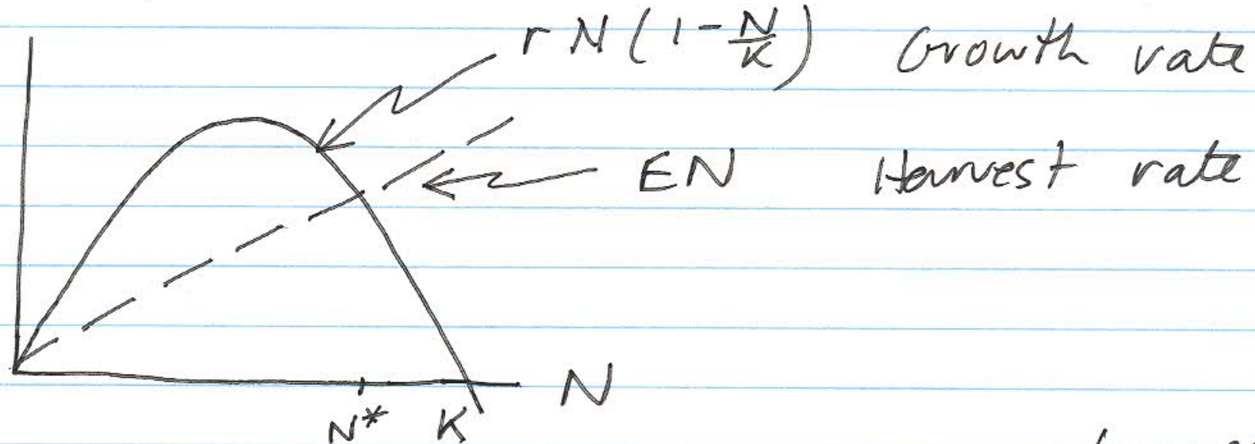
Fishery Harvesting Model

$$\dot{N} = \underbrace{rN\left(1-\frac{N}{K}\right)}_{\text{Logistic growth}} - \underbrace{q\hat{E}N}_{\text{Harvesting}} = f(N)$$

N - fish population
r - intrinsic growth rate
K - carrying capacity
q - catchability
$\hat{E}$ - effort

Harvesting $= q\hat{E}N$ is "Catch per unit effort hypothesis"

Let $q\hat{E} = q$

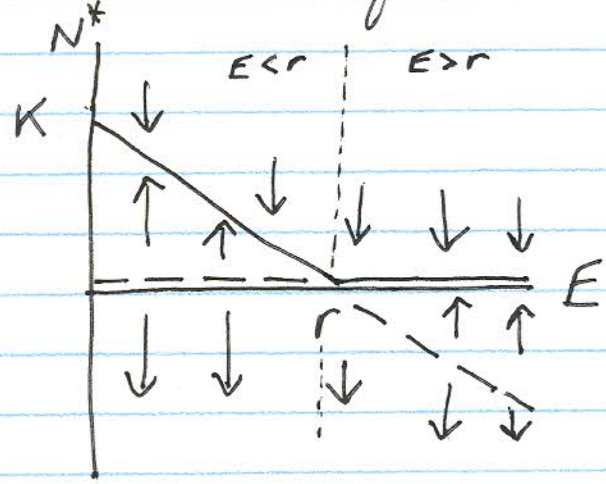


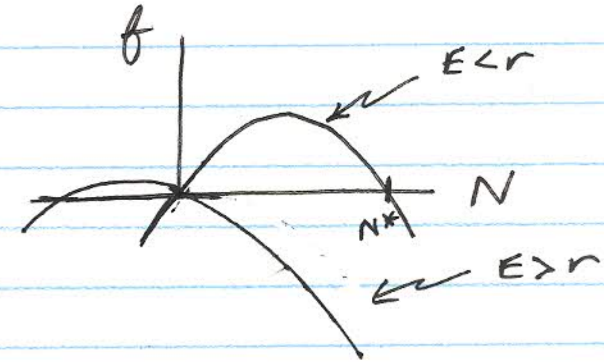
Equilibrium N^* when growth rate = harvest rate

