Global Hopf Bifurcation Analysis of a Neuron Network Model with Time Delays

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Dedicated to Professor G. R. Sell for the occasion of his 70th birthday

Abstract For a two-neuron network with self-connection and time delays, we carry out stability and bifurcation analysis. We establish that a Hopf bifurcation occurs when the total delay passes a sequence of critical values. The stability and direction of the local Hopf bifurcation are determined using the normal form method and center manifold theorem. To show that periodic solutions exist away from the bifurcation points, we establish that local Hopf branches globally extend for arbitrarily large delays.

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1 Introduction

Research has been devoted to rigorous stability and bifurcation analysis of small neural network models with time delays [1–6, 8, 10, 11, 15, 16, 18, 20, 21]. Shayer and Campbell [19] studied bifurcation and multistability in the following two-neuron network with self-connection and time delays:

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$$\begin{cases} \dot{x}_1(t) = -kx_1(t) + \beta \tanh(x_1(t)) + a_{12} \tanh(x_2(t - \tau_2)), \\ \dot{x}_2(t) = -kx_2(t) + \beta \tanh(x_2(t)) + a_{21} \tanh(x_1(t - \tau_1)), \end{cases} \tag{1}$$

where $k > 0$, β, a_{12}, a_{21} are all constants. Their numerical investigation shows that the model possesses very rich dynamics. For a more general class of neural network model

$$\begin{cases} \dot{u}_1(t) = -\mu_1 u_1(t) + a_1 f_1(u_1(t)) + b_1 g_1(u_2(t - \tau_1)), \\ \dot{u}_2(t) = -\mu_2 u_2(t) + a_2 f_2(u_2(t)) + b_2 g_2(u_1(t - \tau_2)), \end{cases} \tag{2}$$

Wei et al. [21] carried out bifurcation analysis for the case $\mu_1 = \mu_2$ and $a_1 f_1 = a_2 f_2$. In this chapter, we carry out complete and detailed analysis on the stability, the bifurcation, and the global existence of periodic solutions for the general two-neuron network (2).

In Sect. 2, we investigate stability and local Hopf bifurcation as we vary the total delay $\tau = \tau_1 + \tau_2$, by analyzing the related characteristic equation for system (2). We show that a sequence of Hopf bifurcations occur at the origin as the total delay increases. In Sect. 3, we establish the direction and stability of the first Hopf bifurcation branch using the center manifold theorem and normal form method. Global extensions of the local Hopf branch are established in Sect. 4, where we apply a global Hopf bifurcation theorem of Wu [22] and higher-dimensional Bendixson–Dulac criteria for ordinary differential equations of Li and Muldowney [14]. Numerical simulations are carried out to support our theoretical results.

2 Stability and Local Hopf Bifurcation

In this section, we investigate the effect of delay on the dynamic behaviors of the two-neuron network model (2).

Let $x_1(t) = u_1(t - \tau_2)$, $x_2(t) = u_2(t)$, and $\tau = \tau_1 + \tau_2$. Then system (2) becomes

$$\begin{cases} \dot{x}_1(t) = -\mu_1 x_1(t) + a_1 f_1(x_1(t)) + b_1 g_1(x_2(t - \tau)), \\ \dot{x}_2(t) = -\mu_2 x_2(t) + a_2 f_2(x_2(t)) + b_2 g_2(x_1(t)). \end{cases} \tag{3}$$

We make the following assumptions.

$$(\mathbf{H}_1)\ f_i, g_i \in C^4,\ xf_i(x) > 0,\ \text{and}\ xg_i(x) > 0\ \text{ for }\ x \neq 0,\ i = 1, 2.$$

Under $(\mathbf{H}_1)$, the origin $(0,0)$ is an equilibrium of system (3). Without loss of generality, we assume that $f_i'(0) = 1$ and $g_i'(0) = 1,\ i = 1, 2$. Then the linearization of system (3) at the origin is

$$\begin{cases} \dot{y}_1(t) = -\mu_1 y_1(t) + a_1 y_1(t) + b_1 y_2(t - \tau), \\ \dot{y}_2(t) = -\mu_2 y_2(t) + a_2 y_2(t) + b_2 y_1(t). \end{cases}$$

Its characteristic equation is

$$\lambda^2 + [(\mu_1 - a_1) + (\mu_2 - a_2)]\lambda + (\mu_1 - a_1)(\mu_2 - a_2) - b_1 b_2 \mathrm{e}^{-\lambda\tau} = 0. \tag{4}$$

Lemma 2.1. *Suppose that there exists a $\tau_0 > 0$ such that (4) with τ_0 has a pair of purely imaginary roots $\pm i\omega_0$, and the root of (4)*

$$\lambda(\tau) = \alpha(\tau) + \mathrm{i}\omega(\tau)$$

satisfies $\alpha(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$. Then $\alpha'(\tau_0) > 0$.

Proof. Substituting $\lambda(\tau)$ into (4) and differentiating with respect to τ, we have

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = -\frac{2\lambda + [(\mu_1 - a_1) + (\mu_2 - a_2)]}{b_1 b_2 \lambda \mathrm{e}^{-\lambda\tau}} - \frac{\tau}{\lambda}.$$

Note that

$$b_1 b_2 \mathrm{e}^{-\lambda\tau} = \lambda^2 + [(\mu_1 - a_1) + (\mu_2 - a_2)]\lambda + (\mu_1 - a_1)(\mu_2 - a_2),$$

and $\lambda(\tau_0) = \mathrm{i}\omega_0$. We have

$$\left(\frac{\mathrm{d}\lambda(\tau_0)}{\mathrm{d}\tau}\right)^{-1} = \frac{[(\mu_1 - a_1) + (\mu_2 - a_2)] + 2\mathrm{i}\omega_0}{[(\mu_1 - a_1) + (\mu_2 - a_2)]\omega_0^2 - \mathrm{i}\omega_0[(\mu_1 - a_1)(\mu_2 - a_2) - \omega_0^2]} - \mathrm{i}\frac{\tau_0}{\omega_0},$$

and thus,

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda(\tau_0)}{\mathrm{d}\tau}\right)^{-1} = \frac{\omega_0^2}{\triangle}[(\mu_1 - a_1)^2 + (\mu_2 - a_2)^2 + 2\omega_0^3],$$

where

$$\triangle = [(\mu_1 - a_1) + (\mu_2 - a_2)]^2\omega_0^4 + \omega_0^2[(\mu_1 - a_1)(\mu_2 - a_2) - \omega_0^2]^2.$$

The conclusion follows. □

We make the following further assumption.
$(\mathbf{H}_2)$ $(\mu_1 + \mu_2) - (a_1 + a_2) > 0$.

Lemma 2.2. *Suppose that assumption $(\mathbf{H}_2)$ is satisfied.*

(i) *If*

$$|b_1 b_2| \leq |(\mu_1 - a_1)(\mu_2 - a_2)| \quad \textit{and} \quad 0 < (\mu_1 - a_1)(\mu_2 - a_2) \neq b_1 b_2,$$

then all the roots of (4) have negative real parts for all $\tau \geq 0$.

(ii) *If*

$$b_1 b_2 > (\mu_1 - a_1)(\mu_2 - a_2),$$

then (4) has at least one root with positive real part for all $\tau \geq 0$. If, in addition,

$$b_1 b_2 > |(\mu_1 - a_1)(\mu_2 - a_2)|,$$

then there exist a sequence values of τ, $\bar{\tau}_0 < \bar{\tau}_1 < \cdots$, such that (4) has a pair of purely imaginary roots $\pm \mathrm{i}\omega_0$ when $\tau = \bar{\tau}_j,\ j = 0, 1, 2, \ldots$.

(iii) *If*

$$b_1 b_2 < -|(\mu_1 - a_1)(\mu_2 - a_2)|,$$

then, for the same sequence $\bar{\tau}_0 < \bar{\tau}_1 < \bar{\tau}_2 < \cdots$ as in (ii)*, all the roots of (4) have negative real parts when $\tau \in [0, \bar{\tau}_0)$; (4) has at least a pair of roots with positive real parts when $\tau > \bar{\tau}_0$; and (4) has a pair of purely imaginary root $\pm \mathrm{i}\omega_0$ when $\tau = \bar{\tau}_j,\ j = 0, 1, 2, \ldots$. Furthermore, all the roots of (4) with $\tau = \bar{\tau}_0$ have negative real parts except $\pm \mathrm{i}\omega_0$.*

Proof. When $\tau = 0$, the roots of (4) are

$$\lambda_{1,2} = \frac{1}{2}\Big\{ -[(\mu_1 - a_1) + (\mu_2 - a_2)] \pm \sqrt{[(\mu_1 - a_1) + (\mu_2 - a_2)]^2 - 4[(\mu_1 - a_1)(\mu_2 - a_2) - b_1 b_2]} \Big\}.$$

This leads to

$$\mathrm{Re}\lambda_{1,2} < 0 \quad \text{when} \quad (\mu_1 - a_1)(\mu_2 - a_2) > b_1 b_2,$$

and

$$\mathrm{Re}\lambda_1 > 0 \ \text{ and } \ \mathrm{Re}\lambda_2 < 0 \ \text{ when } \ (\mu_1 - a_1)(\mu_2 - a_2) < b_1 b_2.$$

Equation (4) has a pair of purely imaginary roots $i\omega$ $(\omega > 0)$ if and only if ω satisfies

$$\begin{cases} (\mu_1 - a_1)(\mu_2 - a_2) - \omega^2 = b_1 b_2 \cos \omega\tau, \\ ((\mu_1 - a_1) + (\mu_2 - a_2))\omega = -b_1 b_2 \sin \omega\tau. \end{cases} \tag{5}$$

It follows from (5) that

$$\omega^4 + ((\mu_1 - a_1)^2 + (\mu_2 - a_2)^2)\omega^2 + [(\mu_1 - a_1)^2(\mu_2 - a_2)^2 - b_1^2 b_2^2] = 0,$$

and thus,

$$\omega^2 = \frac{1}{2}\Big\{ -[(\mu_1 - a_1)^2 + (\mu_2 - a_2)^2] \pm \sqrt{[(\mu_1 - a_1)^2 + (\mu_2 - a_2)^2]^2 - 4[(\mu_1 - a_1)^2(\mu_2 - a_2)^2 - b_1^2 b_2^2]} \Big\}. \tag{6}$$

Clearly, a real number ω does not exist when $|(\mu_1-a_1)(\mu_2-a_2)| \geq |b_1b_2|$. This shows that (4) has no root on the imaginary axis. The conclusion (i) follows.

