Chapter 5. Elementary Bifurcation Theory

5.1. Introduction

Consider a family of differential equations

\[ x' = f(x, \mu), \tag{1} \]

where \( f : \mathbb{R}^{n+1} \to \mathbb{R}^n \) is analytic for \( \mu \in \mathbb{R}, x \in \mathbb{R}^n \). Let \( x = x_0(\mu) \) be a family of equilibria of (1), namely, \( f(x_0(\mu), \mu) = 0 \). Set

\[ z = x - x_0(\mu). \]

Then,

\[ z' = A(\mu)z + O(|z|^2), \quad A(\mu) = \frac{\partial f}{\partial x}(x_0(\mu), \mu). \]

Let \( \lambda_1(\mu), \ldots, \lambda_n(\mu) \) be the eigenvalues of \( A(\mu) \). If, for some \( i \), \( \text{Re}\lambda_i(\mu) \) changes sign at \( \mu = \mu_0 \), we say that \( \mu_0 \) is a bifurcation point of (1). Sometimes, we also call \( (x_0(\mu_0), \mu_0) \) a bifurcation point.

Remarks:

(1) \( f \) is analytic in \( x, \mu \) implies that \( x_0(\mu) \) is analytic in \( \mu \), provided

\[ \det A(\mu) = \det \frac{\partial f}{\partial x}(x_0(\mu), \mu) \neq 0. \]

Analyticity may fail at a bifurcation point since \( \det A(\mu) = \lambda_1(\mu) \cdots \lambda_n(\mu) \).

(2) Being the roots of \( \det(\lambda I - A(\mu)) = 0 \), \( \lambda = \lambda_i(\mu) \) are also analytic in \( \mu \) except possibly at bifurcation points.
Simplifying Assumption: $\lambda_i(\mu) \neq \lambda_j(\mu)$ if $i \neq j$.

We will look at some examples when $n = 2$.

**Example 1.** One-dimensional bifurcation (only one-dimensional eigenspace changes with $\mu$):

Since only one-dimensional eigenspace changes with $\mu$, we may simply assume $n = 1$. Therefore, $f: \mathbb{R}^2 \rightarrow \mathbb{R}^1$, and $x_0(\mu)$ is a real-valued analytic function of $\mu$ provided

$$\lambda_1(\mu) = f_x(x_0(\mu), \mu) = A(\mu) \neq 0.$$ 

Therefore, the equilibrium $x_0(\mu)$ is u.a.s. if $\lambda_1(\mu) < 0$, and unstable if $\lambda_1(\mu) > 0$. This implies that $\mu_0$ is a bifurcation point if $\lambda_1(\mu_0) = 0$. Therefore, bifurcation points $(x_0(\mu_0), \mu_0)$ are solutions of

$$f(x, \mu) = 0, \quad \text{and} \quad f_x(x, \mu) = 0.$$ 

The bifurcation diagram describes the general shape of $x_0(\mu)$ for $\mu$ near the bifurcation point $\mu_0$. At $\mu \neq \mu_0$, $\lambda_1(\mu) = f_x(x, \mu) \neq 0$. By the Implicit Function Theorem, $x = x_0(\mu)$ is the unique solution of $f(x, \mu) = 0$, and

$$\frac{\partial f}{\partial x} \frac{dx}{d\mu} + \frac{\partial f}{\partial \mu} = 0.$$
Since \( \lambda_1(\mu_0) = f_x(x_0(\mu_0), \mu_0) = 0 \), if \( f_\mu(x_0(\mu_0), \mu_0) \neq 0 \), then

\[
\left| \frac{dx}{d\mu} \right| \to \infty, \quad \text{as} \quad \mu \to \mu_0.
\]

Therefore the curve \( x = x_0(\mu) \) has a vertical tangent line when \( \mu = \mu_0 \) if \( f_\mu(x_0(\mu_0), \mu_0) \neq 0 \).

In the subsequent discussions, w.l.o.g., we will assume that the bifurcation point is at \((0, 0)\), namely, \( \mu_0 = 0 \) and \( x_0(\mu_0) = 0 \). The most common bifurcation types are illustrated by the following examples.

**Example 3.** Saddle-Node Bifurcation. Consider

\[
x' = \mu - x^2.
\]

In this case, the bifurcation equations (2) becomes

\[
x^2 = \mu,
\]

\[
2x = 0.
\]

There are two branches of equilibria: \( x_0(\mu) = \pm \sqrt{\mu} \), and the bifurcation point is at \((x, \mu) = (0, 0)\). Since \( f_\mu(0, 0) = 1 \neq 0 \), we should expect a vertical tangent line at \((0, 0)\) for \( x_0(\mu) \). The bifurcation diagram is shown in the following figure.

Therefore, the saddle-node bifurcation can be described as a single branch of equilibria undergoes a change in stability, from being stable to unstable.

