigma$. *Then there is no simple closed rectifiable curve in* $\mathscr{D}$ *whose trace intersects* $\{x \in \mathscr{D} : a(x) > 0\}$ *and which is invariant with respect to* (1.1).

Proof. Suppose there is a simple closed rectifiable curve ψ in $\mathscr{D}$ which is invariant with respect to (1.1) and that $a(\psi(u_0)) > 0$ for some $u_0 \in \mathscr{U}$. Let $u \mapsto \chi(u)$ be a 2-surface in $\mathscr{S}(\psi, \mathscr{D})$ Then, since $\psi(\partial \mathscr{U})$ is invariant, $u \mapsto \varphi_t(\chi(u))$ is a 2-surface in $\mathscr{S}(\psi, \Sigma)$. Moreover, $\frac{\partial \varphi_t}{\partial u_i}(\chi(u)) = \frac{\partial \varphi_t}{\partial x}(\chi(u))\, \frac{\partial \chi}{\partial u_i}(u) = \frac{\partial \varphi_t}{\partial x}|_{\mathscr{T}_{\chi(u)}}\, \frac{\partial \chi}{\partial u_i}(u)$, $i = 1, 2$, imply

$$\frac{\partial \varphi_t}{\partial u_1}(\chi(u)) \wedge \frac{\partial \varphi_t}{\partial u_2}(\chi(u)) = \bigwedge^{2} \left(\frac{\partial \varphi_t}{\partial x}(\chi(u)) \Big|_{\mathscr{T}_{\chi(u)}} \right) \frac{\partial \chi}{\partial u_1}(u) \wedge \frac{\partial \chi}{\partial u_2}(u)$$

and (5.2), (5.6) imply

$$\lim_{t \to \infty} a(\varphi_t(\chi(u))) \left\| \frac{\partial \varphi_t}{\partial u_1}(\chi(u)) \wedge \frac{\partial \varphi_t}{\partial u_2}(\chi(u)) \right\| = 0$$

uniformly with respect to $u \in \overline{\mathscr{U}}$. Hence $\lim_{t \to \infty} \mathscr{A} \varphi_t \circ \chi = 0$ contradicting Proposition 5.1. Therefore no such invariant curve exists in $\mathscr{D}$. ∎

If we choose $V(x, z) = |z|$, where $|\cdot|$ is any norm in $\mathbf{R}^{\binom{n}{m+2}}$, then (5.2) is satisfied with $a(x)$ constant and we may choose $\tilde{\mu}(x) = \mu(\frac{\partial f}{\partial x}^{[m+2]}(x))$ where μ is the Lozinskiĭ measure corresponding to $|\cdot|$ as discussed in Section 4. In particular $\tilde{\mu}$ may be any of the expressions in (4.3), $k = m+2$.

The choice $V(x, z) = e^{v(x)}\,|z|$, where $x \mapsto v(x)$ is any C^1 real-valued function, leads to

$$\tilde{\mu}(x) = v'_{(1.1)}(x) + \mu\left(\frac{\partial f}{\partial x}^{[m+2]}(x)\right). \tag{5.7}$$

More generally, if we consider $V(x, z) = |A(x)\,z|$, where $x \mapsto A(x)$ is a C^1 nonsingular $\binom{n}{m+2} \times \binom{n}{m+2}$ matrix-valued function on $\mathscr{D}$, we find

$$\tilde{\mu} = \mu\left(A'_{(1.1)} A^{-1} + A\,\frac{\partial f}{\partial x}^{[m+2]} A^{-1}\right). \tag{5.8}$$

When $m = n-2$, then $\mu((\partial f/\partial x)^{[m+2]}(x)) = \operatorname{div} f(x)$ in (5.7) so that, for example, with $v(x) = 0$, we may choose $\tilde{\mu}(x) = \operatorname{div} f(x)$ in (5.4) for this case.

A set $\mathscr{D}_0$ is *absorbing* with respect to (1.1) on Σ if each compact subset $\mathscr{B} \subset \Sigma$ satisfies $\varphi_t(\mathscr{B}) \subset \mathscr{D}_0$ for all sufficiently large t.

COROLLARY 5.3. *Suppose V satisfies* (5.2), (5.3) *and*

$$\tilde{\mu}(x) < v(x) \tag{5.9}$$

if $x \in \Sigma$. Then, if Σ has a compact absorbing set, (1.1) *satisfies a Bendixson condition on Σ and no simple closed curve whose trace intersects $\{x \in \Sigma : a(x) > 0\}$ and which lies in a simply connected subset of Σ can be invariant with respect to* (1.1).