A real number ω satisfies (6) when $|(\mu_1-a_1)(\mu_2-a_2)| < |b_1b_2|$. In this case, define

$$\omega_0 = \frac{1}{\sqrt{2}}\Big[-[(\mu_1-a_1)^2+(\mu_2-a_2)^2] \\ +\sqrt{[(\mu_1-a_1)^2-(\mu_2-a_2)^2]^2+4b_1^2b_2^2}\,\Big]^{\frac{1}{2}} \tag{7}$$

and

$$\bar{\tau}_j = \frac{1}{\omega_0}\Big[\arccos\frac{(\mu_1-a_1)(\mu_2-a_2)-\omega_0^2}{b_1b_2}+2j\pi\Big], \quad j=0,1,2,\ldots. \tag{8}$$

Then $\pm i\omega_0$ is a pair of purely imaginary roots of (4) with $\tau=\bar{\tau}_j$. Since (4) with $\tau=0$ has a root with positive real part when $b_1b_2 > (\mu_1-a_1)(\mu_2-a_2)$, conclusion (ii) follows from Lemma 2.1.

Similarly, since the roots of (4) with $\tau=0$ have negative real parts when $b_1b_2 < (\mu_1-a_1)(\mu_2-a_2)$, and $\bar{\tau}_0$ is the first value of $\tau \geq 0$ such that (4) has a root on the imaginary axis, we know that conclusion (iii) follows from Lemma 2.1. □

Applying Lemmas 2.1, 2.2, and a result in Hale [12, Theorem 1.1, p. 147] , we have the following result.

Theorem 2.3. *Suppose that assumptions* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *are satisfied.*

(i) *If*

$$|b_1b_2| \leq |(\mu_1-a_1)(\mu_2-a_2)| \quad \text{and} \quad 0 < (\mu_1-a_1)(\mu_2-a_2) \neq b_1b_2,$$

then the zero solution of system (3) is absolutely stable, that is, the zero solution is asymptotically stable for all $\tau \geq 0$.

(ii) *If*

$$b_1b_2 > (\mu_1-a_1)(\mu_2-a_2),$$

then the zero solution is unstable for all $\tau \geq 0$. *If*

$$b_1b_2 > |(\mu_1-a_1)(\mu_2-a_2)|,$$

then there exist a sequence of values of τ, $\bar{\tau}_0 < \bar{\tau}_1 < \bar{\tau}_2 < \cdots$ *defined in (8), such that system (3) undergoes a Hopf bifurcation at the origin when* $\tau=\bar{\tau}_j$, $j=0,1,2,\ldots$, *where* $\bar{\tau}_j$ *is defined in (8).*

(iii) *If*

$$b_1b_2 < -|(\mu_1-a_1)(\mu_2-a_2)|,$$

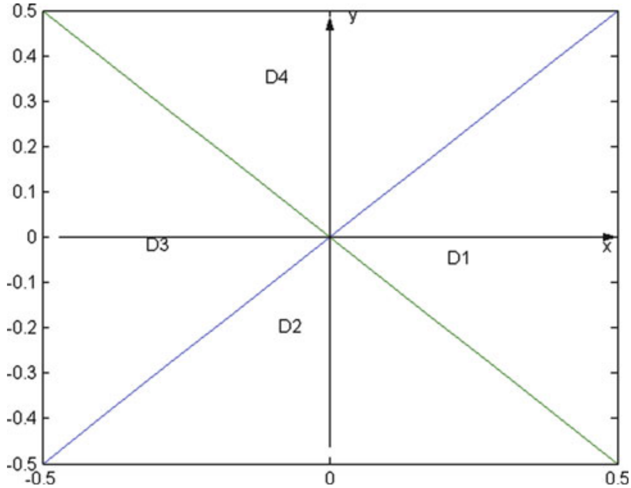

Fig. 1 Illustration of bifurcation sets. The horizontal axis is for values of $x = (\mu_1 - a_1)(\mu_2 - a_2)$, and vertical axis is for $y = b_1 b_2$. The lines $b_1 b_2 = \pm(\mu_1 - a_1)(\mu_2 - a_2)$ divide the plane into four regions, $D_1,\ D_2,\ D_3$, and D_4. D_1 is an absolutely stable region, D_2 is a conditionally stable region, and $D_3 \cup D_4$ is an unstable region

then, for the same sequence, $\bar{\tau}_0 < \bar{\tau}_1 < \bar{\tau}_2 < \cdots$ defined in (8), the zero solution of system (3) is asymptotically stable when $\tau \in [0, \bar{\tau}_0)$ and unstable when $\tau > \bar{\tau}_0$, and system (3) undergoes a Hopf bifurcation at the origin when $\tau = \bar{\tau}_j,\ \ j = 0, 1, 2, \ldots$.

The conclusions of Theorem 2.3 are illustrated in Fig. 1.

3 Direction and Stability of the Local Hopf Bifurcation at $\bar{\tau}_0$

In this section, we derive explicit formula for determining the direction and stability of the Hopf bifurcation at the first critical value $\bar{\tau}_0$, using the normal form and center manifold theory as presented in [13]. From Lemmas 2.1 and 2.2 we know that, if assumption $(\mathbf{H}_2)$ and condition

$$b_1 b_2 < -|(\mu_1 - a_1)(\mu_2 - a_2)|$$

are satisfied, then, at $\tau = \bar{\tau}_0$, all the roots of (4) except $\pm i\omega_0$ have negative real parts, and the transversality condition is satisfied.

We introduce the following change of variables:

$$y_1(t) = x_1(\tau t) \quad \text{and} \quad y_2(t) = x_2(\tau t).$$

Then system (3) becomes

$$\begin{cases} \dot{y}_1(t) = -\mu_1 \tau y_1(t) + a_1 \tau f_1(y_1(t)) + b_1 \tau g_1(y_2(t-1)), \\ \dot{y}_2(t) = -\mu_2 \tau y_2(t) + a_2 \tau f_2(y_2(t)) + b_2 \tau g_2(y_1(t)). \end{cases} \tag{9}$$

The characteristic equation associated with the linearization of system (9) at $(0,0)$ is

$$v^2 + \tau[(\mu_1 - a_1) + (\mu_2 - a_2)]v + \tau^2(\mu_1 - a_1)(\mu_2 - a_2) + \tau^2 b_1 b_2 e^{-v} = 0. \tag{10}$$

Comparing (10) and (4), we see that $v = \lambda\tau$. All the roots of (10) at $\tau = \bar{\tau}_0$ except $\pm i\bar{\tau}_0\omega_0$ have negative real parts, and the root of (10)

$$v(\tau) = \beta(\tau) + i\gamma(\tau)$$

with $\beta(\bar{\tau}_0) = 0$ and $\gamma(\bar{\tau}_0) = \bar{\tau}_0\omega_0$ satisfies

$$\beta'(\bar{\tau}_0) = \bar{\tau}_0\alpha'(\bar{\tau}_0).$$

For convenience of notation, we drop the bar in $\bar{\tau}_0$ and let $\tau = \tau_0 + \nu, \quad \nu \in \mathbb{R}$. Then $\nu = 0$ is a Hopf bifurcation value for system (9). Choose the phase space as $C = C([-1,0],\mathbb{R}^2)$. Under the assumption $(\mathbf{H}_1)$, system (9) can be rewritten as

$$\begin{cases} \dot{y}_1(t) = -(\tau_0+\nu)(\mu_1 - a_1)y_1(t) \\ \qquad + (\tau_0+\nu)a_1\left[\frac{f_1''(0)}{2}y_1^2(t) + \frac{f_1'''(0)}{6}y_1^3(t) + \cdots\right] \\ \qquad + (\tau_0+\nu)b_1\left[y_2(t-1) + \frac{g_1''(0)}{2}y_2^2(t-1) + \frac{g_1'''(0)}{6}y_2^3(t-1) + \cdots\right], \\ \dot{y}_2(t) = -(\tau_0+\nu)(\mu_2 - a_2)y_2(t) \\ \qquad + (\tau_0+\nu)a_2\left[\frac{f_2''(0)}{2}y_2^2(t) + \frac{f_2'''(0)}{6}y_1^3(t) + \cdots\right] \\ \qquad + (\tau_0+\nu)b_2\left[y_1(t) + \frac{g_2''(0)}{2}y_1^2(t) + \frac{g_2'''(0)}{6}y_1^3(t) + \cdots\right]. \end{cases} \tag{11}$$

For $\varphi \in C$, let

$$L_\nu\varphi = -B_0\varphi(0) + B_1\varphi(-1),$$

where

$$B_0 = \begin{bmatrix} (\tau_0+\nu)(\mu_1 - a_1) & 0 \\ -(\tau_0+\nu)b_2 & (\tau_0+\nu)(\mu_1 - a_1) \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & -(\tau_0+\nu)b_1 \\ 0 & 0 \end{bmatrix},$$

and

$$F(\nu,\phi)=\frac{\tau_0+\nu}{2}\begin{bmatrix}a_1f_1''(0)\varphi_1^2(0)+b_1g_1''(0)\varphi_2^2(-1)\\a_2f_2''(0)\varphi_2^2(0)+b_2g_2''(0)\varphi_1^2(-1)\end{bmatrix}$$
$$+\frac{\tau_0+\nu}{6}\begin{bmatrix}a_1f_1'''(0)\varphi_1^3(0)+b_1g_1'''(0)\varphi_2^3(-1)\\a_2f_2'''(0)\varphi_2^3(0)+b_2g_2'''(0)\varphi_1^3(-1)\end{bmatrix}+O(|\varphi|^4).$$

By the Riesz representation theorem, there exists matrix $\eta(\theta,\nu)$, whose components are functions of bounded variation in $\theta\in[-1,0]$, such that

$$L_\nu\varphi=\int_{-1}^{0}\mathrm{d}\eta(\theta,\nu)\varphi(\theta),\ \text{ for }\ \varphi\in C.$$

In fact, we can show

$$\eta(\theta,\nu)=\begin{cases}-B_0, & \theta=0,\\ -B_1\delta(\theta+1), & \theta\in[-1,0).\end{cases}$$

For $\varphi\in C^1([-1,0],\mathbb{R}^2)$, define

$$A(\nu)\varphi=\begin{cases}\dfrac{\mathrm{d}\varphi(\theta)}{\mathrm{d}\theta}, & \theta\in[-1,0),\\ \displaystyle\int_{-1}^{0}\mathrm{d}\eta(t,\nu)\varphi(t), & \theta=0,\end{cases}$$

and

$$R\varphi=\eta(\theta,\nu)=\begin{cases}0, & \theta\in[-1,0),\\ F(\nu,\varphi), & \theta=0.\end{cases}$$

We can rewrite (11) in the following form:

$$\dot{y}_t=A(\nu)y_t+R(\nu)y_t, \tag{12}$$

where $y=(y_1,y_2)^T,\ y_t=y(t+\theta)$ for $\theta\in[-1,0]$. For $\psi\in C^1([0,1],\mathbb{R}^2)$, we define

$$A^*\psi=\begin{cases}-\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s\in(0,1],\\ \displaystyle\int_{-1}^{0}\mathrm{d}\eta^T(t,0)\psi(-t), & s=0.\end{cases}$$

For $\varphi\in C[-1,0]$ and $\psi\in[0,1]$, consider the bilinear form

$$\langle\psi,\varphi\rangle\ =\ \bar{\psi}(0)\varphi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{\psi}(\xi-\theta)\mathrm{d}\eta(\theta)\varphi(\xi)\mathrm{d}\xi,$$

where $\eta(\theta) = \eta(\theta,0)$. Then $A(0)$ and A^* are adjoint operators. Since $\pm i\tau_0\omega_0$ are eigenvalues of $A(0)$, they are also eigenvalues of A^*.