IE $f(N^*) = 0$ so that $EN^* = rN^*(1 - N^*/K)$

$\Rightarrow N^* = 0$ or $N^* = K(1 - E/r)$

Bifurcation Diagram





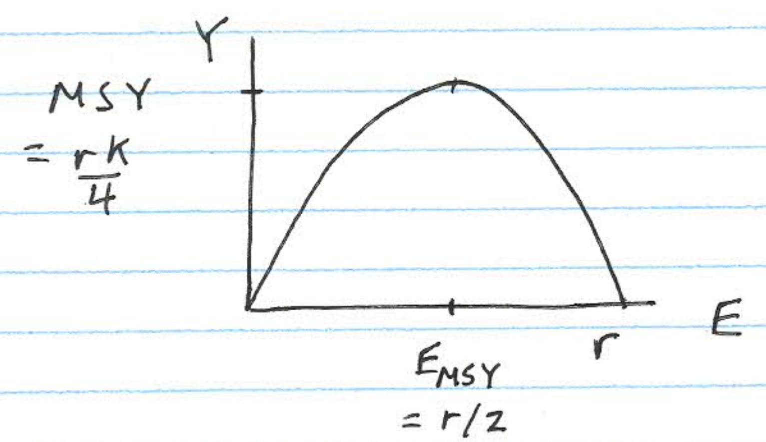
Transcritical bifurcation

$\uparrow \equiv \dot{N} > 0$

$\downarrow \equiv \dot{N} < 0$

The yield to fishermen & women is given by

$$Y = EN^* = EK(1 - E/r)$$

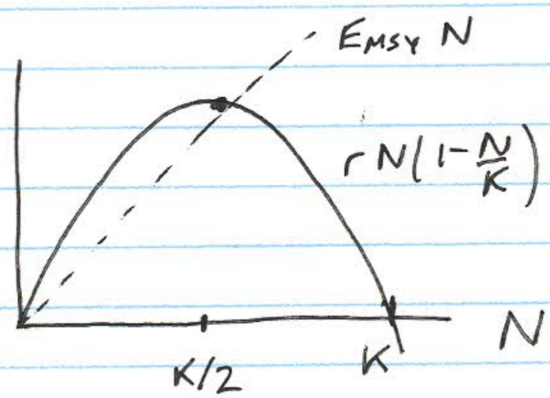


MSY ≡ Maximum Sustainable Yield

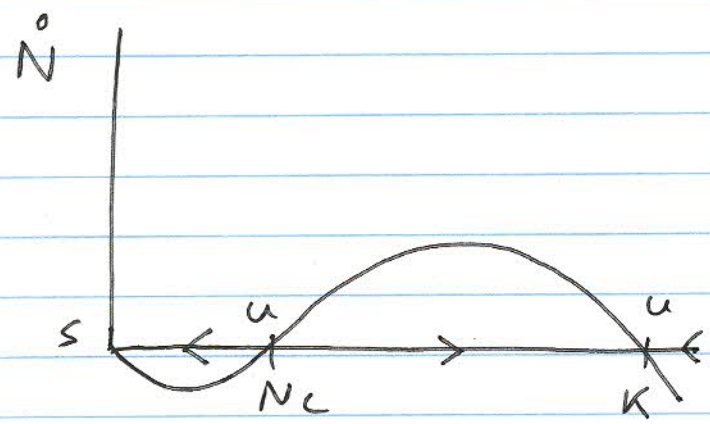
When $E = E_{MSY}$ the equilibrium population level is

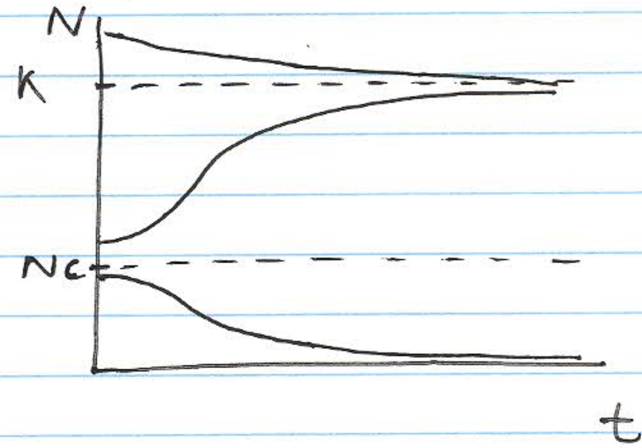
$$N^* = K(1 - E/r)\Big|_{E=r/2}$$

$$= K/2$$



What is the fish population exhibits an Allee effect?

