Lecture Notes for Math 524

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Chapter 1. Existence and Uniqueness Theorems

1.1. Initial value problems

Let D be an open set in $\mathbf{R}^{n+1}$, and $f \in C(D \to \mathbf{R}^n)$. Consider an ODE in $\mathbf{R}^n$

$$x' = f(t, x). \tag{1}$$

A *solution* to (1) in an interval $\mathcal{I} \subset \mathbf{R}$ is a differentiable function $\varphi : \mathcal{I} \to \mathbf{R}^n$ such that

$$\varphi'(t) = f(t, \varphi(t)).$$

Example 1. The equation $x' = 1$ has solutions defined in $\mathbf{R}$ of form

$$x(t) = t + c,$$

where c is an arbitrary constant. In general, an ODE has infinitely many solutions.

We would like to solve equation (1) subject to conditions

$$x(t_0) = x_0 \tag{2}$$

for $(t_0, x_0) \in D$. Equations (1) and (2) are called an **Initial Value Problem**, an IVP for short.

In this chapter, we investigate the following two questions:

1. Existence of solutions to the IVP (1)-(2).
2. Uniqueness of solutions to the IVP (1)-(2).

We first examine some examples.

Example 2. The IVP $x' = x,\ x(t_0) = x_0$ has solution

$$x(t) \;=\; x_0\, e^{(t-t_0)}$$

which exists for all $t \in \mathbf{R}$ and is unique.

Example 3. The ODE $x' = x^2$ has a solution

$$x(t) \;=\; -\frac{1}{t+c},$$

where c can be uniquely determined by the initial condition $x(t_0) = x_0$, for $x_0 \neq 0$. This solution only exists near t_0 and is unique as long as it exists. When $x_0 = 0$ the IVP has a unique solution $x(t) = 0$.

Example 4. Consider the IVP $x' = |x|^{1/2},\ x(t_0) = x_0$. For $x_0 > 0$, we have

$$\frac{x'}{x^{1/2}} = 1, \quad 2x^{1/2} = t + c, \quad x(t) = \frac{1}{4}(t+c)^2,$$

where $c = 2x_0^{1/2} - t_0$. This solution exists in $\mathbf{R}$. When $x_0 = 0$, we have $x(t) = 0$ for all t. Solutions to the IVP are not unique over $\mathbf{R}$. However a solution is locally unique if $x_0 \neq 0$.

1.2. An equivalent problem

The IVP (1)-(2) is equivalent to the following integral equation

$$x(t) \;=\; x_0 + \int_{t_0}^{t} f(s, x(s))\, ds. \tag{3}$$

Namely, the IVP and the integral equation has the same set of solutions.

The following useful result is a version of Gronwall's lemma.

Lemma 1.1 *Let $g(t)$ be a continuous real-valued function and $g(t) \geq 0$ for $t \in [t_0, b]$. If, for $K > 0, C \geq 0$,*

$$g(t) \leq C + K \int_{t_0}^{t} g(s)\, ds$$

for $t \in [t_0, b]$, then

$$g(t) \leq C e^{K(t-t_0)} \tag{4}$$

for $t \in [t_0, b]$.

Proof. Let $G(t) = C + K \int_{t_0}^{t} g(s) ds$. Then $G(t) \geq g(t)$ and $G(t) \geq 0$ for all $t \in [t_0, b]$. Moreover,

$$G'(t) = K g(t) \leq K G(t).$$

Thus

$$G'(t) - KG(t) \leq 0$$

and

$$G'(t)e^{-Kt} - G(t)Ke^{-Kt} \leq 0,$$

namely

$$[G(t)e^{-Kt}]' \leq 0,$$

and thus $G(t)e^{-Kt} \leq G(t_0)e^{-Kt_0} = Ce^{-Kt_0}$, it follows $g(t) \leq G(t) \leq Ce^{K(t-t_0)}$. □

1.3. Uniqueness under Lipschitz conditions

A function $f \in C(D \to \mathbf{R}^n)$ is said to satisfy *Lipschitz condition* with respect to x if there exists (Lipschitz) constant $K > 0$, such that

$$|f(t, x_1) - f(t, x_2)| \leq K |x_1 - x_2| \tag{5}$$

for all (t, x_1), $(t, x_2) \in D$. Here, $|\cdot|$ denote any norm in $\mathbf{R}^n$.

Example 5. $f(x) = \sin x$ is Lipschitz continuous on $(-\infty, +\infty)$.

Example 6. $f(x) = x^2$ is not globally Lipschitz continuous over $\mathbf{R}$. It is, however, Lipschitz continuous over any compact subset of $\mathbf{R}$. In this case, we say it is *locally* Lipschitz continuous over $\mathbf{R}$.