**Example 4.** Transcritical Bifurcation. Consider

\[
x' = \mu x - x^2.
\]

The bifurcation equation (2) becomes \( \mu x - x^2 = 0, \mu - 2x = 0 \). This gives two branches of equilibria \( x_0 = 0 \), and \( x_0 = \mu \). For the branch \( x_0 = 0 \), we have \( \lambda_1 = \mu \) and thus the stability changes from stable to unstable as \( \mu \) increases cross 0, and \( \mu_0 = 0 \) is the bifurcation point. For the second branch, \( x_0 = \mu \), we have \( \lambda_1 = -\mu \). Therefore, this branches changes stability in the opposite direction to the first branch, and the bifurcation point is also \( \mu_0 = 0 \). See the bifurcation diagram below.
The transcritical bifurcation can be described as two branches of equilibria intersect and exchange stability type at the bifurcation point.

**Example 5.** Pitchfork Bifurcation. Consider

\[ x' = \mu x - x^3. \]

From the bifurcation equations, we find that there are three branches of equilibria: \( x_0 = 0, \ x_0 = \sqrt{\mu}, \) and \( x_0 = -\sqrt{\mu}. \) The corresponding \( \lambda_1 \) for the three branches are \( \lambda_1 = \mu, -2\mu, \) and \( -2\mu, \) respectively. It is easy to see that \((0, 0)\) is the bifurcation point for all three branches, and the bifurcation diagram is as shown below.

Case 1: Subcritical pitchfork.

Case 2: Supercritical pitchfork.

The subcritical (or supercritical) pitchfork bifurcation can be described as a branch of equilibria which changes stability type at the bifurcation point is intersected there by a stable (or unstable) branch.

The following theorem establishes that the three type of bifurcations observed in Examples 3-5 are indeed the only generic ones there are.

**Theorem 5.1** Suppose

(i) \( \mathbf{f}(x, u) \) is an analytic function of \( (x, \mu) \) near \((0, 0)\).
(ii) \((x, \mu) = (0, 0)\) is a bifurcation point (namely \(0 = f(0, 0) = f_x(0, 0)\)).

Then

(a) If \(f_\mu(0, 0) \neq 0\) and \(f_{xx}(0, 0) \neq 0\), then there exists, in a neighborhood of \((0, 0)\), a single branch of critical points which has a saddle node bifurcation at \((0, 0)\).

(b) If \(f_\mu(0, 0) = 0\), let \(D = \begin{vmatrix} f_\mu & f_{xx} \\ f_x & f_{xx} \end{vmatrix}_{(0,0)} = f_\mu f_{xx} - f_{x\mu}^2\), then

1. If \(D > 0\), then \((0, 0)\) is an isolated critical point.
2. If \(D < 0\), then there are two branches of critical points which intersect at \((0, 0)\). The bifurcation is either transcritical or pitchfork.

**Proof.** From the analyticity of \(f\) we have

\[
f(x, \mu) = \alpha \mu + \frac{1}{2} (a \mu^2 + 2b \mu x + c x^2) + O(x^3, x^2 \mu, x \mu^2, \mu^3),
\]

where \(\alpha = f_\mu(0, 0), a = f_{\mu\mu}(0, 0), b = f_{\mu x}(0, 0), c = f_{xx}(0, 0)\). Then \(D = ac - b^2\). Therefore

\[
f_x(x, \mu) = bx + O(x^2, x \mu, \mu^2)
\]

\[
f_\mu(x, \mu) = \alpha + bx + O(x^2, x \mu, \mu^2).
\]

In case (a), we have \(0 \neq \alpha = f_\mu(0, 0)\). Then the IFT implies \(f(x, \mu) = 0\) can be solved uniquely in the form \(\mu = \mu(x)\) near \((0, 0)\), and \(\mu(0) = 0\). Now (3) implies that \(\mu(x) = O(x^2)\), near \((0, 0)\), and thus

\[
\mu(x) = -\frac{c}{2\alpha} x^2 + O(x^3).
\]

From (4) we know

\[
\lambda_1 = f_x(x, \mu(x)) = b(-\frac{c}{2\alpha})x^2 + bx + O(x^2) = cx + O(x^2).
\]

Since \(c = f_{xx}(0, 0) \neq 0\), \(\lambda_1(x)\) changes sign at \(x = 0\), and thus \((0, 0)\) is the bifurcation point.

It is easy to see \(\frac{\partial \mu}{\partial x} = 0\) at the bifurcation point. In summary, \(\mu = \mu(x)\) gives a single branch of equilibria which changes stability type at the bifurcation point; we have a saddle-node bifurcation in the case.

In case (b), we have \(\alpha = f_\mu(0, 0) = 0\). Then

\[
f(x, \mu) = \frac{1}{2} (a \mu^2 + 2b \mu x + c x^2) + O(x^3, x^2 \mu, x \mu^2, \mu^3).
\]

If \(D > 0\), then \(\frac{1}{2}(a \mu^2 + 2b \mu x + c x^2)\) is a positive definite quadratic form, and \((x, \mu) = (0, 0)\) is the only critical point in a neighborhood of \((0, 0)\).