Evidently, (5.9) implies that (5.4) is satisfied uniformly with respect to x in any compact subset of Σ.

Consider the following system

$$\begin{aligned} x_1' &= -x_2 \\ x_2' &= x_1 \\ x_3' &= -x_3. \end{aligned} \tag{5.10}$$

The $x_1 x_2$-plane is invariant and defined by $g(x) = x_3 = 0$. Since $m = 1$ and $g'_{(5.1)}(x) = -g(x)$, we have $v(x) = N(x) = -1$. Noting that $n = m+2$ for system (5.1) and using the remark after (5.8), $\tilde{\mu}(x) = \operatorname{div} f(x) = -1 = v(x)$, for all $x \in \mathbf{R}^3$. Therefore the strict inequality in the Bendixson condition (5.9) does not hold. It is easy to see that (5.10) has a family of concentric periodic orbits on the invariant $x_1 x_2$-plane. This demonstrates the sharpness of our Bendixson conditions. We also remark that the negative divergence in this example does not give a Bendixson condition, since the invariant

x_1x_2-plane is not a level surface of a first integral, as is required in the result of Demidowitsch.

6. CONVERGENCE THEOREMS

It was shown by R. A. Smith [18] that his Bendixson condition for dissipative systems in $\mathbf{R}^n$ has an even stronger implication than the non-existence of periodic orbits other than equilibria. The alpha or omega limit set of any precompact semi-orbit in such a system is a single equilibrium. Li and Muldowney [13] extend this result to general systems in $\mathbf{R}^n$ satisfying their Bendixson–Dulac conditions and further show that the Hausdorff dimension of any compact invariant set in such a system is at most 1. McCluskey and Muldowney [14] give an elementary proof that the classical Bendixson condition for planar systems implies that every bounded solution converges to an equilibrium. Here we will prove a similar assertion for systems that satisfy a Bendixson condition (5.2), (5.3), (5.4) on an invariant manifold Σ.

THEOREM 6.1. *Suppose that the invariant manifold Σ is simply connected and that the system* (1.1) *satisfies a global Bendixson condition on Σ. Then, if $x_0 \in \Sigma$ and $\gamma_+(x_0)$ is precompact,* $\lim_{t\to\infty} \varphi_t(x_0) = p$ *where p is an equilibrium whose stable manifold with respect to the flow on Σ has codimension* 1 *at most.*

The restriction of the C^1 vector field f to the invariant manifold Σ will also be denoted by f. A point $p \in \Sigma$ is *wandering* with respect to the flow $(t, x) \mapsto \varphi_t(x)$ on Σ if there exists a neighbourhood $\mathcal{N}$ in Σ of p and $T > 0$ such that $\varphi_t(\mathcal{N}) \cap \mathcal{N}$ is empty if $t \geqslant T$. Any alpha or omega limit point, for example, is non-wandering. The C^1 Closing Lemma of Pugh [16], as proved by Pugh and Robinson [17] and formulated by Hirsch in [6], states that, if a non-equilibrium p is non-wandering with respect to the flow of a C^1 vector field f on a manifold Σ and the orbit of p has compact closure, then every neighbourhood of f in the space of C^1 vector fields on Σ contains a field $\hat{f}$ having a periodic orbit through p. Moreover, $\hat{f}$ can be chosen to agree with f outside a given neighbourhood $\mathcal{N}$ of p.

Proof of Theorem 6.1. We first use the Closing Lemma to prove that every $p \in \omega(x_0) = \bigcap_{t \geqslant 0} \bar{\gamma}_+(\varphi_t(x_0))$ is an equilibrium. Suppose $f(p) \neq 0$. Then p is not a periodic point of the vector field f because this is precluded by the global Bendixson condition. Let $\mathcal{N}_0$ be a precompact neighbourhood of p, $B = \sup_{x \in \mathcal{N}_0} \|b(x)\|/\|a(x)\|$ and let $K > C$; see (5.2) and (5.6). Let $\mathcal{S}_0$ be a codimension 1 transverse section through p of the flow of f on Σ and consider for the Closing Lemma a neighbourhood $\mathcal{N} \subset \mathcal{N}_0$ of p,