By direct computation, we obtain that

$$q(\theta) = \begin{bmatrix} 1 \\ \frac{\mu_1 - a_1 + \mathrm{i}\omega_0}{b_1}\mathrm{e}^{\mathrm{i}\tau_0\omega_0} \end{bmatrix} \mathrm{e}^{\mathrm{i}\tau_0\omega_0\theta}$$

is the eigenvector of $A(0)$ for $\mathrm{i}\tau_0\omega_0$, and

$$q^*(s) = D\begin{bmatrix} 1 \\ \frac{b_1}{\mu_2 - a_2 - \mathrm{i}\omega_0}\mathrm{e}^{\mathrm{i}\tau_0\omega_0} \end{bmatrix} \mathrm{e}^{\mathrm{i}\tau_0\omega_0 s}$$

is the eigenvector of A^* for $-\mathrm{i}\tau_0\omega_0$, where

$$D = \left[1 + \frac{\mu_1 - a_1 - \mathrm{i}\omega_0}{\mu_2 - a_2 - \mathrm{i}\omega_0} + \tau_0(\mu_1 - a_1 - \mathrm{i}\omega_0)\right]^{-1}.$$

Moreover, $\langle q^*(s), q(\theta)\rangle = 1$, $\langle q^*, \bar{q}\rangle = 0$.

Using the same notations as in [13], we first compute the center manifold $\mathcal{C}_0$ at $\nu = 0$. Let y_t be the solution of system (11) with $\nu = 0$. Define

$$z(t) = \langle q^*, y_t\rangle \quad \text{and} \quad W(t,\theta) = y_t(\theta) - z(t)q(\theta) - \bar{z}(t)\bar{q}(\theta).$$

Let z and $\bar{z}$ be the local coordinates for the center manifold $\mathcal{C}_0$ in the direction of $q^*$and $\bar{q^*}$. Then, on $\mathcal{C}_0$, we have $W(t,\theta) = W(z,\bar{z},\theta)$, where $W(z,\bar{z},\theta) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots$.

Note that W is real if y_t is real. We only consider real-valued solutions. For solution $y_t \in \mathcal{C}_0$ of system (11), since $\nu = 0$,

$$\begin{aligned}
\dot{z}(t) &= \mathrm{i}\tau_0\omega_0 z + \langle q^*(s), F(W + 2\mathrm{Re}\{z(t)q(0)\})\rangle \\
&= \mathrm{i}\tau_0\omega_0 z + \bar{q^*}(0)F(W(z,\bar{z},0) + 2\mathrm{Re}\{zq(0)\}) \\
&\stackrel{def}{=} \mathrm{i}\tau_0\omega_0 z + \bar{q^*}(0)F_0(z,\bar{z}).
\end{aligned}$$

We rewrite this as

$$\dot{z}(t) = \mathrm{i}\tau_0\omega_0 z + g(z,\bar{z}), \tag{13}$$

where

$$\begin{aligned}
g(z,\bar{z}) &= \bar{q^*}(0)F(W(z,\bar{z},0) + 2\mathrm{Re}\{z(t)q(0)\}) \\
&= g_{20} + \frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots.
\end{aligned}$$

From (12) and (13) we have

$$\dot{W} = \dot{y}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} AW - 2\mathrm{Re}\{\bar{q^*}(0)F_0 q(\theta)\}, & \theta \in [-1,0), \\ AW - 2\mathrm{Re}\{\bar{q^*}(0)F_0 q(\theta)\} + F_0, & \theta = 0. \end{cases}$$
$$\stackrel{def}{=} AW + H(z,\bar{z},\theta),$$

where

$$H(z,\bar{z},\theta) = H_{20}\frac{z^2}{2} + H_{11}z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots. \tag{14}$$

Expanding the above series and comparing the coefficients, we get

$$(A - 2i\tau_0\omega_0 I)W_{20} = -H_{20}(\theta),\ \ AW_{11}(\theta) = -H_{11}(\theta),\ldots. \tag{15}$$

Note

$$y_1(t) = W^{(1)}(0) + z + \bar{z},$$
$$y_2(t) = W^{(2)}(0) + \frac{\mu_1 - a_1 + \mathrm{i}\omega_0}{b_1}\mathrm{e}^{\mathrm{i}\tau_0\omega_0}z + \frac{\mu_1 - a_1 - \mathrm{i}\omega_0}{b_1}\mathrm{e}^{-\mathrm{i}\tau_0\omega_0}\bar{z},$$
$$y_2(t-1) = W^{(2)}(-1) + \frac{\mu_1 - a_1 + \mathrm{i}\omega_0}{b_1}z + \frac{\mu_1 - a_1 - \mathrm{i}\omega_0}{b_1}\bar{z},$$

where

$$W^{(1)}(0) = W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots,$$
$$W^{(2)}(0) = W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots,$$
$$W^{(2)}(-1) = W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + \cdots,$$

and

$$F_0 = \frac{\tau_0}{2}\begin{bmatrix} a_1 f_1''(0)y_1^2(t) + b_1 g_1''(0)y_2^2(t-1) \\ a_2 f_2''(0)y_2^2(t) + b_2 g_2''(0)y_1^2(t) \end{bmatrix}$$
$$+ \frac{\tau_0}{6}\begin{bmatrix} a_1 f_1'''(0)y_1^3(t) + b_1 g_1'''(0)y_2^3(t-1) \\ a_2 f_2'''(0)y_2^3(t) + b_2 g_2'''(0)y_1^3(t), \end{bmatrix} + \cdots.$$

Denote

$$M_1 = \frac{\mu_1 - a_1 + \mathrm{i}\omega_0}{b_1} \quad \text{and} \quad M_2 = \frac{b_1}{\mu_1 - a_2 + \mathrm{i}\omega_0}.$$

Then

$$q^*(0) = D\begin{bmatrix} 1 \\ \bar{M}_2 e^{i\tau_0\omega_0} \end{bmatrix},$$

$$y_2(t) = W^2(0) + M_1 e^{i\tau_0\omega_0} z + \bar{M}_1 e^{-i\tau_0\omega_0}\bar{z},$$

$$y_2(t-1) = W^2(-1) + M_1 z + \bar{M}_1\bar{z},$$

and

$$F_0 = \tau^0 \begin{bmatrix} \frac{a_1}{2} f_1''(0)(W^{(1)}(0)+z+\bar{z})^2 + \frac{b_1}{2} g_1''(0)(W^{(2)}(-1)+M_1 z+\bar{M}_1\bar{z})^2 \\ +\frac{a_1}{6} f_1'''(0)(W^{(1)}(0)+z+\bar{z})^3 + \frac{b_1}{6} g_1'''(0)(W^{(2)}(-1)+M_1 z+\bar{M}_1\bar{z})^3 \\ \frac{a_2}{2} f_2''(0)(W^{(2)}(0)+M_1 e^{i\tau_0\omega_0} z+\bar{M}_1 e^{-i\tau_0\omega_0}\bar{z})^2 \\ +\frac{b_1}{2} g_2''(0)(W^{(1)}(0)+z+\bar{z})^2 \\ +\frac{a_2}{6} f_2'''(0)(W^{(2)}(0)+M_1 e^{i\tau_0\omega_0} z+\bar{M}_1 e^{-i\tau_0\omega_0}\bar{z})^3 \\ +\frac{b_2}{6} g_2'''(0)(W^{(1)}(0)+z+\bar{z})^3 \end{bmatrix}$$

$$+\cdots\cdots$$

$$= \tau^0 \begin{bmatrix} a_1 f_1''(0) + b_1 g_1''(0) M_1^2 \\ a_2 f_2''(0) M_1^2 e^{2i\tau_0\omega_0} + b_2 g_2''(0) \end{bmatrix} \frac{z^2}{2} + \tau_0 \begin{bmatrix} a_1 f_1''(0) + b_1 g_1''(0)|M_1|^2 \\ a_2 f_2''(0)|M_1|^2 + b_2 g_2''(0) \end{bmatrix} z\bar{z}$$

$$+ \tau_0 \begin{bmatrix} a_1 f_1''(0) + b_1 g_1''(0)\bar{M}_1^2 \\ a_2 f_2''(0)\bar{M}_1^2 e^{-2i\tau_0\omega_0} + b_2 g_2''(0) \end{bmatrix} \frac{\bar{z}^2}{2}$$

$$+ \tau_0 \begin{bmatrix} a_1 f_1''(0)(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) + b_1 g_1''(0)(2W_{11}^{(2)}(-1)M_1 \\ +W_{20}^{(2)}(-1)\bar{M}_1) + a_1 f_1'''(0) + b_1 g_1'''(0)|M_1|^2 M_1 \\ a_2 f_2''(0)(2W_{11}^{(2)}(0)M_1 e^{i\tau_0\omega_0} + W_{20}^{(2)}(0)\bar{M}_1 e^{-i\tau_0\omega_0}) + b_2 g_2''(0)(2W_{11}^{(1)}(0) \\ +W_{20}^{(1)}(0)) + a_2 f_2'''(0)|M_1|^2 M_1 e^{i\tau_0\omega_0} + b_2 g_2'''(0) \end{bmatrix} \frac{z^2\bar{z}}{2}$$

$$+\cdots\cdots.$$

Here

$$\begin{aligned} g(z,\bar{z}) &= \bar{q}^*(0)F_0 = \bar{D}(1, M_2 e^{-i\tau_0\omega_0})F_0 \\ &= \bar{D}\tau_0\Big[(a_1 f_1''(0) + b_1 g_1''(0)M_1^2 + a_2 f_2''(0)M_1^2 M_2 e^{i\tau_0\omega_0} + b_2 M_2 g_2''(0)e^{-i\tau_0\omega_0})\frac{z^2}{2} \\ &\quad + (a_1 f_1''(0) + b_1 g_1''(0)|M_1|^2 + a_2 f_2''(0)|M_1|^2 M_2 e^{-i\tau_0\omega_0} + b_2 M_2 g_2''(0)e^{-i\tau_0\omega_0})z\bar{z} \\ &\quad + (a_1 f_1''(0) + b_1 g_1''(0)\bar{M}_1^2 + a_2 f_2''(0)\bar{M}_1^2 M_2 e^{-3i\tau_0\omega_0} + b_2 M_2 g_2''(0)e^{-i\tau_0\omega_0})\frac{\bar{z}^2}{2} \end{aligned}$$

$$+ (a_1 f_1''(0)(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) + b_1 g_1''(0)(2W_{11}^{(2)}(-1)M_1 + W_{20}^{(2)}(-1)\bar{M}_1)$$
$$+ a_1 f_1'''(0) + b_1 g_1'''(0)|M_1|^2 M_1 + a_2 f_2''(0) M_2 (2W_{11}^{(2)}(0)M_1 + W_{20}^{(2)}(0)\bar{M}_1 \mathrm{e}^{-2\mathrm{i}\tau_0\omega_0})$$
$$+ b_2 g_2''(0) M_2 \mathrm{e}^{-\mathrm{i}\tau_0\omega_0}(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) + a_2 f_2'''(0)|M_1|^2 M_1 M_2$$
$$+ b_2 g_2'''(0) M_2 \mathrm{e}^{-\mathrm{i}\tau_0\omega_0}) \frac{z^2\bar{z}}{2} \Big] + \cdots\cdots.$$