Now we assume \(D < 0\). First suppose \(c = f_{xx}(0, 0) \neq 0\). Then \((x, \mu) = (\pi, 0)\) is not a solution of \(f(x, \mu) = 0\) if \(\pi \neq 0\). Set

\[
x = \mu z, \quad \text{and} \quad g(z, \mu) = \frac{f(\mu z, \mu)}{\mu^2}.
\]
Then \( f(x, \mu) = 0, \mu \neq 0 \) if and only if \( g(z, \mu) = 0 \). From (5) we have

\[
g(z, \mu) = \frac{1}{2}(a + 2b z + c z^2) + O(\mu).
\]

Therefore, \( g(z, 0) = 0 \) has two distinct solutions

\[
z = -\frac{b \pm \sqrt{-D}}{c} = \gamma_1, \gamma_2
\]

and \( g_\iota(\gamma_1, 0) = \pm \sqrt{-D} \neq 0 \). The IFT implies that \( g(z, \mu) = 0 \) has two distinct branches of analytic solutions \( z = z_1(\mu), z = z_2(\mu), z_i(0) = \gamma_i, \ i = 1, 2 \). Therefore, \( f(x, \mu) = 0 \) has two distinct branches of solutions \( x = \mu z_1(\mu) \) and \( x = \mu z_2(\mu) \) which intersect at the bifurcation point, and \( \gamma_1, \gamma_2 \) are slopes of these two curves at \( (0, 0) \).

To see that these branches change stability types, observe

\[
\lambda_{1,2}(\mu) = f_x(x_{1,2}(\mu), \mu) = b \mu + c \mu z_{1,2}(\mu) + O(\mu^2), \quad \text{from (5)}
\]

\[
= b \mu + c \mu \gamma_{1,2} + O(\mu^2) \quad \text{since } z_{1,2}(\mu) = \gamma_{1,2} + \pi_{1,2} \mu + O(\mu^2)
\]

\[
= \pm \mu \sqrt{-D} + O(\mu^2)
\]

Therefore, these two branches have opposite stability types and exchange stability at the bifurcation point; namely, this is a transcritical bifurcation.

Next consider the case \( c = f_{xx}(0, 0) = 0 \) (together with \( \alpha = f_\mu(0, 0) = 0 \) and \( D < 0 \)). In this case

\[
g(z, \mu) = \frac{1}{2}(a + 2b z) + O(\mu).
\]

Therefore \( g(z, 0) = 0 \) has exactly one solution \( z = -\frac{a}{2b} \) (\( b \neq 0 \) since \( D = ac - b^2 < 0 \) and \( c = 0 \)), and \( g_\iota(-\frac{a}{2b}, 0) = b \neq 0 \). The IFT implies that there exists a unique analytic solution \( z = z(\mu) \) of \( g(z, \mu) = 0, z(0) = -\frac{a}{2b} \). Hence, there is a branch of critical points of the form \( x = x(\mu) = \mu z(\mu) \), with \(-\frac{a}{2b}\) as its slope at \( \mu = 0 \). From (4), we have \( \lambda(\mu) = b \mu + O(\mu^2) \). So this branch changes stability type at the bifurcation point. To obtain the second branch, let

\[
\mu = x \omega, \quad \text{and} \quad h(x, \omega) = \frac{f(x, x \omega)}{x^2}.
\]

Then

\[
h(x, \omega) = \frac{1}{2}(a \omega^2 + 2b \omega) + O(x),
\]

and \( h(0, 0) = 0 \) and \( h_\omega(0, 0) = b \neq 0 \). The IFT implies that \( h(x, \omega) = 0 \) has a unique analytic solution \( \omega = \omega(x) \) with \( \omega(0) = 0 \). Therefore, there is a second branch of critical points \( \mu = \mu(x) = x \omega(x) = \beta x^2 + O(x^3) \). For the stability of this branch, observe that, from (4) (with \( c = 0 \)),

\[
\lambda_1 = b \beta x^2 + O(x^3).
\]

Therefore, this branch is stable if \( b \beta < 0 \) (supercritical pitchfork) and unstable if \( b \beta > 0 \) (subcritical pitchfork).
If $\beta = 0$, then the sign of the first nonzero term in $\mu(x)$ will determine the sign of $\lambda_1$. Therefore, in case (b), the bifurcation is always transcritical or pitchfork.

**Example.** Consider
\[ x' = x(\mu - x^2)(x - \mu + 2). \] (7)
The critical points are $x = 0$, $x^2 = \mu$, and $x = \mu - 2$. Bifurcation points are (i) $(0, 0)$, (ii) $(2, 0)$, (iii) $(1, -1)$, and (iv) $(4, 2)$. Also,
\[ f_x(x, \mu) = (\mu - x^2)(x - \mu - 2) - 2x^2(x - \mu - 2) + x(\mu - x^2), \]
and thus
\[
\lambda_1 = \begin{cases} 
\mu(2 - \mu), & \text{if } x = 0, \\
-2x^2(2 + x - x^2), & \text{if } x^2 = \mu, \\
x(2 + x - x^2), & \text{if } x = \mu - 2,
\end{cases}
\]

Therefore, a supercritical pitchfork at (i), and transcritical at (ii), (iii), and (iv).