Example 7. $f(x) = \frac{1}{x^2}$ is globally Lipschitz continuous on $[\alpha, \infty)$ for any $\alpha > 0$, but only locally Lipschitz continuous on $(0, \infty)$.

Theorem 1.2 (Uniqueness) *Suppose*

(1) $f \in C(D \to \mathbf{R}^n)$.

(2) f *satisfies Lipschitz condition in* D *with respect to* x.

Then the IVP has at most one solution in D.

More specifically, if there are two solutions $\varphi_1(t)$ and $\varphi_2(t)$ to the IVP, both defined for $t \in [a, b]$, such that $(t, \varphi_i(t)) \in D$ for $t \in [a, b]$ and $i = 1, 2$, then $\varphi_1(t) = \varphi_2(t)$ for all $t \in [a, b]$.

Proof. (Proof # 1) Suppose $\varphi_1(t)$ and $\varphi_2(t)$ are both solutions to the IVP. Then

$$\varphi_i(t) = x_0 + \int_{t_0}^{t} f(s, \varphi_i(s))\, ds, \quad i = 1, 2.$$

and thus

$$\begin{array}{rcl} |\varphi_1(t) - \varphi_2(t)| & = & |\int_{t_0}^{t} [f(s, \varphi_2(s)) - f(s, \varphi_2(s))]\, ds| \\ & \leq & \int_{t_0}^{t} |f(s, \varphi_2(s)) - f(s, \varphi_2(s))|\, ds \quad (t \geq t_0) \\ & \leq & \int_{t_0}^{t} K\, |\varphi_1(s) - \varphi_2(s)|\, ds \quad \text{(Lipschitz condition)} \end{array} \tag{6}$$

Set $l(t) = |\varphi_1(t) - \varphi_2(t)| \geq 0$. Then, if $t \geq t_0$,

$$l(t) \leq \int_{t_0}^{t} Kl(s)\, ds,$$

which implies $l(t) \leq 0$ for $t \geq t_0$ by the Gronwall's lemma. Since $l(t_0) \geq 0$, we have $l(t) = 0$ for all $t \geq t_0$. A similar argument can show $l(t) = 0$ for all $t \leq t_0$. □

Proof. (Proof # 2) Set $\delta = max_{t\in[t_0,b]}|\varphi_1(t) - \varphi_2(t)|$. From (6), we have

$$|\varphi_1(t) - \varphi_2(t)| \leq \int_{t_0}^{t} K\delta\, ds = K\delta(t - t_0).$$

Using (6) again, we get

$$|\varphi_1(t) - \varphi_2(t)| \leq \int_{t_0}^{t} K^2\delta(s - t_0)\, ds = \delta K^2(t - t_0)^2/2.$$

Repeat the process, we get, for any integer n,

$$|\varphi_1(t) - \varphi_2(t)| \leq \delta K^n(t - t_0)^n/n! \to 0 \quad \text{as } n \to \infty.$$

This implies $\varphi_1(t) = \varphi_2(t)$ for $t \in [t_0, b]$. (Proof for $t \in [a, t_0]$ is similar.) □

1.4. Picard's Method of Successive Approximation and local existence under Lipschitz conditions

We prove the following result.

Theorem 1.3 (Picard Local Existence Theorem) *Suppose*

(1) $f \in C(D \to \mathbf{R}^n)$.

(2) f *satisfies Lipschitz condition in* D *with respect to* x.

Then there exists $\alpha > 0$ *such that the IVP has a solution for* $t \in [t_0 - \alpha, t_0 + \alpha]$.

Proof. Let $(t_0, x_0) \in D$. Then there exists $a_1, a_2 > 0$ such that the rectangle $R = \{(t, x) : |t - t_0| < a_1, |x - x_0| < a_2\} \subset D$ (D is open). Let

$$M = \max\{|f(t, x)| \quad : \quad (t, x) \in R\}.$$

We will show that $\alpha = \min\{a_1, \frac{a_2}{M}\}$ satisfies the conclusion in the theorem. The following method is called Picard Method of Successive Approximation.

Consider the equivalent integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds. \tag{7}$$

Step 1: Set up the successive approximation

$$\begin{array}{rcl} x_0(t) & = & x_0 \\ x_{n+1}(t) & = & x_0 + \int_{t_0}^{t} f(s, x_n(s))\, ds, \quad n = 0, 1, 2, \cdot. \end{array} \tag{8}$$

Step 2: Verify (use induction) that the sequence $\{x_n(t)\}$ are well-defined for $|t - t_0| < \alpha$, and that $(t, x_n(t)) \in R$.

$$|x_1(t) - x_0| \leq \int_{t_0}^{t} |f(s, x_0)| ds \leq M|t - t_0| \leq a_2,$$

$$|x_n(t) - x_0| \leq \int_{t_0}^{t} |f(s, x_{n-1}(s))|\, ds \leq M|t - t_0| \leq a_2, \quad n = 2, 3, \cdots,$$

if $|t - t_0| \leq a_2/M$.