$\mathcal{N} = \bigcup_{t \in (-\alpha, \alpha)} \varphi_t(\mathcal{S}_0)$. For any preassigned $T > 0$, $\mathcal{N}$ may be chosen sufficiently small that $\varphi_t(\mathcal{S}_0)$ does not intersect $\mathcal{S}_\alpha$, $0 < t < T$, where $\mathcal{S}_\alpha = \varphi_\alpha(\mathcal{S}_0)$; otherwise p would be a non-equilibrium periodic point contradicting the Bendixson condition. Choose $\mathcal{N}$ so that T is large enough to ensure from (5.4) that

$$K\, B \exp\left(\tfrac{1}{2} \int_0^t m(\varphi_s(x))\, ds\right) < 1, \qquad \text{for} \quad t \geqslant T, \tag{6.1}$$

if $x \in \mathcal{S}_\alpha$ and $m(x) = \tilde{\mu}(x) - \nu(x)$. Let $u \mapsto \chi(u)$ be a rectifiable 2-surface in Σ. From (5.6) we find that

$$a(\varphi_t(\chi))\, \|w(t)\| \leqslant C\, b(\chi) \exp\left(\int_0^t m(\varphi_s(\chi))\, ds\right) \|w(0\|, \tag{6.2}$$

where $\chi = \chi(u)$ and $w(t) = \frac{\partial}{\partial u_1} \varphi_t(\chi) \wedge \frac{\partial}{\partial u_2} \varphi_t(\chi)$, $u \in \mathcal{U}$.

We will show that the assumption $f(p) \neq 0$ leads to the conclusion that, when a vector field $\hat{f}$ of the Closing Lemma is sufficiently C^1 close to f, then

$$\lim_{t \to \infty} a(\hat{\varphi}_t(\chi))\, \|\hat{w}(t)\| = 0, \tag{6.3}$$

where $\chi = \chi(u)$, $\hat{\varphi}_t$ is the flow of $\hat{f}$ and $\hat{w}(t) = \frac{\partial}{\partial u_1} \hat{\varphi}_t(\chi) \wedge \frac{\partial}{\partial u_2} \hat{\varphi}_t(\chi)$. Further, it will be shown that $a(\hat{\varphi}_t(\chi))\, \|\hat{w}(t)\|$ is uniformly bounded with respect to $u \in \mathcal{U}$ and consequently $\lim_{t \to \infty} \mathcal{A} \hat{\varphi}_t \circ \chi = 0$ as in the proof of Theorem 5.2. This leads to the conclusion that the vector field $\hat{f}$ has no periodic orbits contradicting the Closing Lemma and hence $f(p) = 0$ as asserted.

To prove (6.3), just note that this is obvious for orbits $\hat{\gamma}_+ = \{\hat{\varphi}_t(x) \colon t \geqslant 0\}$ that enter $\mathcal{N}$ at most finitely many times. This follows from (5.4), (6.2) since $\hat{\varphi}_t(y) = \varphi_t(y)$ if $y = \hat{\varphi}_{t_0}(x)$ and $\hat{\varphi}_t(x) \notin \mathcal{N}$, $t \geqslant t_0$, and $\hat{f}$ agrees with f outside $\mathcal{N}$. Now choose the vector field $\hat{f}$ in the Closing Lemma sufficiently C^1 close to f so that $\hat{f}(x) \neq 0$, if $x \in \mathcal{N}$, and that

$$\begin{gathered} K\, B \exp\left(\tfrac{1}{2} \int_0^t m(\hat{\varphi}_s(x))\, ds\right) < 1, \qquad \text{if} \quad t \geqslant T \\ a(\hat{\varphi}_t(x))\, \|\hat{w}(t)\| \leqslant K\, b(x) \exp\left(\int_0^t m(\hat{\varphi}_s(x))\, ds\right) \|\hat{w}(0)\| \end{gathered} \tag{6.4}$$

when $x \in \mathcal{S}_\alpha$ and $\hat{\varphi}_s(x) \notin \mathcal{S}_\alpha$, $0 < s < t$. This follows from (6.1), (6.2), $K > C$ and the fact that $\hat{\varphi}_s(x) = \varphi_s(x)$, $0 \leqslant s \leqslant t$, except possibly for $s \in [t - \delta, t]$ when $\hat{\varphi}_s(x) \in \mathcal{N}$. The length $\delta \geqslant 0$ of this interval is bounded, the bound is arbitrarily small relative to T, dependent only on α and the C^1 proximity of $\hat{f}$ to f. Let $\{t_k : k=1, 2, ...\}$ be such that $\hat{\varphi}_{t_k}(x) \in \mathscr{S}_\alpha$ and $\hat{\varphi}_t(x) \notin \mathscr{S}_\alpha$, $t_k < t < t_{k+1}$. Note that the set t_k depends on $\chi = \chi(u)$ and for a given u may be empty, finite (some $t_k = \infty$) or infinite. Then (6.4) shows that