### 5.3. One-dimensional bifurcations $(n = 2)$

Consider a two-dimensional system
\[
\begin{align*}
x' &= f(x, y, \mu) \\
y' &= g(x, y, \mu),
\end{align*}
\] (8)
where $f, g$ are analytic in $(x, y, \mu)$. Suppose that $(x, y, \mu) = (0, 0, 0)$ is a bifurcation point. Rewrite system (8) in the following form
\[
\begin{bmatrix}
x' \\ y'
\end{bmatrix} = \begin{bmatrix} a(\mu) \\ b(\mu) \end{bmatrix} + A_0(\mu) \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} F(x, y, \mu) \\ G(x, y, \mu) \end{bmatrix},
\] (9)
where $F, G = O(r^2)$, as $r \to \infty$, $r^2 = x^2 + y^2$. Let $\gamma_1(\mu)$ and $\gamma_2(\mu)$ be the eigenvalues of $A_0(\mu)$. Then, the assumption that $(0, 0, 0)$ is a bifurcation point implies that $a(0) = b(0) = 0$, $\gamma_1(0) = 0$, and $\gamma_2(0) < 0$ (or $> 0$). Therefore, near $\mu = 0$, $\gamma_1(\mu)$ and $\gamma_2(\mu)$ are analytic, $\gamma_2(\mu) < 0$ (or $> 0$). W.o.l.g., we assume
\[
A_0(\mu) = \begin{bmatrix} \gamma_1(\mu) & 0 \\ 0 & \gamma_2(\mu) \end{bmatrix},
\]
and thus
\[
\begin{align*}
f(x, y, \mu) &= a(\mu) + \gamma_1(\mu)x + F(x, y, \mu), \\
g(x, y, \mu) &= b(\mu) + \gamma_2(\mu)y + G(x, y, \mu),
\end{align*}
\]
with \( g(0,0,0) = 0, g_y(0,0,0) = \gamma_2(0) \neq 0 \). Therefore \( g(x,y,\mu) = 0 \) has a unique branch of solutions \( y = Y(x,\mu) \) analytic in \( (x,\mu) \). Furthermore

\[
Y(x,\mu) = -\frac{b'(0)}{\gamma_2(0)} \mu + O(x^2, \mu x, \mu^2).
\]

(Note: \( Y_x(0,0) = 0 \).) Substitute \( Y(x,\mu) \) into \( f(x,y,\mu) \) and let

\[
\hat{f}(x,\mu) = f(x,Y(x,\mu),\mu) = a(\mu) + \gamma_1(\mu)x + F(x,Y(x,\mu),\mu),
\]

we have

\[
\hat{f}_x = f_x + f_y Y_x = \gamma_1(\mu) + F_x + F_y Y_x, \\
\hat{f}_\mu = f_\mu + f_y Y_\mu = a'(\mu) + \gamma'_1(\mu)x + F_\mu + F_y Y_\mu.
\]

Therefore, \( \hat{f}(0,0) = \hat{f}_x(0,0) = 0 \). This is similar to the case of \( n = 1 \). Let

\[
D = \begin{vmatrix} \hat{f}_{x\mu} & \hat{f}_{x\mu} \\ \hat{f}_{x\mu} & \hat{f}_{xx} \end{vmatrix}.
\]

Now, the equilibria of the system lie on the surface \( x = X(\mu), y = Y(x,\mu) \), where \( X(\mu) \) is any solution of \( \hat{f}(x,\mu) = 0 \). The determination of the number of branches of solutions \( x = X(\mu) \) is exactly the same as in the Theorem 5.1, replacing \( f(x,\mu) \) by \( \hat{f}(x,\mu) \). The stability of an equilibrium is determined from the eigenvalues \( \lambda_1(\mu), \lambda_2(\mu) \) of \( A(\mu) \), the linearization about the equilibrium.

(Note: \( A(\mu) = A_0(\mu) \) if the equilibrium is \( (0,0,0) \), but not necessarily otherwise.)

Since \( \lambda_2(0) = \gamma_2(0) < 0 \), we have \( \lambda_2(\mu) < 0 \) for \( \mu \) near \( 0 \) : the stability character is therefore determined by \( \lambda_1(\mu) \). Observe

\[
\lambda_1 \lambda_2 = f_x g_y - f_y g_x \quad (x = X(\mu), \ y = Y(x,\mu)),
\]

and that \( g(x,Y,\mu) = 0 \) implies \( g_x + g_y Y_x = 0 \), we have

\[
\lambda_1 \lambda_2 = g_y(f_x + f_y Y_x) = g_y \hat{f}_x.
\]

But \( g_y(0,0,0) = \gamma_2(0) = \lambda_2(0) < 0 \), we thus have

\[
\lambda_1 = \frac{\hat{f}_x}{1 + O(\mu,x)}.
\]

Therefore, near \( (0,0) \) the sign of \( \lambda_1 \) is the same as that of \( \hat{f}_x \). We thus have the following result.

