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On Bendixson's Criterion*

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For autonomous differential equations in $\mathbb{R}^n$ criteria are developed which preclude the existence of invariant closed curves such as periodic or homoclinic trajectories. The technique is based on the study of functionals on 2-surfaces. Results generalize to higher dimensions a criterion of Bendixson for the nonexistence of nonconstant periodic solutions in the case $n=2$. As an example, an application to the Lorenz system in $\mathbb{R}^3$ is given. © 1993 Academic Press, Inc.

1. INTRODUCTION

Suppose the function $x \mapsto f(x)$ is C^1 with x in an open subset D_0 of $\mathbb{R}^n$ and $f(x) \in \mathbb{R}^n$. We investigate conditions for the nonexistence of nonconstant periodic solutions of the differential equation

$$\frac{dx}{dt} = f(x). \tag{1.1}$$

More generally, we give criteria for the nonexistence of simple closed curves which are invariant under the dynamics of (1.1).

Results given here generalize to higher dimensions the statement of Bendixson [1] that when $n=2$, (1.1) has no nonconstant periodic solutions if $\operatorname{div} f \neq 0$ on $\mathbb{R}^2$. A result of Dulac [6] generalizes this to the statement that if $n=2$ and $\operatorname{div}(\alpha f) \neq 0$ on a simply connected open subset D of $\mathbb{R}^2$, where α is a continuous real-valued function on D, then there is no closed path of (1.1) which lies entirely in D.

Smith [10, 11] seems to be the first to give a generalization of the results of Bendixson and Dulac for equations (1.1) of arbitrary dimension n. For example, Theorem 7 of [11] states that if $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ are the eigenvalues of $\frac{1}{2}((\partial f/\partial x)^* + \partial f/\partial x)$, where $\partial f/\partial x$ is the Jacobian matrix of f and the asterisk denotes transposition, then all bounded semi-orbits of (1.1)

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tend to an equilibrium if $\lambda_1 + \lambda_2 < 0$ on $\mathbb{R}^n$. In particular, no nonconstant periodic solution can exist under these circumstances. Smith also treats situations where $\lambda_1 + \lambda_2 < 0$ holds only on subsets D of $\mathbb{R}^n$. More generally, Theorem 2 of [11] shows that the Hausdorff dimension of any compact invariant set of (1.1) is less than 2, if $\lambda_1 + \lambda_2 < 0$, and Theorem 5 then implies that no simple closed contour is invariant if (1.1) is dissipative. Thus, for example, a dissipative system satisfying Smith's condition $\lambda_1 + \lambda_2 < 0$ can have no homoclinic orbits.

In [9] it was shown that if $\mu((\partial f/\partial x)^{[2]}) < 0$ or $\mu(-(\partial f/\partial x)^{[2]}) < 0$, then (1.1) has no nonconstant periodic solutions. Here $(\partial f/\partial x)^{[2]}$ is a $\binom{n}{2} \times \binom{n}{2}$ matrix, the second additive compound discussed in Section 2 of this paper, and μ is a "logarithmic norm" corresponding to a vector norm $|\cdot|$ as defined by Lozinskiĭ [8] and Dahlquist [5] (see Coppel [4, p. 41]). When $|y| = (y^* y)^{1/2}$, $\mu((\partial f/\partial x)^{[2]}) = \lambda_1 + \lambda_2$ so that $\mu((\partial f/\partial x)^{[2]}) < 0$ is Smith's condition in this case; here $\mu(-(\partial f/\partial x)^{[2]}) < 0$ means $\lambda_{n-1} + \lambda_n > 0$.

As can be seen from Theorem 3.3 and Section 4 of this paper, other choices of norm often lead to expressions $\mu(\pm(\partial f/\partial x)^{[2]})$ which are easier to calculate or estimate than eigenvalues. The resulting criteria also reduce to that of Bendixson when $n = 2$.

The argument of [9] that the condition $\lambda_1 + \lambda_2 < 0$ implies the nonexistence of closed paths is roughly as follows. First, this condition implies that the area of a surface decreases as it evolves under the dynamics of (1.1). By considering, in a certain generalized sense, a surface of "minimum area" whose boundary is C, a closed path (and therefore an invariant set) of (1.1), we find that its boundary continues to be C and that its area decreases as it evolves over a short time interval. The minimality of the original surface area is thus contradicted so that no such closed path can exist. The condition $\lambda_{n-1} + \lambda_n > 0$ similarly implies that surface areas increase in the system (1.1) and the same conclusion may be deduced from a time reversal. The result for general Lozinskiĭ norms μ of [9] is obtained in the same way by considering different measures of surface area in this argument.

In the present paper, we consider more general functionals than areas for surfaces with a fixed boundary C. By examining the behaviour of these functionals under the dynamics of (1.1), we arrive at new criteria for the nonexistence of invariant closed curves. Even in the case $n = 2$ our approach yields a slightly more flexible formulation of the results of Bendixson and Dulac than the traditional ones.

The present paper was mainly motivated by the work of Smith [11]. It has however some contact with other papers. The results given here may be extended in the spirit of Theorem 1 of Lloyd [7] which is a result of the Dulac type and gives a bound on the number of closed paths based on the connectivity of a region where Dulac's condition holds. The other results of

[7] develop a different technique for obtaining estimates on the number of limit cycles in a planar system. Busenberg and van den Driessche [2] give conditions which preclude the possibility of certain oriented loops occurring in the dynamics of (1.1) and are not confined to finite dimensional spaces. In [3] Butler, Schmid, and Waltman investigate a generalized divergence condition for n-dimensional systems and derive a condition like Dulac's which guarantees that invariant sets in certain population models have Lebesgue measure zero. This is somewhat related to work on the Hausdorff dimension of invariant sets; see Smith [11] and Temam [13].