Step 3: Show that $\{x_n(t)\}$ is a Cauchy sequence for $|t - t_0| \leq \alpha$.

$$\begin{array}{rcl} |x_{n+1}(t) - x_n(t)| & \leq & \left| \int_{t_0}^{t} |f(s, x_n(s)) - f(s, x_{n-1}(s))|\, ds \right| \\ \text{(Lipschitz)} & \leq & K \left| \int_{t_0}^{t} |x_n(s) - x_{n-1}(s)|\, ds \right|, \quad n = 1, 2, ... \end{array}$$

and

$$|x_1(t) - x_0(t)| = |\int_{t_0}^{t} f(s, x_0)\, ds| \leq M|t - t_0|, \quad |t - t_0| \leq \alpha.$$

Thus

$$|x_2(t) - x_1(t)| \leq MK|t - t_0|^2/2!, \quad |t - t_0| \leq \alpha,$$

and

$$|x_3(t) - x_2(t)| \leq MK^2|t - t_0|^3/3!, \quad |t - t_0| \leq \alpha,$$

and, for $n = 1, 2, ...$

$$|x_{n+1}(t) - x_n(t)| \leq \frac{MK^n|t - t_0|^{n+1}}{(n+1)!} \leq \frac{M}{K} \frac{(K\alpha)^{n+1}}{(n+1)!}.$$

Thus, for $0 \leq m \leq n$, we have

$$|x_n(t) - x_m(t)| \leq \frac{M}{K} \sum_{k=m}^{n-1} \frac{(K\alpha)^{k+1}}{(k+1)!}, \quad |t - t_0| \leq \alpha. \tag{9}$$

Step 4: The solution to the IVP is obtained as the uniform limit of $\{x_n(t)\}$.

Cauchy's criterion and (9) imply that

$$\lim_{t \to \infty} x_n(t) = x(t)$$

uniformly for $t \in [t_0, t_0 + \alpha]$, and $x(t)$ is continuous. In the equation (8), let $t \to \infty$, we verify that $x(t)$ is a solution to the integral equation (7). Thus $x(t)$ is differentiable and satisfies the IVP. □

1.5. The existence and uniqueness under Lipschitz conditions (summary)

Summarize the results in Sections 1.3 and 1.4, we have the following theorem.

Theorem 1.4 *Suppose*

(1) $f \in C(D \to \mathbf{R}^n)$.

(2) f *satisfies Lipschitz condition in* D *with respect to* x.

Then there exists $\alpha > 0$ *such that the IVP has a unique solution for* $t \in [t_0 - \alpha,\, t_0 + \alpha]$.

The following is an abstract version of the proof of Picard's existence and uniqueness theorem. Let $(\mathcal{B}, |\cdot|)$ be a Banach space. Let $V = \{\varphi \in \mathcal{B} \;:\; |\varphi| \le b\}$.

Lemma 1.5 *Suppose that* $T : V \to \mathcal{B}$ *is a transformation that satisfies:*

(a) $|T(\varphi)| \le b$ *for* $\varphi \in V$, *i.e.,* $T(V) \subset V$.

(b) $\exists 0 < K_1 < 1$, *such that* $|T(\varphi) - T(\psi)| \le K_1 |\varphi - \psi|$, *for* $\varphi, \psi \in V$.

Then, there exists a unique $\varphi \in V$ *such that* $T(\varphi) = \varphi$, *namely,* T *has a unique fixed point in* V.

Proof. Exercise. (Hint: use Picard successive approximation.)

The Picard's theorem can now be proved using this fixed point result. Let $\mathcal{B} = C[t_0 - a_1, t_0 + a_1]$. Define T using the integral equation (3) as follows:

$$T(\psi)(t) = x_0 + \int_{t_0}^{t} f(s, \psi(s))\, ds.$$

Choose a_1 small enough so that $a_1 K < 1$, where K is the Lipschitz constant of f in D, then we can show that T satisfies

(a) $|T(\varphi)| \le a_2$ if $|\varphi| \le a_2$,

(b) $|T(\varphi) - T(\psi)| \le a_1 K\, |\varphi - \psi|$, for $|\varphi|,\, |\psi| \le a_2$.

Then by the previous lemma, T has a unique fixed point $\varphi \in \mathcal{B}$ and $x = \varphi(t)$ is a solution to the integral equation (3), and hence a solution to the IVP.