**Theorem 5.2** Suppose that \( a(0) = b(0) = 0, \ \gamma_1(0) = 0, \ \text{and} \ \gamma_2(0) < 0 \). Then

1. If \( a'(0) \neq 0, \ \hat{f}_{xx}(0,0) \neq 0 \), there exists near \( (0,0,0) \) a single branch of critical points which has a saddle-node bifurcation at \( (0,0,0) \).
2. If \( a'(0) = 0 \ \text{and} \ D > 0 \), the equilibrium at \( (0,0,0) \) is isolated.
3. If \( a'(0) = 0 \ \text{and} \ D < 0 \), there are two branches of critical points which intersect at \( (0,0,0) \). The bifurcation is either transcritical and pitchfork.
Example. Consider a Lotka-Volterra food-chain model

\[
x' = x(\mu - x) - (x + 1)y^2 \\
y' = y(x - 1).
\] (10)

The equilibria are

1. \(x = 0, \ y = 0\)
2. \(x = \mu, \ y = 0\)
3. \(x = 1, \ 2y^2 = \mu - 1\).

We have

\[
A(\mu) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} \mu - 2x - y^2 & -2(x + 1)y \\ y & x - 1 \end{bmatrix}.
\]

At an equilibrium in (1),

\[
A(\mu) = \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix},
\]

and thus it is stable if \(\mu < 0\) and unstable for \(\mu > 0\).

At an equilibrium in (2),

\[
A(\mu) = \begin{bmatrix} -\mu & 0 \\ 0 & \mu - 1 \end{bmatrix},
\]

and thus it is stable for \(0 < \mu < 1\) and unstable for \(\mu < 0\) and \(\mu > 1\).

At an equilibrium in (3),

\[
A(\mu) = \begin{bmatrix} y^2 - 1 & -4y \\ y & 0 \end{bmatrix}, \ 2y^2 = \mu - 1.
\]

Its eigenvalues are

\[
\frac{1}{2} [y^2 - 1 \pm \sqrt{(y^2 - 9)^2 - 80}] = \frac{1}{4} [(\mu - 3) \pm \sqrt{(\mu - 19)^2 - 320}].
\]

This branch in the plane \(x = 1\) exists only in \(\mu \geq 1\) and it is stable if \(1 < \mu < 3\), and unstable for \(\mu > 3\).

In summary, we see that branches (1) and (2) undergo a transcritical bifurcation at \((0,0,0)\).

branches (2) and (3) undergo a supercritical pitchfork bifurcation at \((1,0,1)\).

5.4. Hopf Bifurcation \((n \geq 2)\)

We first examine an example.

Example. Consider

\[
x' = -y + x(\mu - x^2 - y^2) \\
y' = x + y(\mu - x^2 - y^2).
\] (11)
The only equilibrium lies at \((0, 0, \mu)\), at which
\[
A(\mu) = \begin{bmatrix}
\mu & -1 \\
1 & \mu \\
\end{bmatrix}.
\]
Thus, it is stable if \(\mu < 0\) and unstable if \(\mu > 0\). In polar coordinates, the system can be written as
\[
\begin{align*}
r' &= r(\mu - r^2) \\
\theta' &= 1.
\end{align*}
\]
We see that \(r^2 = \mu\) gives rise a one-parameter family of stable limit cycles.

As \(\mu\) increases through 0, the equilibrium changes its stability character and a periodic limit cycle bifurcates from the equilibrium. This phenomenon is called a Hopf bifurcation.

In general, we consider a 2-dimensional analytic system
\[
\begin{align*}
x' &= f(x, y, \mu) \\
y' &= g(x, y, \mu).
\end{align*}
\]
(12)
The equilibria \(x = x_0(\mu), y = y_0(\mu)\) satisfy
\[
f(x_0(\mu), y_0(\mu)) = g(x_0(\mu), y_0(\mu)) = 0.
\]
with Jacobian matrix
\[
A(\mu) = \begin{bmatrix}
f_x & f_y \\
g_x & g_y \\
\end{bmatrix}_{(x_0(\mu), y_0(\mu), \mu)},
\]
whose eigenvalues are \(\lambda_1(\mu), \lambda_2(\mu)\) such that
\[
\lambda_1(\mu) = \lambda_2(\mu) = \alpha(\mu) + i \beta(\mu).
\]
We assume that \(\alpha(\mu)\) changes sign at \(\mu = 0\) so that \(\mu = 0\) is a bifurcation value.
\[
\det(A(\mu)) = \alpha^2(\mu) + \beta^2(\mu) \neq 0
\]
near \(\mu = 0\). Change variables from \(x\) to \(x + x_0(\mu)\) and \(y\) to \(y + y_0(\mu)\), we obtain
\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = A(\mu) \begin{bmatrix}
x \\
y
\end{bmatrix} + \begin{bmatrix}
F(x, y, \mu) \\
G(x, y, \mu)
\end{bmatrix},
\]
(13)
the equilibrium remains at \((0, 0, \mu)\), and \(F, G = O(r^2)\), as \(r = \sqrt{x^2 + y^2} \to 0\). W.l.o.g., we assume
\[
A(\mu) = \begin{bmatrix}
\alpha(\mu) & \beta(\mu) \\
-\beta(\mu) & \alpha(\mu)
\end{bmatrix}.
\]
Therefore, \((0, 0, \mu)\) is a focus which changes stability character as \(\mu\) passes through 0.
Theorem 5.3 (Poincaré-Andronov-Hopf Bifurcation Theorem) Suppose

(i) The system \((13)\) has an equilibrium at \((0, 0)\) if \(\mu\) is near 0.

