To get full credit solve 3 of the following problems (you are welcome to attempt them all)

1. Suppose that the extension $L/\mathbb{Q}$ is finite Galois with simple non cyclic Galois group. Show that there is no rational prime $p$ such that $(p)$ remains prime in $L$.

2. Let $\alpha$ be a root of $f(X) = X^2 + X + 6$.
   a) Show that $\mathbb{Z}[\alpha]$ is the ring of integers of $\mathbb{Q}(\sqrt{-23})$. Find the prime factorization of $(2) = p_1 p_2$.
   b) Show that there are no elements in $\mathbb{Z}[\alpha]$ with norm 2. Conclude that $p_1, p_2$ are not principal.
   c) What is the class number $h_{\mathbb{Q}(\sqrt{-23})}$?

3. Let $L = \mathbb{Q}(\sqrt{-17})$ and $\mathcal{O}_L = \mathbb{Z}[\sqrt{-17}]$.
   a) Let $p_1 = (2, 1 + \sqrt{-17})$, $p_2 = (3, 1 - \sqrt{-17})$, and $p_3 = (3, 1 + \sqrt{-17})$. Show that $(2) = p_1^2$ and $(3) = p_2 p_3$.
   b) Find the prime decomposition for $(18)$ and $(1 \pm \sqrt{-17})$.
   c) Find all the ideals $a \subset \mathcal{O}_L$ with $N(a) = 18$.

4. Let $L$ be a number field. Recall that a prime $p \in \mathbb{Z}$ is totally ramified if $p \mathcal{O}_L = \mathfrak{B}^r$ where $r = [L : \mathbb{Q}]$.
   a) Show that if $p$ is totally ramified in $\mathcal{O}_L$ then it is also totally ramified in $\mathcal{O}_K$ where $K$ is an intermediate field: $\mathbb{Q} \subset K \subset L$.
   b) Show that if $L_1$ and $L_2$ are two extensions of $\mathbb{Q}$ such that there is a prime $p \in \mathbb{Z}$ with $p$ totally ramified in $L_1$ and unramified in $L_2$, then $L_1 \cap L_2 = \mathbb{Q}$.
   c) Use the previous result to show that if $p_1, \ldots, p_r$ are different primes in $\mathbb{Z}$ then $\mathbb{Q}[\sqrt{p_1}, \ldots, \sqrt{p_r}]$ is an extension of $\mathbb{Q}$ of degree $2^r$.

5. Let $A$ a Dedekind domain, $K$ its field of fractions. $K \subset L \subset M$ finite extensions, $B$ and $C$ the integral closures of $A$ in $L$ and $M$ respectively. Let $p \subset b \subset c$ be three prime ideals os $A$, $B$, and $C$ respectively. Prove that $e$ and $f$ are multiplicative, i.e.,
   $$e(c/p) = e(c/b)e(b/p)$$
   $$f(c/p) = f(c/b)f(b/p)$$

6. Let $S$ be a compact, convex, symmetric set in $\mathbb{R}^n$ such that $\mu(S) \geq m2^n$ for some positive integer $m$. Show that $S$ contains at least $2m$ points in $\mathbb{Z}^n$.

7. Show that $\mathbb{Z}[\sqrt{223}]$ has three ideal classes.

8. Show that the ideal class group of $\mathbb{Z}[\sqrt{-14}]$ is cyclic of order 4.