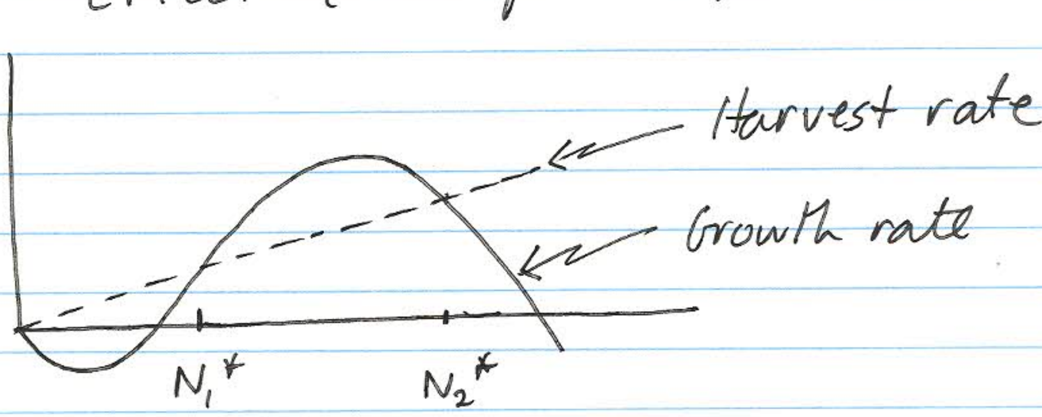


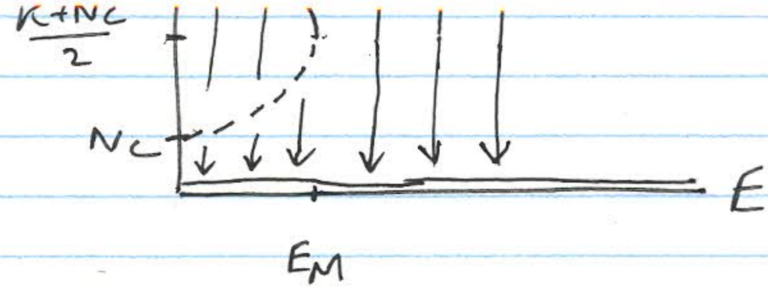


Here N_c is the minimum viable pop'n size and K is the carrying capacity.

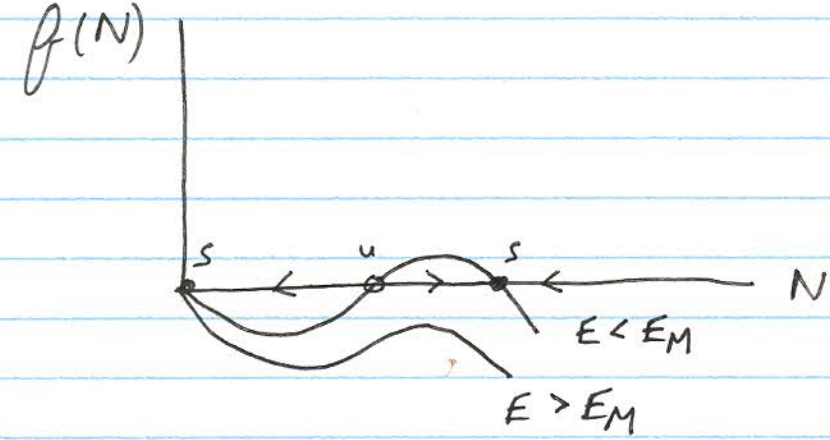
Adding Harvesting

$$\dot{N} = \underbrace{\rho N \left(\frac{N}{N_c} - 1\right)\left(1 - \frac{N}{K}\right)}_{\text{Pop'n growth with Allee effect \{cubic function\}}} - \underbrace{EN}_{\text{Harvesting}} = f(N)$$