(ii) The eigenvalues \(\lambda_{1,2}(\mu) = \alpha(\mu) \pm i \beta(\mu)\) of \(A(\mu)\) satisfies \(\alpha(0) = 0, \beta(0) \neq 0\).

(iii) \(\alpha'(0) > 0\).

(iv) The equilibrium at \((0, 0)\) is asymptotically stable when \(\mu = 0\).

Then, as \(\mu\) increases through 0, the stable equilibrium at \((0, 0)\) bifurcates to an unstable equilibrium at \((0, 0)\) surrounded by a stable limit cycle. Furthermore, the period of the limit cycle is

\[
\frac{2\pi}{\beta(0)} + O(\mu), \quad \mu \to 0.
\]

Remark. The bifurcation stated in Theorem 5.3 is called a supercritical Hopf bifurcation. If we change the assumption (iv) of the theorem to

(iv’) The equilibrium at \((0, 0)\) is negatively asymptotically stable (as \(t \to -\infty\)).

then the conclusion will be: as \(\mu\) decreases through 0, the unstable equilibrium at \((0, 0)\) bifurcates into a stable equilibrium at \((0, 0)\) surrounded by an unstable limit cycle, of period \(\frac{2\pi}{|\beta(0)|} + O(\mu)\). In this case, the bifurcation is called a subcritical Hopf bifurcation. [See, for example, D. K. Arrowsmith and C. M. Mace, An Introduction to Dynamical Systems, Cambridge University Press, 1990.]

Proof. We will rewrite \((13)\) in complex form

\[
z' = (\alpha(\mu) - i \beta(\mu)) z + N(z, \bar{z}, \mu),
\]

namely

\[
z' = \lambda z + N(z, \bar{z}, \mu),
\]

where \(z = x + iy, \bar{z} = x - iy, \lambda(\mu) = \alpha(\mu) - i \beta(\mu), \) and \(N(z, \bar{z}, \mu) = F(x, y, \mu) + i G(x, y, \mu).\)

Therefore, \(x = (z + \bar{z})/2, y = (z - \bar{z})/(2i), \) and \(N(z, \bar{z}, \mu) = O(|z|^2)\).

Proposition 5.4 There exists a ‘near-identity’ analytic transformation

\[
S = z + S(z, \bar{z}, \mu),
\]

with \(S = O(|z|^2), \) as \(|z| \to 0, \) so that the system takes the normal form

\[
S' = (\alpha(\mu) - i \beta(\mu)) S + (\gamma(\mu) + i \delta(\mu)) |S|^2 S + O(|S|^4),
\]

where \(\gamma(\mu), \delta(\mu)\) are analytic in \(\mu.\)
Proof of Proposition 5.4.

\[ N(z, \bar{z}, \mu) = \frac{1}{2} n_1 z^2 + n_2 z \bar{z} + \frac{1}{2} n_3 \bar{z}^2 + O(|z|^3). \]

Let

\[ w = z + Q(z, \bar{z}, \mu), \]

where

\[ Q(z, \bar{z}, \mu) = \frac{1}{2} q_1 z^2 + q_2 z \bar{z} + \frac{1}{2} q_3 \bar{z}^2, \]

and \( q_1, q_2, \) and \( q_3 \) are to be determined so that all quadratic terms in the transformed equation are 0. The inverse transformation is

\[ z = w - Q(w, \bar{w}, \mu) + O(|w|^3). \] (15)

We thus have

\[ w' = z' + (q_1 z + q_2 \bar{z}) z' + (q_2 z + q_3 \bar{z}) \bar{z}', \]

\[ = (\alpha - i \beta)[w - \frac{1}{2} q_1 w^2 - q_2 w \bar{w} - \frac{1}{2} q_3 \bar{w}^2 + O(|w|^3)] \]

\[ + \frac{1}{2} n_1 w^2 + n_2 w \bar{w} + \frac{1}{2} n_3 \bar{w}^2 + O(|w|^3) \]

\[ + (q_1 w + q_2 \bar{w})(\alpha - i \beta)w + (q_2 w + q_3 \bar{w})(\alpha + i \beta)\bar{w} + O(|w|^3). \] (from (15))

Therefore

\[ w' = (\alpha - i \beta)w + \frac{1}{2} \hat{n}_1 w^2 + \hat{n}_2 w \bar{w} + \frac{1}{2} \hat{n}_3 \bar{w}^2 + O(|w|^3), \]

where

\[ \hat{n}_1 = n_1 + (\alpha - i \beta)q_1, \quad \hat{n}_2 = n_2 + (\alpha + i \beta)q_2, \quad \hat{n}_3 = n_3 + (\alpha + 3i \beta)q_3. \]

Since \( \alpha(0) = 0 \) but \( \beta(0) \neq 0 \), near \( \mu = 0 \), analytic \( q_1, q_2, \) and \( q_3 \) may be chosen such that \( \hat{n}_1 = \hat{n}_2 = \hat{n}_3 = 0 \). Therefore

\[ w' = (\alpha - i \beta)w + M(w, \bar{w}, \mu), \]

and \( M(w, \bar{w}, \mu) = O(|w|^3) \) near \( \mu = 0 \).