This gives that

$$\begin{aligned}
g_{20} &= \tau_0 \bar{D}[a_1 f_1''(0) + b_1 g_1''(0) M_1^2 + a_2 f_2''(0) M_1^2 M_2 \mathrm{e}^{\mathrm{i}\tau_0\omega_0} + b_2 M_2 g_2''(0) \mathrm{e}^{-\mathrm{i}\tau_0\omega_0}];\\
g_{11} &= \tau_0 \bar{D}[a_1 f_1''(0) + b_1 g_1''(0)|M_1|^2 + a_2 f_2''(0)|M_1|^2 M_2 \mathrm{e}^{-\mathrm{i}\tau_0\omega_0} + b_2 M_2 g_2''(0) \mathrm{e}^{-\mathrm{i}\tau_0\omega_0}];\\
g_{02} &= \tau_0 \bar{D}[a_1 f_1''(0) + b_1 g_1''(0) \bar{M}_1^2 + a_2 f_2''(0) \bar{M}_1^2 M_2 \mathrm{e}^{-3\mathrm{i}\tau_0\omega_0} + b_2 M_2 g_2''(0) \mathrm{e}^{-\mathrm{i}\tau_0\omega_0}];\\
g_{21} &= \tau_0 \bar{D}[a_1 f_1''(0)(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) + b_1 g_1''(0)(2W_{11}^{(2)}(-1)M_1 + W_{20}^{(2)}(-1)\bar{M}_1)\\
&\quad + a_1 f_1'''(0) + b_1 g_1'''(0)|M_1|^2 M_1\\
&\quad + a_2 f_2''(0) M_2 (2W_{11}^{(2)}(0) M_1 + W_{20}^{(2)}(0)\bar{M}_1 \mathrm{e}^{-2\mathrm{i}\tau_0\omega_0})\\
&\quad + b_2 g_2''(0) M_2 \mathrm{e}^{-\mathrm{i}\tau_0\omega_0}(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) + a_2 f_2'''(0)|M_1|^2 M_1 M_2\\
&\quad + b_2 g_2'''(0) M_2 \mathrm{e}^{-\mathrm{i}\tau_0\omega_0}].
\end{aligned} \tag{16}$$

We need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}. Comparing the coefficients of

$$\begin{aligned}
H(z,\bar{z},\theta) &= -2\mathrm{Re}\{\bar{q}^*(0)F_0 q(\theta)\} = -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta)\\
&= -\left(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots\right)q(\theta)\\
&\quad -\left(\bar{g}_{20}\frac{\bar{z}}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \cdots\right)\bar{q}(\theta),
\end{aligned}$$

with those in (14), we obtain

$$H_{20}\theta = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta) \ \text{ and } \ H_{11}\theta = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta).$$

It follows from (15) that

$$\dot{W}_{20}(\theta) = 2\mathrm{i}\tau_0\omega_0 W_{20}(\theta) - g_{20}q(0)\mathrm{e}^{\mathrm{i}\tau_0\omega_0\theta} - \overline{g}_{02}\overline{q}(0)\mathrm{e}^{-\mathrm{i}\tau_0\omega_0\theta}.$$

Solving for $W_{20}(\theta)$ we obtain

$$W_{20}(\theta) = -\frac{\mathrm{i}g_{20}}{\tau_0\omega_0}q(0)\mathrm{e}^{\mathrm{i}\tau_0\omega_0\theta} + \frac{\mathrm{i}\overline{g}_{02}}{3\tau_0\omega_0}\overline{q}(0)\mathrm{e}^{-\mathrm{i}\tau_0\omega_0\theta} + E_1\mathrm{e}^{2\mathrm{i}\tau_0\omega_0\theta}. \tag{17}$$

Similarly,

$$W_{11}(\theta) = -\frac{\mathrm{i}g_{11}}{\tau_0\omega_0}q(0)\mathrm{e}^{\mathrm{i}\tau_0\omega_0\theta} + \frac{\mathrm{i}\overline{g}_{11}}{\tau_0\omega_0}\overline{q}(0)\mathrm{e}^{-\mathrm{i}\tau_0\omega_0\theta} + E_2,$$

where E_1 and E_2 are both two-dimensional constant vectors and can be determined by setting $\theta = 0$ in H. In fact, since

$$H(z,\bar{z},0) = -2\mathrm{Re}\{\bar{q}^*(0)F_0q(0)\} + F_0,$$

we have

$$H_{20}(0) = -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + \tau_0\begin{bmatrix} a_1f_1''(0) + b_1g_1''(0)M_1^2 \\ a_2f_2''(0)M_1^2\mathrm{e}^{2\mathrm{i}\tau_0\omega_0} + b_2g_2''(0)\end{bmatrix} \tag{18}$$

and

$$H_{11}(0) = -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + \tau_0\begin{bmatrix} a_1f_1''(0) + b_1g_1''(0)|M_1|^2 \\ a_2f_2''(0)M_1^2 + b_2g_2''(0)\end{bmatrix}.$$

From (15) and the definition of A, we have

$$\tau_0\begin{bmatrix} -\mu_1+a_1 & 0 \\ b_1 & -\mu_2+a_2\end{bmatrix}W_{20}(0) + \tau_0\begin{bmatrix} 0 & b_1 \\ 0 & 0\end{bmatrix}W_{20}(-1) = 2\mathrm{i}\tau_0\omega_0W_{20}(0) - H_{20}(0), \tag{19}$$

and

$$\tau_0\begin{bmatrix} -\mu_1+a_1 & 0 \\ b_1 & -\mu_2+a_2\end{bmatrix}W_{11}(0) + \tau_0\begin{bmatrix} 0 & b_1 \\ 0 & 0\end{bmatrix}W_{11}(-1) = -H_{11}(0).$$

Substituting (17) into (19) and noticing that

$$\tau_0\begin{bmatrix} -\mu_1+a_1-\mathrm{i}\omega_0 & b_1\mathrm{e}^{-\mathrm{i}\tau_0\omega_0} \\ b_2 & -\mu_2+a_2-\mathrm{i}\omega_0\end{bmatrix}q(0) = 0,$$

we have

$$\tau_0\begin{bmatrix} -\mu_1+a_1-2\mathrm{i}\omega_0 & b_1\mathrm{e}^{-2\mathrm{i}\tau_0\omega_0} \\ b_2 & -\mu_2+a_2-2\mathrm{i}\omega_0\end{bmatrix}E_1 = -g_{20}q(0) - \bar{g}_{20}\bar{q}(0) - H_{20}(0).$$

Substituting (18) into this relation we get

$$\begin{bmatrix} -\mu_1+a_1-2\mathrm{i}\omega_0 & b_1\mathrm{e}^{-2\mathrm{i}\tau_0\omega_0} \\ b_2 & -\mu_2+a_2-2\mathrm{i}\omega_0\end{bmatrix}E_1 = -\begin{bmatrix} a_1f_1''(0) + b_1g_1''(0)M_1^2 \\ a_2f_2''(0)M_1^2\mathrm{e}^{2\mathrm{i}\tau_0\omega_0} + b_2g_2''(0)\end{bmatrix}.$$

Solving the equation for $E_1 = (E_1^{(1)}, E_2^{(2)})$ we get $E_1 = (\frac{\triangle_1^{(1)}}{\triangle_1}, \frac{\triangle_1^{(2)}}{\triangle_1})$, where

$$\begin{aligned}
\triangle_1 &= (\mu_1 - a_1 + 2\mathrm{i}\omega_0)(\mu_2 - a_2 + 2\mathrm{i}\omega_0) - b_1 b_2 \mathrm{e}^{-2\mathrm{i}\tau_0\omega_0},\\
\triangle_1^{(1)} &= (\mu_2 - a_2 + 2\mathrm{i}\omega_0)(a_1 f_1''(0) + b_1 g_1''(0) M_1^2)\\
&\quad + b_1 \mathrm{e}^{-2\mathrm{i}\tau_0\omega_0}(a_2 f_2''(0) M_1^2 \mathrm{e}^{2\mathrm{i}\tau_0\omega_0} + b_2 g_2''(0)),\\
\triangle_1^{(2)} &= (\mu_1 - a_1 + 2\mathrm{i}\omega_0)(a_2 f_2''(0) M_1^2 \mathrm{e}^{2\mathrm{i}\tau_0\omega_0} + b_2 g_2''(0))\\
&\quad + b_2(a_1 f_1''(0) + b_1 g_1''(0) M_1^2).
\end{aligned}$$

Similarly, we can get

$$\begin{bmatrix} -\mu_1 + a_1 & b_1 \\ b_2 & -\mu_2 + a_2 \end{bmatrix} E_2 = -\begin{bmatrix} a_1 f_1''(0) + b_1 g_1''(0)|M_1|^2 \\ a_2 f_2''(0)|M_1|^2 + b_2 g_2''(0) \end{bmatrix},$$

and thus $E_2 = \left(\frac{\triangle_2^{(1)}}{\triangle_2}, \frac{\triangle_2^{(2)}}{\triangle_2}\right)$, where

$$\begin{aligned}
\triangle_2 &= (\mu_1 - a_1)(\mu_2 - a_2) - b_1 b_2,\\
\triangle_2^{(1)} &= (\mu_2 - a_2)(a_1 f_1''(0) + b_1 g_1''(0)|M_1|^2)\\
&\quad + b_1(a_2 f_2''(0)|M_1|^2 + b_2 g_2''(0)),\\
\triangle_2^{(2)} &= (\mu_1 - a_1)(a_2 f_2''(0)|M_1|^2 + b_2 g_2''(0))\\
&\quad + b_2(a_1 f_1''(0) + b_1 g_1''(0)|M_1|^2).
\end{aligned}$$

Based on the above analysis, we see that each g_{ij} in (16) can be determined by the parameters and delay in (3). Therefore, we can compute the following quantities:

$$\begin{aligned}
c_1(0) &= \frac{\mathrm{i}}{2\tau_0\omega_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},\\
\nu_2 &= -\frac{\mathrm{Re}\{c_1(0)\}}{\tau_0\alpha'(\tau_0)},\\
\beta_2 &= 2\mathrm{Re}\{c_1(0)\},\\
T_2 &= -\frac{\mathrm{Im}\{c_1(0)\} + \tau_0\nu_2\,\mathrm{Im}\{\lambda'(\tau_0)\}}{\tau_0\omega_0},
\end{aligned}$$

which determine the properties of bifurcating periodic solutions at the critical value $\bar{\tau}_0$. More specifically, parameter ν_2 determines the direction of the Hopf bifurcation: if $\nu_2 > 0$ ($\nu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_0$ ($\tau < \tau_0$); parameter β_2 determines the stability of the bifurcating periodic solutions: they are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); and parameter T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

Consider a special case for system (2),

$$\begin{cases} \dot{x}_1(t) = -\mu x_1(t) + af(x_1(t)) + b_1 f(x_2(t-\tau_1)), \\ \dot{x}_2(t) = -\mu x_2(t) + af(x_2(t)) + b_2 f(x_1(t-\tau_2)). \end{cases} \tag{20}$$

When $a = 0$ and $b_1 = b_2$, system (20) has been studied in Chen and Wu [4].