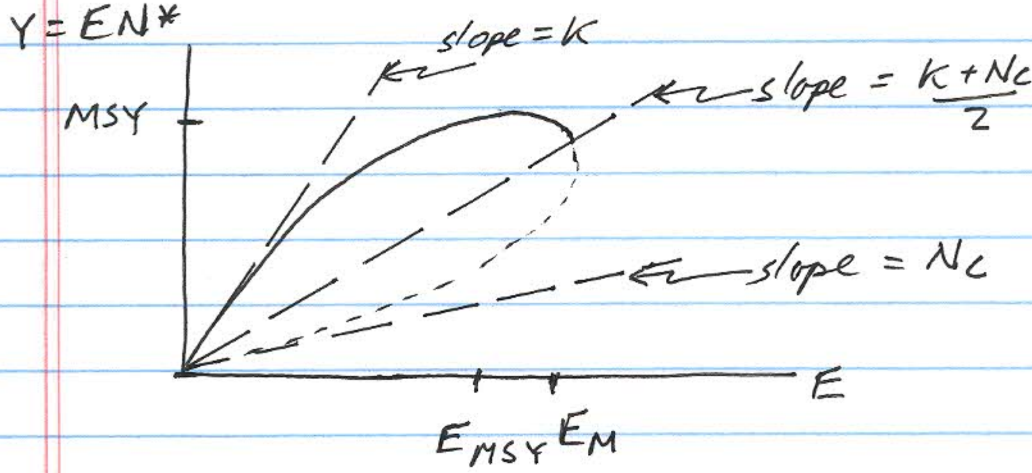


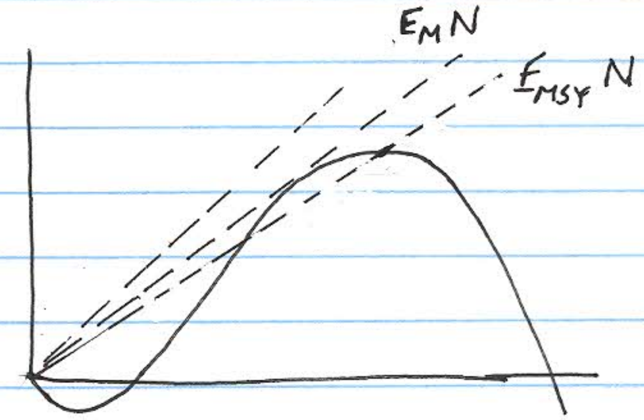


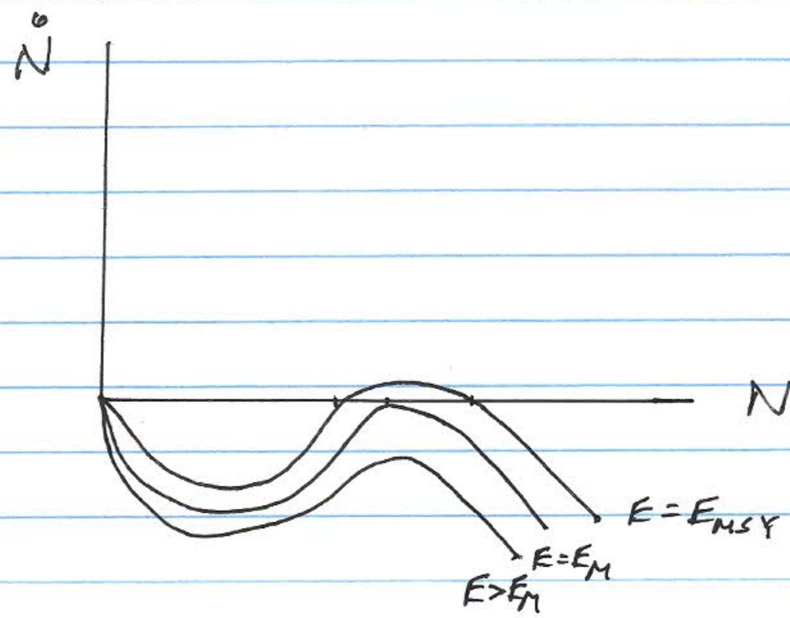
bifurcation



Catastrophic Yield Curve.







A small change in effort may result in a population crash for the fish.

Some examples of catastrophic extinction (or almost extinction) via harvesting:

- passenger pigeon – hunted to extinction in 19^{th} & early 20^{th} C.
- Blue whale – Antarctic Blue whale – harvested to verge of extinction by the early 1960s.