Next, we will consider another “near-identity” transformation to get rid of most of the cubic terms. Let

\[ M(w, \bar{w}) = \frac{1}{3} m_1 w^3 + m_2 w^2 \bar{w} + m_3 w \bar{w}^2 + \frac{1}{3} m_4 \bar{w}^3, \]

and set

\[ S = w + R(w, \bar{w}, \mu), \]

where

\[ R(w, \bar{w}, \mu) = \frac{1}{3} r_1 w^3 + r_2 w^2 \bar{w} + r_3 w \bar{w}^2 + \frac{1}{3} r_4 \bar{w}^3. \]

Then

\[ S' = w' + (r_1 w^2 + 2r_2 w \bar{w} + r_3 \bar{w}^2)w' + (r_2 w^2 + 2r_3 w \bar{w} + r_4 \bar{w}^2)\bar{w}', \]
and thus
\[ S' = (\alpha - i \beta)[S - \frac{1}{3}r_2 S^3 - r_2 S^2 \mathcal{S} - r_3 \mathcal{S}^2 - \frac{1}{3}r_4 \mathcal{S}^3 + O(|S|^4)] \]
\[ + \frac{1}{3} m_1 S^3 + m_2 S^2 \mathcal{S} + m_3 \mathcal{S}^2 + \frac{1}{3} m_4 S^3 + O(|S|^4) \]
\[ + (r_1 S^2 + 2r_2 S \mathcal{S} + r_3 \mathcal{S}^2)(\alpha - i \beta) S \]
\[ + (r_2 S^2 + 2r_3 S \mathcal{S} + r_4 \mathcal{S}^2)(\alpha + i \beta) \mathcal{S} + O(|S|^4). \]

Therefore
\[ S' = (\alpha - i \beta)S + \frac{1}{3} \hat{m}_1 S^3 + \hat{m}_2 S^2 \mathcal{S} + \hat{m}_3 \mathcal{S}^2 + \frac{1}{3} \hat{m}_4 S^3 + O(|S|^4), \]
where
\[ \hat{m}_1 = m_1 + 2(\alpha - i \beta)r_1, \quad \hat{m}_2 = m_2 + 2\alpha r_2, \quad \hat{m}_3 = m_3 + 2(\alpha + i \beta)r_3, \quad \hat{m}_4 = m_4 + 2(\alpha + 2i \beta)r_4. \]

Again, since \( \alpha(0) = 0, \beta(0) \neq 0 \) we can choose \( r_1, r_2, \) and \( r_3 \) such that \( \hat{m}_1 = \hat{m}_3 = \hat{m}_4 = 0, \)
and that
\[ S' = (\alpha - i \beta)S + m_2 |S|^2 S + O(|S|^4), \quad (16) \]
with \( m_2 = \gamma + i \delta. \)

Remarks.

1. Linearized equation at \( S = 0, \mu = 0 \) is
\[ z' = (\alpha(0) - i \beta(0))z = -i\beta(0)z, \]
whose solution is
\[ z(t) = z(0)e^{-i\beta(0)t}. \]

2. \( m_2 \) is given by the formula
\[ m_2 = n_3 - \frac{n_1 n_2}{\alpha - i \beta} - \frac{n_1 n_2}{2(\alpha + i \beta)} - \frac{|n_2|^2}{\alpha + i \beta} - \frac{|n_3|^2}{2(\alpha + 3i \beta)} = \gamma + i \delta. \]

3. \( 16\gamma(0) = [(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + \frac{1}{\beta}(f_{xy}(f_{xx} + f_{yy})) \]
\[ - g_{xy}[g_{xx} + g_{yy}] - f_{xx}g_{xx} + f_{yy}g_{yy}] \bigg|_{(x,y)=(0,0)}. \]

4. Write \( z = re^{i\theta} \), and \( S = Re^{i\phi} \). Then \( S = z + O(|z|^2) \) can be written as
\[ R = r + O(r^2) \]
\[ \phi = \theta + O(r). \]
Cubic approximation of system (16) is
\[ R' = \alpha(\mu)R + \gamma(\mu)R^3, \]
\[ \phi' = -\beta(\mu) + \delta(\mu)R^2. \]
If $\gamma(0) \neq 0$, then there exist two branches of critical points for the $R$ equation:

$$R = 0, \quad R^2 = -\frac{\alpha(\mu)}{\gamma(\mu)} = -\frac{\alpha'(0)}{\gamma(0)} \mu + O(\mu^2).$$

$\alpha(0) = 0, \alpha'(0) \neq 0$ means stability character of $0$ equilibrium changes as $\mu$ passes through $0$. There is a pitchfork bifurcation at $R = 0, \mu = 0$.