$$K\,B \exp\left(\tfrac{1}{2}\int_{t_k}^{t_{k+1}} m(\hat{\varphi}_s(\chi)\,ds\right) < 1$$

$$a(\hat{\varphi}_t(\chi))\,\|\hat{w}(t)\| \leqslant K\,b(\hat{\varphi}_{t_k}(\chi)) \exp\left(\int_{t_k}^{t} m(\hat{\varphi}_s(\chi))\,ds\right) \|\hat{w}(t)\|$$

$t_k < t < t_{k+1}$. Therefore, when $\chi = \chi(u)$ and $t_k \leqslant t \leqslant t_{k+1}$,

$$\begin{aligned} a(\hat{\varphi}_t(\chi))\,\|\hat{w}(t)\| &\leqslant K^{k+1}B^k \exp\left(\int_0^t m(\hat{\varphi}_s(\chi))\,ds\right)\|\hat{w}(0)\| \\ &\leqslant K^2\,B\,b(\chi) \exp\left(\int_0^{t_1} m(\hat{\varphi}_s(\chi))\,ds \right. \\ &\quad \left. + \tfrac{1}{2}\int_{t_1}^{t_k} m(\hat{\varphi}_s(\chi))\,ds + \int_{t_k}^{t} m(\hat{\varphi}_s(\chi))\,ds\right)\|\hat{w}(0)\| \\ &\leqslant K^2\,B\,b(\chi) \exp\left(\int_0^{t_1} m(\hat{\varphi}_s(\chi))\,ds + \int_{t_k}^{t} m(\hat{\varphi}_s(\chi))\,ds\right) \\ &\quad \times (K\,B)^{-k}\,\|\hat{w}(0)\| \\ &\leqslant K^2\,B\,b(\chi) \exp\left(\int_0^{t_1} m(\hat{\varphi}_s(\chi))\,ds\right)(K\,B)^{-k-1}\,\|\hat{w}(0)\|, \end{aligned}$$

if $t - t_k \geqslant T$, from (6.4). This establishes (6.3) and the uniform boundedness assertion on $a(\hat{\varphi}_t(\chi))\,\|\hat{w}(t)\|$, $\chi = \chi(u)$, completing the proof that p is an equilibrium.

It remains only to prove that $\omega(x_0) = \{p\}$, a single equilibrium. From the preceding discussion each $p \in \omega(x_0)$ is an equilibrium in Σ and hence of the unrestricted vector field f. The linearization of (1.1) at p is the autonomous system $\dot{y} = \frac{\partial f}{\partial x}(p)\,y$. Let $\lambda_{m+1}, ..., \lambda_n$ be the eigenvalues of $\frac{\partial f}{\partial x}(p)$ corresponding to its invariant subspace $\mathscr{T}_p$ with $\operatorname{Re}\lambda_{m+1} \geqslant \operatorname{Re}\lambda_{m+2} \geqslant \cdots \geqslant \operatorname{Re}\lambda_n$. Then (5.4) and (5.6) with $x = p$ imply that $\lim_{t\to\infty} \wedge^2 (\frac{\partial \varphi_t}{\partial x}(p)|_{\mathscr{T}_p}) = 0$ and therefore the eigenvalues $\lambda_i + \lambda_j$, $m+1 \leqslant i < j \leqslant n$ of $\frac{\partial f^{[2]}}{\partial x}(p)$ corresponding to its invariant subspace $\wedge^2 \mathscr{T}_p$ all satisfy $\operatorname{Re}(\lambda_i + \lambda_j) < 0$. We can thus conclude that $\operatorname{Re}\lambda_i < 0$, $i = m+2, ..., n$ and only λ_{m+1} can possibly have nonnegative real part. Thus the stable manifold of p on Σ has at most codimension 1 with respect to Σ. If $\omega(x_0)$ has more than one point then, since every $q \in \omega(x_0)$ is an equilibrium and $\omega(x_0)$ is connected, $\lambda_{m+1} = 0$ and p has a local centre manifold in Σ of dimension 1. Moreover, in a sufficiently small neighbourhood of p, the local centre manifold consists entirely of equilibria each with a codimension 1 stable manifold. Since every orbit that intersects this neighbourhood is asymptotic to an orbit in the centre manifold, cf. [21] page 48, it follows that $\lim_{t\to\infty} \varphi_t(x_0) = p$ negating the existence of more than one point in $\omega(x_0)$ and thus $\{p\} = \omega(x_0)$ as asserted. ▮

Bendixson's condition also places strong limitations on general compact invariant sets. Let $\dim_H K$ denote the Hausdorff dimension of a compact set $K \subset \mathbf{R}^n$. A further development of the proof of Theorem 6.1 along the lines of the proof of Theorem 2.10 in [13] leads to the following result.

