# Some useful probability distributions

1. A Bernoulli r.v. has only two possible values: 0 (failure) or 1 (success)

$$P_0 = 1-p \qquad P_1 = p$$

where $p = Pr\{\text{success}\}$

Eg The outcome from flipping a fair coin is given by $N=0$ (tails) or $N=1$ (heads). $N$ is distributed as a Bernoulli r.v. with $p = 1/2$.

2. The sum of $m$ independent identically distributed Bernoulli r.v.s is a Binomial r.v. The probability of $n$ successes in $m$ trials is:

$$P_n = \binom{m}{n} p^n (1-p)^{m-n} \qquad n = 0, 1, 2, \ldots, m$$

Note
$$\sum_{n=0}^{m} P_n = \sum_{n=0}^{m} \binom{m}{n} p^n (1-p)^{m-n}$$

$$= (p + (1-p))^m = 1$$

Eg The probability of 7 heads in 10 coin flips is

$$P_7 = \binom{10}{7} \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3$$

3 If each trial is an independent identically distributed Bernoulli r.v. then the probability of the $k^{th}$ success occuring on the $n^{th}$ trial is

$$p_n = Pr(N=n)$$

$$= Pr\{k-1 \text{ successes on the first } n-1 \text{ trials}\}$$

$$\cdot Pr\{\text{success}\}$$

$$= \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} \cdot p$$

$$= \binom{n-1}{k-1} p^{k} (1-p)^{n-k} \qquad n = k, k+1, k+2, \ldots$$

Note $\sum_{n=k}^{\infty} p_n = 1$

Eg The probability of the $7^{th}$ head occuring on coin flip 10 is

$$P_{10} = \binom{9}{6} \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3$$

# Random Variables: Measures of Variability

- Let $N$ be a random variable indicating the number of individuals ($N \in \mathbb{Z}^+$)
- Let $P_n = Pr\{N=n\}$, $n = 0, 1, 2, \ldots$, $\sum_{n=0}^{\infty} P_n = 1.$

First Moment (Expected value of $N$ or mean value of $N$)

$$E(N) = \sum_{n=0}^{\infty} n P_n = M_1 \text{ , the first moment of } P_n$$

Second Moment (Expected value of $N^2$ or mean value of $N^2$)

$$E(N^2) = \sum_{n=0}^{\infty} n^2 P_n = M_2 \text{, the second moment of } P_n.$$

Variance (Second moment of $P_n$, measured about $E(N)$).

$$\begin{aligned} var(N) &= E\left((N - E(N))^2\right) \\ &= \sum_{n=0}^{\infty} [n - E(N)]^2 P_n = \sum_{n=0}^{\infty} n^2 P_n - 2E(N) \sum_{n=0}^{\infty} n P_n \\ &\quad + E^2(N) \underbrace{\sum_{n=0}^{\infty} P_n}_{=1} \\ &= M_2 - 2M_1^2 + M_1^2 = M_2 - M_1^2 = \sigma^2 \end{aligned}$$

Standard Deviation $\qquad \sigma = \sqrt{var(N)}$

Coefficient of Variation $\qquad CV = \dfrac{\sigma}{E(N)}$ = measure of variation of solution about the mean.

Given a random variable $N$ describing the outcome of a given experiment over $\mathbb{Z}^+$.

① <u>Mean</u> the value of $N$ averaged over an arbitrarily large replications of the experiment

$$\boxed{M_1 = \sum_{n=0}^{\infty} n P_n.}$$

② <u>Mode</u> the most likely value for $N$

$$\boxed{\text{mode} = \max_n \{P_n\}.}$$

③ <u>Median</u> value of $N$ for which it is just as likely to have a larger value of $N$ as a ~~smaller or equal value of~~ $N$.

$$\boxed{\text{median is value of } n \text{ satisfying } \sum_{K=0}^{n} P_K = 1/2}$$

Eg $P_0 = 0.05$, $P_1 = 0.25$, $P_2 = 0.20$, $P_3 = 0.15$, $P_4 = 0$, $P_5 = 0.15$, $P_6 = 0.20$

$M_1 = 3.05$

Mode $n = 1$

Median $n = 2$.

<u>Defn</u> <u>Expected value</u>. Given a random variable $N$ with probability distribution $P_n$ so that

$$P_n = Pr\{N=n\} \qquad n=0,1,2,\ldots$$

and a function $f$, then

$$E(f(N)) = \sum_{n=0}^{\infty} P_n f(n)$$

<u>Note</u> Expected value is a linear operator with properties

$$E(f(N)+g(N)) = E(f(N)) + E(g(N))$$

$$E(c f(N)) = c\, E(f(N)) \qquad (c \text{ constant})$$

$$E(c) = c$$

<u>Means and Variances for some useful probability distributions</u>

<u>Bernoulli</u>
$$M_1 = E(N) = 0\cdot P_0 + 1\cdot P_1$$
$$= 0\cdot(1-p) + 1\cdot p = p$$

$$M_2 = E(N^2) = 0^2\cdot P_0 + 1^2\cdot P_1 = p.$$

$$\sigma^2 = M_2 - M_1^2 = p(1-p)$$

<u>Binomial</u>
$$M_1 = E(N) = \sum_{n=0}^{m} n \binom{m}{n} p^n (1-p)^{m-n}$$

$$= \sum_{n=1}^{m} \frac{m!}{(n-1)!\,(m-n)!}\, p^{n} (1-p)^{m-n}.$$

$$= mp \sum_{n=1}^{m} \binom{m-1}{n-1} p^{n-1} (1-p)^{m-n}.$$

Let $n_1 = n-1$ and $m_1 = m-1$. Then

$$M_1 = mp \sum_{n_1=0}^{m_1} \binom{m_1}{n_1} p^{n_1} (1-p)^{m_1-n_1} = \underline{\underline{mp}}$$

(compare with <u>Bernoulli</u>)

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$$E(N(N-1)) = \sum_{n=0}^{m} n(n-1) \binom{m}{n} p^{n} (1-p)^{m-n}$$

$$= \sum_{n=2}^{m} \frac{m!}{(n-2)!\,(m-n)!}\, p^{n} (1-p)^{m-n}$$

$$= m(m-1)\,p^2 \sum_{n=2}^{m} \binom{m-2}{n-2} p^{n-2} (1-p)^{m-n}.$$

$$= m(m-1)\,p^2 \sum_{n_1=0}^{m_1} \binom{m_1}{n_1} p^{n_1} (1-p)^{m_1}.$$

where $n_1 = n-2$
$m_1 = m-2.$

$$= m(m-1)\,p^2.$$

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So $M_2 = E(N(N-1)) + E(N) = m(m-1)p^2 + mp$

$$\sigma^2 = M_2 - M_1^2 = m(m-1)p^2 + mp - m^2p^2$$

$$= mp(1-p)$$

(compare with <u>Bernoulli</u>.)