References
- R.M. May (1977) Nature 269: 471–7
- Ludwig et al (1978) J. Anim Ecol 47: 315–32
- C. Clark (1990) Mathematical Bioeconomics.

Bifurcation Theory

1) The solution to $\dot{N} = f(N)$, $N(0) = N_0$ can be viewed as function of two variables $\phi(t, N_0)$. $\phi(t, N_0)$ is called the <u>flow</u> of $\dot{N} = f(N)$.

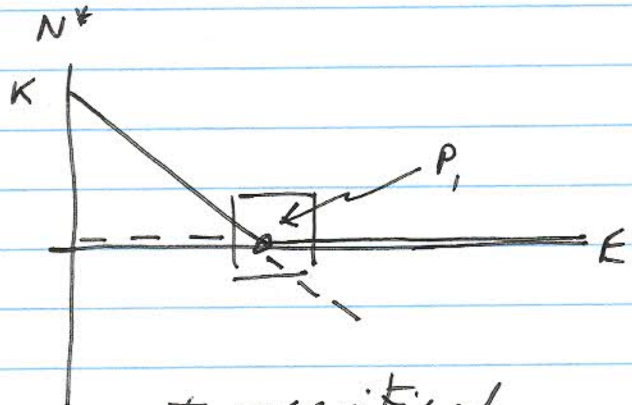
2) Let E be a parameter so that $\dot{N} = f(N; E)$. $\dot{N} = f(N, E)$ is said to have a <u>stable orbit structure</u> at $E = E_c$ if the qualitative structure of the flow does not change for small variations of E about E_c.

3) A parameter value for which the flow does not have a stable orbit structure is called a <u>bifurcation value</u> and for this value the equation is said to be at a <u>bifurcation point</u>.

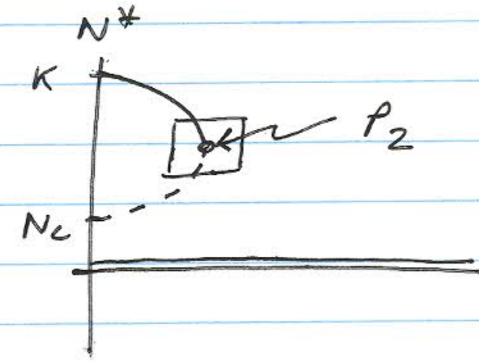
Bifurcation Theory

- Given $\dot{N} = f(N;E)$ is there a systematic way of finding bifurcation points?

- Consider the two bifurcation diagrams we have drawn so far.



Transcritical



Saddle Node

$$f = rN\left(1-\frac{N}{K}\right) - EN \qquad f = rN\left(\frac{N}{N_c}-1\right)\left(1-\frac{N}{K}\right) - EN$$

$$P_1 = (E,N^*) = (r,0) \qquad P_2 = (E,N^*) = \left(\frac{r\,[K-N_c]^2}{4N_cK}, \frac{K+N_c}{2}\right)$$

$$f_N = r - E - \frac{2r}{K}N \qquad f_N = \frac{-3r}{KN_c}N^2 + 2\left(\frac{1}{K}+\frac{1}{N_c}\right)N - (r+E)$$