2. Evolution of Surface Functionals

For an $n \times n$ matrix $A = [a_{ij}]$, the *second additive compound* $A^{[2]}$ is the $\binom{n}{2} \times \binom{n}{2}$ matrix defined as follows. For any integer $i = 1, ..., \binom{n}{2}$, let $(i) = (i_1, i_2)$ be the ith member in the lexicographic ordering of integer pairs (i_1, i_2) such that $1 \leqslant i_1 < i_2 \leqslant n$. Then the element in the i-row and the j-column of $A^{[2]}$ is

$$\begin{array}{rl} a_{i_1 i_1} + a_{i_2 i_2}, & \text{if } (j) = (i) \\ (-1)^{r+s} a_{i_r j_s}, & \text{if exactly one entry } i_r \text{ of } (i) \text{ does not} \\ & \text{occur in } (j) \text{ and } j_s \text{ does not occur in } (i) \\ 0, & \text{if neither entry from } (i) \text{ occurs in } (j). \end{array}$$

In the case $n = 3$, for example, $(1) = (1, 2)$, $(2) = (1, 3)$, and $(3) = (2, 3)$. Thus

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad A^{[2]} = \begin{bmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{bmatrix}.$$

An important connection between second additive compound matrices and differential equations is that if $z_1(t)$, $z_2(t)$ are solutions of a system $dz/dt = A(t)\, z$, then their Grassman product $y(t) = z_1(t) \wedge z_2(t)$ is a solution of $dy/dt = A^{[2]}(t)\, y$ (see [9]). In particular, we know that if $x(t, x_0)$ is the solution of (1.1) such that $x(0, x_0) = x_0$, then $z(t) = (\partial x/\partial x_0)(t, x_0)\, c$, $c \in \mathbb{R}^n$, satisfies the variational equation

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}(x(t, x_0))\, z. \tag{2.1}$$

Therefore $y(t) = z_1(t) \wedge z_2(t)$ satisfies

$$\frac{dy}{dt} = \frac{\partial f^{[2]}}{\partial x}(x(t, x_0))\, y, \tag{2.2}$$

if $z_1(t)$, $z_2(t)$ are solutions of (2.1).

These statements may be interpreted in the following way. In the dynamics of (1.1), points (elementary 0-forms) x_0 evolve in time according to the semigroup $x_0 \mapsto x(t, x_0)$, $t \in \mathbb{R}$; oriented infinitesimal line segments (elementary 1-forms) z_0 at x_0 evolve as $z_0 \mapsto z(t, z_0)$, solutions of (2.1), and oriented infinitesimal areas (elementary 2-forms) y_0 at x_0 evolve as $y_0 \mapsto y(t, y_0)$, solutions of (2.2).

Higher order forms have similar dynamics. A discussion may be found in [9] and its references.

Let $U = B^2(0, 1)$, the euclidean unit ball in $\mathbb{R}^2$ and let $\bar{U}$ and ∂U be its closure and boundary, respectively. If $D \subset \mathbb{R}^n$, a function $\varphi \in \operatorname{Lip}(\bar{U} \to D)$ will be described as a *simply connected rectifiable* 2-*surface in D*; a function $\psi \in \operatorname{Lip}(\partial U \to D)$ is a *closed rectifiable curve in D* and will be called *simple* if it is one-to-one. Moreover, if ψ is the restriction to ∂U of a function φ: $\bar{U} \to D$, we denote this by $\psi = \partial\varphi$. If D is an open, simply connected set, then

$$\sum(\psi, D) = \{\varphi \in \operatorname{Lip}(\bar{U} \to D),\ \varphi(\partial U) = \psi(\partial U)\}$$

is nonempty for each simple closed rectifiable curve ψ in D. To see this, let (r, θ) be polar coordinates in $\mathbb{R}^2$. Since $\psi(\partial U)$ is homotopic to a point in D there is a continuous function $(r, \theta) \mapsto \tilde{\varphi}(r, \theta) \in D$ with $\tilde{\varphi}(r, 0) = \tilde{\varphi}(r, 2\pi)$ and $\tilde{\varphi}(1, \theta) = \psi(1, \theta)$, $0 \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$. To find $\varphi \in \Sigma(\psi, D)$, we construct a lipschitzian approximation to $\tilde{\varphi}$ as follows. We partition $\bar{U}$ into triangular regions and let $\varphi(u) = \tilde{\varphi}(u) = \psi(u)$ if $u \in \partial U$ and $\varphi(u) = \tilde{\varphi}(u)$ if u is a vertex in the interior of U; by interpolating linearly in the triangles we find $\varphi \in \operatorname{Lip}(\bar{U} \to \mathbb{R}^n)$ such that $\varphi = \partial\psi$. Moreover, since D is open, $\varphi(\bar{U}) \subset D$ by continuity if the triangular partition is fine enough so that $\varphi \in \Sigma(\psi, D)$.

