Stochastic Models

- Deterministic models (such as $\dot{N} = rN$) are unrealistic in that they ignore the element of chance. This element may be especially important in low population sizes

- Stochastic Elements

Environmental - Eg Storms, temperature, humidity give unpredictable birth & death rates

Demographic - Even if average birth & death rates are known, populations contain a discrete number of individuals whose behaviours are unpredictable

Overview

- Linear stochastic models
  - analytical progress has been made
  - simple birth processes studied by Yule (1924) & Furey (1937)
  - simple birth & death processes studied by Feller (1939)
  - Galton & Watson studied extinction of family names - even if $b > d$ there is a probability of extinction

- Nonlinear Stochastic Models
  - include density-dependent probabilities for birth, death etc.
  - without immigration or emigration, equilibrium solutions "statistically stable states" cannot be found
  - if you cannot grow unboundedly you will eventually suffer a string of bad luck & die out
  - on a fast timescale one may find a "quasi-stationary state but as $t \to \infty$ this QSS "leaks" to zero.

Stochastic Processes

Defn ① If S is a sample space (collection of all possible outcomes of an experiment) with a probability measure and N is a real-valued function defined over the elements of S then N is a random variable

② A family $\{N(t)\}$ indexed by a parameter t is called a stochastic process

③ A stochastic process $\{N(t)\}$ is called a Markov process if it is history-independent. In the case with t discrete it is a one-step memory process

$$\Pr\{N(t_i)=n_1 \mid N(t_{i-1})=n_2 \wedge N(t_{i-2})=n_3 \wedge \ldots\}$$
$$= \Pr\{N(t_i)=n_1 \mid N(t_{i-1})=n_2\}$$

Linear Birth Process

- Markov process, continuous time, integer sample space

$N(t) =$ # individuals at time t (random variable)

$P_n(t) = Pr\{N(t) = n\}$ $n = 0, 1, 2, \ldots$

For one individual

$$Pr\{1 \text{ birth in } (t, t+\Delta t]\} = b\Delta t + O(\Delta t^2)$$

$$Pr\{>1 \text{ birth in } (t, t+\Delta t]\} = O(\Delta t^2)$$

$$Pr\{0 \text{ births in } (t, t+\Delta t]\} = 1 - b\Delta t + O(\Delta t^2)$$

For n individuals

$$Pr\{1 \text{ birth in } (t, t+\Delta t]\} = nb\Delta t(1 - b\Delta t)^{n-1}$$
$$= nb\Delta t + O(\Delta t^2)$$

$m \le n$ $$Pr\{m \text{ births in } (t, t+\Delta t]\} = \binom{n}{m}(b\Delta t)^m(1-b\Delta t)^{n-m}$$
$$= O(\Delta t^2)$$

(A similar calculation shows $O(\Delta t^2)$ when $m > n$)

$$Pr\{0 \text{ births in } (t, t+\Delta t]\} = 1 - nb\Delta t + O(\Delta t^2)$$

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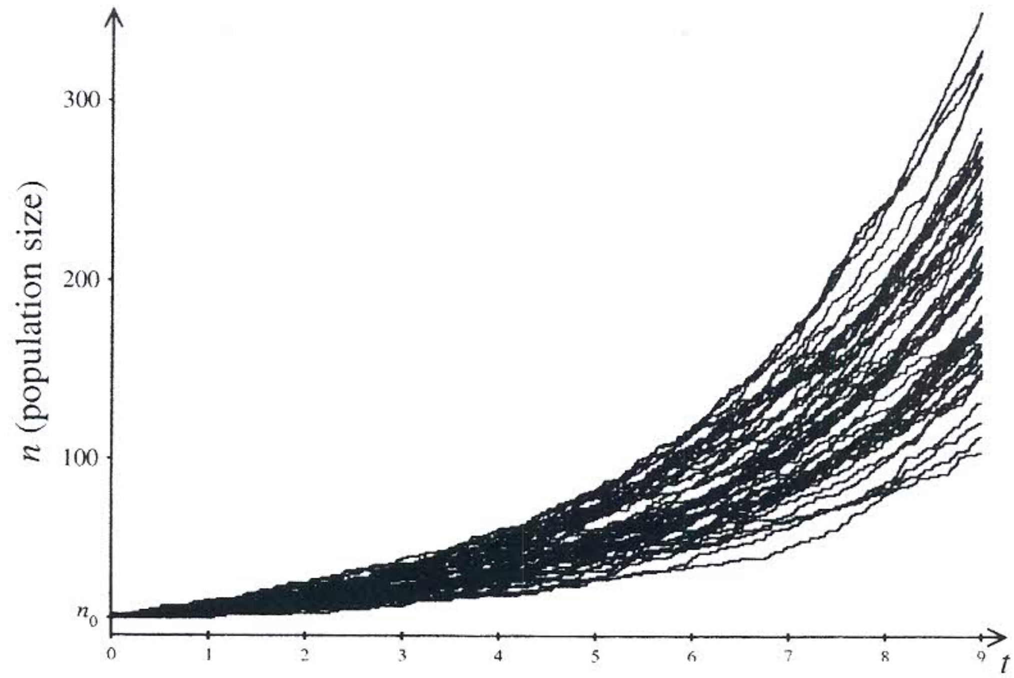


Figure 5.7. *Stochastic simulation of the pure birth process given in* (5.51). *Parameters are* $b = 1/3$ *and* $n_0 = 10$. *Fifty different trajectories are given. They differ only in the seed for the random number generator used.*