$$\lambda_1 = f_N\big|_{P_1} = 0 \qquad \lambda_2 = f_N\big|_{P_2} = 0$$

Recall $\dot{N} = f$ is <u>hyperbolic</u> providing

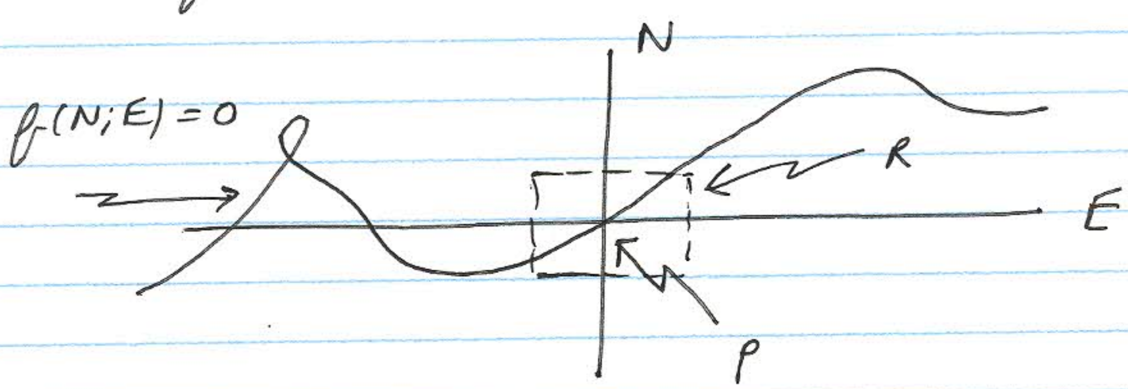
$$\lambda = f_N\big|_P \neq 0$$

(1) The bifurcation points have the property that $N^*(E)$ is not single-valued at a small neighborhood about the bifurcation point

(2) Also note that $f_N|_{P_1} = 0$ and $f_N|_{P_2} = 0$ in the above examples.

Implicit Function Theorem:

Consider a point P on the curve $f(N;E) = 0$. If $f \in C^1$ and $f_N|_P \neq 0$ then $\exists$ a rectangle R about P such that there is a unique $N(E)$ in R.



Also, along the curve $f(N;E) = 0$ we have

$$0 = \frac{df}{dE} = f_E + f_N \frac{dN}{dE} \Rightarrow \frac{dN}{dE} = -f_E / f_N$$

This holds at any point on the curve $f(N;E) = 0$

so $\left.\dfrac{dN}{dE}\right|_P = \dfrac{-f_E|_P}{f_N|_P}$ which exists providing the point is hyperbolic

In other words, a hyperbolic equilibrium satisfies the Implicit Function Theorem.

Note:

The Implicit Function Theorem states that if f is hyperbolic at p then f is single valued at p

In other words

statement (2) false $\Rightarrow$ statement (1) false

$$((2)' \Rightarrow (1)')$$

This is logically equivalent to

$$(1) \Rightarrow (2)$$

which, in words is:

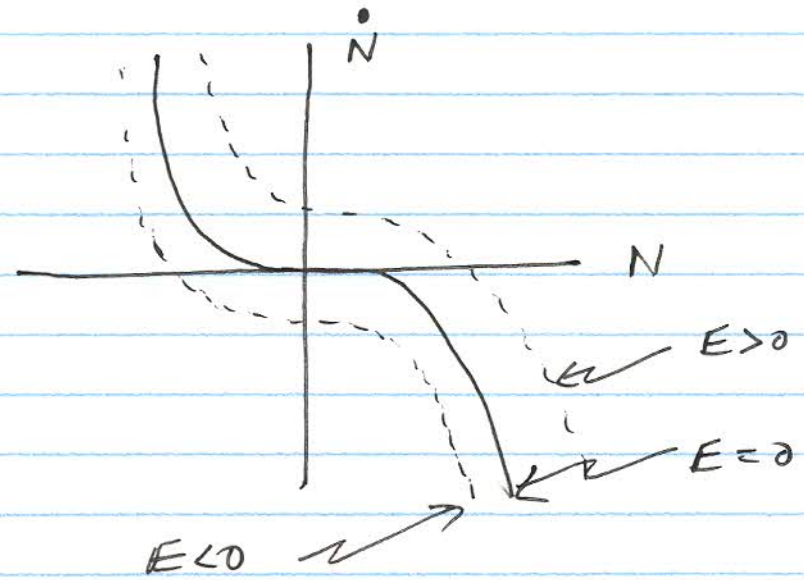
If $N^*(E)$ is not single valued at p then f is not hyperbolic at p.