1.6. Peano existence theorem

In Theorem 1.4, if the assumption (2) on the Lipschitz condition is removed, the conclusion still holds, except for the uniqueness.

Theorem 1.6 *Suppose that* $f \in C(D \to \mathbf{R}^n)$. *Then the IVP has a solution which exists on a maximal interval* $(\omega_-,\, \omega_+)$.

Note: ω_- and ω_+ depend on t_0, x_0 in general.

To establish this result, we need some preliminary propositions.

Let $\mathcal{F}$ be a family of real-valued (or $\mathbf{R}^n$ valued) functions on a closed interval $\mathcal{I} \subset \mathbf{R}$.

(a) $\mathcal{F}$ is *uniformly bounded* if there exists $M > 0$ such that

$$|f(t)| \le M, \quad \forall f \in \mathcal{F}, \quad \forall\, t \in \mathcal{I}.$$

(b) $\mathcal{F}$ is *equi-continuous* if for any $\epsilon > 0$, there exists $\delta > 0$ such that $t_1, t_2 \in \mathcal{I}$, $|t_1 - t_2| < \delta$, $f \in \mathcal{F}$ imply

$$|f(t_1) - f(t_2)| < \epsilon.$$

Proposition 1.7 (Arzelà-Ascoli Theorem) *Suppose that the family $\mathcal{F}$ is uniformly bounded and equi-continuous. Then for any sequence $\{f_k\} \subset \mathcal{F}$, there exists a subsequence that is uniformly convergent on $\mathcal{I}$.*

A more abstract way to state the Arzelà-Ascoli Theorem is: a subset F of the Banach space $C[a, b]$ is precompact iff F satisfies properties (a) and (b).

Proof. Let $\{r_1, r_2, \cdots\} = \mathcal{I} \cap \mathbf{Q}$ (rationals in $\mathcal{I}$). Consider the sequence in $\mathbf{R}$:

$$\{f_k(r_1)\} = \{f_1(r_1),\, f_2(r_1),\, f_3(r_1), \cdots\}. \tag{10}$$

It is bounded and therefore has a convergent subsequence:

$$\{f_{11}(r_1),\, f_{12}(r_1),\, f_{13}(r_1), \cdots\}.$$

The sequence

$$\{f_{11}(r_2),\, f_{12}(r_2),\, f_{13}(r_2), \cdots\}.$$

is bounded and thus has a convergent subsequence

$$\{f_{21}(r_2),\, f_{22}(r_2,\, f_{23}(r_2), \cdots\}.$$

The subsequence $\{f_{2k}(r)\}$ is now convergent for $r = r_1, r_2$. Proceeding by induction, we construct $\{f_{mk}\}$, $m = 1, 2, 3, \cdots$ where

- $\{f_{mk}\}$ is a subsequence of $\{f_{(m-1)k}\}$, $m = 2, 3, 4, \cdots$, and
- $\{f_{mk}(r)\} = \{f_{m1}(r), f_{m2}(r), f_{m2}(r), \cdots\}$ is convergent (as $k \to \infty$) for $r = r_1, r_2, \cdots, r_m$.

Therefore, the sequence $\{f_{kk}(r)\}$ is convergent for every rational $r \in \mathcal{I}$. We will show that $\{f_{kk}(t)\}$ is convergent for every $t \in \mathcal{I}$.

Let $\epsilon > 0$. Choose $\delta > 0$ such that $t_1, t_2 \in \mathcal{I}$, $|t_1 - t_2| < \delta$, $f \in \mathcal{F}$ imply

$$|f(t_1) - f(t_2)| < \epsilon/3. \tag{11}$$

Choose $r_1, r_2, \cdots, r_N$ such that $\{(r_i - \delta, r_i + \delta) \ : \ i = 1, \cdots, N\}$ covers $\mathcal{I}$ ($\mathcal{I}$ compact). There exists H such that for $m, n \geq H$,

$$|f_{mm}(r_i) - f_{nn}(r_i)| \leq \epsilon/3, \quad i = 1, 2, \cdots, N. \tag{12}$$

If $t \in \mathcal{I}$, choose r_i such that $|r_i - t| < \delta$. Then $m, n \geq H$ implies

$$\begin{aligned}
&|f_{mm}(t) - f_{nn}(t)| \\
&\leq |f_{mm}(t) - f_{mm}(r_i)| + |f_{mm}(r_i) - f_{nn}(r_i)| + |f_{nn}(r_i) - f_{nn}(t)| \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \quad \text{by (11), (12).}
\end{aligned}$$

Therefore, $\{f_{kk}\}$, a subsequence of $\{f_k\}$ is uniformly convergent on $\mathcal{I}$ (Cauchy criterion). □

Proposition 1.8 (Peano Local Existence Theorem) *Suppose that* $f \in C(D \to \mathbf{R}^n)$. *Then the IVP has a solution which exists in a neighborhood of* t_0.