THEOREM 6.2. *Suppose that the invariant manifold Σ is simply connected and that* (1.1) *satisfies a global Bendixson condition on Σ. Then, if K is any compact invariant subset of Σ,*

$$\dim_H K \leqslant 1.$$

7. AN APPLICATION TO AN EPIDEMIC MODEL

Consider the following system of differential equations

$$\begin{aligned} s' &= b - bs - \lambda is + \alpha is + \delta r \\ e' &= \lambda is - (\varepsilon + b)\, e + \alpha ie \\ i' &= \varepsilon e - (\gamma + \alpha + b)\, i + \alpha i^2 \\ r' &= \gamma i - (b + \delta)\, r + \alpha ir, \end{aligned} \tag{7.1}$$

which arises from the study of a mathematical model for the spread of an infectious disease in a population with a varying total size. For the biological background and the derivation of the system, we refer the reader to [4], and to [10] for a special case. The variables s, e, i, and r represent fractions of the population that are susceptible, exposed (in the latent period), infectious, and recovered, respectively. All parameters are assumed to be nonnegative, and we assume that $\varepsilon > 0$ and $\gamma > 0$. The biological feasible region for system (7.1) is the following invariant simplex in the positive cone of $\mathbf{R}^4$

$$\Gamma = \{(s, e, i, r) \in \mathbf{R}^4_+ : s + e + i + r = 1\} \tag{7.2}$$

including all of its lower dimensional boundaries. Mathematically, system (7.1) will be regarded as a system in $\mathbf{R}^4_+$ with an invariant manifold Γ of dimension 3. The invariance of Γ follows from

$$(s+e+i+r-1)' = (\alpha i - b)(s+e+i+r-1). \tag{7.3}$$

It is also clear from (7.3) that $g(x) = s+e+i+r-1$, $N(x) = v(x) = \alpha i - b$, where $x = (s, e, i, r) \in \mathbf{R}^4_+$, and $m = \text{rk}(\frac{\partial g}{\partial x}) = 1$. Let $\bar{\Gamma}$ and $\mathring{\Gamma}$ denote the closure and the interior of Γ in the hyperplane $s+e+i+r=1$, respectively. Set

$$\mathscr{R}_0 = \frac{\lambda \varepsilon}{(\varepsilon + b)(\gamma + b + \alpha)}.$$

The following result follows from Theorem 2.3 of [4].

PROPOSITION 7.1. (a) *If $\mathscr{R}_0 < 1$, then the equilibrium $P_0 = (1, 0, 0, 0)$ of (7.1) is globally stable in $\bar{\Gamma}$.* (b) *If $\mathscr{R}_0 > 1$ and if $\delta < \min\{\varepsilon, \gamma\}$, then P_0 is unstable, and there exists a unique interior equilibrium $P^* = (s^*, e^*, i^*, r^*) \in \mathring{\Gamma}$ and P^* is locally asymptotically stable. Moreover, (7.1) is uniformly persistent in $\mathring{\Gamma}$ if $\mathscr{R}_0 > 1$.*

The equilibrium P_0 corresponds to the population being disease-free, and P^* to the disease being endemic. The uniform persistence assertion in Proposition 7.1 follows from the part (b) of Proposition 7.1 and can be proved using the same argument as in the proof of Proposition 3.3 in [10]. The uniform persistence and the boundedness of Γ implies the existence of a compact absorbing set $K \subset D$ such that each compact subset K_1 of $\mathring{\Gamma}$ satisfies $\varphi_t(K_1) \subset K$ for sufficiently large t. Equivalently, there exists a constant $c > 0$ such that

$$s(t) > c, \qquad e(t) > c, \qquad i(t) > c, \qquad r(t) > c \tag{7.4}$$

if $t > T = T(K_1)$, for all solutions $x(t) = (s(t), e(t), i(t), r(t))$ such that $x(0) \in K_1$.