If D_0 is the domain of f in (1.1), we consider functionals $\mathscr{S}$ on the surfaces $\operatorname{Lip}(\bar{U} \to D_0)$ of the form

$$\mathscr{S}\varphi = \int_U S\left(\varphi(u), \frac{\partial}{\partial u_1}\varphi(u) \wedge \frac{\partial}{\partial u_2}\varphi(u)\right) du, \tag{2.3}$$

where $u = (u_1, u_2)$ and $(x, y) \mapsto S(x, y)$ is a real-valued function with $x \in D_0$, $y \in \mathbb{R}^N$, $N = \binom{n}{2}$. We require also that S be locally lipschitzian on its domain and that $\lim_{h \to 0^+}(1/h)[S(x + ha, y + hb) - S(x, y)]$ exists for all

$(x, y) \in D \times \mathbb{R}^N$ and all $(a, b) \in \mathbb{R}^n \times \mathbb{R}^N$. The integral in (2.3) exists since the partial derivatives of φ exist almost everywhere in $\bar{U}$ and are bounded by the lipschitz constant of φ.

We define $\dot{S}$ by

$$\dot{S}(x, y) = \lim_{h \to 0+} \frac{1}{h} \left[S(x + hf(x), y + h(\partial f/\partial x)^{[2]} (x) y) - S(x, y)\right]. \qquad (2.4)$$

Thus $\dot{S} = (\partial S/\partial x)^* f + (\partial S/\partial y)^* (\partial f/\partial x)^{[2]} y$ almost everywhere, since S is lipschitzian and therefore differentiable almost everywhere. When $n = 2$, $\dot{S} = (\partial S/\partial x)^* f + (\partial S/\partial y)(\operatorname{div} f)\, y$.

PROPOSITION 2.1 *Suppose* $\varphi_0 \in \operatorname{Lip}(\bar{U} \to \mathbb{R}^n)$ *and* $\varphi_t(u) = x(t, \varphi_0(u))$. *Then* $\varphi_t \in \operatorname{Lip}(\bar{U} \to \mathbb{R}^n)$, *the right-hand derivative* $D_t^+ \mathscr{S} \varphi_t$ *exists and*

$$D_t^+ \mathscr{S} \varphi_t = \int_U \dot{S}\left(\varphi_t, \frac{\partial}{\partial u_1} \varphi_t \wedge \frac{\partial}{\partial u_2} \varphi_t\right)$$

as long as $\varphi_t(u)$ *exists for each* $u \in \bar{U}$.

Proof. For each $u \in \bar{U}$, $\varphi_t(u)$ is a solution of (1.1). Therefore $z_i(t) = (\partial/\partial u_i)\, \varphi_t(u) = (\partial x/\partial x_0)(t, \varphi_0(u))(\partial/\partial u_i)\, \varphi_0(u)$ satisfies $dz_i/dt = (\partial f/\partial x)(\varphi_t(u))\, z_i$, $i = 1, 2$, and $y(t) = (\partial/\partial u_i)\, \varphi_t(u) \wedge (\partial/\partial u_2)\, \varphi_t(u)$ satisfies $dy/dt = (\partial f/\partial x)^{[2]} (\varphi_t(u))\, y$. It follows that

$$\lim_{h \to 0+} \frac{1}{h} \left[S\left(\varphi_{t+h}, \frac{\partial}{\partial u_1} \varphi_{t+h} \wedge \frac{\partial}{\partial u_2} \varphi_{t+h}\right) - S\left(\varphi_t, \frac{\partial}{\partial u_1} \varphi_t \wedge \frac{\partial}{\partial u_2} \varphi_t\right)\right]$$

exists and equals $\dot{S}(\varphi_t, (\partial/\partial u_1)\, \varphi_t \wedge (\partial/\partial u_2)\, \varphi_t)$. From this and the Lebesgue Dominated Convergence Theorem we deduce Proposition 2.1. ∎

The surface area, counting multiplicities, of $\varphi_t(\bar{U})$ is $\mathscr{S}\varphi_t = \int_U |(\partial/\partial u_1)\, \varphi_t \wedge (\partial/\partial u_2)\, \varphi_t|$. Here $S(x, y) = |y| = (y^* y)^{1/2}$ is the l^2 norm of $y \in \mathbb{R}^N$. For this functional

$$\dot{S}(x, y) = S(x, y)^{-1} \frac{1}{2} y^* \left(\frac{\partial f^*}{\partial x} + \frac{\partial f}{\partial x}\right)^{[2]} y$$

and therefore

$$(\lambda_{n-1}(x) + \lambda_n(x))\, S(x, y) \leqslant \dot{S}(x, y) \leqslant (\lambda_1(x) + \lambda_2(x))\, S(x, y), \qquad (2.5)$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ are the eigenvalues of $\frac{1}{2}((\partial f/\partial x)^* + \partial f/\partial x)$, since the eigenvalues of $\frac{1}{2}((\partial f/\partial x)^* + \partial f/\partial x)^{[2]}$ are $\lambda_i + \lambda_j$, $1 \leqslant i < j \leqslant n$ (see [9]).

Thus the surface area of $\varphi_t(\bar{U})$ increases (decreases) in t as long as $\varphi_t(\bar{U})$ lies in a set where $\lambda_{n-1}+\lambda_n>0$ ($\lambda_1+\lambda_2<0$).