We make the following assumptions.

(**P**) $f \in C^3$, $xf(x) > 0$ for $x \neq 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) \neq 0$, $\mu - a > 0$, and $b_1 b_2 < -(\mu - a)^2$.

Let

$$\tau_0 = \frac{1}{\omega_0} \arccos \frac{(\mu - a)^2 - \omega_0^2}{b_1 b_2},$$

and

$$\omega_0 = [-(\mu - a)^2 + |b_1 b_2|]^{\frac{1}{2}}.$$

We have the following result.

Theorem 3.1. *If the hypothesis* (P) *is satisfied, then there exists* $\tau_0 > 0$ *such that the zero solution of system (20) is asymptotically stable for* $\tau \in (0, \tau_0]$, *and unstable for* $\tau > \tau_0$, *and system (20) undergoes a Hopf bifurcation at the origin when* $\tau = \tau_0$. *Moreover, the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are determined by the sign of* $f'''(0)$. *In fact, if* $f'''(0) < 0$ *(*$f'''(0) > 0$*), then the Hopf bifurcation is supercritical (subcritical), and the bifurcating periodic solutions are orbitally asymptotically stable (unstable).*

The conclusions on stability of the zero solution and the existence of Hopf bifurcation follow from (iii) in Theorem 2.3. Using the fact that $f''(0) = 0$ and relation (16), we have

$$g_{20} = g_{11} = g_{02} = 0,$$

and

$$g_{21} = \tau_0 f'''(0) \bar{D} [a(1 + |M_1|^2 M_1 M_2) + b_1 |M_1| M_2 + b_2 M_2 \mathrm{e}^{-\mathrm{i}\tau_0 \omega_0}], \tag{21}$$

where

$$M_1 = \frac{\mu - a + \mathrm{i}\omega_0}{b_1}, \quad M_2 = \frac{b_1}{\mu - a + \mathrm{i}\omega_0}, \quad D = (2 + \tau_0(\mu - a - \mathrm{i}\omega_0))^{-1},$$

and

$$\mathrm{e}^{-\mathrm{i}\tau_0 \omega_0} = \frac{[\mathrm{i}\omega_0 + (\mu - a)]^2}{b_1 b_2}.$$

Substituting M_1, M_2, D, and $e^{-\mathrm{i}\tau_0\omega_0}$ into (21), we obtain

$$\mathrm{Re}\{g_{21}\} = \frac{\tau_0}{\triangle}(1 + \frac{(\mu - a)^2 + \omega_0^2}{b_2^2})[\mu(2 + \tau_0(\mu - a)) + \tau_0 \omega_0^2] f'''(0),$$

where

$$\triangle = [2+\tau_0(\mu-a)]^2+\omega_0^2.$$

Hence,

$$\beta_2 = 2\mathrm{Re}\{c_1(0)\} < 0\ (>0) \ \text{ when } \ f'''(0) < 0 \quad (>0),$$

and

$$\nu_2 = -\frac{\mathrm{Re}\{c_1(0)\}}{\tau_0\alpha'(\tau_0)} > 0\ (<0) \ \text{ when } \ f'''(0) < 0 \quad (>0).$$

The conclusions of the theorem follow from the standard Hopf bifurcation results [13].

Example 3.2. Let $f(x) = \tanh(x)$ in (20); we arrive at the neural network model

$$\begin{cases} \dot{u}_1(t) = -\mu u_1(t) + a\tanh(u_1(t)) + b_1\tanh(u_2(t-\tau_1)), \\ \dot{u}_2(t) = -\mu u_2(t) + a\tanh(u_2(t)) + b_2\tanh(u_1(t-\tau_2)), \end{cases} \tag{22}$$

where $\mu, a, b_1, b_2, \tau_1 > 0$, and $\tau_2 > 0$ are all constants. Noting that $f'''(0) = -2$, by Theorems 2.3 and 3.1, we obtain the following result, which generalizes results in [1] and [20] where system (22) was investigated when $a = 0$.

Corollary 3.3. *Suppose $\mu - a > 0$ and $b_1b_2 < -(\mu-a)^2$. Then there exists $\tau_0 > 0$ such that the zero solution of (22) is asymptotically stable when $\tau \in [0,\tau_0)$ and unstable when $\tau > \tau_0$. Furthermore, (22) undergoes a supercritical Hopf bifurcation at the origin when $\tau = \tau_0$, and the bifurcating periodic solutions are orbitally asymptotically stable.*

Theorem 2.3 shows that under the assumptions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, if

$$|b_1b_2| > |(\mu_1-a_1)(\mu_2-a_2)|$$

is satisfied, then there exists sequence

$$0 < \bar{\tau}_0 < \bar{\tau}_1 < \bar{\tau}_2 < \cdots < \bar{\tau}_j < \cdots$$

such that system (3) undergoes a Hopf bifurcation at the origin when $\tau = \bar{\tau}_j,\ j = 0,1,2,\ldots$. We have only investigated properties of the bifurcation at $\tau = \bar{\tau}_0$ when $b_1b_2 < 0$. Using a similar procedure, we can investigate the direction and stability of the Hopf bifurcations occurring at $\tau = \bar{\tau}_j$ for $j > 0$. In fact, for system (20), we can show that the Hopf bifurcations at $\tau = \bar{\tau}_j$ $(j \geq 0)$ are supercritical (resp. subcritical), with nontrivial periodic solution orbits stable (resp. unstable) on the center manifold if $f'''(0) < 0$ (resp. $f'''(0) > 0$).

4 Global Existence of Periodic Solutions

In this section, we show that periodic solutions of system (2) exist when the total delay $\tau = \tau_1 + \tau_2$ is away from the bifurcation points. We apply a global Hopf bifurcation theorem of Wu [7, 22] to establish global extension of local Hopf branches. A key step of the proof is to establish that system (2) has no periodic solutions of period 2τ. This is equivalent to show that a four-dimensional ordinary differential equation has no nonconstant periodic solutions. This will be done by applying high-dimensional Bendixson–Dulac criteria developed by Li and Muldowney [14], which we briefly describe in the following.

Consider an n-dimensional ordinary differential equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f \in C^1. \tag{23}$$

Let $x = x(t, x_0)$ be the solution to (23) such that $x(0, x_0) = x_0$. The second compound equation of (23) with respect to $x(t, x_0)$

$$z'(t) = \frac{\partial f}{\partial x}^{[2]}(x(t,x_0))z(t) \tag{24}$$

is a linear system of dimension $\binom{n}{2}$, where $\frac{\partial f}{\partial x}^{[2]}$ is the second additive compound matrix of the Jacobian matrix $\frac{\partial f}{\partial x}$ [9, 17]. System (24) is said to be equi-uniformly asymptotically stable with respect to an open set $D \subset \mathbb{R}^n$, if it is uniformly asymptotically stable for each $x_0 \in D$, and the exponential decay rate is uniform for x_0 in each compact subset of D. The equi-uniform asymptotic stability of (24) implies the exponential decay of the surface area of any compact two-dimensional surface in D. If D is simply connected, this precludes the existence of any invariant simple closed rectifiable curve in D, including periodic orbits. In particular, the following result is proved in [15].

Proposition 4.1. *Let $D \subset \mathbb{R}^n$ be a simply connected open set. Assume that the family of linear systems*

$$z'(t) = \frac{\partial f}{\partial x}^{[2]}(x(t,x_0))z(t), \quad x_0 \in D$$

is equi-uniformly asymptotically stable. Then:

(a) *D contains no simple closed invariant curves including periodic orbits, homoclinic orbits, and heteroclinic cycles.*
(b) *Each semi-orbit in D converges to a simple equilibrium.*

In particular, if D is positively invariant and contains a unique equilibrium $\bar{x}$, then $\bar{x}$ is globally asymptotically stable in D.

The uniform asymptotic stability requirement for the family of linear systems (24) can be verified by constructing suitable Lyapunov functions. For instance, (24) is equi-uniformly asymptotically stable if there exists a positive definite function $V(z)$, such that $\frac{\mathrm{d}V(z)}{\mathrm{d}t}|_{(32)}$ is negative definite, and V and $\frac{\mathrm{d}V}{\mathrm{d}t}|_{(32)}$ are both independent of x_0.

For a 4×4 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix},$$

its second additive compound matrix $A^{[2]}$ is [14, 17],

$$A^{[2]} = \begin{bmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{bmatrix}. \tag{25}$$

Consider the ODE system

$$\begin{cases} \dot{x}_1 = -\mu_1 x_1 + a_1 f_1(x_1) + b_1 g_1(x_4), \\ \dot{x}_2 = -\mu_2 x_2 + a_2 f_2(x_2) + b_2 g_2(x_1), \\ \dot{x}_3 = -\mu_1 x_3 + a_1 f_1(x_3) + b_1 g_1(x_2), \\ \dot{x}_4 = -\mu_2 x_4 + a_2 f_2(x_4) + b_2 g_2(x_3). \end{cases} \tag{26}$$

We make the following assumptions:

($\mathbf{H}_3$) There exists $L>0$ such that $|f_i(x)| \le L$ and $|g_i(x)| \le L$ for $x \in \mathbb{R}$ and $i = 1,2$.

($\mathbf{H}_4$) There exist $\alpha_j > 0,\ j = 1,2,3,4,5$, such that

$$\begin{aligned} \sup_{x\in\mathbb{R}^4} \Big\{ & -(\mu_1+\mu_2 - a_1 f_1'(x_1) - a_2 f_2'(x_2)) + \frac{\alpha_1}{\alpha_5}|b_1 g_1'(x_4)|, \\ & -(2\mu_1 - a_1 f_1'(x_1) - a_1 f_1'(x_3)) + \frac{\alpha_2}{\alpha_1}|b_1 g_1'(x_2)| + \alpha_2 |b_1 g_1'(x_4)|, \\ & -(\mu_1+\mu_2 - a_1 f_1'(x_1) - a_2 f_2'(x_2)) + \frac{\alpha_3}{\alpha_2}|b_2 g_2'(x_3)|, \\ & -(\mu_1+\mu_2 - a_1 f_1'(x_3) - a_2 f_2'(x_2)) + \frac{\alpha_4}{\alpha_2}|b_2 g_2'(x_1)|, \\ & -(2\mu_2 - a_2 f_2'(x_2) - a_2 f_2'(x_4)) + \frac{\alpha_5}{\alpha_3}|b_2 g_2'(x_1)| + \frac{\alpha_5}{\alpha_4}|b_2 g_2'(x_3)|, \\ & -(\mu_1+\mu_2 - a_1 f_1'(x_3) - a_2 f_2'(x_4)) + \frac{1}{\alpha_5}|b_1 g_1'(x_2)| \Big\} < 0. \end{aligned} \tag{27}$$

Proposition 4.2. *Suppose that assumptions* $(\mathbf{H}_1)$*,* $(\mathbf{H}_3)$*, and* $(\mathbf{H}_4)$ *are satisfied. Then system (26) has no nonconstant periodic solutions. Furthermore, the unique equilibrium* $(0,0,0,0)$ *is globally asymptotically stable in* $\mathbb{R}^4$.