The question of whether P^* is globally stable with respect to $\mathring{\Gamma}$ when $\mathscr{R}_0 > 1$ is of great biological interest and was left unresolved in [4]. Using the theory developed in the previous sections and Theorem 6.1 in particular, we prove the following global stability result. Note that $\mathscr{R}_0 > 1$ implies $\lambda > \alpha$.

THEOREM 7.2. *Assume that $\mathscr{R}_0 > 1$ and that $\alpha \leqslant \varepsilon$. Then the unique endemic equilibrium P^* is globally asymptotically stable in $\mathring{\Gamma}$ when $0 \leqslant \delta \leqslant \min\{\gamma, \varepsilon, (\lambda - \alpha)\, a_2 c^2 / a_1\}$, where $a_1 = (\gamma + \alpha)/(\gamma - \delta) > 0$, $a_2 = (\lambda - \alpha)/\lambda > 0$.*

Proof. It suffices to show that each positive semiorbit in $\mathring{\Gamma}$ converges to an equilibrium. Let $f(x)$ denote the vector field defined by system (7.1) and $x=(s, e, i, r)$. Then the system (5.1) for (7.1) is

$$\dot{x}=f(x), \qquad \dot{z}=\frac{\partial f^{[3]}}{\partial x}(x)\, z, \tag{7.5}$$

where $z=(z_1, z_2, z_3, z_4)\in \mathbf{R}^4 \cong \mathbf{R}^{\binom{4}{3}}$. Using the Appendix, the third additive compound $\frac{\partial f^{[3]}}{\partial x}$ for (7.1) can be written as

$$\frac{\partial f^{[3]}}{\partial x}=(-3b-\lambda i+3\alpha i)\, I+\Psi$$

and Ψ is the following matrix

$$\begin{bmatrix} -\varepsilon-\gamma-\alpha+\alpha i & 0 & 0 & \delta \\ \gamma+\alpha r & -\varepsilon-\delta & \lambda s+\alpha e & \lambda s-\alpha s \\ 0 & \varepsilon & -\gamma-\delta-\alpha+\alpha i & 0 \\ 0 & 0 & \lambda i & \lambda i-\varepsilon-\gamma-\delta-\alpha+\alpha i \end{bmatrix}. \tag{7.6}$$

Let

$$V(x, z)=\max\left\{a_1\,|z_1|+|z_2|, \frac{e}{i}\,(|z_3|+a_2\,|z_4|)\right\}, \tag{7.7}$$

where a_1, a_2 are as stated in the theorem. Then $V(x, z)\geqslant a\,|z|$ for $(x, z)\in K\times \mathbf{R}^4\backslash\{0\}$ for some constant $a>0$, since $e\geqslant c$ and $i\geqslant c$ for x in the compact absorbing set $K\subset\mathring{\Gamma}$. Thus, the function V satisfies the condition (5.2). Let $(x(t), z(t))$ be a solution to (7.5) and set $V(t)=V(x(t), z(t))$. Then $\dot{V}(t)=V'_{(7.5)}(x(t), z(t))$ for almost all t. The following differential inequalities follow from (7.5) and (7.6).

$$\begin{aligned} D_+ a_1\,|z_1(t)| &\leqslant -(3b+\lambda i+\varepsilon+\gamma+\alpha-4\alpha i)\, a_1\,|z_1(t)|+\delta\, a_1\,|z_4(t)| \\ &\leqslant -(3b+\lambda i+\varepsilon+\gamma-3\alpha i)\, a_1\,|z_1(t)|+\frac{a_1\delta i}{a_2 e}\frac{e}{i}\, a_2\,|z_4(t)| \end{aligned} \tag{7.8}$$

$$\begin{aligned} D_+\,|z_2(t)| &\leqslant (\gamma+\alpha r)\,|z_1(t)|-(3b+\lambda i+\varepsilon+\delta-3\alpha i)\,|z_2(t)| \\ &\quad +(\lambda s+\alpha e)\,|z_3(t)|+(\lambda-\alpha)\, s\,|z_4(t)| \\ &\leqslant (\gamma-\delta)\, a_1\,|z_1(t)|-(3b+\lambda i+\varepsilon+\delta-3\alpha i)\,|z_2(t)| \\ &\quad +\left(\frac{\lambda is}{e}+\alpha i\right)\frac{e}{i}\,|z_3(t)|+\frac{\lambda is}{e}\frac{e}{i}\, a_2\,|z_4(t)| \end{aligned} \tag{7.9}$$