If $x \mapsto H(x)$ is a C^1 real symmetric $\binom{n}{2}\times\binom{n}{2}$ matrix-valued function and $S(x, y)=y^*H(x)\,y$, then

$$\dot{S}(x, y)=y^*(H_f+(\partial f/\partial x)^{[2]*}\,H+H(\partial f/\partial x)^{[2]})\,y, \tag{2.6}$$

where H_f is the matrix obtained by replacing each entry h_{ij} of H by $(h_{ij})_f=(\partial h_{ij}/\partial x)^*f$, its directional derivative in the direction f. In this case $\mathscr{S}\varphi_t$ increases (decreases) if $\varphi_t(\bar{U})$ lies in a set where $H_f+(\partial f/\partial x)^{[2]*}\,H+H(\partial f/\partial x)^{[2]}$ is positive (negative) definite.

A general class of functionals $\mathscr{S}$ in which we are interested is given by $S(x, y)=|A(x)\,y|$, where $|\cdot|$ is any norm on $\mathbb{R}^N$, $N=\binom{n}{2}$, and $x\mapsto A(x)$ is a C^1 nonsingular real $N\times N$ matrix-valued function. In this case, it follows from (2.4) that

$$-\mu(-B)\,S\leqslant\dot{S}\leqslant\mu(B)\,S, \tag{2.7}$$

where $B=A_fA^{-1}+A(\partial f/\partial x)\,A^{-1}$ and μ is the Lozinskiĭ measure or logarithmic "norm" corresponding to $|\cdot|$ defined by $\mu(B)=\lim_{h\to 0+}(1/h)(|I+hB|-1)$ [4, pp. 41, 58]. When $A=I$, $B=\partial f/\partial x$, and $\mu(B)=\lambda_1+\lambda_2$, $-\mu(-B)=\lambda_{n-1}+\lambda_n$ in the case that $|y|=(y^*y)^{1/2}$. When $n=2$, $-\mu(-B)=\mu(B)=\operatorname{div} f$ so that (2.7) gives $\dot{S}=(\operatorname{div} f)\,S$, the familiar formula of Liouville–Jacobi [4, p. 44].

We now consider functionals $\mathscr{A}$ on $\operatorname{Lip}(\bar{U}\to\mathbb{R}^n)$ defined by

$$\mathscr{A}\varphi=\int_U\left|\frac{\partial}{\partial u_1}\varphi\wedge\frac{\partial}{\partial u_2}\varphi\right|^p, \tag{2.8}$$

where $|\cdot|$ is any norm on $\mathbb{R}^N$ and $p\geqslant 1$. For example, if $|y|=(y^*y)^{1/2}$ and $p=1$, $\mathscr{A}$ is the surface area of $\varphi(\bar{U})$. We show that if $\partial\varphi$ is a simple closed curve, then $\mathscr{A}\varphi$ has a positive lower bound which depends only on $|\cdot|$, p, and $\partial\varphi$.

PROPOSITION 2.2. *Suppose ψ is a simple closed rectifiable curve in $\mathbb{R}^n$. Then there exists $\delta>0$ such that*

$$\mathscr{A}\varphi\geqslant\delta$$

for all $\varphi\in\Sigma(\psi, \mathbb{R}^n)$.

Proof. Since all norms in $\mathbb{R}^N$ are equivalent, it is sufficient to prove the proposition in the case that $|y|=(y^*y)^{1/2}$. It also suffices to prove the statement for $\varphi\in\Sigma(\psi, K)$, where K is the convex hull of $\psi(\partial U)$. This follows from the fact that if Π is any $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ which does not intersect $\psi(\partial U)$ and $\varphi \in \Sigma(\psi, \mathbb{R}^n)$, by orthogonal projection onto Π, if necessary, we can find $\tilde{\varphi} \in \Sigma(\psi, \mathbb{R}^n)$ such that $\mathscr{A}\tilde{\varphi} \leqslant \mathscr{A}\varphi$ and $\tilde{\varphi}(\bar{U})$ does not cross Π. Next, observe that $\int_0^{2\pi} |\psi'|^2 > 0$, where $\psi(\theta) = \psi(\cos\theta, \sin\theta)$, since ψ is one-to-one. Choose the continuous function b from ∂U to $\mathbb{R}^n$ such that if $b(\theta) = (b \circ \psi)(\theta)$, $\int_0^{2\pi} b^* \psi'$ is sufficiently close to $\int_0^{2\pi} |\psi'|^2$ to ensure $\int_0^{2\pi} b^* \psi' > 0$. Then the function b may be extended continuously to $\mathbb{R}^n$ and b may be approximated on $\mathbb{R}^n$ by a C^1 function a such that $\alpha\varphi(\partial U) = \alpha\psi(\partial U)$ is sufficiently close to $\int_0^{2\pi} b^* \psi'$ to ensure $\alpha\varphi(\partial U) > 0$, where α is the 1-form defined by $\alpha = \sum_i a_i(x)\, dx_i$. But Stokes' Theorem implies

$$\begin{aligned} \alpha\varphi(\partial U) &= d\alpha\, \varphi(\bar{U}) \\ &= \int_{\varphi(\bar{U})} \sum_{i<j} \left(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j} \right) dx_i \wedge dx_j \\ &= \int_{\bar{U}} z^*(u)\, y(u)\, du, \end{aligned}$$