Proof. (Proof # 1, Tonelli). We show the existence of a solution on a right half-neighborhood of t_0. (left half-neighborhood is similar)

For $\epsilon > 0$, let

$$\begin{aligned}
x_\epsilon(t) &= x_0, \quad t_0 - \epsilon \leq t \leq t_0, \\
x_\epsilon(t) &= x_0 + \int_{t_0-\epsilon}^{t-\epsilon} f(s, x_\epsilon(s))\, ds, \quad t_0 < t
\end{aligned} \tag{13}$$

Let a_1, a_2, M, α be as in the Picard Local Existence Theorem (Theorem1.3). Then $x_\epsilon(t)$ is well defined by (13) for $t_0 \leq t \leq t_0 + \alpha$, since

$$|x_\epsilon(t) - x_0| = |\int_{t_0-\epsilon}^{t-\epsilon} f(s, x_\epsilon(s))\, ds| \leq M|t - t_0| \leq M\, a_1 \leq a_2.$$

This also shows that $\{x_\epsilon \ : \ 0 < \epsilon$ is uniformly bounded on $[t_0, t_0 + \alpha]$. The set of functions is also equi-continuous, since

$$|x_\epsilon(t_1) - x_\epsilon(t_2)| = |\int_{t_2-\epsilon}^{t_1-\epsilon} f(s, x_\epsilon(s))\, ds| \leq M|t_1 - t_2| \leq \eta, \quad \text{if } |t_1 - t_2| < \eta/M.$$

By Arzelà-Ascoli Theorem, there exists sequence $\epsilon_k \to 0^+$, such that $\{x_{\epsilon_k}\}$ is uniformly convergent on $[t_0, t_0 + \alpha]$. Therefore (13) implies that the limit function $x(t)$ satisfies

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

and $x(t)$ is a solution of the IVP on $[t_0, t_0 + \alpha]$. □

Proof. (Proof #2, Peano). Construct an ϵ-approximate solution (Euler polygon) as follows:
Let $t_i = t_0 + i\epsilon$, $i = 0, 1, 2, \cdots$.
If $t_i \leq t \leq t_{i+1}$, let
$x_\epsilon(t) = x_i + (t - t_i) f(t_i, x_i)$,
where $x_i = x_\epsilon(t_i)$, $i = 1, 2, \cdots$.

Thus $x_\epsilon(t) = x_0 + \int_{t_0}^t F_\epsilon(s)\,ds$,
where $F_\epsilon(s) = f(t_i, x_i),\ t_i \le s \le t_{i+1}$.
Note

$$\begin{aligned} |x_1 - x_0| &= |f(t_0, x_0)|(t_1 - t_0) \le M(t_1 - t_0), \\ |x_{i+1} - x_0| &= |x_\epsilon(t_{i+1}) - x_0| \le |x_\epsilon(t_{i+1}) - x_\epsilon(t_i)| + \cdots + |x_\epsilon(t_1) - x_0| \\ &\le M\,(t_{i+1} - t_0) \le a_2, \quad t_0 \le t_i \le t_0 + \alpha. \end{aligned}$$

Thus (use induction) $(t_i, x_i) \in R$ for $i = 1, 2, \cdots$.

Show

(i) $|x_\epsilon(t) - x_0| \le M(t - t_0), \quad t_0 \le t \le t_0 + \alpha$;

(ii) There is a sequence $\epsilon_k \to 0^+$ such that $\{x_{\epsilon_k}(t)\}$ is uniformly convergent on $[t_0, t_0 + \alpha]$ to a function $x(t)$;

(iii) $F_{\epsilon_k}(t)$ converges uniformly on $[t_0, t_0 + \alpha]$ to $f(t, x(t))$;

(iv) $x'(t) = f(t, x(t)), \quad x(t_0) = x_0$.

□

The rest of the section concerns the maximal interval of existence.

Proposition 1.9 *Suppose*

(i) $f \in C(D \to \mathbf{R}^n)$;

(ii) $K \subset D$ *compact;*

(iii) $(t, x(t)) \in K,\ t_0 \le t < \omega < \infty$ *for some solution* $x(t)$ *of the IVP.*

Then $x_1 = \lim_{t\to\omega^-} x(t)$ *exists and* $(\omega, x_1) \in K$. *Furthermore,* $[t_0, \omega)$ *is* **not** *a right-maximal interval of existence for* $x(t)$.