$$\begin{aligned} D_+ |z_3(t)| &\leqslant \varepsilon |z_2(t)| - (3b + \lambda i + \gamma + \delta + \alpha - 4\alpha i) |z_3(t)| \\ &\leqslant \varepsilon |z_2(t)| - (3b + \gamma + \delta + \alpha - 3\alpha i) |z_3(t)| \end{aligned} \tag{7.10}$$

$$\begin{aligned} D_+ a_2 |z_4(t)| &\leqslant \lambda a_2 i |z_3(t)| - (3b + \varepsilon + \gamma + \delta + \alpha - 4\alpha i) a_2 |z_4(t)| \\ &\leqslant (\lambda i - \alpha i) |z_3(t)| - (3b + \gamma + \delta + \alpha - 3\alpha i) a_2 |z_4(t)|, \end{aligned} \tag{7.11}$$

since $i < 1$, $\alpha i < \alpha \leqslant \varepsilon$, and $\lambda > \alpha$. Set

$$v_1(t) = a_1 |z_1(t)| + |z_2(t)| \qquad \text{and} \qquad v_2(t) = \frac{e(t)}{i(t)} (|z_3(t)| + a_2 |z_4(t)|). \tag{7.12}$$

Then using (7.8), (7.9) we have

$$D_+ v_1(t) \leqslant -(3b + \lambda i + \varepsilon + \delta - 3\alpha i) v_1(t) + \left(\frac{\lambda i s}{e} + \alpha i + \frac{\delta a_1}{a_2 e} \right) v_2(t). \tag{7.13}$$

From (7.10), (7.11) we derive

$$\begin{aligned} D_+ v_2(t) &= \left(\frac{e'}{e} - \frac{i'}{i} \right) v_2(t) + \frac{e}{i} D_+(|z_3(t)| + a_2 |z_4(t)|) \\ &\leqslant \frac{\varepsilon e}{i} |z_2(t)| + \left(\frac{e'}{e} - \frac{i'}{i} - 3b - \gamma - \alpha - \delta + 3\alpha i \right) v_2(t) \\ &\leqslant \frac{\varepsilon e}{i} v_1(t) + \left(\frac{e'}{e} - \frac{i'}{i} - 3b - \gamma - \alpha - \delta + 3\alpha i \right) v_2(t). \end{aligned} \tag{7.14}$$

Using (7.13) and (7.14) we can show

$$D_+ V(t) \leqslant \tilde{\mu}(t) V(t), \tag{7.15}$$

where $\tilde{\mu}(t) = \max\{ g_1(t),\ g_2(t) \}$ and

$$g_1(t) = -3b - \lambda i - \varepsilon - \delta + 4\alpha i + \left(\frac{\lambda i s}{e} + \frac{\delta a_1}{a_2 e} \right) \tag{7.16}$$

$$g_2(t) = \frac{\varepsilon e}{i} + \frac{e'}{e} - \frac{i'}{i} - 3b - \gamma - \alpha - \delta + 3\alpha i. \tag{7.17}$$

Rewriting (7.1) we find

$$\frac{\lambda si}{e}+\alpha i=\frac{e'}{e}+\varepsilon+b, \tag{7.18}$$

$$\frac{\varepsilon e}{i}+\alpha i=\frac{i'}{i}+\gamma+\alpha+b \tag{7.19}$$

$$\frac{r'}{r}-\frac{\gamma i}{r}=-b-\delta+\alpha i. \tag{7.20}$$

Recall that $v(t)=\alpha i(t)-b$. We thus have from (7.16)–(7.20),

$$\begin{aligned}\tilde{\mu}(t)-v(t)&\leqslant\frac{e'(t)}{e(t)}-b-\delta+\alpha i(t)+\max\left\{0,\ -(\lambda-\alpha)\, i(t)+\frac{\delta a_1}{a_2 e(t)}\right\}\\&\leqslant\frac{e'(t)}{e(t)}+\frac{r'(t)}{r(t)}-\frac{\gamma i(t)}{r(t)}+\max\left\{0,\ -(\lambda-\alpha)\, c+\frac{\delta a_1}{a_2 c}\right\},\end{aligned}$$

for all $t>T=T(K_1)$ and solutions $x=x(t)$ such that $x(0)\in K_1$, by (7.4). Set $\bar{\delta}=\min\{\gamma, \varepsilon, (\lambda-\alpha)\, a_2 c^2/a_1\}>0$. Then, if $\delta\leqslant\bar{\delta}$,

$$\int_0^t (\tilde{\mu}(\tau)-v(\tau))\, d\tau\leqslant\log e(t)+\log r(t)-\int_0^t \frac{\gamma i(\tau)}{r(\tau)}\, d\tau\leqslant 2\, |\log c|-\gamma ct$$

for $t>T$. Thus $V(x, z)$ also satisfies conditions (5.4), and Theorem 7.1 follows from Theorem 6.1. ∎