where $y(u) = (\partial/\partial u_1)\, \varphi(u) \wedge (\partial/\partial u_2)\, \varphi(u)$, $z_i(u) = (\partial a_{i_2}/\partial x_{i_1})(x) - (\partial a_{i_1}/\partial x_{i_2})(x)$, $x = \varphi(u)$, $(i) = (i_1, i_2)$, $i = 1, ..., N = \binom{n}{2}$. Since a is C^1 on $\mathbb{R}^n$ and $\varphi(u) \in K$, there is a constant M independent of φ such that $|z(u)| \leqslant M$ for all $u \in \bar{U}$. Thus Hölder's Inequality implies

$$\begin{aligned} 0 < \alpha\psi(\partial U) = \alpha\varphi(\partial U) &\leqslant \int_{\bar{U}} |z|\, |y| \\ &\leqslant \left(\int_{\bar{U}} |z|^q \right)^{1/q} \left(\int_{\bar{U}} |y|^p \right)^{1/p} \\ &\leqslant \pi^{1/q} M (\mathscr{A}\varphi)^{1/p}. \end{aligned}$$

We conclude that Proposition 2.2 holds with $\delta = [\alpha\psi(\partial U)/\pi^{1/q} M]^p$. ∎

DEFINITION 2.3. A functional $\mathscr{S}$ of the form (2.3) is *strongly decreasing* with respect to (1.1) on $D \subset \mathbb{R}^n$ if there exist constants p, a, b with $p \geqslant 1$, $a \geqslant 0$, $b \geqslant 0$, and $a + b > 0$ such that

$$\dot{S}(x, y) \leqslant -(a + b\, |y|^p) \tag{2.9}$$

if $x \in D$ and $y \in \mathbb{R}^N$.

It follows from Propositions 2.1, 2.2 that

$$D_t^+ \mathscr{S}\varphi_t \leqslant -(a\pi + b\delta) \tag{2.10}$$

if $\varphi_t \in \Sigma(\psi, D)$ and D is a set where (2.9) holds.

3. Bendixson's Criterion

A subset D of D_0 is *invariant* with respect to (1.1) if $x(t, D) = D$ for all $t \in (-\infty, \infty)$. A simple closed rectifiable curve ψ in D_0 is *invariant with respect to* (1.1) if $\psi(\partial U)$ is invariant with respect to (1.1).

Criterion 3.1. Suppose that ψ is a simple closed rectifiable curve in D_0 which is invariant with respect to (1.1). Then there cannot exist a functional $\mathscr{S}$ of the form (2.3) such that (a) and (b) are satisfied:

(a) $-\infty < m = \inf\{\mathscr{S}\varphi : \varphi \in \Sigma(\psi, D_0)\}$.

(b) There is a sequence of surfaces $\varphi^k \in \Sigma(\psi, D_0)$ such that $m = \lim_{k \to \infty} \mathscr{S}\varphi^k$ and $\mathscr{S}$ is *strongly decreasing with respect to* (1.1) on $\{x(t, \varphi^k(\bar{U})) : t \in [0, \varepsilon], k = 1, 2, \ldots\}$ for some $\varepsilon > 0$.

Criterion 3.2. Suppose that ψ is a simple closed rectifiable curve in D_0 which is invariant with respect to (1.1). Then there cannot exist a functional $\mathscr{S}$ such that (a) and (b) are satisfied:

(a) $-\infty < m = \inf\{\mathscr{S}\varphi : \varphi \in \Sigma(\psi, D_0)\}$.

(b) There is a surface $\varphi_0 \in \Sigma(\psi, D_0)$ such that $\mathscr{S}$ is *strongly decreasing with respect to* (1.1) on $\{x(t, \varphi_0(\bar{U})) : t \in [R, \infty)\}$ for some $R > 0$.

To establish these criteria, observe that the invariance of ψ implies $\varphi_t \in \Sigma(\psi, D_0)$ if $\varphi_0 \in \Sigma(\psi, D_0)$ as long as $\varphi_t(u)$ exists for each $u \in \bar{U}$. From Proposition 2.1 and (2.10) the conditions of Criterion 3.1 imply $\mathscr{S}\varphi^k_\varepsilon \leqslant \mathscr{S}\varphi^k - (a\pi + b\delta)\,\varepsilon$, where $\delta^k_t(u) = x(t, \varphi^k(u))$, $k = 1, 2, \ldots$. Since $\varphi^k \in \Sigma(\psi, D_0)$ implies $\varphi^k_\varepsilon \in \Sigma(\psi, D_0)$ and $\limsup_{k \to \infty} \mathscr{S}\varphi^k_\varepsilon \leqslant m - (a\pi + b\pi)\,\varepsilon < m$ we have a contradiction of (a).

Similarly, the conditions of Criterion 3.2 imply $\mathscr{S}\varphi_t \leqslant \mathscr{S}\varphi_R - (a\pi + b\delta)(t - R)$, $R \leqslant t < \infty$, so that $\varphi_t \in \Sigma(\psi, D_0)$ and $\lim_{t \to \infty} \mathscr{S}\varphi_t = -\infty < m$, again contradicting (a).

We now deduce more concrete expressions from these criteria, beginning with six conditions in $\mathbb{R}^n$ each of which is Bendixson's criterion when $n = 2$.

