Mahler measures as values of regulators

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Mahler measure

Definition 1 *For* $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$*, the (logarithmic)* Mahler measure *is defined by*

$$m(P) = \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n}. \quad (1)$$

The simplest examples in several variables are due to Smyth [16]

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} \mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1)$$

where

$$\mathrm{L}(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}$$

is the Dirichlet L-series in the character of conductor 3, and

$$m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3) \quad (2)$$

involving the Riemann zeta function (see [1]).

Other examples include the family studied by Boyd [2], Deninger [7], and Rodriguez-Villegas [14]:

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} - k\right) \stackrel{?}{=} \frac{\mathrm{L}'(E_k, 0)}{B_k} \qquad k \in \mathbb{N}, \quad k \neq 4 \quad (3)$$

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} - 4\right) = 2\mathrm{L}'(\chi_{-4}, -1) \quad (4)$$

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} - 4\sqrt{2}\right) = \mathrm{L}'(A, 0) \quad (5)$$

$$A : y^2 = x^3 - 44x + 112.$$

The first formula (3) indicates that the result is only known numerically, i.e., the Mahler measure is related in many decimal places to a value of the L-series of the elliptic curve determined by the zeros of the polynomial. When $k = 4$ we obtain (4), a degenerate case (the curve has genus zero). In some cases like (5) when the elliptic curve has complex multiplication, it is possible to prove the equality.

An algebraic integration for Mahler measure

In order to understand such equalities, Deninger [7] proposed the following. We write the Mahler measure as an integral of a certain $\mathbb{R}(n-1)$-valued smooth $n-1$-form in $X(\mathbb{C})$, the variety determined by the zeroes of the polynomial.

$$m(P) = m(P^*) + \frac{1}{(-2\pi \mathrm{i})^{n-1}} \int_\Gamma \eta_n(n)(x_1, \dots, x_n)$$

where

$$\Gamma = \{P(x_1, \dots, x_n) = 0\} \cap \{|x_1| = \dots = |x_{n-1}| = 1, |x_n| \geq 1\}.$$

Philosophy of Beilinson's conjectures

The important point about $\eta_n(n)$ is that it is a regulator. Regulators appear in the context of something known as Beilinson's conjectures.

The idea is to obtain global information from local information through L-functions. There are many statements (theorems and conjectures) predicting this, including the Dirichlet class number formula, the Birch–Swinnerton-Dyer conjecture, and more generally, Bloch's and Beilinson's conjectures.

Typically, we have a collection of objects,

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F a number field)
- L-function ($\mathrm{L}_F = \zeta_F$)
- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)
- Regulator map $\mathrm{reg} : K \to \mathbb{R}$ ($\mathrm{reg} = \log|\cdot|$)

Then, the conjectures state, for instance, when K has rank 1, that

$$\mathrm{L}'_X(0) \sim_{\mathbb{Q}^*} \mathrm{reg}(\xi)$$

for some non-zero element $\xi \in K$.

For instance, for a number field F, Dirichlet class number formula states that

$$\lim_{s \to 1} (s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} h_F \mathrm{reg}_F}{\omega_F \sqrt{|D_F|}}.$$

If F is real quadratic, the statement can be rephrased as $\zeta'_F(0) \sim_{\mathbb{Q}^*} \log|\epsilon|$, $\epsilon \in \mathcal{O}_F^*$.

An algebraic integration for Mahler measure: two variables

The two-variable case was studied in detail by Rodriguez-Villegas (1997). Let us see the case of Smyth's polynomial. We consider a modified version of it:

$$P(x, y) = y + x - 1 \qquad X = \{P(x, y) = 0\}.$$

Now we compute the Mahler measure

$$m(P) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log|y + x - 1| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}.$$

By Jensen's equality,

$$= \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log^+ |1-x| \frac{\mathrm{d}x}{x},$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \log |y| \frac{\mathrm{d}x}{x} = -\frac{1}{2\pi \mathrm{i}} \int_{\gamma} \eta(x,y),$$

where

$$\gamma = X \cap \{|x| = 1, |y| \geq 1\}$$

and

$$\eta(x,y) = \log|x| \mathrm{d} \mathrm{i} \arg y - \log|y| \mathrm{d} \mathrm{i} \arg x.$$

At this point, let us note some properties of $\eta(x,y)$:

- $\eta(x,y) = -\eta(y,x)$,
- $\eta(x_1 x_2, y) = \eta(x_1, y) + \eta(x_2, y)$.
- $\mathrm{d}\eta(x,y) = \mathrm{i}\,\mathrm{Im}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}\right)$

For us, the crucial result is

Theorem 2

$$\eta(x, 1-x) = \mathrm{d}\mathrm{i}D(x).$$

Here D stands for the Bloch–Wigner dilogarithm:

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|.$$

Where

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad |x| < 1$$

is the classical dilogarithm function.