APPENDIX

The third additive compound matrix $A^{[3]}$ for a 4×4 matrix $A=(a_{ij})$ is

$$A^{[3]}=\begin{bmatrix} a_{11}+a_{22}+a_{33} & a_{34} & -a_{24} & a_{14}\\ a_{43} & a_{11}+a_{22}+a_{44} & a_{23} & -a_{13}\\ -a_{42} & a_{32} & a_{11}+a_{33}+a_{44} & a_{12}\\ a_{41} & -a_{31} & a_{21} & a_{22}+a_{33}+a_{44}\end{bmatrix}.$$

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REFERENCES

1. W. A. Coppel, "Stability and Asymptotic Behavior of Differential Equations," Heath, Boston, 1965.
2. W. B. Demidowitsch, Eine Verallgemeinerung des Kriteriums von Bendixson, *Z. Angew. Math. Mech.* **46** (1966), 145–146.
3. M. Fiedler, Additive compound matrices and inequality for eigenvalues of stochastic matrices, *Czech. Math. J.* **99** (1974), 392–402.
4. D. Greenhalgh, Hopf bifurcation in epidemic models with a latent period and nonpermanent immunity, *Math. Comput. Modelling* **25** (1997), 85–107.
5. P. Hartman, "Ordinary Differential Equations," 2nd ed., Birkhäuser, Boston, 1982.
6. M. W. Hirsch, Systems of differential equations that are competitive or cooperative. VI: A local C^r closing lemma for 3-dimensional systems, *Ergodic Theory Dynam. Systems* **11** (1991), 443–454.
7. M. Y. Li, "Geometrical Studies on the Global Asymptotic Behaviour of Dissipative Dynamical Systems," Ph.D. thesis, University of Alberta, 1993.
8. M. Y. Li, Dulac criteria for autonomous systems having an invariant affine manifold, *J. Math. Anal. Appl.* **199** (1996), 374–390.
9. M. Y. Li, Bendixson's criterion for autonomous systems with an invariant linear subspace, *Rocky Mountain J. Math.* **25** (1995), 351–363.
10. M. Y. Li, J. R. Graef, L. Wang, and J. Karsai, Global dynamics of a SEIR model with varying total population size, *Math. Biosci.* **160** (1999), 191–213.
11. Y. Li and J. S. Muldowney, On Bendixson's criterion, *J. Differential Equations* **106** (1994), 27–39.
12. Y. Li and J. S. Muldowney, Lower bounds for the Hausdorff dimension of attractors, *J. Dynam. Differential Equations* **7** (1995), 457–469.
13. Y. Li and J. S. Muldowney, On R. A. Smith's autonomous convergence theorem, *Rocky Mountain J. Math.* **25** (1995), 365–379.
14. C. C. McCluskey and J. S. Muldowney, Stability implications of Bendixson's criterion, *SIAM Rev.* **40** (1998), 931–934.
15. J. S. Muldowney, Compound matrices and ordinary differential equations, *Rocky Mountain J. Math.* **20** (1990), 857–872.
16. C. C. Pugh, The closing lemma, *Amer. J. Math.* **89** (1967), 956–1009.
17. C. C. Pugh and C. Robinson, The C^1 closing lemma including Hamiltonians, *Ergodic Theor. Dynam. System* **3** (1983), 261–313.
18. R. A. Smith, Some applications of Hausdorff dimension inequalities for ordinary differential equations, *Proc. Roy. Soc. Edinburgh Sect. A* **104** (1986), 235–259.
19. W. Szlenk, "An Introduction to the Theory of Smooth Dynamical Systems," Wiley, Chichester, 1984.
20. R. Temam, "Infinite-Dimensional Dynamical Systems in Mechanics and Physics," Applied Mathematical Sciences, Vol. 68, Springer-Verlag, New York, 1988.
21. S. Wiggins, "Global Bifurcations and Chaos. Analytical Methods," Applied Mathematical Sciences, Vol. 73, Springer-Verlag, New York, 1988.