Theorem 3.3 (Bendixson's Criterion in $\mathbb{R}^n$). *A simple closed rectifiable curve which is invariant with respect to* (1.1) *cannot exist if any one of the following conditions is satisfied on* $\mathbb{R}^n$:

$$\text{(i)}\quad \sup\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q \neq r, s}\left(\left|\frac{\partial f_q}{\partial x_r}\right| + \left|\frac{\partial f_q}{\partial x_s}\right|\right) : 1 \leqslant r < s \leqslant n\right\} < 0,$$

$$\text{(ii)}\quad \sup\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q \neq r, s}\left(\left|\frac{\partial f_r}{\partial x_q}\right| + \left|\frac{\partial f_s}{\partial x_q}\right|\right) : 1 \leqslant r < s \leqslant n\right\} < 0,$$

(iii) $\lambda_1 + \lambda_2 < 0$,

(iv) $\inf\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q \neq r, s}\left(\left|\frac{\partial f_q}{\partial x_r}\right| + \left|\frac{\partial f_q}{\partial x_s}\right|\right): 1 \leqslant r < s \leqslant n\right\} > 0$,

(v) $\inf\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} - \sum_{q \neq r, s}\left(\left|\frac{\partial f_r}{\partial x_q}\right| + \left|\frac{\partial f_s}{\partial x_q}\right|\right): 1 \leqslant r < s \leqslant n\right\} > 0$,

(vi) $\lambda_{n-1} + \lambda_n > 0$.

Proof. This result may be deduced from either of the preceding criteria. However, since solutions of (1.1) do not necessarily exist globally, the technicalities in using Criterion 3.1 are fewer. If $y \in \mathbb{R}^N$ and $|y| = \sup_i |y_i|$, $\sum_i |y_i|$, or $(y^*y)^{1/2}$, then the Lozinskiĭ measure $\mu((\partial f/\partial x)^{[2]})$ discussed in Section 2 is the expression in (i), (ii), or (iii) and $-\mu(-(\partial f/\partial x)^{[2]})$ is the expression in (iv), (v), or (vi), respectively. It follows from (2.7) with $A = I$ that if $S(x, y) = |y|$, $\mathcal{S}$ is strongly decreasing with respect to (1.1) on any compact subset of $\mathbb{R}^n$ if the corresponding condition (i), (ii), or (iii) holds and that $\mathcal{S}$ is strongly decreasing on compacta with respect to (1.1) with reversed time if (iv), (v), or (vi) holds. Since $\mathcal{S}\varphi \geqslant 0$ for every φ, it remains only to show that if ψ is a simple closed curve, there is a sequence in $\Sigma(\psi, D)$ which minimizes $\mathcal{S}$ over $\Sigma(\psi, \mathbb{R}^n)$, where $D = \{x\colon |x_i| \leqslant c_i\}$ satisfies $\psi(\partial U) \subset D$. This follows from the observation that if $\varphi \in \Sigma(\psi, \mathbb{R}^n)$ and $\tilde{\varphi}_i(u) = -c_i$, $\varphi_i(u)$, c_i as $\varphi_i(u) \in (-\infty, -c_i]$, $(-c_i, c_i)$, $[c_i, \infty)$, $i = 1, ..., n$, then $\tilde{\varphi} \in \Sigma(\psi, D)$ and $\mathcal{S}\tilde{\varphi} \leqslant \mathcal{S}\varphi$, since $|(\partial/\partial u_1)\,\tilde{\varphi} \wedge (\partial/\partial u_2)\,\tilde{\varphi}| \leqslant |(\partial/\partial u_1)\,\varphi \wedge (\partial/\partial u_2)\,\varphi|$.

Any of the criteria (i)–(vi) of Theorem 3.3 may be modified, following Smith [10], by replacing f by αf, where $x \mapsto \alpha(x)$ is a positive C^1 scalar-valued function. This amounts to a change of the independent variable t and gives a generalization of Theorem 3.3 in the spirit of Dulac (see [9, Remark (d)]). This arbitrary function introduced into the criteria may be replaced by $\binom{n}{2}^2$ such functions if we consider the functional $\mathcal{S}$ defined by (2.3) with $S(x, y) = |A(x, y)|$, where $x \mapsto A(x)$ is a C^1 nonsingular $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function on D_0. Here $\mathcal{S}\varphi = \int_{\bar{U}} |A(\varphi)(\partial/\partial u_1)\,\varphi \wedge (\partial/\partial u_2)\,\varphi|$. It follows from (2.7) that $\dot{S}(x, y) < 0$ whenever $\mu(B) < 0$ where μ is the Lozinskiĭ measure corresponding to $|\cdot|$ and that S is strongly decreasing on sets where $\mu(B) \leqslant -b < 0$. Similarly $\dot{S}(x, y) > 0$ when $-\mu(-B) > 0$. ∎

We will say that D_0 has the *minimum property with respect to* S if, for each simple closed rectifiable curve ψ in D_0, there is a minimizing sequence $\varphi^k \in \Sigma(\psi, D_0)$ for $\mathcal{S}$ such that $\bigcup_k \varphi^k(\bar{U})$ has compact closure in D_0. It follows that if D_0 has this property and $\mu(B) < 0$, then $\varphi^k_t(u)$ exists for each $u \in \bar{U}$ and $k = 1, 2, ..., 0 \leqslant t \leqslant \varepsilon$, for some $\varepsilon > 0$, and that the conditions (a) and (b) of Criterion 3.1 are satisfied for each simple closed rectifiable curve ψ in D_0. Thus we have the following theorem.

THEOREM 3.4. *Suppose that*

(a) D_0 *has the minimum property with respect to* $S(x, y) = |A(x)\,y|$.