Proof. To show $x_1 = \lim_{t\to\omega^-} x(t)$ exists, we show that $x(t)$ is uniformly continuous on $[t_0, \omega)$. From the integral equation we have

$$|x(t_1) - x(t_2)| = |\int_{t_1}^{t_2} f(t, x(t))dt| \le M|t_1 - t_2|$$

for all $t_1, t_2 \in [t_0, \omega)$, where $M = \max\{|f(t, x)| \ : \ (t, x) \in K\}$. The Peano Existence Theorem (Theorem1.8) implies that $x(t)$ can be continued to the right of ω. □

Exercise. Suppose that $x_1(t)$ is a solution of the IVP

$$x'(t) = f(t, x), \quad x(t_0) = x_0,$$

in $[t_0, a)$ and that $\lim_{t\to a} x_1(t) = x_1$ exists and that $(a, x_1) \in D$. Let $x_2(t)$ be a solution to the IVP

$$x' = f(t, x), \quad x(a) = x_1$$

in $[a,b)$. Define $x(t)=x_1(t)$ for $t\in[t_0,a)$ and $x(t)=x_2(t)$ for $t\in[a,b)$ Show that $x(t)$ is a solution of the IVP

$$x'=f(t,x),\quad x(t_0)=x_0$$

in $[t_0,b)$.

Proposition 1.10 *Suppose*

(i) $f\in C(D\to\mathbf{R}^n)$;

(ii) $x(t)$ *is a solution of the IVP;*

(iii) $t_0<t_k<\omega,\ \lim_{k\to\infty}t_k=\omega,\ \lim_{k\to\infty}x(t_k)=x_1$;

(iv) $(\omega,x_1)\in D$.

Then $x_1=\lim_{t\to\omega^-}x(t)$ *and* $[t_0,\omega)$ *is* **not** *a right-maximal interval of existence for* $x(t)$.

Proof. Let $\epsilon>0$ be sufficiently small that $\mathcal{J}_\epsilon\subset D$, where $\mathcal{J}_\epsilon=\{(t,x)\ :\ |t-\omega|\le\epsilon,\ |x-x_1|\le\epsilon\}$. Let $M_\epsilon=\max\{|f(t,x)|\ :\ (t,x)\in\mathcal{J}_\epsilon\}$. Then $\exists N\ \ni\ 0<\omega-t_N<\epsilon/2M_\epsilon$ and $|x(t_N)-x_1|<\epsilon/2$. We claim that

$$|x(t)-x(t_N)|<M_\epsilon(\omega-t_N)<\epsilon/2,\quad t_N\le t<\omega. \tag{14}$$

Then, $t_N\le t<\omega$ implies

$$|x(t)-x_1|\le|x(t)-x(t_N)|+|x(t_N)-x_1|<\epsilon/2+\epsilon/2=\epsilon.$$

and thus $\lim_{t\to\omega^-}x(t)=x_1$. Moreover, $(\omega,x_1)\in D$ implies that $[t_0,\omega)$ is not the right-maximal interval of existence.

To show claim (14): suppose it is false. Then there exists a **least** $T>t_N$ such that

$$|x(T)-x(t_N)|=M_\epsilon(\omega-t_N)<\epsilon/2. \tag{15}$$

Therefore,

$$|x(t)-x_1|\le|x(t)-x(t_N)|+|x(t_N)-x_1|<\epsilon/2+\epsilon/2=\epsilon,$$

for $t_N\le t<T$. Thus $(t,x(t))\in\mathcal{J}_\epsilon,\ t_N\le t<T$. Therefore

$$|x(T)-x(t_N)|=|\int_{t_N}^{T}f(s,x(s))ds|\le\int_{t_N}^{T}|f(s,x(s))|ds \tag{16}$$

$$\le M_\epsilon(T-t_N)<M_\epsilon(\omega-t_N) \tag{17}$$

which contradicts (15), proving the claim. (Question: why can't we use (refBBB) to prove (14) directly?) □

Corollary 1.11 *Assume that* $D=(a,b)\times\mathbf{R}^n$, $t_0\in(a,b)$, *and* (ω_-,ω_+) *is a maximal interval of existence for a solution of* $x'=f(t,x)$. *Then*

$$\omega_-=a\quad\text{or}\quad\lim_{t\to\omega_-^+}|x(t)|=\infty,$$

$$\omega_+=b\quad\text{or}\quad\lim_{t\to\omega_+^-}|x(t)|=\infty.$$

Proof. Suppose that $\omega_+ < b$. Then the uniform continuity of $x(t)$ in $[t_0, \omega_+)$, now a finite interval, implies that $\lim_{t\to\omega_+^-} |x(t)|$ exists (though may not be finite). Suppose that $\lim_{t\to\omega_+^-} |x(t)| < \infty$. Then the uniform continuity of $x(t)$ again implies that there exists $x_1 \in \mathbf{R}^n$ such that $\lim_{t\to\omega_+^-} x(t) = x_1$, and $(\omega_+, x_1) \in D = (a, b) \times \mathbf{R}^n$. Therefore $x(t)$ can be extended to the closed interval $[t_0, \omega_+]$, contradicting the assumption that $[t_0, \omega_+)$ is the right maximal interval of existence. The proof for the case of ω_- can be proved similarly. □