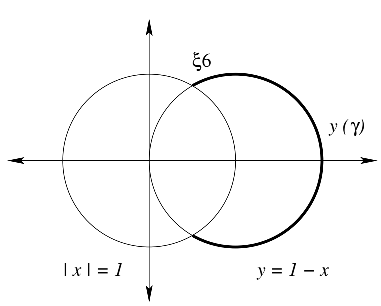
Back to our example, we use Stokes Theorem:

$$m(P) = -\frac{1}{2\pi \mathrm{i}} \int_{\gamma} \eta(x, 1-x) = -\frac{1}{2\pi} D(\partial\gamma).$$

We need to understand the boundary of γ. Writing $x = \mathrm{e}^{2\pi \mathrm{i}\theta}$, we obtain

$$y(\gamma(\theta)) = 1 - \mathrm{e}^{2\pi \mathrm{i}\theta}, \quad \theta \in [1/6\,; 5/6]$$

$$\partial\gamma = [\bar{\xi}_6] - [\xi_6]$$



Then we finally get

$$2\pi m(x+y+1) = D(\xi_6) - D(\bar{\xi_6}) = 2D(\xi_6) = \frac{3\sqrt{3}}{2}\mathrm{L}(\chi_{-3}, 2).$$

In general, we have a polynomial $P(x,y) \in \mathbb{C}[x,y]$ and let $X := \{P(x,y) = 0\}$. We write

$$m(P) = m(P^*) - \frac{1}{2\pi \mathrm{i}} \int_\gamma \eta(x,y).$$

Now in order to get exactness, the condition is

$$x \wedge y = \sum_j r_j \;\; z_j \wedge (1 - z_j)$$

in $\bigwedge^2(\mathbb{C}(X)^*) \otimes \mathbb{Q}$. This can be rephrased as $\{x,y\} = 0$ in $K_2(\mathbb{C}(X)) \otimes \mathbb{Q}$.

Then

$$\int_\gamma \eta(x,y) = \sum r_j D(z_j)|_{\partial\gamma}.$$

The big picture is as follows:

$$\ldots \to (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \to K_2(X, \partial\gamma) \to K_2(X) \to \ldots$$

$$\partial\gamma = X \cap \mathbb{T}^2$$

- If $\{x,y\} = 0$ in $K_2(X)$, then $\eta(x,y)$ is exact and $\{x,y\} \in K_3(\partial\gamma)$. We have $\partial\gamma \neq \emptyset$ and we use Stokes' Theorem.

  $\rightsquigarrow$ dilogarithms, zeta function.

- If $\partial\gamma = \emptyset$, then $\{x,y\} \in K_2(X)$. We have $\eta(x,y)$ is not exact.

  $\rightsquigarrow$ L-series of a curve.

We may get combinations of both situations.

The Mahler measure of $x + \frac{1}{x} + y + \frac{1}{y} - k$ for $k \neq 4$, studied by Boyd [1], Deninger [7], and Rodriguez-Villegas [14] is an example of the non-exact case.

Another application is seen in identities like the one discovered numerically by Boyd [2] and proved by Rodriguez-Villegas [15]

$$7m(y^2 + 2xy + y - x^3 - 2x^2 - x) = 5m(y^2 + 4xy + y - x^3 + x^2).$$

Using regulators in this way it is also possible to prove the following equation due to Rogers

$$m(4n^2) + m\left(\frac{4}{n^2}\right) = 2m\left(2n + \frac{2}{n}\right),$$

where

$$m(k) := m\left(x + \frac{1}{x} + y + \frac{1}{y} - k\right).$$

The idea of how the non-exact case works is as follows. Under certain circumstances (when the tame symbols are trivial), we have $\{x,y\} \in K_2(E)$. Then

$$r(\{x,y\}) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \eta(x,y)$$

where γ is a generator of $H_1(E,\mathbb{Z})^-$ (the subgroup of $H_1(E,\mathbb{Z})$ where complex conjugation acts by -1). The idea is to identify $\mathbb{T}^1$ with γ in the homology. In that way, the right-hand-side is the Mahler measure. On the other hand,

$$r(\{x,y\}) = D^E((x)\diamond(y))$$

if $(x),(y)$ supported on $E_{tors}(\bar{\mathbb{Q}})$. Writing $E(\mathbb{C})\cong \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ we have $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}\cong\mathbb{C}^*/q^{\mathbb{Z}}$ where z mod $\Lambda=\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ is identified with $\mathrm{e}^{\mathrm{i}\pi z}$. Then the elliptic dilogarithm is defined by

$$D^E(x) := \sum_{n\in\mathbb{Z}} D(xq^n)$$

where $q=\mathrm{e}^{\mathrm{i}\pi\tau}$, and D is the Bloch-Wigner dilogarithm.