Proof. First of all, we verify that the solutions of (26) are uniformly ultimately bounded. Let

$$V(x_1,x_2,x_3,x_4)=\frac{1}{2}\left[x_1^2+x_2^2+x_3^2+x_4^2\right].$$

Then the derivative of V along a solution of (26) is

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(34)} &= -\mu_1 x_1^2-\mu_2 x_2^2-\mu_1 x_3^2-\mu_2 x_4^2\\ &\quad +a_1x_1f_1(x_1)+b_1x_1g_1(x_4)+a_2x_2f_2(x_2)+b_2x_2g_2(x_1)\\ &\quad +a_1x_3f_1(x_3)+b_1x_3g_1(x_2)+a_2x_4f_2(x_4)+b_2x_4g_2(x_3).\end{aligned}$$

Using $(\mathbf{H}_3)$ we have

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(34)} \le -\mu\,\Sigma_{i=1}^4 x_i^2+2aL\Sigma_{i=1}^4|x_i|,$$

where $\mu=\min\{\mu_1,\mu_2\}$, and $a=\max_{1\le i\le 2}\{|a_i|,|b_i|\}$. Then there exists $M>1$ such that $\left.\frac{dV}{dt}\right|_{(34)}<0$ for $\Sigma_{i=1}^4 x_i^2\ge M^2$. As a consequence, solutions of (26) are uniformly ultimately bounded.

Let $x=(x_1,x_2,x_3,x_4)$ and

$$\begin{aligned}f(x)=(&-\mu_1x_1+a_1f_1(x_1)+b_1g_1(x_4),-\mu_2x_2+a_2f_2(x_2)+b_2g_2(x_1),\\ &-\mu_1x_3+a_1f_1(x_3)+b_1g_1(x_2),-\mu_2x_4+a_2f_2(x_4)+b_2g_2(x_3))^T.\end{aligned}$$

Then $\frac{\partial f}{\partial x}$ is given as follows:

$$\begin{bmatrix}-\mu_1+a_1f_1'(x_1) & 0 & 0 & b_1g_1'(x_4)\\ b_2g_2'(x_1) & -\mu_2+a_2f_2'(x_2) & 0 & 0\\ 0 & b_1g_1'(x_2) & -\mu_1+a_1f_1'(x_3) & 0\\ 0 & 0 & b_2g_2'(x_3) & -\mu_2+a_2f_2'(x_4)\end{bmatrix}.$$

By (25),

$$\frac{\partial f}{\partial x}^{[2]}(x)=(m_{ij})_{6\times 6}$$

with

$$\begin{aligned}&m_{11}=-(\mu_1+\mu_2)+a_1f_1'(x_1)+a_2f_2'(x_2), m_{12}=m_{13}=m_{14}=0,\\ &\quad m_{15}=-b_1g_1'(x_4),\ m_{16}=0;\end{aligned}$$

$$
\begin{aligned}
& m_{21} = b_1 g_1'(x_2),\ m_{22} = -2\mu_1 + a_1 f_1'(x_1) + a_1 f_1'(x_3), \\
& \quad m_{23} = m_{24} = m_{25} = 0,\ m_{26} = -b_1 g_1'(x_4); \\
& m_{31} = 0,\ m_{32} = b_2 g_2'(x_3),\ m_{33} = -(\mu_1 + \mu_2) + a_1 f_1'(x_1) + a_2 f_2'(x_4), \\
& \quad m_{34} = m_{35} = m_{36} = 0; \\
& m_{41} = 0,\ m_{42} = b_2 g_2'(x_1),\ m_{43} = 0,\ m_{44} = -(\mu_1 + \mu_2) + a_2 f_2'(x_2) + a_1 f_1'(x_3), \\
& \quad m_{45} = m_{46} = 0; \\
& m_{51} = m_{52} = 0,\ m_{53} = b_2 g_2'(x_1),\ m_{54} = b_2 g_2'(x_3), \\
& \quad m_{55} = -2\mu_2 + a_2 f_2'(x_2) + a_2 f_2'(x_4),\ m_{56} = 0; \\
& m_{61} = m_{62} = m_{63} = m_{64} = 0,\ m_{65} = b_1 g_1'(x_2), \\
& \quad m_{66} = -(\mu_1 + \mu_2) + a_1 f_1'(x_3) + a_2 f_2'(x_4).
\end{aligned}
$$

The second compound system

$$
\dot{Z} = \frac{\partial f}{\partial x}^{[2]}(x) Z, \qquad Z = (z_1, \ldots, z_6), \tag{28}
$$

is

$$
\begin{cases}
\dot{z}_1 = -(\mu_1 + \mu_2 - a_1 f_1'(x_1(t)) - a_2 f_2'(x_2(t))) z_1 - b_1 g_1'(x_4(t)) z_5, \\
\dot{z}_2 = b_1 g_1'(x_2(t)) z_1 - (2\mu_1 - a_1 f_1'(x_1(t)) - a_1 f_1'(x_3(t))) z_2 - b_1 g_1'(x_4(t)) z_6, \\
\dot{z}_3 = b_2 g_2'(x_3(t)) z_2 - (\mu_1 + \mu_2 - a_1 f_1'(x_1(t)) - a_2 f_2'(x_4(t))) z_3, \\
\dot{z}_4 = b_2 g_2'(x_1(t)) z_2 - (\mu_1 + \mu_2 - a_2 f_2'(x_2(t)) - a_1 f_1'(x_3(t))) z_4, \\
\dot{z}_5 = b_2 g_2'(x_1(t)) z_3 + b_2 g_2'(x_3(t)) z_4 - (2\mu_2 - a_2 f_2'(x_2(t)) - a_2 f_2'(x_4(t))) z_5, \\
\dot{z}_6 = b_1 g_1'(x_2(t)) z_5 - (\mu_1 + \mu_2 - a_1 f_1'(x_3(t)) - a_2 f_2'(x_4(t))) z_6,
\end{cases}
$$

where $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T$ is a solution of system (26) with $x(0) = x_0 \in \mathbb{R}^4$. Set

$$
W(t) = \max\{\alpha_1 |z_1|, \alpha_2 |z_2|, \alpha_3 |z_3|, \alpha_4 |z_4|, \alpha_5 |z_5|, |z_6|\}.
$$

Then direct calculation leads to the following inequalities:

$$
\begin{aligned}
\frac{\mathrm{d}^+}{\mathrm{d}t} \alpha_1 |z_1| &\le -(\mu_1 + \mu_2 - a_1 f_1'(x_1(t)) - a_2 f_2'(x_2(t))) \alpha_1 |z_1| + \frac{\alpha_1}{\alpha_5} |b_1 g_1'(x_4(t))| \alpha_5 |z_5|, \\
\frac{\mathrm{d}^+}{\mathrm{d}t} \alpha_2 |z_2| &\le -(2\mu_1 - a_1 f_1'(x_1(t)) - a_1 f_1'(x_3(t))) \alpha_2 |z_2| \\
&\quad + \frac{\alpha_2}{\alpha_1} |b_1 g_1'(x_2(t))| \alpha_1 |z_1| + \alpha_2 |b_1 g_1'(x_4(t))| |z_6|,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}^+}{\mathrm{d}t}\alpha_3|z_3| &\le -(\mu_1+\mu_2-a_1f_1'(x_1(t))-a_2f_2'(x_4(t)))\alpha_3|z_3|+\frac{\alpha_3}{\alpha_2}|b_2g_2'(x_3(t))|\alpha_2|z_2|,\\
\frac{\mathrm{d}^+}{\mathrm{d}t}\alpha_4|z_4| &\le -(\mu_1+\mu_2-a_1f_1'(x_3(t))-a_2f_2'(x_2(t)))\alpha_4|z_4|+\frac{\alpha_4}{\alpha_2}|b_2g_2'(x_1(t))|\alpha_2|z_2|,\\
\frac{\mathrm{d}^+}{\mathrm{d}t}\alpha_5|z_5| &\le -(2\mu_2-a_2f_2'(x_2(t))-a_2f_2'(x_4(t)))\alpha_5|z_5|\\
&\quad+\frac{\alpha_5}{\alpha_3}|b_2g_2'(x_1(t))|\alpha_3|z_3|+\frac{\alpha_5}{\alpha_4}|b_2g_2'(x_3(t))|\alpha_4|z_4|,\\
\frac{\mathrm{d}^+}{\mathrm{d}t}|z_6| &\le -(\mu_1+\mu_2-a_1f_1'(x_3(t))-a_2f_2'(x_4(t)))|z_6|+\frac{1}{\alpha_5}|b_1g_1'(x_2(t))|\alpha_5|z_5|,
\end{aligned}
$$

where $\frac{\mathrm{d}^+}{\mathrm{d}t}$ denotes the right-hand derivative. Therefore,

$$
\frac{\mathrm{d}^+}{\mathrm{d}t}W(Z(t)) \le \mu(t)W(Z(t)),
$$

with

$$
\begin{aligned}
\mu(t) = \max\Big\{ &-(\mu_1+\mu_2-a_1f_1'(x_1(t))-a_2f_2'(x_2(t)))+\frac{\alpha_1}{\alpha_5}|b_1g_1'(x_4(t))|,\\
&-(2\mu_1-a_1f_1'(x_1(t))-a_1f_1'(x_3(t)))+\frac{\alpha_2}{\alpha_1}|b_1g_1'(x_2(t))|+\alpha_2|b_1g_1'(x_4(t))|,\\
&-(\mu_1+\mu_2-a_1f_1'(x_1(t))-a_2f_2'(x_2(t)))+\frac{\alpha_3}{\alpha_2}|b_2g_2'(x_3(t))|,\\
&-(\mu_1+\mu_2-a_1f_1'(x_3(t))-a_2f_2'(x_2)(t))+\frac{\alpha_4}{\alpha_2}|b_2g_2'(x_1(t))|,\\
&-(2\mu_2-a_2f_2'(x_2(t))-a_2f_2'(x_4(t)))+\frac{\alpha_5}{\alpha_3}|b_2g_2'(x_1(t))|+\frac{\alpha_5}{\alpha_4}|b_2g_2'(x_3(t))|,\\
&-(\mu_1+\mu_2-a_1f_1'(x_3(t))-a_2f_2'(x_4(t)))+\frac{1}{\alpha_5}|b_1g_1'(x_2(t))|\Big\}.
\end{aligned}
$$

Thus, under assumption ($\mathbf{H}_4$), and by the boundedness of solution to (26), there exists a $\delta>0$ such that $\mu(t)\le-\delta<0$, and hence

$$
W(Z(t)) \le W(Z(s))\mathrm{e}^{-\delta(t-s)},\ \ t\ge s>0.
$$

This establishes the equi-uniform asymptotic stability of the second compound system (28), and hence the conclusions of Proposition 4.2 follow from Proposition 4.1. □

Now we are in the position to state the main result of this section.