In Summary: If $f \in C(D \to \mathbf{R}^n)$, then the IVP has a solution and every solution $x(t)$ of the IVP exists on a maximal interval (ω_-, ω_+) which depends on the solution. Moreover,

$$\lim_{t\to\omega_\pm} d((t, x(t)),\ \partial D) = 0, \tag{18}$$

where $d(x, A) = \inf_{a\in A} |x - a|$.

To see the relation (18), suppose that $\lim_{t\to\omega_+} d((t, x(t)),\ \partial D) \neq 0$. Then there exist $\delta_0 > 0$ and $t_n \to \omega_+$ such that $\lim_{n\to\infty} d((t, x(t_n)),\ \partial D) \geq \delta_0$. First of all, this relation implies that $x(t_n)$ is bounded, since otherwise, D must be unbounded and $\{\infty\} \subset \partial D$, and there exists a subsequence t_k such that $\lim_{k\to\infty} |x(t_k)| = \infty$. This implies that $\lim_{k\to\infty} d((t, x(t_k)),\ \partial D) = 0 < \delta_0$. Now $|x(t_n)|$, being bounded, must have a convergent subsequence, $x(t_m)$ such that $t_m \to \omega_+$ and $x(t_m) \to x_1$, and thus $d((\omega_+, x_1),\ \partial D) \geq \delta_0$. This implies that $(\omega_+, x_1) \in D$, and the solution can then be further continued. This contradiction shows that (18) must hold on a maximal interval.

To further demonstrate relation (18), we examine the following example.

Example. Consider $x'(t) = \frac{1}{t^2}\cos\frac{1}{t}$. Here $f(t, x) = \frac{1}{t^2}\cos\frac{1}{t}$ is continuous in $D = (0, \infty) \times \mathbf{R}$. A solution to the DE $x(t) = c - \sin\frac{1}{t}$ has a left maximal interval $(0, t_0]$. We see that $\lim_{t\to 0_+} d(x(t), \partial D) = 0$, but $\lim_{t\to 0_+} x(t)$ does not exist.

However, for a solution $x(t)$ on its maximal interval of existence (ω_-, ω_+), the limit $\lim_{t\to\omega_\pm} x(t)$ may not exist, as the following example shows.

Example. Consider

$$x'(t) = \frac{1}{t^2}\ \cos\frac{1}{t}.$$

The function $f(t, x) = \frac{1}{t^2}\cos\frac{1}{t}$ is continuous in $D = (0, \infty) \times \mathbf{R}$. A solution $x(t) = c - \sin\frac{1}{t}$ has a left maximal interval $(0, t_0)$. We see that $\lim_{t\to 0^+} d((t, x(t)), \partial D) \to 0$, but $\lim_{t\to 0^+} x(t)$ does not exist.

1.7. Continuous dependence on initial conditions and parameters

Consider the IVP

$$x' = f(t, x, \mu) \quad x(t_0) = x_0, \tag{19}$$

where $t \in \mathbf{R}, x \in \mathbf{R}^n, \mu \in \mathbf{R}^m$. We make the following assumptions:

(i) $f \in C(G \to \mathbf{R}^n)$, G open in $\mathbf{R}^{n+m+1}$.

(ii) Solutions to the IVP is **unique** for each $(t_0, x_0, \mu) \in G$.

Denote the solution to the IVP as $x = x(t; t_0, x_0, \mu)$.

We will show that $x(t; t_0, x_0, \mu)$ is continuous in its domain.

Theorem 1.12 *Assume that* $x(t; t_0, x_0, \mu_0)$ *exists for* $t \in [a, b]$. *Then,* $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ *such that*

$$|t_0^* - t_0| < \delta, \quad |x_0^* - x_0| < \delta, \quad |\mu_0^* - \mu_0| < \delta$$

imply

(1) $x(t; t_0^*, x_0^*, \mu_0^*)$ *exists for* $t \in [a, b]$.

(2) $\sup_{a \le t \le b} |x(t; t_0^*, x_0^*, \mu_0^*) - x(t; t_0, x_0, \mu_0)| < \epsilon$.