Finally, the last step is to relate πD^E to $L(E,2)$ and that is HARD.

Properties of $\eta_n(n)(x_1,\ldots,x_n)$

Here are some properties of $\eta_n(n)(x_1,\ldots,x_n)$:

- $\eta_n(n)$ is multiplicative in each variable and anti-symmetric. Hence it can be thought as a function on $\bigwedge^n(\mathbb{C}(X)^*)_{\mathbb{Q}}$.
- $\mathrm{d}\eta_n(n)(x_1,\ldots,x_n) = \widehat{\mathrm{Re}}_n\left(\frac{\mathrm{d}x_1}{x_1}\wedge\ldots\wedge\frac{\mathrm{d}x_n}{x_n}\right)$
- There is an $\mathbb{R}(n-2)$-valued smooth $n-2$-form in $X(\mathbb{C})$ such that

$$\eta_n(n)(x,1-x,x_1,\ldots,x_{n-2}) = \mathrm{d}\eta_{n-1}(n)(x,x_1,\ldots,x_{n-2})$$

In the two-variable case we have

$$\eta_2(2)(x,1-x) = \mathrm{d}\mathrm{i}D(x).$$

The forms for $n=3$ are

$$\eta_3(3)(x,y,z) = \log|x|\left(\frac{1}{3}\mathrm{d}\log|y|\wedge\mathrm{d}\log|z| + \mathrm{d}\mathrm{i}\arg y\wedge\mathrm{d}\mathrm{i}\arg z\right)$$

$$+\log|y|\left(\frac{1}{3}\mathrm{d}\log|z|\wedge\mathrm{d}\log|x| + \mathrm{d}\mathrm{i}\arg z\wedge\mathrm{d}\mathrm{i}\arg x\right)+\log|z|\left(\frac{1}{3}\mathrm{d}\log|x|\wedge\mathrm{d}\log|y| + \mathrm{d}\mathrm{i}\arg x\wedge\mathrm{d}\mathrm{i}\arg y\right),$$

$$\eta_3(3)(x,1-x,y) = \mathrm{d}\eta_3(2)(x,y),$$

$$\eta_3(2)(x,y) = \mathrm{i}D(x)\mathrm{d}\mathrm{i}\arg y + \frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|).$$

Now the first variable of $\eta_n(n-1)$ behaves like the five-term relation.

As before, there is a $\mathbb{R}(n-3)$-valued smooth $n-3$-form in $X(\mathbb{C})$ such that

$$\eta_n(n-1)(x,x,x_1,\ldots,x_{n-3}) = \mathrm{d}\eta_n(n-2)(x,x_1,\ldots,x_{n-3}).$$

The first variable in $\eta_n(n-2)$ behaves like the functional equations of the trilogarithm.

And so on...

Finally, the second to last form satisfies

$$\eta_n(2)(x,x) = \mathrm{d}\eta_n(1)(x),$$

with

$$\eta_n(1)(x) = \widehat{\mathcal{L}_n}(x).$$

Examples in three variables

Here are some examples in three variables that can be also computed using regulators.

- Smyth[17]:

$$\pi^2 m(1 + x + y^{-1} + (1 + x + y)z) = \frac{14}{3}\zeta(3),$$

- Condon [5]:

$$\pi^2 m\left(z - \left(\frac{1-x}{1+x}\right)(1+y)\right) = \frac{28}{5}\zeta(3),$$

- D'Andrea & L. [6]:

$$\pi^2 m\left(z(1-xy)^2 - (1-x)(1-y)\right) = \frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)}.$$

New examples

Using this method we have been able to prove the following examples which were originally computed numerically by Boyd

$$m(x^2 + 1 + (x+1)y + (x-1)z) = \frac{\mathrm{L}(\chi_{-4},2)}{\pi} + \frac{21}{8\pi^2}\zeta(3),$$

$$m(x^2 + x + 1 + (x+1)y + z) = \frac{\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) + \frac{19}{6\pi^2}\zeta(3).$$

An example in four variables

In [12] we computed this example

$$\pi^3 m\left(1 + x + \left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right) = 2\pi^2\mathrm{L}(\chi_{-4},2) + 8\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\frac{(-1)^{j+k+1}}{(2j+1)^3 k}.$$

With this method we have been able to prove that

$$= 24\mathrm{L}(\chi_{-4},4).$$

In particular this implies

$$\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\frac{(-1)^{j+k+1}}{(2j+1)^3 k} = 3\mathrm{L}(\chi_{-4},4) - \frac{\pi^2}{4}\mathrm{L}(\chi_{-4},2)$$

More generally, by using the Hurwitz zeta function we have been able to prove

$$\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\frac{(-1)^{j+k+1}}{(2j+1)^m k} = m\mathrm{L}(\chi_{-4}, m+1) + \sum_{h=1}^{\frac{m-1}{2}} \frac{(-1)^h \pi^{2h}(2^{2h}-1)}{(2h)!} B_{2h}\mathrm{L}(\chi_{-4}, m-2h+1),$$

for m odd.