(b) $\mu(A_f A^{-1} + A(\partial f/\partial x)^{[2]} A^{-1}) < 0$ *on* D_0.

Then no simple closed rectifiable curve in D_0 *is invariant with respect to* (1.1).

The condition (b) may be replaced by $\mu(-A_f A^{-1} - A(\partial f/\partial x)^{[2]} A^{-1}) < 0$ by using a time reversal argument.

With $A = I$ and $D_0 = \mathbb{R}^n$, we obtain Theorem 3.3 if $|\cdot|$ is any of the three norms mentioned in the proof of that theorem. In fact the proof is simply a demonstration that $\mathbb{R}^n$ has the minimum property with respect to $S(x, y) = |y|$. The same argument applies equally well to any absolute norm, where $|\cdot|$ is said to be *absolute* if $|y|$ is unchanged by replacing the components y_i of y by $|y_i|$.

COROLLARY 3.5. *Suppose that one of*

$$\mu\left(\frac{\partial f^{[2]}}{\partial x}\right) < 0, \qquad \mu\left(-\frac{\partial f^{[2]}}{\partial x}\right) < 0$$

holds on $\mathbb{R}^n$ *where* μ *is a Lozinskiĭ measure corresponding to an absolute norm* $|\cdot|$ *on* $\mathbb{R}^N$, $N = \binom{n}{2}$. *Then no simple closed rectifiable curve in* $\mathbb{R}^n$ *is invariant with respect to* (1.1).

If solutions of (1.1) exist for all $t \geq 0$, a subset D_1 of D_0 is said to be *absorbing* with respect to (1.1) if each bounded subset D of D_0 satisfies $x(t, D) \subset D_1$ for all sufficiently large t. It follows from Criterion 3.2 that if $\mathscr{S}$ satisfies (a) and (1.1) has an absorbing set D_1 on which $\mathscr{S}$ is strongly decreasing, then no simple closed curve ψ in D_0 for which $\Sigma(\psi, D_0)$ is non-empty can be invariant with respect to (1.1). This gives the following result.

THEOREM 3.6. *Suppose that*

(a) D_0 *is simply connected.*

(b) $\mu(A_f A^{-1} + A(\partial f/\partial x)^{[2]} A^{-1}) \leq b < 0$ *on a set* D_1 *which is absorbing with respect to* (1.1).

Then there is no simple closed rectifiable curve in D_0 *which is invariant with respect to* (1.1).

For example, if (1.1) has a compact global attractor K (see [13, p. 21]) on which $\mu(A_f A^{-1} + A(\partial f/\partial x)^{[2]} A^{-1}) < 0$, then there exists an absorbing neighbourhood D_1 of K on which (b) holds.

If $S(x, y) = \alpha(x)$, where α is C^1, then $\dot{S} = (\partial\alpha/\partial x)^* f$ and $\mathscr{S}$ is strongly decreasing on any set where $(\partial\alpha/\partial x)^* f \leqslant -b < 0$. Our criteria then translate into a weak form of the familiar observation that no nonconstant periodic solution can exist if there is a real-valued function α which decreases along trajectories of (1.1).

Finally we observe that when $n = 2$, Dulac's criterion follows from consideration of $S(x, y) = \alpha(x)\, y$. Then, for any simple closed curve ψ with $\varphi \in \Sigma(\psi, D_0)$, $\mathscr{S}\varphi = \int_D \alpha$, where D is the region bounded by $\psi(\partial U)$, and so $\mathscr{S}\varphi$ depends only on ψ. If ψ is invariant and $D \subset D_0$, we may find a function $\varphi \in \Sigma(\psi, D)$ such that $\partial\varphi/\partial u_1 \wedge \partial\varphi/\partial u_2 = \partial(\varphi_1, \varphi_2)/\partial(u_1, u_2) \geqslant 0$ and the constant sequence $\varphi^k = \varphi$, $k = 1, 2, \ldots$, is minimizing for $\mathscr{S}$. Also $\mathscr{S}$ is strongly decreasing on D if $(\partial\alpha/\partial x)^* f + \alpha \operatorname{div} f < 0$. This is Dulac's condition. Criteria 3.1 and 3.2 now both imply Dulac's criterion. Here it was not necessary to assume $\alpha(x)$ was of one sign. In contrast, the higher dimensional analogue needed to consider $S(x, y) = |A(x)\, y|$ where $A(x)$ is nonsingular. This condition may be relaxed somewhat if a more general definition of "strongly decreasing" is given, replacing the constants a, b of (2.9) by functions $a(x)$, $b(x)$ in which case one also needs an extension of Proposition 2.2 to a functional $\mathscr{A} = \mathscr{S}$ determined by (2.3) with $S(x, y) = b(x)\, |y|^p$.

4. An Example

We consider the Lorenz model for fluid convection in a two dimensional layer heated from below and establish a region which contains no invariant closed curves and, in particular, no periodic trajectories. The system is (see [13, Sect. 2.3])

$$\begin{aligned} x_1' &= -\sigma x_1 + \sigma x_2, \\ x_2' &= r x_1 - x_2 - x_1 x_3, \\ x_3' &= -b x_3 + x_1 x_2, \end{aligned} \tag{4.1}$$

where σ, r, b are three positive numbers.

This dissipative system has been the object of intensive numerical investigation which indicates that it has a strange attractor for a large range of values of the parameters and that it has many periodic orbits (see [12, p. 21]). Any set not containing periodic trajectories could therefore not completely contain the attractor.