Thus

$$P_n(t+\Delta t) = P_{n-1}(t) \cdot Pr\{1 \text{ birth in } (t, t+\Delta t]\}$$
$$+ P_n(t) \cdot Pr\{0 \text{ births in } (t, t+\Delta t]\}$$
$$+ \mathcal{O}(\Delta t^2)$$
$$= P_{n-1}(t)(n-1)b\Delta t + P_n(t)(1 - bn\Delta t) + \mathcal{O}(\Delta t^2)$$

$$\Rightarrow \quad \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = b\{(n-1)P_{n-1}(t) - n P_n(t)\}$$

As $\Delta t \to 0$ we have an infinite sytem of ODEs

$$\frac{d}{dt} P_n(t) = b\{(n-1)P_{n-1}(t) - n P_n(t)\}$$

$n = 0, 1, 2, 3 \ldots \qquad P_{-1} = 0 \qquad$ with initial data

$$P_n(0) = \begin{cases} 1 & \text{if } n = n_0 \\ 0 & \text{otherwise.} \end{cases}$$

<u>Note</u>: As only birth is possible (i.e. no death)

$P_n(t) = 0$ for all $n < n_0$ and $t \geq 0$.

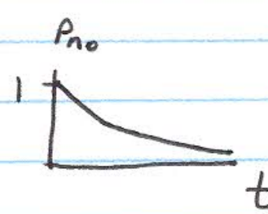
This can be expressed as an infinite dimensional system

$$\dot{p} = A p \qquad p = \begin{pmatrix} p_0 \\ p_1 \\ \vdots \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & -b & 0 & 0 & \cdots \\ 0 & b & -2b & 0 & \cdots \\ 0 & 0 & 2b & -3b & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

However, as $P_n(t)$ depends only upon $P_{n-1}(t)$ & $P_n(t)$ we can solve the ODEs in a chain

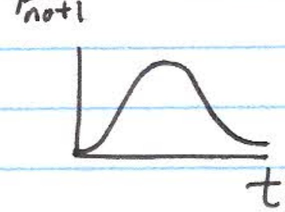
1. $$\left.\begin{aligned} \frac{d}{dt} P_{n_0} &= -b n_0 P_{n_0} \\ P_{n_0}(0) &= 1 \end{aligned}\right\} \Rightarrow P_{n_0}(t) = e^{-b n_0 t}$$



2. $$\frac{d}{dt} P_{n_0+1} = -b(n_0+1) P_{n_0+1} + b n_0 e^{-b n_0 t}$$

$$P_{n_0+1}(0) = 0$$

$$\Longrightarrow P_{n_0+1}(t) = n_0 e^{-b n_0 t} \left(1 - e^{-bt}\right)$$



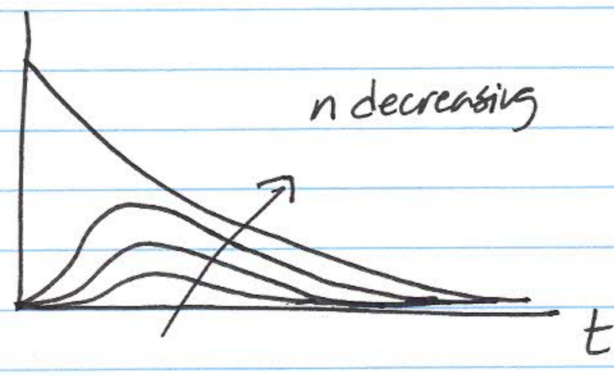
⋮

3. $$P_{n_0+k}(t) = \binom{n_0+k-1}{n_0-1} e^{-bn_0t}\left(1-e^{-bt}\right)^k$$

writing $n = n_0 + k$ gives

$$P_n(t) = \binom{n-1}{n_0-1} e^{-bn_0t}\left(1-e^{-bt}\right)^{n-n_0}$$

Negative binomial distribution with parameter $\phi = e^{-bt}$



$$M_1 = E(N(t)) = \sum_{n=n_0}^{\infty} n P_n(t) = \sum_{n=n_0}^{\infty} n \binom{n-1}{n_0-1} e^{-bn_0t}\left(1-e^{-bt}\right)^{n-n_0}$$

$$= \sum_{k=0}^{\infty} (k+n_0)\binom{n_0+k-1}{k} e^{-bn_0t}\left(1-e^{-bt}\right)^k$$

$$= n_0 e^{-bn_0t} \sum_{k=0}^{\infty} \binom{k+n_0}{k}\left(1-e^{-bt}\right)^k$$

$$= n_0 e^{-bn_0t}\left[e^{-bt}\right]^{-(n_0+1)} = n_0 e^{bt}$$ exponential growth.

The variance can be calculated as

$$\sigma^2 = n_0 e^{bt}(e^{bt} - 1)$$

$$CV = \frac{\sigma}{M_1} = \sqrt{\frac{1 - e^{-bt}}{n_0}} \rightarrow \frac{1}{\sqrt{n_0}} \text{ as } t \to \infty$$

An alternative method for calculating M_1 & σ^2 uses

$$\frac{d}{dt}\{E(N(t))\} = \sum_{n=n_0}^{\infty} n \, \dot{p}_n(t)$$

$$= \sum_{n=n_0}^{\infty} n \left[b(n-1) p_{n-1}(t) - b n p_n(t) \right]$$

$$= \dots = b \, E(N(t))$$

$$\Rightarrow E(N(t)) = E(N(0)) \, e^{bt} = n_0 e^{bt}$$

A similar differential equation can be derived for M_2 & hence σ^2 can be calculated.