Exploring the n-variable world

- For

$$z = \left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)$$

Theorem 3 *[13] Both $\eta_{n+1}(n+1)$ and $\eta_{n+1}(n)$ are exact.*

- Now we consider the general mixed sparse resultant. $X := \{\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n} = 0\} \subset \mathbb{C}^k$.

Theorem 4 *[6] $\eta_k(k)$ is exact.*

Generalized Mahler measure

Introduced by Gon & Oyanagi [8]

For $f_1, \dots, f_r \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$,

$$m(f_1, \dots, f_r) = \frac{1}{(2\pi \mathrm{i})^n}\int_{\mathbb{T}^n} \max\{\log|f_1|, \dots, \log|f_r|\}\frac{\mathrm{d}x_1}{x_1}\dots\frac{\mathrm{d}x_n}{x_n}$$

Note that in particular,

$$m(f_1, f_2) = m(f_1 + zf_2).$$

Examples

There is a particular case. Fix $P \in \mathbb{C}[x]$ and set $f_j = P(x_j)$.

Gon & Oyanagi [8] computed the following example

$$m(1-x_1, \dots, 1-x_{2m}) = \frac{(-1)^{m+1}(2m)!}{\pi^{2m}}\zeta(2m+1)$$

$$+(2m)!\sum_{j=1}^{m}(-1)^j\frac{(1-2^{2j})}{(2m-2j)!(2\pi)^{2j}}\zeta(2j+1),$$

$$m(1-x_1, \dots, 1-x_{2m-1}) = (2m-1)!\sum_{j=1}^{m-1}(-1)^j\frac{(1-2^{2j})}{(2m-2j-1)!(2\pi)^{2j}}\zeta(2j+1).$$

Some particular cases are:

$$m(1-x_1, 1-x_2) = m(1-x_1+z(1-x_2)) = \frac{7}{2\pi^2}\zeta(3),$$

$$m(1-x_1,1-x_2,1-x_3)=\frac{9}{2\pi^2}\zeta(3),$$

$$m(1-x_1,1-x_2,1-x_3,1-x_4)=-\frac{93}{2\pi^4}\zeta(5)+\frac{9}{\pi^2}\zeta(3).$$

This example can be also computed using regulators. Using that $|P(x)|$ is monotonous when $0\le \arg x\le\pi$ (in this case, $|P(x)|=2\left|\sin\frac{\arg x}{2}\right|$)

$$m(P(x_1),\dots,P(x_n))=\frac{n!}{(\pi\mathrm{i})^n}\int_{0\le\arg x_n\le\ldots\le\arg x_1\le\pi}\eta(P(x_1),x_1,\dots,x_n)$$

We have been able to also compute this example

$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_{2m}}{1+x_{2m}}\right)=\frac{(-1)^{m+1}(2m)!(2^{2m+1}-1)}{(2\pi)^{2m}}\zeta(2m+1)$$

$$+(2m)!\sum_{j=1}^{m}(-1)^j\frac{(1-2^{2j+1})}{(2m-2j)!(2\pi)^{2j}}\zeta(2j+1),$$

$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_{2m-1}}{1+x_{2m-1}}\right)=(2m-1)!\sum_{j=1}^{m-1}(-1)^j\frac{(1-2^{2j+1})}{(2m-2j-1)!(2\pi)^{2j}}\zeta(2j+1).$$

Some particular cases:

$$m\left(\frac{1-x_1}{1+x_1},\frac{1-x_2}{1+x_2}\right)=m\left(\frac{1-x_1}{1+x_1}+z\left(\frac{1-x_2}{1+x_2}\right)\right)=\frac{7}{\pi^2}\zeta(3),$$

$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_3}{1+x_3}\right)=\frac{21}{2\pi^2}\zeta(3),$$

$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_4}{1+x_4}\right)=-\frac{93}{\pi^4}\zeta(5)+\frac{21}{\pi^2}\zeta(3).$$

Finally, we computed the following

$m(1+x_1-x_1^{-1},\dots,1+x_n-x_n^{-1})=$ combination of polylogarithms.

Some particular cases include

$$m(1+x_1-x_1^{-1})=-\log(\varphi),$$

$$m(1+x_1-x_1^{-1},1+x_2-x_2^{-1})=\frac{1}{\pi^2}\,\mathrm{Re}(\mathrm{Li}_3(\varphi^2)-\mathrm{Li}_3(-\varphi^2)+\mathrm{Li}_3(\varphi^{-2})-\mathrm{Li}_3(-\varphi^{-2}))$$

for $\varphi=\frac{-1+\sqrt{5}}{2}$.

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