In this case the Jacobian matrix $\partial f/\partial x$ of the right-hand side of (4.1) and its second additive compound $(\partial f/\partial x)^{[2]}$ are

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - x_3 & -1 & -x_1 \\ x_2 & x_1 & -b \end{bmatrix}, \qquad \left(\frac{\partial f}{\partial x}\right)^{[2]} = \begin{bmatrix} -\sigma - 1 & -x_1 & 0 \\ x_1 & -\sigma - b & \sigma \\ -x_2 & r - x_3 & -b - 1 \end{bmatrix}.$$

If we choose $S(x, y) = |y|$, where $|y| = \sup\{\sqrt{y_1^2 + y_2^2}, |y_3|\}$, we find that $\mu((\partial f/\partial x)^{[2]}) \leqslant \sup\{-1, -b, -b-1+|x_2|+|x_3-r|\}$ so that $\mu((\partial f/\partial x)^{[2]}) < 0$ if

$$|x_2| + |x_3 - r| < b + 1, \tag{4.2}$$

which determines a cylinder D_0 parallel to the x_1-axis. In this case the functional $\mathscr{S}$ defined by (2.3) is

$$\mathscr{S}\varphi = \int_{\bar{U}} \sup\left\{\left[\frac{\partial(\varphi_1, \varphi_2)}{\partial(u_1, u_2)}^2 + \frac{\partial(\varphi_1, \varphi_3)}{\partial(u_1, u_2)}^2\right]^{1/2}, \left|\frac{\partial(\varphi_2, \varphi_3)}{\partial(u_1, u_2)}\right|\right\}. \tag{4.3}$$

The cylinder D_0 has the minimum property with respect to S. To see this, suppose ψ is a simple closed rectifiable curve in the half-space D: $x_2 + (x_3 - r) \leqslant c$. Let $\varphi \in \Sigma(\psi, \mathbb{R}^3)$ and suppose that $\varphi(\bar{U})$ crosses the plane Π: $x_2 + (x_3 - r) = c$, specifically $\varphi_2(u) + (\varphi_3(u) - r) > c$ if $u \in U_0 \subset \bar{U}$. Now consider the surface $\tilde{\varphi} \in \Sigma(\psi, D)$ obtained by modifying φ so that this portion is reflected in Π:

$$\tilde{\varphi}(u) = \varphi(u), \qquad \text{if } u \in U \backslash U_0$$

$$\tilde{\varphi}_1(u) = \varphi_1(u), \qquad \tilde{\varphi}_2(u) = c + r - \varphi_3(u), \qquad \tilde{\varphi}_3(u) = c - \varphi_2(u), \text{ if } u \in U_0.$$

Then $\tilde{\varphi} \in \Sigma(\psi, D)$, $\tilde{\varphi}(\bar{U})$ does not cross Π and, from (4.3), $\mathscr{S}\tilde{\varphi} = \mathscr{S}\varphi$. A similar argument may be applied to planes $\pm x_2 \pm (x_3 - r) = c$ and $x_1 = k$ to deduce that if ψ is a simple closed rectifiable curve in D_0, then there is a compact box in D_0 which contains the curve and a minimizing sequence for $\mathscr{S}$ in $\Sigma(\psi, D_0)$.

We conclude from Theorem 3.4 that the Lorenz system (4.1) has no invariant rectifiable closed curve in the cylinder defined by (4.2).

This statement may be improved by a more judicious choice of S. Let $S(x, y) = |Ay|$ where $A = \operatorname{diag}(1, 1, \alpha)$ and α is a positive constant; in fact $S(x, y) = \sup\{\sqrt{y_1^2 + y_2^2}, \alpha\,|y_3|\}$, another norm. Now $\mu(A(\partial f/\partial x)^{[2]} A^{-1}) \leqslant \sup\{-1 + (1/\alpha - 1)\,\sigma,\ -b + (1/\alpha - 1)\,\sigma,\ -b - 1 + \alpha(|x_2| + |x_3 - r|)\}$. By optimizing the choice of the constant α we find $\mu(A(\partial f/\partial x)^{[2]} A^{-1}) < 0$ if

$$|x_2| + |x_3 - r| < (b+1)\inf\left\{\frac{\sigma+1}{\sigma}, \frac{\sigma+b}{\sigma}\right\}, \tag{4.4}$$

a larger cylinder than that defined by (4.2).

A similar argument to the one given previously now shows that *the Lorenz system* (4.1) *has no invariant rectifiable closed curve in the cylinder defined by* (4.4).

In conclusion we note that the choice $S(x, y) = |y| = \sup\{\sqrt{y_1^2 + y_2^2}, |y_3|\}$ leading to the "surface area" $\mathscr{S}$ in (4.3) gives stronger results than the more usual norms $|y| = \sup\{|y_1|, |y_2|, |y_3|\}$, $|y| = |y_1| + |y_2| + |y_3|$, or $|y| = (y_1^2 + y_2^2 + y_3^2)^{1/2}$. The first two, by easy computations, lead to the conditions (i), (ii) of Theorem 3.3 which, for the Lorenz system, hold on smaller sets than that specified by (4.2). The third norm requires an estimation of the region where the expression $\lambda_1 + \lambda_2$ of Theorem 3.3 (iii) is negative and this may be implemented using the approach of Smith [11] but also leads to a smaller set than (4.4).

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