Theorem 4.3. *Suppose that assumptions* $(\mathbf{H}_1)$ – $(\mathbf{H}_4)$ *and the condition*

$$|b_1b_2| > |(\mu_1 - a_1)(\mu_2 - a_2)|$$

are satisfied. Let τ_j *be defined in (8).*

(i) *If* $b_1b_2 > 0$*, then system (3) has at least* $j+1$ *nonconstant periodic solutions for* $\tau > \bar{\tau}_j,\ j \geq 0$.
(ii) *If* $b_1b_2 < 0$*, then system (3) has at least* j *nonconstant periodic solutions for* $\tau > \bar{\tau}_j,\ j \geq 1$.

Proof. We regard (τ, p) as parameters and apply Theorem 3.3 in Wu [22]. By $(\mathbf{H}_1)$ we know that the origin is an equilibrium of system (3). Hence, $(0, \tau, p)$ is a stationary point of (3), and the corresponding characteristic function is

$$\Delta_{(0,\tau,p)}(\lambda) = \lambda^2 + [(\mu_1 - a_1) + (\mu_2 - a_2)]\lambda + (\mu_1 - a_1)(\mu_2 - a_2) - b_1b_2\mathrm{e}^{-\lambda\tau}.$$

Clearly, $\Delta_{(0,\tau,p)}(\lambda)$ is continuous in $(\tau, p, \lambda) \in R_+ \times R_+ \times C$. To locate centers, we consider

$$\begin{aligned}\Delta_{(0,\tau,p)}\left(\mathrm{i}\frac{2m\pi}{p}\right) = &-\left(\frac{2m\pi}{p}\right)^2 + \mathrm{i}\,[(\mu_1 - a_1) + (\mu_2 - a_2)]\,\frac{2m\pi}{p} \\ &+ (\mu_1 - a_1)(\mu_2 - a_2) - b_1b_2\mathrm{e}^{-\mathrm{i}\frac{2m\pi}{p}\tau}.\end{aligned}$$

Using the conclusion (ii) in Lemma 2.2 we know that $(0, \tau, p)$ is a center if and only if $m = 1, \tau = \bar{\tau}_j$ and $p = \frac{2\tau}{\omega_0}$. In particular, $(0, \bar{\tau}_j, \frac{2\tau}{\omega_0})$ is a center, and all the centers are isolated. In fact, the set of centers is countable and can be expressed as

$$\left\{\left(0, \bar{\tau}_j, \frac{2\tau}{\omega_0}\right) \;:\; j = 0, 1, 2, \ldots\right\},$$

where ω_0 and $\bar{\tau}_j$ are defined in (7) and (8), respectively.

Consider $\Delta_{(0,\tau,p)}(\lambda)$ with $m = 1$. By Lemmas 2.1 and 2.2, for fixed j, there exist $\varepsilon, \delta > 0$ and a smooth curve $\lambda : (\bar{\tau}_j - \delta, \bar{\tau}_j + \delta) \to C$, such that $\Delta_{(0,\tau,p)}(\lambda(\tau)) = 0$, $|\lambda(\tau) - \mathrm{i}\omega_0| < \varepsilon$ for all $\tau \in (\bar{\tau}_j - \delta, \bar{\tau}_j + \delta)$, and

$$\lambda(\bar{\tau}_j) = \mathrm{i}\omega_0, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Re}\lambda(\tau)|_{\tau=\bar{\tau}_j} > 0.$$

Let

$$\Omega_\varepsilon = \left\{(v, p) \;:\; 0 < v < \varepsilon,\ \left|p - \frac{2\tau}{\omega_0}\right| < \varepsilon\right\}.$$

Clearly, if $|\tau - \bar{\tau}_j| < \delta$ and $(v, p) \in \partial\Omega_\varepsilon$ such that $q(v + \mathrm{i}\frac{2\pi}{p}) = 0$, then $\tau = \bar{\tau}_j, v = 0$, and $p = \frac{2\pi}{\omega_0}$. This verifies the hypothesis (A_4) for $m = 1$ in Theorem 3.3 of Wu [22]. Moreover, if we set

$$H_m^{\pm}\left(0,\bar{\tau}_j,\frac{2\tau}{\omega_0}\right)(v,p)=\Delta_{(0,\bar{\tau}_j\pm\delta,p)}\left(v+\mathrm{i}m\frac{2\pi}{p}\right),$$

then, at $m=1$, we have

$$\begin{aligned}\gamma_m\left(0,\bar{\tau}_j,\frac{2\tau}{\omega_0}\right)&=\deg_B\left(H_m^{-}\left(0,\bar{\tau}_j,\frac{2\tau}{\omega_0}\right),\Omega_\varepsilon\right)-\deg_B\left(H_m^{+}\left(0,\bar{\tau}_j,\frac{2\tau}{\omega_0}\right),\Omega_\varepsilon\right)\\&=-1.\end{aligned}\tag{29}$$

If (3) has another equilibrium, say (x_1^*,x_2^*), then the characteristic equation associated with the linearization of (3) at (x_1^*,x_2^*) is

$$\begin{aligned}&\lambda^2+[(\mu_1-a_1f_1'(x_1^*))+(\mu_2-a_2f_2'(x_2^*))]\lambda\\&\quad+(\mu_1-a_1f_1'(x_1^*))(\mu_2-a_2f_2'(x_2^*))-b_1b_2g_1'(x_2^*)g_2'(x_1^*)\mathrm{e}^{-\lambda\tau}=0.\end{aligned}\tag{30}$$

Suppose that equation (30) has a pair of purely imaginary roots $\pm\mathrm{i}\omega^*$ when $\tau=\tau^*$. Denote

$$\lambda(\tau)=\alpha^*(\tau)+\mathrm{i}\omega^*(\tau)$$

be the root of (30) satisfying $\alpha^*(\tau^*)=0$ and $\omega^*(\tau^*)=\omega^*$. Similar to Lemma 2.1, we have

$$\left.\frac{\mathrm{d}\alpha^*(\tau)}{\mathrm{d}\tau}\right|_{\tau=\tau^*}>0.$$

Similar to the discussion above, we know $((x_1^*,x_2^*),\tau^*,\frac{2\pi}{\omega^*})$ is an isolate center of (3), and the crossing number, at $m=1$, is

$$\gamma_m\left(\left(x_1^*,x_2^*\right),\tau^*,\frac{2\pi}{\omega^*}\right)=-1.$$

Let

$$\Sigma=\mathrm{cl}\{(x,\tau,p)\ :\ x \text{ is a } p\text{-periodic solution of (3)}\}.$$

By Theorem 3.3 in [22], we conclude that the connected component $\mathcal{C}(0,\bar{\tau}_j,\frac{2\pi}{\omega_0})$ through $(0,\bar{\tau}_j,\frac{2\pi}{\omega_0})$ in Σ is nonempty. Meanwhile, (29) and (30) imply that the first crossing number of each center is always -1. Therefore, we conclude that $\mathcal{C}(0,\bar{\tau}_j,\frac{2\pi}{\omega_0})$ is unbounded by Theorem 3.3 of [22].

Now, we prove that periodic solutions of (3) are uniformly bounded. Let

$$\mu=\min\{\mu_1,\mu_2\},\quad M\geq\max\{1,\ L(|a_1+b_1|+|a_2+b_2|)/\mu\},$$

and

$$r(t)=\sqrt{x_1^2(t)+x_2^2(t)}.$$

Differentiating $r(t)$ along a solution of (3) we have

$$\begin{aligned}
\dot{r}(t) &= \frac{1}{r(t)}[x_1(t)\dot{x}_1(t)+x_2(t)\dot{x}_2(t)]\\
&= \frac{1}{r(t)}[-(\mu_1 x_1^2(t)+\mu_2 x_2^2(t))+a_1x_1(t)f_1(x_1(t))+b_1x_1(t)g_1(x_2(t-\tau))\\
&\quad +a_2x_2(t)f_2(x_2(t))+b_2x_2(t)g_2(x_1(t))]\\
&\le \frac{1}{r(t)}[-\mu(x_1^2(t)+x_2^2(t))+L(|a_1+b_1||x_1(t)|+|a_2+b_2||x_2(t)|)].
\end{aligned}$$

If there exists $t_0>0$ such that $r(t_0)=A\ge M$, we have

$$\dot{r}(t_0)\le \frac{1}{A}[-\mu A^2+AL(|a_1+b_1|+|a_2+b_2|)] = -\mu A+L(|a_1+b_1|+|a_2+b_2|)<0.$$

It follows that if $x(t)=(x_1(t),x_2(t))^T$ is a periodic solution of (3), then $r(t)<M$ for all t. This shows that the periodic solutions of (3) are uniformly bounded.

Next, we establish that system (3) has no 2τ-periodic solutions. Suppose $x(t)=(x_1(t),x_2(t))^T$ is a 2τ-periodic solution of system (3). Let

$$x_3(t)=x_1(t-\tau),\quad x_4(t)=x_2(t-\tau).$$

Then $(x_1(t),x_2(t),x_3(t),x_4(t)))$ is a nonconstant periodic solution to system (26). This contradicts to the conclusion of Proposition 4.2 and implies that system (3) has no 2τ-periodic solutions.

By the definition of $\bar{\tau}_j$ in (8), we have that $\bar{\tau}_j<\bar{\tau}_{j+1},\ j\ge 0$, and

$$\omega_0\bar{\tau}_0 = \arcsin\left(-\frac{[(\mu_1-a_1)+(\mu_2-a_2)]\omega_0}{b_1b_2}\right)\in(\pi,2\pi),$$

when $b_1b_2>0$. Hence, $\frac{2\pi}{\omega_0}<2\bar{\tau}_0$. Thus, there exists an integer m such that $\frac{2\bar{\tau}_0}{m+1}<\frac{2\pi}{\omega_0}<\frac{2\bar{\tau}_0}{m}$. Since system (3) has no 2τ-periodic solutions, it has no $\frac{2\tau}{n}$-periodic solutions for any integer n. This implies that the period p of a periodic solution on the connected component $\mathcal{C}(0,\bar{\tau}_0,\frac{2\pi}{\omega_0})$ satisfies $\frac{2\tau}{m+1}<p<\frac{2\tau}{m}$. Therefore, the periods of the periodic solutions of system (3) on $\mathcal{C}(0,\bar{\tau}_0,\frac{2\pi}{\omega_0})$ are uniformly bounded for $\tau\in[0,\bar{\tau})$, where $\bar{\tau}$ is fixed.