(5) If $\gamma(0) = 0$, we can continue the procedure to obtain the approximation

$$R' = \alpha(\mu) R + \gamma(\mu) R^2 \mu + O(R^3),$$
$$\phi' = -\beta(\mu) + \delta(\mu) R^2.$$

The sign of $\lambda_1(\mu) = \alpha(\mu)$ determines the stability of $R = 0$. The sign of

$$\lambda_1(\mu) = \left. \frac{\partial}{\partial R} (\alpha R + r R^3) \right|_{R^2 = \alpha(\mu)/\gamma(\mu)} = -2\alpha(\mu)$$

determines the stability of the other equilibrium.

(6)

$$\phi(t) = \phi(0) - (\beta(\mu) + \delta(\mu) \frac{\alpha(\mu)}{\gamma(\mu)} ) t = \phi_0 + (w_0 + \sigma(1)) t,$$

where $w_0 = -\beta(0)$.

---

Proof of P-A-H Theorem ($\gamma(0) \neq 0$). From (16), we have

$$R' = \alpha(\mu) R + \gamma(\mu) R^2 \mu + O(R^3),$$
$$\phi' = -\beta(\mu) + \delta(\mu) R^2 + O(R^3).$$

Let $R = \epsilon \rho, \mu = \epsilon^2 \nu$, with $\epsilon$ being a small parameter, and $w_0 = -\beta(0)$. Then

$$\frac{d \rho}{d \phi} = \epsilon^2 H(\rho, \phi, \epsilon),$$

where

$$w_0 H(\rho, \phi, \epsilon) = \alpha'(0) \nu \rho + \gamma(0) \rho^2 + O(\epsilon).$$

Now

$$\rho(\phi) = \rho_0 + \epsilon^2 \int_0^\phi H(\rho(s), s, \epsilon) ds, \quad \rho_0 = \rho(0),$$
is periodic if and only if 
\[ \rho(2\pi, \rho_0, \epsilon) = \rho_0 \]
if and only if 
\[ P(\rho_0, \epsilon) := \int_0^{2\pi} H(\rho(s), s, \epsilon) \, ds = 0. \]
Observe the following
\[ \lim_{\epsilon \to 0^+} P(\rho_0, \epsilon) = P_0(\rho_0) := \frac{2\pi}{w_0} [\alpha'(0)\nu \rho_0 + \gamma(0)\rho_0^3], \]
and \( P_0(\rho_0) = 0 \) if and only if \( \rho_0 = 0 \) or \( \rho_0^2 = -\frac{\alpha'(0)\nu}{\gamma(0)} \), and
\[ \frac{\partial P}{\partial \rho_0}(\rho_0, 0) = \frac{2\pi}{w_0} [-2\alpha'(0)\nu] \neq 0 \]
if \( \nu \neq 0 \). Applying the IFT, we obtain that there exists a unique \( \rho_0 = \rho_0(\epsilon) \), \( \rho_0(0)^2 = -\frac{\alpha'(0)\nu}{\gamma(0)} \) such that \( P(\rho_0(\epsilon), \epsilon) = 0 \), and thus \( \rho_0 = \rho_0(\epsilon) \) gives rise to a family of periodic solutions, whose period is obtained from the equation for \( \phi \).
\[ \square \]

**Example.** Consider
\[ \begin{align*}
x' &= \mu x + y \\
y' &= -x + \mu y - x^2 y
\end{align*} \]
The complex form is
\[ z' = (\mu - i)z - \frac{1}{8}(z^3 + z^2 \overline{z} - z \overline{z}^2 - \overline{z}^3). \]
Since the quadratic terms are already zero, there is a transformation \( S = z + R(z, \overline{z}, \mu) \) which changes the equation to normal form
\[ S' = (\mu - i)S - \frac{1}{8}|S|^2 S + O(|S|^4), \]
or in polar form
\[ \begin{align*}
R' &= \mu R - \frac{1}{8}R^3 + O(R^4), \\
\phi' &= -1 + O(R^2).
\end{align*} \]
Therefore, there exists a unique family of periodic solutions (through Hopf bifurcation)
\[ R^2 = 8\mu + O(\mu^2). \]

**Remark.** A somewhat technical further development of the proof of the P-A-H Theorem leads to the result that (when \( \alpha'(0) > 0 \) \( R'(\mu) > 0 \) implies the periodic orbit is stable and \( R'(\mu) < 0 \)
implies it is unstable [see J. K. Hale and H. Koçak, Dynamics and Bifurcations, Springer-Verlag, 1991, page 353]. The periodic solution branch may thus undergo further bifurcation.

**Example.** Consider
\[
\begin{align*}
x' &= y + x(\mu + cr^2 - r^4) \\
y' &= -x + y(\mu + cr^2 - r^4),
\end{align*}
\]
where \( r^2 = x^2 + y^2 \). In polar coordinates, we have
\[
\begin{align*}
r' &= r(\mu + cr^2 - r^4) \\
\theta' &= 1.
\end{align*}
\]
When \( c > 0 \), the branch \( \mu = -cr^2 + r^4 \) undergoes a saddle-node bifurcation when \( \mu = -\frac{3c^2}{16} \).