MATH 228 (R1) MIDTERM

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INSTRUCTOR: MATILDE N. LALÍN

NAME:

SID:

1. This exam consists of 8 pages, including this one.
2. No books, notes, or electronic devices are allowed.
3. Ensure that **your *full* name and student number** appear on this page.
4. Read the questions carefully before starting to work.
5. **Justify your answers. Show all your work**
6. Continue on the back of the page if you run out of space.

Question:	1	2	3	4	5	6	7	Total
Points:	15	15	15	15	15	15	15	105
Score:								

1. (15 points) Find all the solutions $x \in \mathbb{Z}_{35}$ of the equation $20x = 15$.

Solution: By factorization we see that $d = (20, 35) = 5$ which divides 15. The equation must have 5 solutions. We find a solution using the Euclidean algorithm

$$35 = 20 \cdot 1 + 15$$

$$20 = 15 \cdot 1 + 5$$

$$15 = 5 \cdot 3$$

Therefore $5 = 20 - 15 = 20 - (35 - 20) = 20 \cdot 2 - 35$. Multiplying by 3, we get $15 = 20 \cdot 6 - 35 \cdot 3$. Therefore, $x = 6$ is a solution. Since $\frac{n}{d} = \frac{35}{5} = 7$, all the solutions are $x = 6 + 7k$, in other words, $x = 6, 13, 20, 27, 34$ in $\mathbb{Z}_{35}$.

2. (15 points) Solve the following system

$$x \equiv 3 \pmod 7$$

$$x \equiv 1 \pmod 9$$

Solution: By applying Euclidean algorithm,

$$9 = 7 \cdot 1 + 2$$

$$7 = 2 \cdot 3 + 1$$

Therefore, $(9, 7) = 1 = 7 - 2 \cdot 3 = 7 - (9 - 7) \cdot 3 = 7 \cdot 4 - 9 \cdot 3$. Thus we take $x = 1 \cdot 7 \cdot 4 - 3 \cdot 9 \cdot 3 = 28 - 81 = -53$. The general solution for the system is $x \equiv 10 \pmod{63}$ and is unique by the Chinese remainder theorem.

3. (15 points) Prove by induction that

$$[7]^n = [6n+1] \quad \text{in } \mathbb{Z}_9.$$

Solution: For $n = 0$, we have that $7^0 = 1 = 6 \cdot 0 + 1$ in $\mathbb{Z}_9$. Assume the statement is true for $n = k$, i.e., that $7^k = 6k + 1$ in $\mathbb{Z}_9$. Let $n = k + 1$, we have $7^{k+1} = 7 \cdot 7^k = 7 \cdot (6k + 1) = 42k + 7 = 6k + 7 = 6(k + 1) + 1$ in $\mathbb{Z}_9$, which completes the induction.

4. Define a relation on the vector space $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a, b) | a, b \in \mathbb{R}\}$ by the following

$$(a, b) \sim (c, d) \Leftrightarrow a + d = b + c$$

   (a) (6 points) Find three vectors that are equivalent to $(2, -3)$

   (b) (9 points) Show that $\sim$ defines an equivalence relation in $\mathbb{R}^2$.

Solution: (a) Notice that the condition for the relation may be written as $a - b = c - d$. In the case of $(2, -3)$ we have $2 - (-3) = 5$, so we need to find other vectors whose coordinates satisfy this difference. For example $(3, -2), (4, -1), (5, 0)$.

(b) It is reflexive: $(a, b) \sim (a, b)$ since $a + b = b + a$. It is symmetric: $(a, b) \sim (c, d)$ iff $a + d = b + c$, iff $c + b = d + a$, iff $(c, d) \sim (a, b)$. It is transitive: $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ iff $a + d = b + c$ and $c + f = d + e$. Summing both expressions, we get $a + d + c + f = b + c + d + e$, which means $a + f = b + e$, which implies $(a, b) \sim (e, f)$. A relation that is reflexive, symmetric, and transitive is an equivalence relationship.

5. (15 points) Show that the subset $S = \{0, 2, 4, 6, 8\}$ of $\mathbb{Z}_{10}$ is a subring. Does S have an identity?

Solution: Notice that S is the set of all elements that may be written as $2k$. Let $2k, 2j \in S$. Then $2k + 2j = 2(k + j)$ so clearly the sum is in S. Also $(2k)(2j) = 4kj = 2(2kj)$ and the product is in S. We also see that 0, the additive identity for $\mathbb{Z}_{10}$, is in S by definition. Finally, if $2k \in S$, then $-2k \in S$. Therefore S is a subring of $\mathbb{Z}_{10}$.

S has an identity, namely 6. We check that $2 \cdot 6 = 12 = 2$, $4 \cdot 6 = 24 = 4$, $6 \cdot 6 = 36 = 6$ and $8 \cdot 6 = 48 = 8$. Since the ring is commutative, we do not have to check any other equality.

6. The following are the addition table and the multiplication table for a ring with three elements.

$+$	r	s	t
r	r	s	t
s	s	t	r
t	t	r	s

$\cdot$	r	s	t
r	r	r	r
s	r	t	s
t	r	s	t

Answer the following questions (with justification):

(a) (3 points) What is the additive identity?

(b) (3 points) What is the additive inverse of s?

(c) (3 points) What is the additive inverse of t?

(d) (6 points) Is this a commutative ring? Does it have a multiplicative identity?

Solution: (a) The additive identity is r since from the additive table we see $r+r = r$, $r + s = s$ and $r + t = t$.

(b) The additive table shows $s + t = r$ and r is the additive identity. Therefore, the additive inverse of s is t.

(c) As before, the additive table shows $t + s = r$, then the additive inverse of t is s.

(d) It is a commutative ring since the multiplicative table is symmetric about the main diagonal, showing that $xy = yx$ for all x, y.

Also we see that $tr = rt = r$, $ts = st = s$ and $t^2 = t$, then t is the multiplicative identity.

7. (15 points) Let $a, b \in \mathbb{Z}$. Prove that $(a, b) | (a + b, a - b)$.

Solution: This is problem 1.2.34a of the suggested problems.

Since $(a, b)|a$ and $(a, b)|b$, then $a = (a, b)a_1$ and $b = (a, b)b_1$. Then $a + b = (a, b)(a_1 + b_1)$ and $a - b = (a, b)(a_1 - b_1)$. Therefore, $(a, b)|a + b$ and $(a, b)|a - b$, which implies that $(a, b)|(a + b, a - b)$ since any common divisor of two numbers is also a divisor of the greatest common divisor of the two numbers.