The inequality (27) implies that

$$-[(\mu_1-a_1f_1'(x))+(\mu_2-a_2f_2'(x))]<0 \ \text{ for } \ (x_1,x_2)\in\mathbb{R}^2,$$

and hence

$$\begin{aligned}
&\frac{\partial}{\partial x_1}[-\mu_1x_1+a_1f_1(x_1)+b_1g_1(x_2)]+\frac{\partial}{\partial x_2}[-\mu_2x_2+a_2f_2(x_2)+b_2g_2(x_1)]\\
&\quad = -[(\mu_1-a_1f_1'(x_1)+(\mu_2-a_2f_2'(x_2))]<0
\end{aligned}$$

for all $(x_1,x_2) \in \mathbb{R}^2$. This shows that system (3) with $\tau = 0$ has no nonconstant periodic solutions, by the classical Bendixson's criterion. Thus, the projection of $\mathcal{C}(0,\bar{\tau}_0,\frac{2\pi}{\omega_0})$ onto the τ-space must be an interval $[T,\infty)$ with $0 < T \leq \bar{\tau}_0$. This shows that for any $\tau > \bar{\tau}_0$, system (3) has at least one nonconstant periodic solution on $\mathcal{C}(0,\bar{\tau}_0,\frac{2\pi}{\omega_0})$.

Similarly, we can show that, for any $\tau > \bar{\tau}_j,\ j \geq 1$, system (3) has at least one nonconstant periodic solution on $\mathcal{C}(0,\bar{\tau}_j,\frac{2\pi}{\omega_0})$. Therefore, for any $\tau > \bar{\tau}_j$, system (3) has at least $j+1$ nonconstant periodic solutions in the case of $b_1b_2 > 0$. The proof of (i) is complete.

The proof of (ii) is similar and is omitted. □

Example 4.4. Consider the neural network model

$$\begin{cases} \dot{u}_1(t) = -\mu u_1(t) + a\tanh(u_1(t)) + b_1\tanh(u_2(t-\tau_1)), \\ \dot{u}_2(t) = -\mu u_2(t) + a\tanh(u_2(t)) + b_2\tanh(u_1(t-\tau_2)). \end{cases} \tag{31}$$

For $j = 0,1,2,\ldots$, let

$$\bar{\tau}_j = \frac{1}{\omega_0}\left[\arccos\frac{(\mu-a)^2-\omega_0^2}{b_1b_2} + 2j\pi\right], \quad j = 0,1,2,\ldots,$$

and

$$\omega_0 = [-(\mu-a)^2 + |b_1b_2|]^{\frac{1}{2}}.$$

We have the following result.

Corollary 4.5. *Suppose that $a > 0,\ b_1b_2 > (\mu-a)^2$ and*

$$\mu - a > \max\left\{|b_1|/\sqrt{2},\ |b_2|/\sqrt{2}\right\}. \tag{32}$$

Then for any $\tau > \bar{\tau}_j$ $j = 0,1,2,\ldots$, system (31) has at least $j+1$ nonconstant periodic solutions.

It is sufficient to verify that $(\mathbf{H}_4)$ is satisfied. Noting that $f_1 = f_2 = g_1 = g_2 = \tanh$ and $0 < \tanh'(x) \leq 1$ and taking $\alpha_1 = \alpha_3 = \alpha_4 = 1$, we have

$$-(2\mu - a\tanh'(x_1) - a\tanh'(x_2)) + \frac{1}{\alpha_5}|b_1\tanh'(x_4)| \leq -2(\mu-a) + \frac{1}{\alpha_5}|b_1|,$$

$$-(2\mu - a\tanh'(x_1) - a\tanh'(x_3)) + \alpha_2|b_1\tanh'(x_2)| + \alpha_2|b_1\tanh'(x_4|$$
$$\leq -2(\mu-a) + 2\alpha_2|b_1|,$$

$$-(2\mu - a\tanh'(x_1) - a\tanh'(x_2)) + \frac{1}{\alpha_2}|b_2\tanh'(x_3)| \leq -2(\mu-a) + \frac{1}{\alpha_2}|b_2|,$$

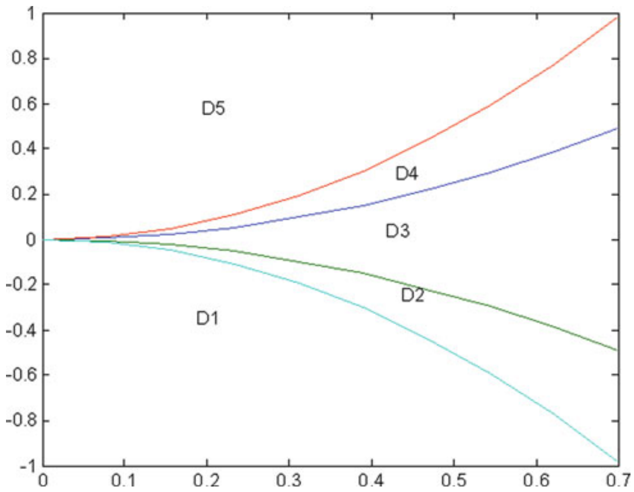

Fig. 2 The curves $b_1b_2 = \pm(\mu - a)^2$ and $b_1b_2 = \pm 2(\mu - a)^2$ divide the right half plane into five regions, D_1, D_2, D_3, D_4, and D_5. D_3 is an absolutely stable region, $D_1 \cup D_2$ is a conditionally stable region, and $D_4 \cup D_5$ is an unstable region. Values of $\mu - a$ are plotted on the horizontal axis and b_1b_2 plotted on the vertical axis. If $(\mu - a, b_1b_2) \in D_4$ (resp. D_2) and (32) is satisfied, then system (31) has at least one nonconstant periodic solution for $\tau > \bar{\tau}_0$ (resp. $\tau > \bar{\tau}_1$)

$$
\begin{aligned}
&-(2\mu - a\tanh'(x_2) - a\tanh'(x_3)) + \frac{1}{\alpha_2}|b_2\tanh'(x_1)| \le -2(\mu - a) + \frac{1}{\alpha_2}|b_2|,\\
&-(2\mu - a\tanh'(x_2) - a\tanh'(x_4)) + \alpha_5|b_2\tanh'(x_1)| + \alpha_5|b_2\tanh'(x_3)|\\
&\quad \le -2(\mu - a) + 2\alpha_5|b_2|,\\
&-(2\mu - a\tanh'(x_3) - a\tanh'(x_4)) + \frac{1}{\alpha_5}|b_1\tanh'(x_2)| \le -2(\mu - a) + \frac{1}{\alpha_5}|b_1|.
\end{aligned}
$$

Let $\alpha_2 = \alpha_5 = \frac{1}{\sqrt{2}}$. Then (32) implies that

$$
\begin{aligned}
&-2(\mu - a) + \frac{1}{\alpha_5}|b_1| < 0, \qquad &&-2(\mu - a) + \alpha_2|b_1| < 0,\\
&-2(\mu - a) + \frac{1}{\alpha_2}|b_2| < 0, \qquad &&-2(\mu - a) + \alpha_5|b_2| < 0.
\end{aligned}
$$

Therefore, $(\mathbf{H}_4)$ is satisfied. The conclusion of Corollary 4.5 is illustrated in Fig. 2.

To demonstrate the Hopf bifurcation results in Theorems 3.1 and 4.3, we carry out numerical simulations on system (31). The simulations are done using Mathematica with different values of μ, a, b_i, and τ_i and different initial values for u_i. The simulations consistently show the bifurcating periodic solution being asymptotically stable and global existence of periodic solution: existence of periodic solutions for values $\tau = \tau_1 + \tau_2$ near $\bar{\tau}_0$ and far away from $\bar{\tau}_k$. In Fig. 3, we show one

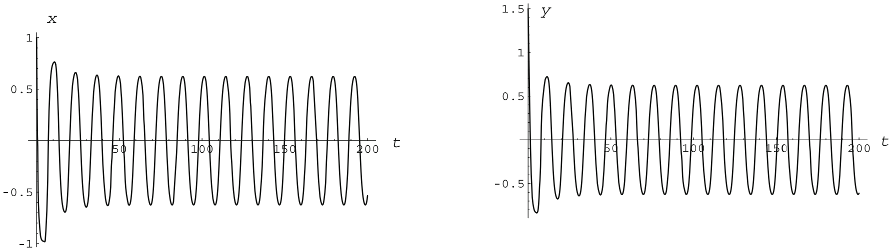

Fig. 3 Mathematical simulations of a periodic solution to system (31) with $\mu = 2$, $a = 0.8$, $b_1 = -1.2$, $b_2 = 1.5$, $\tau_1 = 2.8$, and $\tau_2 = 2.2$. The total delay $\tau = \tau_1 + \tau_2 = 5$ is greater than the first Hopf bifurcation value $\bar{\tau}_0 = 3.6905$

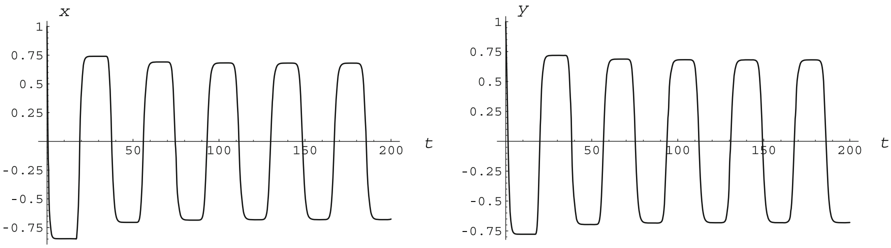

Fig. 4 Mathematical simulations show that an asymptotically stable periodic solution to system (31), with $\mu = 2$, $a = 0.8$, $b_1 = -1.2$, $b_2 = 1.5$, $\tau_1 = 9$, and $\tau_2 = 8$, continues to exist when the total delay $\tau = \tau_1 + \tau_2 = 17$ is between the two consecutive Hopf bifurcation values $\bar{\tau}_1 = 14.1625$ and $\bar{\tau}_2 = 24.6345$

of the simulations using $\mu = 2$, $a = 0.8, b_1 = -1, b_2 = 1.5$ such that (32) is satisfied and $(\mu - a, b_1 b_2) \in D_2$. In this case, it can be calculated that $\omega_0 = 0.6$ and for $k = 0, 1, 2, \ldots,$ $\bar{\tau}_k = 3.6905 + 10.472 \times k$. The delays are chosen as $\tau_1 = 2.8,\ \tau_2 = 2.2$ so that $\tau = \tau_1 + \tau_2 = 5$ is larger than $\bar{\tau}_0 = 3.6905$. An asymptotically stable periodic solution is shown to exist in Fig. 3. Similarly, in Fig. 4, the parameters μ, a, and b_i are chosen as above; the delays are chosen as $\tau_1 = 9,\ \tau_2 = 8$ so that $\tau = \tau_1 + \tau_2 = 17$ is between the two Hopf bifurcation values $\bar{\tau}_1 = 14.1625$ and $\bar{\tau}_2 = 24.6345$. An asymptotically stable periodic solution is shown in Fig. 4.

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