W.o.l.g., we may drop the dependence on μ. The IVP

$$\begin{cases} x' &= f(t, x, \mu) \\ x(t_0) &= x_0, \end{cases} \qquad x = x(t; t_0, x_0, \mu)$$

is equivalent to the IVP

$$\begin{cases} x' &= f(t, x, y) \\ y' &= 0 \\ x(t_0) &= x_0 \\ y(t_0) &= \mu, \end{cases} \qquad \begin{cases} x &= x(t; t_0, x_0, \mu) \\ y &= y_0 \ (= \mu). \end{cases}$$

Proof of Theorem 1.12. Equivalently, we prove the following. Let

$$(t_k, x_k) \to (t_0, x_0), \quad \text{as} \quad k \to \infty.$$

Then for $k > K$,

(1) $x(t; t_k, x_k)$ exists for $t \in [a, b]$.

(2) $\lim_{k\to\infty} \sup_{a \le t \le b} |x(t; t_k, x_k) - x(t; t_0, x_0)| = 0$, or equivalently, $x(t; , t_k, x_k) \to x(t, t_0, x_0)$ as $k \to \infty$ uniformly for $t \in [a, b]$.

Step 1. Choose $D_0 \subset \overline{D_0} \subset D_1 \subset \overline{D_1} \subset D$ such that $(t, x(t)) \in D_0$ for $t \in [a, b]$. Set

$$M > \sup_{(t,x)\in\overline{D_1}} |f(t, x)|.$$

Choose α such that

$$R_\alpha(t_0, x_0) = \{(t, x)\ :\ |t - t_0| \le \alpha,\ |x - x_0| \le M\alpha\} \subset D_1.$$

Then $x(t; t_0, x_0)$ is defined for $[t_0 - \alpha,\ t_0 + \alpha]$, and $\exists K > 0$, such that $k > K$ implies $(t_k, x_k) \in R_\alpha(t_0, x_0)$ and

$$R_\alpha(t_k, x_k) \subset D_1,$$

and thus $x_k(t) = x(t; t_k, x_k)$ is defined for $t \in [t_0 - \alpha,\ t_0 + \alpha]$. From the integral equation

$$x_k(t) = x_k + \int_{t_k}^{t} f(s, x_k(s))\, ds$$

we conclude that $\{x_k(t)\}$ is uniformly bounded and equi-continuous in $[t_0 - \alpha,\ t_0 + \alpha]$. Arzelà-Ascoli Theorem implies that any subsequence of $\{x_k(t)\}$ has a subsequence uniformly convergent to the same limit $x(t, t_0, x_0)$ (uniqueness of solutions). This implies that

$$x(t; t_k, x_k) \to x(t; t_0, x_0) \quad \text{as } k \to \infty$$

uniformly in $[t_0 - \alpha,\ t_0 + \alpha]$.

Step 2. Suppose $[a, b] \not\subset [t_0 - \alpha,\ t_0 + \alpha]$. Let $t_0^1 = t_0 + \alpha$, $x_0^1 = x(t_0^1)$. Choose α_1 such that

$$R_{\alpha_1}(t_0^1, x_0^1) \subset D_1.$$

Repeating the preceding argument, we conclude: $\exists\, K_1 > 0$ such that $k > K_1$ implies $x(t; t_k, x_k)$ is defined for $t \in [t_0,\ t_0 + \alpha + \alpha_1]$, and

$$x(t; t_k, x_k) \to x(t; t_0, x_0) \quad \text{as } k \to \infty$$

uniformly in $[t_0,\ t_0 + \alpha + \alpha_1]$. Proceed like this, after <u>finite</u> number of steps, we obtain $[a_1, b_1] \supset [a, b]$ and $K > 0$ such that $k > K$ implies

(1) $x(t; t_k, x_k)$ exists for $t \in [a_1, b_1]$.

(2) $\lim_{k\to\infty} \sup_{a_1 \le t \le b_1} |x(t; t_k, x_k) - x(t; t_0, x_0)| = 0$.

Remark. The result also include the continuous dependence on the vector field $f(t, x)$. Suppose $x_k(t)$ solves the IVP

$$x' = f_k(t, x), \quad x(t_k) = x_k,$$

and, as $k \to \infty$, $(t_k, x_k) \to (t_0, x_0)$, $f_k(t, x) \to f(t, x)$ on any compact subset of $D \subset \mathbf{R}^{n+1}$. Then, for $k > K > 0$, $x(t; t_k, x_k, f_k)$ exists in $[a, b]$ and

$$\lim_{k\to\infty} \sup_{a \le t \le b} |x(t; t_k, x_k, f_k) - x(t; t_0, x_0, f)| = 0.$$