However, the converse is not true.

f nonhyperbolic does not necessarily imply a bifurcation point

Eg $\quad \dot{N} = f(N;E) = E - N^3$

Eg (Cont'd)



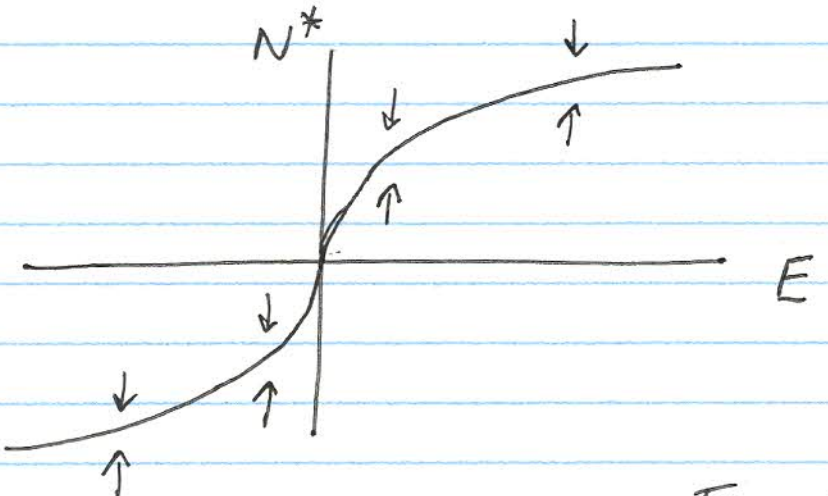
$P = (0,0)$

$f(0;0) = 0$

$\frac{\partial f}{\partial N}(0;0) = 0$

nonhyperbolic at P.

If $f(N^*;E) = 0$, $E = N^{*3}$



no qualitative change in the flow as E passes through zero

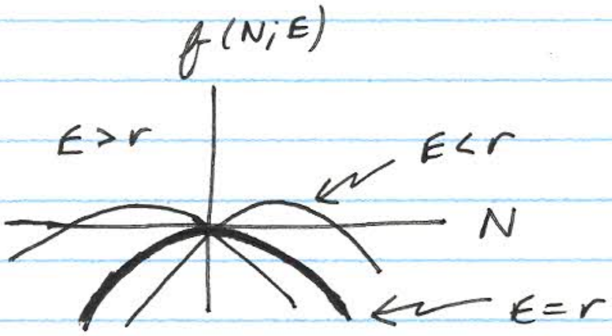
In general...

P nonhyperbolic is a necessary but not sufficient condition for a bifurcation.

In other words if f is hyperbolic at p then the Implicit Function Theorem is satisfied and $N^*(E)$ is single-valued (no bifurcation).

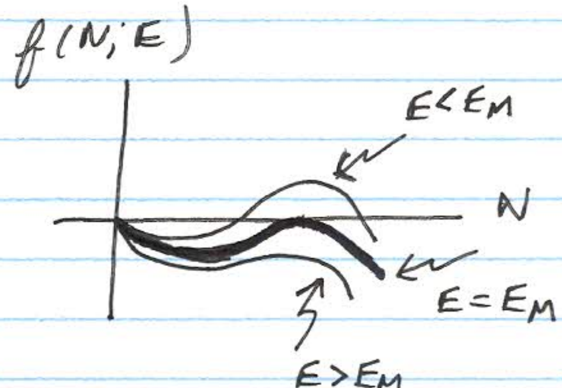
Hence, the first step in finding bifurcation points is to determine values of E for which f is nonhyperbolic.

Eg



Logistic growth & harvesting

$f_N = 0$ at $N=0$ when $E=r$



Allee effect & harvesting

$f_N = 0$ at $N = N^*$ when $E = E_M$

A nonhyperbolic equilibrium is a necessary but not sufficient condition for a bifurcation

- What is the difference between P_1 & P_2?

P_1: $f_N|_{P_1} = 0$ $f_E|_{P_1} = 0$ singular point

there is no function $N(E)$ or $E(N)$ in nbhd of P_1

P_2: $f_N|_{P_2} = 0$ $f_E|_{P_2} \neq 0$ regular point

There is no function $N(E)$ but there is a function $E(N)$ in nbhd of P_2

Defn: If $\frac{dE}{dN}$ changes sign at P then P is a *turning point*.

Theorem: At a regular turning point, one side of point is a stable equilibrium and at the other side of the point is an unstable equilibrium.

Note: Along the curve $f(N,E) = 0$

$$0 = \frac{df}{dN} = f_N + f_E \frac{dE}{dN} \quad \text{so}$$

$$\lambda = f_N = -\underbrace{f_E}_{\neq 0} \underbrace{\frac{dE}{dN}}_{\text{changes sign at } P.}$$